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On cohomological obstructions for the existence of log-symplectic structures

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We prove that a compact log-symplectic manifold has a class in the second cohomology group whose powers, except maybe for the top, are nontrivial. This result gives cohomological obstructions for the existence of log-symplectic structures similar to those in symplectic geometry.

A Poisson structure π on a smooth manifold M of dimension $2n$ is called a *log-symplectic* structure if the map

$$\wedge^n\pi : M \longrightarrow \bigwedge^{2n} TM, \quad x \mapsto \wedge^n\pi(x)$$

is transverse to the zero section.

These structures were initially studied in the framework of deformation quantization in [4], where they are called *b-symplectic* structures. Later, their complex analogue was considered in [1], where they were first given the name log symplectic. In the context of Poisson geometry, this class of Poisson structures was introduced on two-dimensional surfaces in [5] (under the name of *topologically stable Poisson structures*) where a complete classification was obtained. In higher dimensions, a systematic investigation of the geometric properties of log-symplectic structures appeared in [3]. Their integrations by symplectic groupoids were studied in [2].

Our interest in log-symplectic structures comes from the fact that these can be used to construct regular corank-one Poisson structures. First, the singular locus of a log-symplectic structure $Z := (\wedge^n\pi)^{-1}(0)$, if nonempty, carries a regular corank-one Poisson structure with a very special property: it has a transverse Poisson vector field [3]. Secondly, a log-symplectic structure can be used to construct a regular corank-one Poisson on $M \times S^1$, simply given by

$$\pi + X \wedge \frac{\partial}{\partial\theta},$$

where X is a representative of the modular vector field of (M,π). However, our result excludes the possibility of using this procedure to construct corank-one Poisson structures in some interesting examples, e.g., on $S^4 \times S^1$.

Our result is the following:

Theorem. *Let (M^{2n},π) be a compact log-symplectic manifold. Then there exists a class $c \in H^2(M)$ such that $c^{n-1} \in H^{2n-2}(M)$ is nonzero.*

Proof. Denote by $Z := (\wedge^n \pi)^{-1}(0)$ the singular locus of π. If $Z = \emptyset$, we can apply the usual argument from symplectic geometry. Assume that $Z \neq \emptyset$.

We first assume that M is orientable. Let μ be a volume form on M and denote by $t := \langle \pi^n, \mu\rangle$. The singular locus becomes $Z = \{t = 0\}$. The log-symplectic condition implies that t is a submersion along Z, so we can find a retraction $r : U \to Z$, where U is an open around Z, such that $(r,t) : U \xrightarrow{\sim} Z \times (-\delta,\delta)$ is a diffeomorphism. Since Z is a Poisson submanifold (it is fixed by all Poisson automorphisms, hence all Hamiltonians are tangent to Z), in this open set U we can write

$$\pi|_U = t\partial/\partial t \wedge X_t + w_t$$

for a vector field X_t and a bivector w_t on Z, both depending smoothly on t. Since $(1/t)\pi^n = n\partial/\partial t \wedge X_t \wedge w_t^{n-1}$ is nowhere vanishing, we have that the bivector $\partial/\partial t \wedge X_t + w_t$ is invertible. Denote its inverse by $\alpha_t \wedge dt + \beta_t$, with α_t and β_t forms on Z depending smoothly on $t \in (-\delta,\delta)$. Then $\omega := \pi|_{M\setminus Z}^{-1}$ can be written as

$$\omega|_{U\setminus Z} = \alpha_t \wedge dt/t + \beta_t.$$

Since ω is closed we get that α_0 and β_0 are closed, and since $dt \wedge \alpha_0 + \beta_0$ is invertible, it follows that $\alpha_0 \wedge \beta_0^{n-1}$ is a volume form on Z. Since Z is compact, this implies that β_0^{n-1} cannot be exact. We will construct a closed 2-form ω' on M whose pullback to Z is β_0; hence $c := [\omega']$ will satisfy the conclusion of the theorem.

Let $\chi : (-\delta,\delta) \to \mathbb{R}$ be a bump function that takes the value 1 for $|t| \leq \delta/4$, and 0 for $|t| \geq \delta/2$. Consider the 2-form ω' on $M\setminus Z$ that coincides with ω outside of U and on $U\setminus Z$ it is given by

$$\omega'|_{U\setminus Z} = (\alpha_t - \chi(t)\alpha_0) \wedge dt/t + \beta_t.$$

The form ω' extends smoothly to Z, since for $|t| \leq \delta/4$ it can be written as $\omega' = \lambda_t \wedge dt + \beta_t$, where $\lambda_t = \int_0^1 \dot{\alpha}_{ts} ds$, or equivalently $\alpha_t = \alpha_0 + t\lambda_t$. So ω' is a closed 2-form on M whose pullback to Z is β_0; thus $[\omega']^{n-1} \neq 0$.

If M is not orientable, consider $p : \widetilde{M} \to M$ the orientable double cover, and let $\gamma : \widetilde{M} \xrightarrow{\sim} \widetilde{M}$ be the corresponding deck transformation. We first construct a tubular neighborhood $(\widetilde{r}, t) : \widetilde{U} \xrightarrow{\sim} \widetilde{Z} \times (-\delta, \delta)$ of the singular locus $\widetilde{Z} := p^{-1}(Z)$ of $\widetilde{\pi} := p^*(\pi)$, with $\widetilde{U} = p^{-1}(U)$, and such that the action of γ corresponds to $\gamma(z, t) = (\gamma(z), -t)$, for $(z, t) \in \widetilde{Z} \times (-\delta, \delta)$. The map $\widetilde{r} : \widetilde{U} \to \widetilde{Z}$ can be constructed by lifting a retraction $r : U \to Z$. Consider a volume form μ_0, and denote by f the smooth function satisfying $\gamma^*(\mu_0) = -e^f \mu_0$. Then the volume form $\mu := e^{f/2}\mu_0$ satisfies $\gamma^*(\mu) = -\mu$. Thus, by shrinking U, we can use $t := \langle \widetilde{\pi}^n, \mu \rangle$ to construct the desired tubular neighborhood. As before, on $\widetilde{Z} \times (-\delta, \delta)$ we can write $p^*(\omega|_{U \setminus Z}) = \alpha_t \wedge dt/t + \beta_t$. Invariance under γ implies that $(\gamma|_{\widetilde{Z}})^*(\alpha_t) = \alpha_{-t}$ and $(\gamma|_{\widetilde{Z}})^*(\beta_t) = \beta_{-t}$. In particular α_0 and β_0 are invariant. Thus, choosing the function $\chi(t)$ from the construction from the orientable case to satisfy $\chi(t) = \chi(-t)$, we obtain an invariant closed 2-form ω' on $\widetilde{M}$ that satisfies $[\omega']^{n-1} \neq 0$. Invariance implies that $\omega' = p^*(\omega'')$ for a closed 2-form ω'' on M; hence $c := [\omega'']$ satisfies the conclusion. □

Remark. Observe that for $Z \neq \emptyset$ the proof of the theorem uses only the compactness of Z and not that of M.

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