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HOW INFLATIONARY SPACETIMES MIGHT EVOLVE INTO SPACETIMES OF FINITE TOTAL MASS*

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On the occasion of the first author's seventieth birthday

Abstract. We consider how the finite mass shock wave cosmology introduced by the authors in [6] could connect up with Guth's original theory of inflation.

Key words. shock wave, inflation, cosmology, white/black hole

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1. Introduction. In [Smoller-Temple, *Shock wave cosmology inside a black hole*, Proc. Natl. Acad. Sci., September 2003] the authors constructed an exact shock wave solution of the Einstein equations by matching a $k = 0$ Friedmann-Robertson-Walker (FRW) metric to what we termed a Tolman-Oppenheimer-Volkoff (TOV) metric *inside the black hole* across a subluminal, entropy-satisfying shock wave out beyond one Hubble length in the FRW metric[3]. We needed $\frac{2M}{\bar{r}} > 1$ in order for the solution to contain a sufficiently large region of uniform expansion near the center of the explosion consistent with observations of the expanding universe of galaxies. In this construction we identifed the TOV metric *inside the black hole* as the simplest matter-filled spacetime metric that can cut off the FRW mass at a finite value via shock matching. The matching of these two metrics across a shock wave interface in [6], resulted in a physically plausible scenario in which the Big Bang appears something like a classical explosion of finite mass with a shock wave at the leading edge of the FRW expansion.

However, although there is a shock wave and the total mass is finite, the solution isn't *exactly* like a classical explosion because the spacetime still begins with a "Big Bang" spacetime singularity at $t = 0$. In our exact solution, the density and pressure on both sides of the shock wave, (that separates the FRW from the TOV metrics), tend to the same value, and both tend to the same (correct) equation of state, ($p = c^2/3\,\rho$), in the limit $t \to 0$. This suggests that our exact shock wave solution represents an idealization of a *finite mass, shock wave cosmology* in which the Big Bang begins like the standard Big Bang, (in the sense that immediately after the singularity, the density and pressure are everywhere constant throughout a spacelike hypersurface at fixed time), but the total mass is finite at each time. To be consistent with observations, on the $t = const.$ hypersurfaces, the spacetime should be FRW in a region near the center, but would asymptote out to a TOV metric somewhere out beyond one Hubble

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[3]A TOV metric *inside the black hole* is a metric in standard Schwarzschild coordinates in which the non-angular metric entries depend only on $\bar{r}$, where $\bar{r} = Rr$ measures arclength distance at each fixed time on the FRW side of the shock wave, but is the timelike coordinate on the TOV side when $\frac{2M}{\bar{r}} > 1$. Here R is the cosmolgical scale factor and r is the FRW radial coordinate, [6].

length's distance from the center, because *the total mass is finite*. As a consequence, far out from center, the pressure and density would evolve differently from the pressure and density in the FRW universe that surrounds the center. Our exact solution shows that the pressure and density drop *faster* in the TOV part of the solution than in the FRW part, and so, in such a finite mass cosmology, a gradient in the density and pressure would develop as time evolves, eventually breaking into a strong outgoing wave. We expect that the resulting spacetime would evolve qualitatively like the exact solution constructed in [6], where the wave is modeled exactly by a pure shock wave. Said differently, to get an exact solution that represents such a finite mass cosmology, in [6] we take the finite mass cutoff of the FRW metric at each fixed time to be exactly the constant mass TOV spacetime *inside the black hole*, and we determine the pressure in the TOV solution exactly so that the wave is modeled by a single shock wave interface between the two metrics. The shock wave must lie beyond one Hubble length from the center of the FRW metric in order to make the radial coordinate timelike, which is required for the total mass, (a function of the radial coordinate), to provide a finite mass cutoff, *constant at each fixed time*. Since $2M/r > 1$ beyond one Hubble length in the standard model, it follows as a result that in a finite mass shock wave cosmology, the Big Bang begins inside a *time-reversed black hole*.

In this paper we discuss how such a finite mass shock wave cosmology might connect up with inflation. Our main conclusion is that, due to the high degree of symmetry of the inflationary spacetime, the spacelike slice at fixed time in the universe at the end of inflation could just as well be a spacelike slice of finite total mass. In this case, the spacetime thereafter evolves into a finite mass cosmological model as described above, because the constant density and pressure at the end of inflation would then develop a gradient which would develop into a wave by the same mechanism as in the above scenario without inflation.

In our original exact shock wave solution [6], the solution decays time asymptotically to a classical $k = 0$ Oppenheimer-Snyder solution, a finite ball of mass expanding into empty space *outside the black hole*, something like a gigantic supernova. That is, as a consequence of the entropy condition, the TOV spacetime decays to the empty space Schwarzschild spacetime at the event horizon of the black hole, while the FRW spacetime continues as an expanding matter-filled universe outside the black hole[4]. Here we consider how the finite mass shock wave solution contructed in [6] connects up with the theory of inflation; that is, we consider how a finite mass shock wave cosmology might arise from an inflationary spacetime. We restrict attention to the original inflationary model of Guth, [1]. Although there is a large literature on inflation, Guth's original inflationary model is the simplest, and is the point of departure for later developments.

Inflation is an epoch of explosive increase in scale incorporated into the standard model of cosmology. The interest in inflation rests on the fact that an explosive increase in the scale factor $R(t)$ of the standard FRW metric at a very early time resolves several problems with the standard model of cosmology, the foremost being the flatness problem, the problem of why the universe is so close to flat, [1, 7]. In Guth's original model of inflation, the universe is modeled by an FRW metric before, during and after the inflationary epoch which occurs between time 10^{-35} to 10^{-30} seconds after the Big Bang. During the inflationary epoch the spacetime evolves as an

[4]Actually, the entropy condition selects the white hole explosion over its time reversal, the black hole collapse, but in this paper we loosely use the term black hole to refer to the region where $2M/\bar{r} > 1$.

FRW metric with inflationary source term taking the form of a cosmological constant $T_{ij} = \lambda g_{ij}$, where the energy density $\rho_* = -\lambda > 0$, [7]. A mathematically rigorous discussion of how inflation resolves the flatness problem is given in [7], and exact fomulas for inflationary FRW metrics are derived. The fact that there was a period of rapid expansion early on in the universe is compelling in light of the fact that the microwave background radiation coming from different directions is at the same temperature even though, in the standard model, radiation from opposite directions was some seventy or more horizon distances apart at the time the radiation was emitted, [1]. Indeed, the idea that there was a period of rapid expansion early on might be more compelling than any of the theories for the mechanisms that might have caused this expansion. In this paper we do not discuss any of these mechanisms, but consider the more modest problem of how our shock wave solutions in [6] might naturally emerge from the simplest inflationary FRW spacetime—the inflationary spacetime in Guth's original model. In particular, we assume only that the universe emerged from an inflationary epoch, making no assumption about the spacetime before inflation.

To connect the shock wave solution in [6] with Guth's inflationary spacetime, it is natural to first construct the inflationary FRW and TOV metrics, match them across a radial interface, and then ask how this matched solution might evolve to the correspondingly matched perfect fluid FRW-TOV shock wave solution in [6], at the end of inflation. To understand this, note that the difference between the inflationary regime and the post inflationary regime is that the stress tensor changes from the inflationary form $T_{ij} = -\rho_* g_{ij}$ to that of a perfect fluid $T_{ij} = (\rho + p)u_i u_j + p g_{ij}$. The main difference between these two source terms is that the perfect fluid defines a special coordinate frame, the frame co-moving with the fluid velocity $\mathbf{u}$; but the inflationary source term, being proportional to the metric tensor itself, has all of the Lorentz invariance properties of the vacuum, and thus defines no preferred coordinate frame. For example, in the standard model of cosmology after inflation, the FRW coordinates are special because the perfect fluid is co-moving with respect to the FRW time slices, and the flatness problem is to explain why these spacelike 3-surfaces turn out to be flat–that is, why $k \approx 0$. There are no such preferred frames in the inflationary epoch. Thus, for a solution defined by the matching of an inflationary FRW metric to an inflationary TOV metric to evolve into a corresponding shock wave solution of this form at the end of inflation, the perfect fluid that evolves out of the inflationary regime would have to become co-moving with respect to the spacelike slices of the matched inflationary FRW–TOV metrics at the end of the inflationary epoch. Our original plan was to construct the class of metrics consisting of an inflationary FRW metric, (constructed in [7]), attached on the inside to an inflationary TOV metric, (constructed in Section 2 below), such that there is a smooth matching at the interface, and then to consider how it could be that the perfect fluid which emerges from inflation might become co-moving with respect to the spacelike time slices of this FRW-TOV inflationary spacetime. However, what we found was that the inflationary FRW and TOV metrics can be matched smoothly across any surface. We then discovered that this is because all the inflationary k-FRW metrics, as well as the inflationary TOV metric, are equivalent to the same underlying metric, the Einstein-de Sitter metric, represented in different coordinate systems. That is, all of these metrics represent different time-slicings of the same underlying metric. Thus the matching of a k-inflationary metric to an inflationary TOV metric can be achieved across any interface. Assuming the interface is radial and satisfies $2M/\bar{r} > 1$ as in [6], the matched metric simply defines a different time slicing of the de Sitter spacetime; namely, the constant curvature surfaces of the k-FRW metric inside the interface, and the constant mass surfaces of the TOV metric beyond the interface. This then leads to the question as to what are the spacelike time slices that one would expect the perfect fluid to become co-moving with respect to, at the end of inflation? In this paper our main objective is to point out that there are many natural time slices of the standard inflationary spacetime, and if the perfect fluid which emerges from the inflationary epoch is co-moving with respect to the simplest time slices of finite total mass, (the time slices of the matched inflationary FRW-TOV metrics), then the universe will evolve from inflation as a finite mass explosion similar to the one constructed in [6].

We begin Section 2 by reviewing the exact formulas for the inflationary FRW metrics—that is, metrics of FRW type, assuming an inflationary source term of the form $T_{ij} = -\rho_* g_{ij}$, where the energy density ρ_* is constant. In Section 3 we derive exact formulas for the inflationary TOV metrics. In Section 4 we show that all of the inflationary FRW and TOV metrics are coordinate equivalent to the Einstein-de Sitter metric. We do this by deriving the class of inflationary metrics in standard Schwarzschild coordinates, which we show depend on only two parameters—λ and M_0, (c.f. [1, 4] and [8] page 116). In particular, the two parameter family of inflationary spacetimes in standard Schwarzschild coordinates represents an interpolation between the inflationary FRW metrics and the Schwarzschild metric. We then transform to standard Schwarzschild coordinates to show that all the inflationary FRW and TOV metrics[5] are coordinate equivalent to the case $M_0 = 0$. Then since all of the inflationary FRW metrics represent different time slices of the same inflationary spacetime, it follows that there are constant curvature, homogeneous, isotropic spacelike 3-surfaces of arbitrary curvature passing through every spacetime point of Guth's standard inflationary spacetime, (c.f. Corollary 1 of Theorem 6 below). We use these results to make the point that a further argument is required to explain which of these 3-surfaces the perfect fluid becomes co-moving with respect to, when it emerges at the end of inflation.

In the resolution of the flatness problem, (c.f. [7]), it is implicitly assumed that at the end of inflation the perfect fluid becomes co-moving with respect to $k = 0$ FRW time slices—in a sense, the argument replaces the assumption that $k = 0$ at present time by the assumption that $k = 0$ at the end of inflation. If on the other hand, at the end of inflation, the perfect fluid should become co-moving with respect to different 3-surfaces in different regions of spacetime, then waves would be generated at the interfaces. Thus we argue that if the perfect fluid becomes co-moving with respect to the $k = 0$ FRW spacelike time slices, (as in the standard inflationary scenario), inside some radius $\bar{r} < \bar{r}_0$, but co-moving with respect to the constant time surfaces $\bar{r} = \bar{r}_0$ of the TOV coordinates beyond the FRW metric[6], where $\bar{r}_0$ is large enough so that $2M/\bar{r}_0 > 1$, then our shock wave solution in [6] would approximate the spacetime that would emerge at the end of inflation. Since $2M/\bar{r} > 1$ implies that $\bar{r}$ is the timelike coordinate in a TOV metric, and since the mass function M depends only on $\bar{r}$ in a TOV metric, it follows that the spacelike 3-surface $\bar{r} = \bar{r}_0$ in a TOV metric is equivalent to the 3-surface $M = M_0 \equiv M(\bar{r}_0)$; that is, in TOV metric *inside the black hole*, the 3-surface $\bar{r} = \bar{r}_0$ it is a spacelike 3-surface of constant mass. Such a surface is close to isotropic when the radial variable $\bar{r}_0$ is sufficiently large. It follows

[5]Hawking and Gibbons [4] showed that the inflationary TOV metric has a Kruskal extension, c.f. [2]

[6]Recall that $\bar{r}$ is a spacelike coordinate measuring arclength distance at fixed time in the FRW metric, but is the timelike coordinate of the TOV metric when $2M/\bar{r} > 1$.

that when $2M/\bar{r}_0 > 1$, the time slices of the matched inflationary FRW-TOV metrics can be viewed as the simplest time slices of the de Sitter spacetime on which the mass function M is bounded. This shows that there are natural finite mass time slices of the inflationary de Sitter spacetime, and if the perfect fluid at the end of inflation should become co-moving with respect to these, then a finite mass cosmology similar to the shock wave cosmology given in [6] would emerge from the inflationary epoch. In this sense our shock wave model represents a simple finite mass cosmology which could emerge from an inflationary spacetime, and demonstrates the consistency of the constraint of finite total mass in cosmology by the introduction of a wave.

2. The Inflationary FRW Metric. In this section we summarize the results in [7] regarding the metrics of FRW type that solve the Einstein equations when the stress tensor is the inflationary source term

$$T_{ij} = -\rho_* g_{ij}. \tag{2.1}$$

The stress tensor for a perfect fluid is given by

$$T_{ij} = (\rho c^2 + p)u_i u_j + p g_{ij}, \tag{2.2}$$

so (2.1) is the special case of a perfect fluid when $p = -\rho$. (Note that $T_{ij} = -\rho g_{ij}$ implies that $\rho = const.$ in light of $divT = 0$.) Here ρc^2 is the energy density, p is the pressure, u^i is the $i'th$ component of the 4-velocity of the fluid, g_{ij} is the gravitational metric tensor, and we use the convention that we take the speed of light $c = 1$ and Newton's constant $\mathcal{G}$=1 when convenient. Note that the main difference between the inflationary source and the perfect fluid source is that the inflationary source has all of the Lorentz invariance properties of the metric tensor, and so, like a true vacuum, defines no preferred frame. But the perfect fluid defines the special frame co-moving with the fluid; that is, the frame in which $u = \frac{\partial}{\partial t}$.

The standard FRW metric of cosmology takes the form, [9],

$$ds^2 = -dt^2 + R(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\}, \tag{2.3}$$

where $R(t)$ is the cosmological scale factor, k is the curvature parameter, $\frac{k}{R^2}$ is the spatial curvature at each fixed time t, and $d\Omega^2$ is the standard metric on the unit 2-sphere. The FRW metric is invariant under the scaling

$$\begin{aligned} R &\to \omega R, \\ r &\to \omega^{-1} r \\ k &\to \omega^2 k, \end{aligned} \tag{2.4}$$

for any positive constant ω. Under this scaling both $\bar{r} = rR$ and the scalar curvature

$$\mathcal{K} = k/R^2, \tag{2.5}$$

of a 3-surface $t = const$, are invariant, and we can rescale k to one of the values $\{-1, 0, 1\}$, or we can set $R(t_0) = 1$ at any given time t_0, (but not both unless $k = 0$). Putting (2.3) and (2.2) into the Einstein equations

$$G = \kappa T, \qquad \kappa = \frac{8\pi\mathcal{G}}{c^4}, \tag{2.6}$$

and assuming the fluid is co-moving with the FRW metric, we obtain the FRW equations for R and ρ :

$$H^2 = \frac{\kappa}{3}\rho c^2 - \frac{k}{R^2}, \tag{2.7}$$

$$\dot{\rho} = -3(\rho + p)H, \tag{2.8}$$

where $H \equiv H(t)$ denotes the Hubble constant,

$$H = \frac{\dot{R}}{R}, \tag{2.9}$$

and *overdot* denotes differentiation with respect to FRW time $d/d(ct)$. Equations (2.7), (2.8) close when an equation of state $p = p(\rho)$ is specified. If $p = p(\rho)$ is specified, then a measurment of the density $\rho(t_0)$ and Hubble constant $H(t_0)$ at an initial time $t = t_0$, together with an arbitrary assignment of $R(t_0) = R_0$ determines k through (2.7), and then system (2.7), (2.8) determines a unique evolution for $t \geq t_0$. Note that the curvature $\frac{k}{R^2(t_0)}$ is determined by the density and Hubble constant through (2.7), and these three quantities are invariant under the scaling (2.4)).

In [7] we derived the following exact solution of the FRW equations (2.7), (2.8) for arbitrary values of the curvature parameter $k \in \mathbf{R}$ in the case of inflation, when $T_{ij} = -\rho_* g_{ij}$:

Theorem 1. *The general solution of the FRW equations (2.7), (2.8) when $T_{ij} = -\rho g_{ij}$, $\rho_* > 0$, is given by,*

$$\rho \equiv \rho_* = const. > 0 \tag{2.10}$$

$$R(t) = \frac{k}{4\gamma^2 C}e^{-\gamma ct} + Ce^{\gamma ct}, \tag{2.11}$$

where

$$\gamma = \sqrt{\frac{\kappa\rho_* c^2}{3}}, \tag{2.12}$$

and $\rho_* > 0$ and C are constants of integration.

In the case $k = 0$, (2.11) agrees with the Einstein-de Sitter metric, and Theorem 1 shows that the spacelike slices of the inflationary k-FRW metrics given in (2.11) all decay exponentially to the Einstein-di Sitter metric. A carefull analysis of this in the context of the flatness problem was given in [7].

3. The Inflationary TOV Metric Inside the Black Hole. A TOV metric is a metric of the form

$$ds^2 = -B(\bar{r})d\bar{t}^2 + \frac{1}{A(\bar{r})}d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{3.1}$$

where $A = \frac{1}{1-2M/\bar{r}}$ defines the mass function $M \equiv M(\bar{r})$, c.f. [6]. If we assume that the fluid is co-moving with respect to the metric, (that is, that the fluid is fixed relative to the spacelike coordinates), then the character of solutions of the Einstein equations for (3.1) changes from static to dynamical as the sign of A changes from positive to negative because the timelike vector changes from $\partial/\partial t$ to $\partial/\partial\bar{r}$, respectively, and in this paper we are interested in the case $A < 0$, which is equivalent to $\frac{2M}{\bar{r}} > 1$. Assuming that the stress tensor takes the form (2.2) of a perfect fluid, the Einstein equations for (3.1) in the case $\frac{2M}{\bar{r}} > 1$ are given by,

$$\bar{p}' = \frac{\bar{\rho}+\bar{p}}{2}\frac{N'}{N-1}, \tag{3.2}$$

$$(\bar{r}N)' = -\kappa\bar{p}\bar{r}^2, \tag{3.3}$$

$$\frac{B'}{B} = -\frac{1}{(N-1)}\left\{\frac{N}{\bar{r}} - \kappa\bar{\rho}\bar{r}\right\}. \tag{3.4}$$

Here $N = \frac{2M}{\bar{r}}$, and we use bar to indicate standard Schwarzschild coordinates for the TOV metric, c.f., [6]. The equations close when an equation of state $\bar{p} = \bar{p}(\bar{\rho})$ is given, and we recover the case of a cosmological constant $T_{ij} = -\rho_* g_{ij}$ by setting $\bar{p} = -\bar{\rho}$, in which case (3.2) implies that $\bar{\rho} = \rho_* = const$. We will need the following theorem:

THEOREM 2. *Assume that the equation of state is given by* $\bar{p} = \bar{\sigma}\bar{\rho}$ *for some constant* $\bar{\sigma} \neq 0$. *Then*

$$\bar{p} = \bar{p}_0\left(\frac{N-1}{N_0-1}\right)^{\frac{1+\bar{\sigma}}{2\bar{\sigma}}}, \tag{3.5}$$

where $\bar{p}_0$ *and* N_0 *are values taken at some reference value* $\bar{r}_0$.

Proof. Set $p = \sigma\rho$ and write (3.2) in the form

$$\frac{\bar{p}'}{\bar{p}} = \frac{1+\bar{\sigma}}{2\bar{\sigma}}\frac{(N-1)'}{N-1}.$$

and integrate. □

We consider now the TOV metrics of form (3.1) that solve the Einstein equations when the equation of state is given by $\bar{p} = -\bar{\rho}$. In this case, system (3.2)-(3.4) can be solved exactly. We record the solution in the following theorem:

THEOREM 3. *Assume the conditions for inflation* $\bar{p} = -\bar{\rho}$, $\bar{\rho} > 0$, *and assume that* $A < 0$, $N > 1$. *Then the solution of the TOV equations (3.2), (3.3) and (3.4) is given by,*

$$\bar{p} = -\rho_* = -\bar{\rho}, \tag{3.6}$$

$$N = \frac{N_0}{\bar{r}} + \frac{\kappa}{3}\rho_*\bar{r}^2, \tag{3.7}$$

$$B = B_0\left(N - \frac{N_0}{\bar{r}} - 1\right)\left(\frac{\gamma\bar{r}+1}{\gamma\bar{r}-1}\right)^{\frac{\gamma N_0}{2}} e^{\frac{-N_0}{\bar{r}}}. \tag{3.8}$$

for some constants $\rho_* > 0, B_0 > 0$ and $N_0 \in \mathbf{R}$. In this case, the TOV metric solves the Einstein equations in the regime $N > 1$. In particular, if $N_0 = 0$ then (3.4) has the solution $B = A = N - 1$, and the TOV metric (3.1) reduces to the simple form

$$ds^2 = -\frac{1}{(N-1)}d\bar{r}^2 + (N-1)d\bar{t}^2 + \bar{r}^2 d\Omega^2, \tag{3.9}$$

where $\bar{r}$ is the timelike coordinate because $A < 0$.

We refer to the metric (3.9) as the *inflationary TOV metric.*

Proof. Since $\bar{p} = -\bar{\rho}$ is the case $\sigma = -1$ of Theorem 2 above, it follows from (3.5) that $\bar{p} = -\rho_* = -\bar{\rho}$ is the solution of (3.2), and we restrict here to the case $\rho_* > 0$, the case of a positive energy density. Substituting $\bar{\rho} = \rho_*$ into (3.3) and integrating then gives (3.7). Substituting (3.6) and (3.7) into (3.4) leads to

$$\ln\frac{B}{B_0} = \int_{\bar{r}_0}^{\bar{r}} \frac{\frac{2\kappa\rho_*}{3}\xi}{\frac{\kappa\rho_*}{3}\xi^2 - 1}d\xi - \int_{\bar{r}_0}^{\bar{r}} \frac{-N_0}{\xi^2(\gamma\xi - 1)(\gamma\xi + 1)}d\xi. \tag{3.10}$$

The first integral integrates by the substitution $u = \frac{\kappa\rho_*}{3}\xi^2 - 1$, and the second one integrates by partial fractions to give the result (3.8). When $N_0 = 0$, (3.8) reduces to $B = B_0(N-1)$ where B_0 is the (positive) constant of integration. Finally, rescaling the coordinate $\bar{t}$ we can scale B_0 to unity and obtain (3.9). □

4. Inflationary Spherically Symmetric Spacetimes. To construct a solution analogous to the shock solution in [6] for the inflationary case $T_{ij} = -\rho_* g_{ij}$, the most straightforward construction would be to match the inflationary TOV metric (3.9) to the inflationary FRW metrics (2.11). The next theorem shows that a smooth matching of these metrics for any k can be achieved across any smooth surface because the metrics are all coordinate equivalent to the same metric, the Einstein-de Sitter metric. It follows that there are many inflationary metrics that correspond to our finite mass shock wave solution, and we will discuss this further in the next section. The fact that each inflationary k-FRW metric (2.11), as well as the inflationary TOV metric (3.9), are all coordinate equivalent to the inflationary $k = 0$ Einstein-de Sitter metric is a consequence of the following theorem which characterizes the spherically symmetric inflationary spacetimes in standard Schwarzschild coordinates, (c.f. [8], page 116 and [5]).

THEOREM 4. *Let g be a spherically symmetric metric in standard Schwarzschild coordinates*

$$ds^2 = -A(\bar{t},\bar{r})d\bar{t}^2 + B(\bar{t},\bar{r})d\bar{r}^2 + \bar{r}^2 d\Omega^2. \tag{4.1}$$

Assume g solves the Einstein equations with inflationary source term

$$T_{ij} = -\rho_* g_{ij}.$$

Then under a possible change of time coordinate $\bar{t} \to \phi(\bar{t})$

$$A = -\left(1 - \frac{\kappa}{3}\rho_* \bar{r}^2 - \frac{2M_0}{\bar{r}}\right), \tag{4.2}$$

$$B = \left(1 - \frac{\kappa}{3}\rho_* \bar{r}^2 - \frac{2M_0}{\bar{r}}\right)^{-1}, \tag{4.3}$$

where M_0 is an arbitrary constant.

As a corollary, we obtain the theorem stated in the introduction:

THEOREM 5. *Every inflationary k-FRW metric of the form*

$$ds^2 = -dt^2 + R(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\},$$

including the $k = 0$ de Sitter spacetime, is coordinate equivalent to the inflationary TOV metric, that is, the metric (4.1), (4.2), (4.3) with $M_0 = 0$.

Before giving the proof of Theorem 4, we first show that Theorem 5 is a consequence of Theorem 4.

Proof of Theorem 5. Transforming the k-FRW metric over to standard Schwarzschild coordinates gives: (Set $\bar{r} = Rr$ and use an integrating factor to eliminate the mixed term; c.f. [6]),

$$ds^2 = -\left\{ \frac{1 - kr^2}{1 - kr^2 - H^2 \bar{r}^2} \right\} \frac{d\bar{t}^2}{\psi^2} + \frac{d\bar{r}^2}{1 - kr^2 - H^2 \bar{r}^2}, \tag{4.4}$$

where ψ solves the PDE

$$\frac{\partial}{\partial \bar{r}} \left\{ \psi \frac{1 - kr^2 - H^2 \bar{r}^2}{1 - kr^2} \right\} - \frac{\partial}{\partial \bar{t}} \left\{ \psi \frac{H \bar{r}}{1 - kr^2} \right\} = 0. \tag{4.5}$$

In this case the coordinate $\bar{t}$ in standard Schwarzschild coordinates is given by the differential

$$d\bar{t} = \psi \frac{1 - kr^2 - H^2 \bar{r}^2}{1 - kr^2} dt - \psi \frac{H \bar{r}}{1 - kr^2} d\bar{r}, \tag{4.6}$$

which is exact by (4.5). Using the FRW equation (2.7) leads to

$$ds^2 = -\frac{1}{\psi^2} \left\{ \frac{1 - kr^2}{1 - \frac{\kappa}{3}\rho_* \bar{r}^2} \right\} d\bar{t}^2 + \left\{ \frac{1}{1 - \frac{\kappa}{3}\rho_* \bar{r}^2} \right\} d\bar{r}^2. \tag{4.7}$$

Observe now that the $d\bar{r}^2$ term is independent of k, and using Theorem 4 it must agree with the $d\bar{r}^2$ term in (4.3). This implies that $M_0 = 0$. It follows that the $d\bar{t}^2$ term must agree with the corresponding coefficient in (4.2) under some change of time coordinate. Thus we conclude that all inflationary k-FRW metrics are equivalent to inflationary TOV with $M_0 = 0$, which is equivalent to the Einstein-de Sitter metric. □

Proof of Theorem 4. The Einstein equations for metrics in standard Schwarzschild coordinates are given by, (c.f. [3]),

$$\frac{A}{r^2B}\left\{r\frac{B'}{B}+B-1\right\}=\kappa A^2T^{00} \tag{4.8}$$

$$-\frac{B_t}{rB}=\kappa ABT^{01} \tag{4.9}$$

$$\frac{1}{r^2}\left\{r\frac{A'}{A}-(B-1)\right\}=\kappa B^2T^{11} \tag{4.10}$$

$$-\frac{1}{rAB^2}\{B_{tt}-A''+\Phi\}=\frac{2\kappa r}{B}T^{22} \tag{4.11}$$

where the quantity Φ in the last equation is given by,

$$\Phi=-\frac{BA_tB_t}{2AB}-\frac{B}{2}\left(\frac{B_t}{B}\right)^2-\frac{A'}{r}+\frac{AB'}{rB}+\frac{A}{2}\left(\frac{A'}{A}\right)^2+\frac{A}{2}\frac{A'}{A}\frac{B'}{B},$$

and

$$T^{ij}=-\rho_*g^{ij}.$$

(For notational convenience in the proof, we use unbarred coordinates (t,r) instead of barred coordinates $(\bar{t},\bar{r})$.) Then equation (4.9) gives

$$-\frac{B_t}{rB}=\kappa ABT^{01}=0\Rightarrow\ B\equiv B(r).$$

Equation (4.8) leads to

$$r\frac{B'}{B}=(3ar^2-1)B+1 \tag{4.12}$$

where

$$a=\frac{\kappa\rho_*}{3}.$$

Now the substitution $u=1/B$ in (4.12) leads to the linear equation

$$-r\frac{u'}{u}=(3ar^2-1)\frac{1}{u}+1,$$

or

$$-(ru'+u)=(3ar^2-1),$$

so

$$-(ru)' = 3ar^2 - 1,$$

having the exact solution

$$B = \frac{1}{1 - ar^2 - 2M_0/r}. \tag{4.13}$$

Equation (4.10) gives

$$\frac{1}{r^2}\left\{r\frac{A'}{A} - (B-1)\right\} = -3aB,$$

so that using (4.13) gives

$$r\frac{A'}{A} = \frac{2ar^2 - \frac{2M_0}{r}}{(ar^2-1) + \frac{2M}{r}}.$$

Thus integrating gives

$$\ln|A| = \int \frac{A'}{A} = \int \frac{2ar - \frac{2M_0}{r^2}}{ar^2 - 1 + -\frac{2M_0}{r}} = \int \frac{du}{u},$$

where

$$u = ar^2 - 1 + -\frac{2M_0}{r}.$$

Thus

$$|A| = \phi(t)^2 \left|ar^2 - 1 + 2M_0/r\right|.$$

Since the sign of B is determined in equation (4.3), it follows that A must have the opposite sign. Thus A takes the form (4.2).

Finally, to verify (4.11), note that

$$T^{22} = -\rho_*\frac{1}{r^2} = \frac{-3a}{\kappa}\frac{1}{r^2.} \tag{4.14}$$

Since A and B depend only on r, we have $A_t = 0 = B_t$, and moreover, $AB = 1$. Thus (4.11) reduces to

$$\frac{A''}{rAB^2} - \frac{1}{rAB^2}\Phi = \frac{2\kappa r}{B}\left(-\rho_*\frac{1}{r^2}\right) = -\frac{6a}{Br}, \tag{4.15}$$

so

$$A'' = \Phi - 6a.$$

Also

$$\Phi = -\frac{A'}{r} + \frac{A^2}{r}\left(-\frac{A'}{A^2}\right) + \frac{A}{2}\left(\frac{A'}{A}\right)^2 - \frac{A}{2}\left(\frac{A'}{A}\right)^2. \\ = -2\frac{A'}{r}. \tag{4.16}$$

Thus (4.15) reduces to

$$A'' = -\frac{2A'}{r} - 6a. \tag{4.17}$$

Now using (4.2) we see that (4.17) holds. This completes the proof. □

We use the final theorem, a consequence of Theorem 4, to show that there are homogeneous isotropic 3-surfaces of arbitrary curvature passing through every point of the Einstein-de Sitter spacetime.

THEOREM 6. *For each k and each point P of Einstein-de Sitter spacetime there exists a coordinate system in which the metric takes the form of an inflationary k-FRW metric, (2.3), (2.11), such that P is at the origin of the coordinates, corresponding to $r = 0$, $t = 0$, and $R(0) = 1$.*

Since a spacelike time slice $t = t_0 = const.$ of a k-FRW metric is a homogeneous isotropic space of constant curvature $\frac{k}{R(t_0)^2}$, (c.f. (2.5) and [9]), it follows that if P is the origin of an k-FRW coordinate system with $R(0) = 1$, then the $t = 0$ spacelike time slice passing through P has constant curvature k. In light of this, the following Corollary is a immediate consequence of Theorem 6.

COROLLARY 1. *For every real number $k \in \mathbf{R}$, and every point P of an Einstein-de Sitter spacetime, there exists homogeneous isotropic spacelike 3-surfaces of constant curvature k passing through P.*

Proof of Theorem (6). Consider first an inflationary k-FRW metric g of form (2.3), (2.11). By Theorem (4) we know that there exists a coordinate transformation that transforms g from k-FRW form (2.3) to the inflationary TOV form of the Einstein-de Sitter spacetime. The following lemma gives precise formulas for this transformation:

LEMMA 1. *Let (t, r, ϕ, θ) be the coordinates of an inflationary k-FRW metric (2.3), (2.11). Then the following coordinate transformation $(t, r) \to (\bar{t}, \bar{r})$ maps the k-FRW metric over to the inflationary TOV form, (4.1)-(4.3) with $M_0 = 0$, of the Einstein-de Sitter coordinates:*

$$\bar{r} = Rr \tag{4.18}$$

$$d\bar{t} = \frac{dt}{\sqrt{1-kr^2}} - \sqrt{\frac{\frac{\kappa}{3}\rho_*\bar{r}^2 - kr^2}{1-kr^2}}\,\frac{d\bar{r}}{1-\frac{\kappa}{3}\rho_*\bar{r}^2}. \tag{4.19}$$

In particular, the differential in (4.19) is exact, and therefore defines $\bar{t} = \bar{t}(t, r)$ to within an additive constant.

To verify the lemma, first use (2.7) to show that

$$\psi = \frac{\sqrt{1-kr^2}}{1-kr^2-H^2\bar{r}^2} = \frac{\sqrt{1-kr^2}}{1-\frac{\kappa}{3}\rho_*\bar{r}^2}, \tag{4.20}$$

then use $r = \frac{\bar{r}}{R}$ and a direct calculation to verify that (4.20) solves (4.5) for general k. Using (4.20) in (4.6) gives (4.19). Note that in the case $k = 0$, (4.19) integrates to give

$$\bar{t} = t - \frac{1}{2H} \ln \left|1 - H^2 \bar{r}^2\right| .$$

To prove Theorem 6, it follows from (4.18) and (4.19) that the mapping that takes the inflationary k-FRW to the inflationary TOV form of the Einstein-de Sitter spacetime, maps the center $t = 0, r = 0$ to the center $\bar{t} = 0, \bar{r} = 0$. Thus the inverse mapping of the inflationary TOV form of the Einstein-de Sitter metric over to the $k = 0$ FRW form also takes the center to the center. It follows that the composition of these these two maps takes an inflationary k-FRW form of the Einstein-de Sitter spacetime to the $k = 0$ form of this metric, and the transformation takes the center $t = 0$, $r = 0$ in the k-FRW metric over to the center $t = 0$, $r = 0$ of the $k = 0$ form. Since the center of the $k = 0$ FRW metric clearly can be taken to be any point P of the Einstein-de Sitter spacetime, it follows that the mapping of this metric back to the k-FRW form would also have P at the center. This completes the proof of Theorem 6. □

We see from Theorem 6 that at the end of inflation, there exist homogeneous isotropic spacetimes of arbitrary curvature to which the perfect fluid formed from the energy density ρ_* might naturally become co-moving with respect to at the end of inflation. Thus a further mechanism is required to explain why the perfect fluid would choose a particular flat $k = 0$ time slice at the end of inflation.

5. Conclusion. The implicit assumption in the inflationary scenario is that the inflationary universe eventually evolves into a spacetime modeled by an FRW metric with perfect fluid sources and postive pressure and energy density. Thus at the end of inflation, there must be some mechanism that determines the special coordinate frame co-moving with the perfect fluid. That is, the co-moving frame of the perfect fluid must somehow be determined during the transition from the inflationary epoch to the perfect fluid regime. In the theory of inflation based on the Einstein-de Sitter metric, it is presumed that the fluid after inflation becomes co-moving with respect to the critical $k = 0$ FRW coordinates in order to resolve the flatness problem. But there is another coordinate system of high symmetry available, the inflationary TOV coordinates. If the perfect fluid became co-moving with respect to the TOV coordinates far out where $2M/\bar{r} > 1$, (recall here that $\bar{r}$ measures radial arclength distance at each fixed time in the FRW coordinates, but is the timelike coordinate in the TOV metric), then after inflation the perfect fluid would evolve co-moving with respect to the TOV coordinate system *inside the black hole*, a coordinate system in which the universe has a constant density at each fixed time, as in the FRW coordinates, but in the TOV coordinates, the mass function M is constant at each time $\bar{r}$. Thus it is natural to consider the case when the fluid at the end of inflation becomes co-moving with respect to FRW coordinates inside a ball of radius $\bar{r}$, but co-moving with respect to the TOV coordinates beyond this radius, because this represents the simplest time-slicing of the inflationary de Sitter spacetime for which the total mass in each time slice is finite. We conclude that in this case, we would expect the spacetime after the inflationary epoch, to evolve into a finite mass cosmology with a wave, similar to the shock wave cosmological model we constructed in [6].

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