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L_∞-ALGEBRAS AND HIGHER ANALOGUES OF DIRAC STRUCTURES AND COURANT ALGEBROIDS

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We define a higher analogue of Dirac structures on a manifold M. Under a regularity assumption, higher Dirac structures can be described by a foliation and a (not necessarily closed, non-unique) differential form on M, and are equivalent to (and simpler to handle than) the multi-Dirac structures recently introduced in the context of field theory by Vankerschaver *et al.*

We associate an L_∞-algebra of observables to every higher Dirac structure, extending work of Baez *et al.* on multisymplectic forms. Further, applying a recent result of Getzler, we associate an L_∞-algebra to any manifold endowed with a closed differential form H, via a higher analogue of split Courant algebroid twisted by H. Finally, we study the relations between the L_∞-algebras appearing above.

Contents

1. Introduction

In the Hamiltonian formalism, many classical mechanical systems are described by a manifold, which plays the role of phase space, endowed with a symplectic structure and a choice of Hamiltonian function. However, symplectic structures are not suitable to describe all classical systems. Mechanical systems with symmetries are described by Poisson structures — integrable bivector fields — and system with constraints are described by closed 2-forms. Systems with both symmetries and constraints are described using Dirac structures, introduced by Ted Courant in the early 1990s [**1**]. Recall that, given a manifold M, $TM \oplus T^*M$ is endowed a natural pairing on the fibers and a bracket on its space of sections, called (untwisted) Courant bracket. A Dirac structure is a maximal isotropic and involutive subbundle of $TM \oplus T^*M$.

Given a Dirac manifold M, one defines the notion of Hamiltonian function — in physical terms, an observable for the system — and shows that the set of Hamiltonian functions is endowed with a Poisson algebra structure.

Higher analogues of symplectic structures are given by multisymplectic structures [**2, 3**] (called p-plectic structures in [**4**]), i.e., closed forms $\omega \in \Omega^{p+1}(M)$ such that the bundle map $\tilde{\omega}\colon TM \to \wedge^p T^*M, X \to \iota_X\omega$ is injective. They are suitable to describe certain physical systems arising from classical field theory, as was realized by Tulczyjew in the late 1960s. They are also suitable to describe systems in which particles are replaced by higher dimensional objects such as strings [**4**].

The recent work of Baez *et al.* [**4**] and then Rogers [**5**] shows that on a p-plectic manifold M the observables — consisting of certain differential forms — have naturally the structure of a Lie p-algebra, by which we mean an L_∞-algebra [**6**] concentrated in degrees $-p+1,\dots,0$. This extends the fact, mentioned above, that the observables of classical mechanics form a Lie algebra (indeed, a Poisson algebra).

The first part of the present paper arose from the geometric observation that, exactly as symplectic structures are special cases of Dirac structures, multisymplectic structures are special cases of higher analogues of Dirac structures. More precisely, for every $p \geq 1$ we consider

$$E^p := TM \oplus \wedge^p T^*M,$$

a vector bundle endowed with a $\wedge^{p-1}T^*M$-valued pairing and a bracket on its space of sections. We regard E^p as a higher analogue of split Courant algebroids. We also consider isotropic, involutive subbundles of E^p. When the latter are Lagrangian, we refer to them as higher Dirac structures.

The following diagram displays the relations between the geometric structures mentioned so far:

In the first part of the paper (Sections 2–4) we introduce and study the geometry of isotropic, involutive subbundles of E^p. Examples include Dirac structures, closed forms together with a foliation, and a restrictive class of multivector fields. The main results are

- Theorem 3.12: a description of all regular higher Dirac structures in terms of familiar geometric data: a (not necessarily closed) differential form and a foliation.
- Theorem 4.5: higher Dirac structures are *equivalent* to multi-Dirac structures, at least in the regular case[1].

Recall that multi-Dirac structures were recently introduced by Vankerschaver *et al.* [**7**]. They are the geometric structures that allow to describe the implicit Euler–Lagrange equations (equations of motion) of a large class of field theories, which include the treatment of non-holonomic constraints. By the above equivalence, higher Dirac structures thus acquire a field-theoretic motivation. Further, since higher Dirac structures are simpler to handle than multi-Dirac structures (which contain some redundancy in their definition), we expect our work to be useful in the context of field theory too.

The second part of the paper is concerned with the algebraic structure on the observables, which turns out to be an L_∞-algebra. Further, we investigate an L_∞-algebra that can be associated to a manifold without any geometric structure on it, except for a (possibly vanishing) closed differential form defining a twist. Recall that a closed 2-form on a manifold M (a 2-cocycle for the Lie algebroid TM) can be used to obtain a Lie algebroid structure on $E^0 = TM \times \mathbb{R}$ [**8,** Section 1.1], so the sections of the latter form a Lie algebra. Recall also that Roytenberg and Weinstein [**9**] associated a Lie 2-algebra to every Courant algebroid (in particular to $E^1 = TM \oplus T^*M$ with Courant bracket twisted by a closed 3-form). Recently, Getzler [**10**] gave an algebraic construction which extends Roytenberg and Weinstein's proof. Applying Getzler's result in a straightforward way one can extend the above results to all E^p's.

Our main results in the second part of the paper (Section 5–9) are:

- Theorem 6.7: the observables associated to an isotropic, involutive subbundle of E^p form a Lie p-algebra.

[1]Regularity is a technical assumption and is probably not necessary. The physically most relevant examples of multi-Dirac structures are regular [**7**].

- Propositions 8.1 and 8.4: to $E^p = TM \oplus \wedge^p T^*M$ and to a closed $p+2$ form H on M, one can associate a Lie $p+1$-algebra extending the H-twisted Courant bracket.
- Theorem 7.1: there is a morphism (with one-dimensional kernel) from the Lie algebra associated to E^0 and a closed 2-form into the Lie 2-algebra associated to the Courant algebroid $E^1 = TM \oplus T^*M$ with the untwisted Courant bracket.

Rogers [**11**] observed that there is an injective morphism — which can be interpreted as a prequantization map — from the Lie 2-algebra of observables on a 2-pletic manifold (M, ω) into the Lie 2-algebra associated to the Courant algebroid $E^1 = TM \oplus T^*M$ endowed with the ω-twisted Courant bracket. We conclude the paper with an attempt to put this into context.

2. Higher analogues of split Courant algebroids

Let M be a manifold and $p \geq 0$ an integer. Consider the vector bundle

$$E^p := TM \oplus \wedge^p T^*M,$$

endowed with the symmetric pairing on its fibres

$$\langle \cdot, \cdot \rangle \colon E^p \times E^p \to \wedge^{p-1} T^*M,$$

given by

$$\langle X + \alpha, Y + \beta \rangle = \iota_X \beta + \iota_Y \alpha. \tag{2.1}$$

Endow the space of sections of E^p with the *Dorfman bracket*

$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha. \tag{2.2}$$

The Dorfman bracket satisfies the Jacobi identity and Leibniz rules

$$[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!], \tag{2.3}$$

$$[\![e_1, f e_2]\!] = f [\![e_1, e_2]\!] + (pr_{TM}(e_1) f) e_2, \tag{2.4}$$

$$[\![f e_1, e_2]\!] = f [\![e_1, e_2]\!] - (pr_{TM}(e_2) f) e_1 + df \wedge \langle e_1, e_2 \rangle, \tag{2.5}$$

where $e_i \in \Gamma(E^p)$, $f \in C^\infty(M)$ and $pr_{TM} \colon E^p \to TM$ is the projection onto the first factor.

The decomposition of the Dorfman bracket into its anti-symmetric and symmetric parts is

$$[\![e_1, e_2]\!] = [\![e_1, e_2]\!]_{Cou} + \frac{1}{2} d \langle e_1, e_2 \rangle, \tag{2.6}$$

where

$$[\![X + \alpha, Y + \beta]\!]_{Cou} := [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha)$$

is known as *Courant bracket*.

Remark 2.1. The Dorfman bracket on E^p was already considered by Hawigara [**12,** Section 3.2], Hitchin [**13**] and Gualtieri [**14,** Section 3.8] [**15,** Section 2.1]. $(E^p, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!])$ is an example of *weak Courant–Dorfman algebra* as introduced by Ekstrand and Zabzine in [**16,** Appendix]. When $p = 1$, we recover an instance of split *Courant algebroid* [**17**]. The Courant bracket has been extended to the setting of multivector fields in [**7,** Section 4].

In [**13–15**], it is remarked that closed $p+1$-forms B on M provide symmetries of the Dorfman bracket (and of the pairing), by the gauge transformation $e^B\colon X+\alpha \mapsto X+\alpha+\iota_X B$. Further, the Dorfman bracket may be twisted by a closed $p+2$-form H, just by adding a term $\iota_Y\iota_X H$ to the r.h.s. of equation (2.2). We refer to the resulting bracket as *H-twisted Dorfman bracket* (this notion will not be used until Section 7), and we use the term Dorfman bracket to refer to the untwisted one given by equation (2.2).

3. Higher analogues of Dirac structures

In this section, we introduce a geometric structure that extends the notion of Dirac structure and multisymplectic form. It is given by a subbundle of E^p, which we require to be involutive and isotropic, since this is needed to associate to it an L_∞-algebra of observables in Section 6. Further, we consider subbundles which are Lagrangian (i.e., maximal isotropic) and study their geometry in detail.

Definition 3.1. Let $p \geq 1$. Let L be a subbundle of $E^p = TM \oplus \wedge^p T^*M$.

- L is *isotropic* if for all sections $X_i + \alpha_i$:

$$\langle X_1+\alpha_1, X_2+\alpha_2\rangle = 0. \tag{3.1}$$

  L is *involutive* if for all sections $X_i + \alpha_i$:

$$[\![X_1+\alpha_1, X_2+\alpha_2]\!] \in \Gamma(L),$$

  where $[\![\cdot,\cdot]\!]$ denotes the Dorfman bracket (2.2).
- L is *Lagrangian* if

$$L = L^\perp := \{e \in E^p : \langle e, L\rangle = 0\}.$$

  (In this case, we also refer to L as a *almost Dirac structure of order* p.) L a *Dirac structure of order p* or *higher Dirac structure* if it is Lagrangian and involutive.
- L is *regular* if $pr_{TM}(L)$ has constant rank along M.

3.1. Involutive isotropic subbundles. In this subsection, we make some simple considerations on involutive isotropic subbundles and present some examples.

The involutive, Lagrangian subbundles of E^1 are the *Dirac structures* introduced by Courant [**1**].

When $p = \dim(M)$, isotropic subbundles are forced to lie inside $TM \oplus \{0\}$ or $\{0\} \oplus \wedge^p T^*M$, hence they are uninteresting.

Now, for arbitrary p, we look at involutive, isotropic subbundles that project isomorphically onto the first or second summand of E^p.

Proposition 3.2. *Let $p \geq 1$. Let ω be a closed $p+1$-form on M. Then*

$$graph(\omega) := \{X - \iota_X\omega : X \in TM\}$$

is an isotropic involutive subbundle of E^p. All isotropic involutive subbundles $L \subset E^p$ that project isomorphically onto TM under $pr_{TM} \colon E^p \to TM$ are of the above form.

Proof. The subbundle $graph(\omega)$ is isotropic because $\langle X - \iota_X\omega, Y - \iota_Y\omega\rangle = -\iota_X\iota_Y\omega - \iota_Y\iota_X\omega = 0$. To see that L is involutive, use the fact that since ω is closed $d(\iota_X\omega) = \mathcal{L}_X\omega$ and compute

$$[\![X - \iota_X\omega, Y - \iota_Y\omega]\!] = [X,Y] - \mathcal{L}_X(\iota_Y\omega) + \iota_Y(\mathcal{L}_X\omega) = [X,Y] - \iota_{[X,Y]}\omega.$$

Let $L \subset E^p$ be a subbundle that projects isomorphically onto TM, i.e. $L = \{X + B(X) : X \in TM\}$ for some $B \colon TM \to \wedge^p T^*M$. If L is isotropic then the map

$$TM \otimes TM \to \wedge^{p-1}T^*M, \quad X \otimes Y \mapsto \iota_X(B(Y))$$

is skew in X and Y, so $B(X) = -\iota_X\omega$ defines a unique $p+1$-form ω, which satisfies $graph(\omega) = L$. If L is involutive then the above computation shows that ω is a closed form. $\square$

The following generalization of Proposition 3.2 is proven exactly as in the last paragraph of the proof of Proposition 3.12. It provides a wide class of regular isotropic, involutive subbundles.

Corollary 3.3. *Fix $p \geq 1$. Let $\omega \in \Omega^{p+1}(M)$ be a $p+1$-form and S an integrable distribution on M, such that $d\omega|_{\wedge^3 S \otimes \wedge^{p-1}TM} = 0$. Then*

$$L := \{X - \iota_X\omega + \alpha : X \in S, \alpha \in \wedge^p S^\circ\}$$

is an isotropic, involutive subbundle of E^p.

Proposition 3.4. *Let $1 \leq p \leq \dim(M) - 1$. Let $\pi \in \Gamma(\wedge^{p+1}TM)$ be either a Poisson bivector field, a $\dim(M)$-multivector field or $\pi = 0$. Then*

$$graph(\pi) := \{\iota_\alpha\pi + \alpha : \alpha \in \wedge^p T^*M\}$$

is an isotropic involutive subbundle of E^p.

All isotropic involutive subbundles $L \subset E^p$ that project isomorphically onto $\wedge^p T^*M$ under $pr_{\wedge^p T^*M} \colon E^p \to \wedge^p T^*M$ are of the above form.

Proof. We write $n := p+1$, so π is an n-vector field. Clearly, $graph(\pi)$ is isotropic in the cases $\pi = 0$ and $n = 2$. For the case $n = \dim(M)$ fix a point $x \in M$. We may assume that at x we have $\pi = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$ where $\{x_i\}_{i \leq n}$

is a coordinate system on M. For each i denote $dx_i^C := dx_1 \wedge \ldots \widehat{dx_i} \cdots \wedge dx_n$. For $i \leq j$ at the point x we have

(3.2)
$$\begin{aligned}&\langle \iota_{dx_i^C}\pi + dx_i^C, \iota_{dx_j^C}\pi + dx_j^C\rangle \\ &\quad = ((-1)^{(n-i)+(i-1)} + (-1)^{(n-j)+(j-2)})dx_1 \wedge \ldots \widehat{dx_i} \ldots \widehat{dx_j} \cdots \wedge dx_n = 0,\end{aligned}$$

showing that $graph(\pi)$ is isotropic.

It is known that $graph(\pi)$ is involutive iff π is a Nambu–Poisson multivector field (see [**12,** Section 4.2]). For $n = 2$ the Nambu–Poisson multivector fields are exactly Poisson bivector field, and for $n = \dim(M)$ all n-multivector fields are Nambu–Poisson. This concludes the first part of the proof.

Conversely, assume that $L \subset E^{n-1}$ is an isotropic subbundle that projects isomorphically onto $\wedge^{n-1}T^*M$, i.e., $L = \{A\alpha + \alpha : \alpha \in \wedge^{n-1}T^*M\}$ for some map $A\colon \wedge^{n-1} T^*M \to TM$.

Assume that A is not identically zero, and that $n \neq 2, \dim(M)$. In this case, we obtain a contradiction to the isotropicity of L, as follows. There is a point $x \in M$ with $A_x \neq 0$. Near x choose coordinates $x_1, \ldots, x_{\dim(M)}$ (note that $\dim(M) \geq n+1$). Without loss of generality at x we might assume that $A(dx_1 \wedge \cdots \wedge dx_{n-1})$ does not vanish. It does not lie in the span of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}$ since we assume that L is isotropic, so by modifying the coordinates $x_n, \ldots, x_{\dim(M)}$ we may assume that $A(dx_1 \wedge \cdots \wedge dx_{n-1}) = \frac{\partial}{\partial x_n}$. Then

$$\begin{aligned}&\langle A_x(dx_1 \wedge \cdots \wedge dx_{n-1}) + dx_1 \wedge \cdots \wedge dx_{n-1},\ A_x(dx_3 \wedge \cdots \wedge dx_{n+1}) \\ &\quad + dx_3 \wedge \cdots \wedge dx_{n+1}\rangle \neq 0.\end{aligned}$$

Indeed, the contraction of $A_x(dx_1 \wedge \cdots \wedge dx_{n-1}) = \frac{\partial}{\partial x_n}$ with $dx_3 \wedge \cdots \wedge dx_{n+1}$ contains the summand $(-1)^{n-3} \cdot dx_3 \wedge \cdots \wedge dx_{n-1} \wedge dx_{n+1}$, whereas the contraction of any vector of T_xM with $dx_1 \wedge \cdots \wedge dx_{n-1}$ cannot contain dx_{n+1}. Hence, we obtain a contradiction to the isotropicity.

If $A \equiv 0$, then clearly L is isotropic. In the case $n = 2$, it is known that L is isotropic iff it is the graph of a bivector field π. Now consider the case $n = \dim(M)$. For any i, let $X_i + dx_i^C \in L$. The isotropicity condition implies that $X_i = \lambda_i \frac{\partial}{\partial x_i}$ for some $\lambda_i \in \mathbb{R}$, and a computation similar to (3.2) implies $\lambda_i = (-1)^{n-i}\lambda_n$ for all i, so that $L = \mathrm{graph}(\pi)$ for $\pi = \lambda_n \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$.

Hence, we have shown that L is isotropic iff L is the graph of an n-vector field where $\pi = 0$, $n = 2$ or $n = \dim(M)$. As seen earlier, if $\mathrm{graph}(\pi)$ is involutive then, in the case $n = 2$, π has to be a Poisson bivector field. $\square$

We present a class of isotropic involutive subbundles, that are not necessarily regular:

Corollary 3.5. *Let Ω be an top degree form on M, and $f \in C^\infty(M)$ such that $\Omega_x \neq 0$ at points of $\{x \in M : f(x) = 0\}$. Then*

$$L := \{fX - \iota_X \Omega : X \in TM\}$$

is an involutive isotropic subbundle of $E^{\dim(M)-1}$.

Proof. Let $x \in M$. If $f(x) \neq 0$, then nearby L is the graph of $\frac{1}{f}\Omega$, which being a top-form is closed. Hence, near x, L defines an isotropic involutive subbundle by Proposition 3.2. Now suppose that $f(x) = 0$. Then L_x is just $0 + \wedge^{\dim(M)-1} T_x^* M$, so nearby L is the graph of a top multivector field, and by Proposition 3.4 it is an isotropic involutive subbundle. □

Note that the isotropic subbundles described in Propositions 3.2, 3.4, Corollary 3.5 are all Lagrangian (use Lemma A.1 below).

We end this subsection relating involutive isotropic subbundles with Lie algebroids and Lie groupoids.

Proposition 3.6. *Let $L \subset E^p$ be an involutive isotropic subbundle. Then $(L, [\![\cdot,\cdot]\!], pr_{TM})$ is a Lie algebroid [**18**], where $pr_{TM} \colon E^p \to TM$ is the projection onto the first factor.*

Proof. The restriction of the Dorfman bracket to $\Gamma(L)$ is skew-symmetric because of equation (2.6), and as seen in equation (2.3) the Dorfman bracket satisfies the Jacobi identity. The Leibniz rule holds because of equation (2.4). □

Recall that (integrable) Dirac structures give rise to presymplectic groupoids in the sense of [**19**] and, restricting to the non-degenerate case, that Poisson structures give rise to symplectic groupoids. We generalize this:

Proposition 3.7. *Suppose that the Lie algebroid L of Proposition 3.6 integrates to a source simply connected Lie groupoid Γ. Then Γ is canonically endowed with a multiplicative closed $p+1$-form Ω.*

Further, if L is the graph of a multivector field as in Proposition 3.4 or the graph of a multisymplectic form (see Section 1), then Ω is a multisymplectic form.

Proof. The first statement follows immediately from recent results of Arias Abad–Crainic, applying [**20,** Thm. 6.1] to the vector bundle map $\tau \colon L \to \wedge^p T^* M$ given by the projection onto the second factor, which satisfies the assumptions of the theorem since L isotropic and because the Lie algebroid bracket on L is the restriction of the Dorfman bracket. Concretely, for all $x \in M$ and $e \in L_x$, $X_1, \ldots, X_p \in T_x M$, the multiplicative form Ω is determined by the equation

$$\Omega(e, X_1, \ldots, X_p) = \langle pr_{\wedge^p T^* M}(e), X_1 \wedge \cdots \wedge X_p \rangle. \tag{3.3}$$

Here, on the lhs, we identify the Lie algebroid L with $ker(s_*)|_M$, where $s\colon \Gamma \to M$ is the source map.

Now, assume that L is the graph of a multivector field π as in Proposition 3.3. First, given a non-zero $e \in L$, it follows that $pr_{\wedge^p T^*M}(e)$ is also non-zero, so it pairs non-trivially with some $X_1 \wedge \cdots \wedge X_p \in \wedge^p TM$. Second, given a non-zero $X_1 \in TM$, extend it to a non-zero element $X_1 \wedge \cdots \wedge X_p \in \wedge^p TM$, and choose $\alpha \in \wedge^p T^*M$ so that their pairing is non-trivial. Let $e := \iota_\alpha \pi + \alpha$. Then the expression (3.3) is non-zero. Since $T\Gamma|_M = TM \oplus ker(s_*)|_M$ and $\Omega|_{\wedge^{p+1}TM} = 0$, this shows that Ω is multisymplectic at points of M. To make the same conclusion at every $g \in \Gamma$, use [**19,** equation (3.4)] that the multiplicativity of Ω implies

$$\Omega_g((R_g)_* e, w_1, \dots, w_p) = \Omega_x(e, t_*(w_1), \dots, t_*(w_p))$$

for all $e \in ker(s_*)|_x$ and $w_i \in T_g\Gamma$. Here, $t\colon \Gamma \to M$ is the target map and $x := t(g) \in M$.

Last, assume that L is the graph of a multisymplectic form ω on M. Given a non-zero $e \in L$, say $e = X - \iota_X\omega$, we have by equation (3.3) that $\iota_e\Omega|_{\wedge^p TM} = -\iota_X\omega \neq 0$. Given a non-zero $X_1 \in TM$, there is $X \wedge X_2 \wedge \cdots \wedge X_p \in \wedge^p TM$ with which $\iota_{X_1}\omega$ pairs non-trivially. Let $e := X - \iota_X\omega$. Then the expression (3.3) is non-zero. This shows that Ω is multisymplectic at points of M, and by the argument above on the whole of Γ. $\square$

3.2. Higher Dirac structures. In this subsection, we characterize Lagrangian subbundles $L \subset E^p$ (i.e., almost Dirac structures of order p) and their involutivity.

We start characterizing Lagrangian subbundles at the linear algebra level. Recall first what happens in the case $p = 1$. Let T be a vector space. Any $L \subset T \oplus T^*$ such that $L = L^\perp$ is determined exactly by the subspace $S := pr_T(L)$ and a skew-symmetric bilinear form on it [**21**]. Further $\dim(S)$ can assume any value between 0 and $\dim(T)$. For $p \geq 2$ the description is more involved, however, it remains true that every Lagrangian subspace of $T \oplus \wedge^p T^*$ can be described by means of a subspace $S \subset T$ (satisfying a dimensional constraint) and a (non-unique) $p+1$-form on T.

Proposition 3.8. *Fix a vector space T and an integer $p \geq 1$. There is a bijection between*

- *Lagrangian subspaces $L \subset T \oplus \wedge^p T^*$*
- *pairs*

$$\begin{cases} S \subset T & \text{such that either } dim(S) \leq (dim(T) - p) \text{ or } S = T, \\ \Omega \in \wedge^2 S^* \otimes \wedge^{p-1} T^* & \text{such that } \Omega \text{ is the restriction of an element of} \\ & \wedge^{p+1} T^*. \end{cases}$$

The correspondence is given by

$$L \mapsto \begin{cases} S := pr_T(L) \\ \Omega \textit{ given by } \iota_X\Omega = \alpha|_{S\otimes\bigotimes^{p-1}T} \textit{ for all } X+\alpha \in L \end{cases}$$
$$(S,\Omega) \mapsto L := \{X+\alpha : X \in S, \alpha|_{S\otimes\bigotimes^{p-1}T} = \iota_X\Omega\}.$$

Here, we regard $\wedge^n T^*$ as the subspace of $\bigotimes^n T^* := T^*\otimes\cdots\otimes T^*$ consisting of elements invariant under the odd representation of the permutation group in n elements. Loosely speaking, the restriction on $\dim(S)$ arises as follows: when it is not satisfied $\wedge^p S^\circ = \{0\}$ and $S \neq T$, and one can enlarge L to an isotropic $L' \subset T \oplus \wedge^p T^*$ such that $pr_T(L')$ is strictly larger than S. The proof of Proposition 3.8 is presented in Appendix A.

An immediate corollary of Proposition 3.8, which we present without proof, is:

Corollary 3.9. *Fix a vector space T and an integer $p \geq 1$. For any Lagrangian subspace $L \subset T \oplus \wedge^p T^*$ let (S,Ω) be the corresponding pair as in Proposition* 3.8*, and $\omega \in \wedge^{p+1}T^*$ an arbitrary extension of Ω. Then L can be described in terms of S and ω as*

$$L = \{X + \iota_X\omega + \alpha : X \in S, \alpha \in \wedge^p S^\circ\}.$$

As an immediate consequence of Lemma A.1, we obtain the following dimensional constraints on the singular distribution induced by a Lagrangian subbundle:

Corollary 3.10. *Let $L \subset E^p$ be a Lagrangian subbundle. Denote $S := pr_{TM}(L)$. Then*

(a) *$dim(S_x) \in \{0,1,\dots,dim(M)-p, dim(M)\}$ for all $x \in M$*
(b) *$dim(L_x) = dim(S_x) + \binom{dim(M)-dim(S_x)}{p}$ is constant for all $x \in M$.*

When $p = 1$, so that L is a maximal isotropic subbundle of $TM \oplus T^*M$, the dimensional constraints of Corollary 3.10 do not pose any restriction of $\dim(S_x)$. (It is known, however, that $\dim(S_x)$ *mod* 2 must be constant on M.) When $p \geq 2$, Lagrangian subbundles of E^p are quite rigid.

Example 3.11. Let $p = \dim(M) - 1$, and let L be a Lagrangian subbundle of E^p. Corollary 3.10 a) implies that at every point $\dim(S_x)$ is either 0, 1 or $\dim(M)$. Assume that $p \geq 2$. By Corollary 3.10 b), if $rk(S) = 1$ at one point then $rk(S) = 1$ on the whole of M, and the rank 2 bundle L is equal to $S \oplus \wedge^{\dim(M)-1}S^\circ$. Otherwise, at any point x we have either $S_x = T_xM$ or $L_x = 0 + \wedge^{\dim(M)-1}T^*M$. In the first case by Corollary 3.9 we known that, near x, L is the graph of a top form. In the second case L projects isomorphically onto the second component $\wedge^{\dim(M)-1}T^*M$ near x, so by Proposition 3.4 it must be the graph of a $\dim(M)$-vector field.

Finally, we characterize when a regular Lagrangian subbundle is a higher Dirac structure.

Theorem 3.12. *Let M be a manifold, fix an integer $p \geq 1$ and a Lagrangian subbundle $L \subset TM \oplus \wedge^p T^*M$. Assume that $S := pr_{TM}(L)$ has constant rank along M. Choose a form $\omega \in \Omega^{p+1}(M)$ such that S and ω describe L as in Corollary* 3.9.

Then L is involutive iff S is an involutive distribution and $d\omega|_{\wedge^3 S \otimes \wedge^{p-1} TM} = 0$.

Proof. First, note that, a $p+1$-form ω as above always exists, as it can be constructed as in Lemma A.2 choosing a (smooth) distribution C on M complementary to S. We use the description of L given in Corollary 3.9.

Assume that L is involutive. By Proposition 3.6, S is an involutive distribution. Let X, Y be sections of S. Using $\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X d\omega$ we have

$$\begin{aligned}[\![X + \iota_X\omega, Y + \iota_Y\omega]\!] &= [X, Y] + \mathcal{L}_X(\iota_Y\omega) - \iota_Y(\mathcal{L}_X\omega) + \iota_Y\iota_X d\omega \\ &= [X, Y] + \iota_{[X,Y]}\omega + \iota_Y\iota_X d\omega.\end{aligned}$$

Since this lies in L we have $\iota_Y\iota_X d\omega \in \wedge^p S^\circ$ for all sections X, Y of S, which is equivalent to $d\omega|_{\wedge^3 S \otimes \wedge^{p-1} TM} = 0$.

Conversely, assume the above two conditions on S and $d\omega$. The above computation shows that for all sections X, Y of S, the bracket $[\![X+\iota_X\omega, Y+\iota_Y\omega]\!]$ lies in L. The brackets of $X + \iota_X\omega$ with sections of $\wedge^p S^\circ$ lie in L since, by the involutivity of S, locally $\wedge^p S^\circ$ admits a frame consisting of p-forms α_i, which are closed and which hence satisfy $[\![\alpha_i, \cdot]\!] = 0$. Therefore L is involutive. $\square$

Note that, for $p = 1$ (so $d\omega$ is a 3-form) we obtain the familiar statement that a regular almost Dirac structure L is involutive iff $pr_{TM}(L)$ is an involutive distribution whose leaves are endowed with closed 2-forms (see [**1,** Thm. 2.3.6]).

4. Equivalence of higher Dirac and multi-Dirac structures

Recently, Vankerschaver *et al.* [**7**] introduced the notion of Multi-Dirac structure. In this section we show that, at least in the regular case, it is equivalent to our notion of higher Dirac structure. This section does not affect any of the following ones and might be skipped on a first reading.

We recall some definitions from [**7,** Section 4]. All along we fix an integer $p \geq 1$ and a manifold M. In the following, the indices r, s range from 1 to p. Define

$$P_r := \wedge^r TM \oplus \wedge^{p+1-r} T^*M.$$

Define a pairing $P_r \times P_s \to \wedge^{p+1-r-s} T^*M$ by

$$\langle\langle (Y,\eta), (\bar{Y},\bar{\eta}) \rangle\rangle := \frac{1}{2}\left(\iota_{\bar{Y}}\eta - (-1)^{rs}\iota_Y \bar{\eta}\right).$$

If $V_s \subset P_s$, then $(V_s)^{\perp,r} \subset P_r$ is defined by

$$(V_s)^{\perp,r} := \{(Y,\eta) \in P_r : \langle\langle (Y,\eta)\ ,\ V_s \rangle\rangle = 0\}. \tag{4.1}$$

Definition 4.1. An *almost multi-Dirac structure of degree* p on M consists of subbundles $(D_1, \dots, D_p)$, where $D_r \subset P_r$ for all r, satisfying

$$D_r = (D_s)^{\perp,r} \tag{4.2}$$

for all r, s with $r + s \le p + 1$.

Proposition 4.2. *Fix a manifold M and an integer $p \geq 1$. There is a bijection*

$$\begin{aligned} &\{\textit{almost multi-Dirac structures of degree } p\} \\ &\quad \cong \{\textit{almost Dirac structures } L \textit{ of order } p \textit{ s.t. } L^{\perp,r} \\ &\qquad \textit{is a subbundle for } r = 2, \dots, p\} \\ &(D_1, \dots, D_p) \mapsto D_1. \end{aligned}$$

The proof of Proposition 4.2 uses the following extension of Corollary 3.9:

Lemma 4.3. *Fix a vector space T and an integer $p \geq 1$. Let L be a Lagrangian subspace of $T \oplus \wedge^p T^*$, and define $D_r := (L)^{\perp,r}$ for $r = 1, \dots, p$. Choose $\omega \in \wedge^{p+1} T^*$ so that ω and $S := pr_T(L)$ describe L as in Corollary* 3.9. *Then for all r we have*

$$D_r = \{Y + \iota_Y \omega + \xi : Y \in S \wedge (\wedge^{r-1} T), \xi \in \wedge^{p+1-r} S^\circ\}.$$

Proof. "$\subset$:" We first claim that

$$pr_{\wedge^r T}(D_r) \subset S \wedge (\wedge^{r-1} T).$$

If $S = T$ this obvious. If $S \neq T$, by Proposition 3.8 we have that $\wedge^p S^\circ \subset L$ is non-zero. As $(Y,\eta) \in D_r$ implies $\iota_Y(\wedge^p S^\circ) = 0$, we conclude that $Y \in S \wedge (\wedge^{r-1} T)$.

Let $(Y,\eta) \in D_r$. For all $(X,\alpha) \in L$, we have $\alpha - \iota_X \omega \in \wedge^p S^\circ$ by Corollary 3.9, and since $Y \in S \wedge (\wedge^{r-1} T)$ we obtain $\iota_Y \alpha = \iota_Y(\iota_X \omega)$. Hence zero equals

$$\begin{aligned} \langle\langle (Y,\eta), (X,\alpha) \rangle\rangle &= \iota_X \eta - (-1)^r \iota_Y \alpha = \iota_X \eta - (-1)^r \iota_Y(\iota_X \omega) \\ &= \iota_X(\eta - \iota_Y \omega), \end{aligned} \tag{4.3}$$

that is, $\eta - \iota_Y \omega \in \wedge^{p+1-r} S^\circ$. Note that, in the last equality of equation (4.3) we used the total skew-symmetry of ω.

"$\supset$" follows from equation (4.3). $\square$

Proof of Proposition 4.2. The map in the statement of Proposition 4.2 is well defined by equation (4.2) with $r = s = 1$. It is injective as $D_r = (D_1)^{\perp,r}$ is determined by D_1 for $r = 2, \dots, p$, again by equation (4.2).

We now show that it is surjective. Let L be a Lagrangian subbundle of E^p, and assume that $D_r := (L)^{\perp,r}$ is a smooth subbundle for $r = 1, \dots, p$. We have to show that equation (4.2) holds for all r, s with $r + s \le p + 1$. If $(Y, \eta) \in D_r$ and $(\bar{Y}, \bar{\eta}) \in D_s$, then $\iota_Y \bar{\eta} = \iota_Y(\iota_{\bar{Y}}\omega)$ by Lemma 4.3, showing $\langle\langle D_r, D_s \rangle\rangle = 0$ and the inclusion "$\subset$".

For the opposite inclusion take $(Y, \eta) \in (D_s)^{\perp,r}$ at some point $x \in M$. In particular (Y, η) is orthogonal to $\wedge^{p+1-s} S_x^\circ$ (where $S_x := pr_{T_xM} L$). The latter does not vanish by Proposition 3.8 if $S_x \neq T_xM$, and since $r \le p+1-s$ we conclude that $Y \in S_x \wedge (\wedge^{r-1} T_xM)$. If $S_x = T_xM$, the same conclusion holds. A computation analogue to equation (4.3) implies that for all $(\bar{Y}, \bar{\eta}) \in D_s$ we have $0 = \iota_{\bar{Y}}(\eta - \iota_Y \omega)$. As such $\bar{Y}$ span $S_x \wedge (\wedge^{s-1} T_xM)$ by Lemma 4.3 applied to D_s, from $s \le p + 1 - r$ it follows that $\eta - \iota_Y \omega \in \wedge^{p+1-r} S_x^\circ$. Hence, by Lemma 4.3 $(Y, \eta) \in D_r$. $\square$

In order to introduce the notion of integrability for almost multi-Dirac structures, as in [**7**] define $[\![\cdot, \cdot]\!]_{r,s} : \Gamma(P_r) \times \Gamma(P_s) \to \Gamma(P_{r+s-1})$ by

$$\begin{aligned} &[\![(Y, \eta), (\bar{Y}, \bar{\eta})]\!]_{r,s} \\ &\quad := \left([Y, \bar{Y}],\ \mathcal{L}_Y \bar{\eta} - (-1)^{(r-1)(s-1)} \mathcal{L}_{\bar{Y}} \eta + \frac{(-1)^r}{2} d\left(\iota_{\bar{Y}} \eta + (-1)^{rs} \iota_Y \bar{\eta}\right) \right). \end{aligned}$$

Definition 4.4. An almost multi-Dirac structure $(D_1, \dots, D_p)$ is *integrable* if

$$[\![D_r, D_s]\!]_{r,s} \subset D_{r+s-1} \tag{4.4}$$

for all r, s with $r + s \le p$. In that case it is a *multi-Dirac structure*.

We call an almost multi-Dirac structure $(D_1, \dots, D_p)$ *regular* if $pr_{TM}(D_1)$ has constant rank. By Lemma 4.3, this is equivalent to $pr_{\wedge^r TM}(D_r)$ having constant rank for $r = 1, \dots, p$. Under this regularity assumption, we obtain an equivalence for integrable structures.

Theorem 4.5. *Fix a manifold M and an integer $p \ge 1$. The bijection of Proposition* 4.2 *restricts to a bijection*

$$\begin{aligned} &\{\textit{regular multi-Dirac structures of degree } p\} \\ &\quad \cong \{\textit{regular Dirac structures of order } p\} \end{aligned}$$

Proof. If $(D_1, \dots, D_p)$ is a multi-Dirac structure, by the remark at the end of [**7,** Section 4], D_1 is involutive w.r.t. the Courant bracket. Therefore, it is involutive w.r.t. Dorfman bracket, that is, it is a Dirac structure of order p.

For the converse, note that if L is a regular Dirac structure L then $L^{\perp,r}$ is always a smooth subbundle by Corollary 4.3. So let $(D_1, \dots, D_p)$ be a regular almost multi-Dirac structure with the property that $L := D_1$ is involutive. Choose $\omega \in \Omega^{p+1}(M)$ so that $(\omega, S := pr_{TM}(L))$ describe L as in Corollary 3.9. Such a differential form exists by the regularity assumption. To show that condition (4.4) holds, let $Y \in \Gamma(S \wedge (\wedge^{r-1}T))$ and $\bar{Y} \in \Gamma(S \wedge (\wedge^{s-1}T))$. We have

$$\left[\!\left[Y + \iota_Y\omega, \bar{Y} + \iota_{\bar{Y}}\omega\right]\!\right]_{r,s} = \left([Y, \bar{Y}], \iota_{[Y,\bar{Y}]}\omega + (-1)^r \iota_Y \iota_{\bar{Y}} d\omega\right),$$

see for instance [**7,** Proof of Theorem 4.5]. Now $\iota_Y \iota_{\bar{Y}} d\omega \in \Gamma(\wedge^{p+2-r-s}S^\circ)$ by Theorem 3.12, so the above lies in D_{r+s-1} by Lemma 4.3. Further, the involutivity of S implies that locally $\wedge^{p+1-s}S^\circ$ admits a frame consisting of closed forms α_i. For any choice of functions f_i we have

$$[\![Y + \iota_Y\omega, f_i\alpha_i]\!]_{r,s} = \mathcal{L}_Y(f_i\alpha_i) + (-1)^{r(s+1)} d\iota_Y(f_i\alpha_i) = \iota_Y(df_i \wedge \alpha_i),$$

which lies in $\Gamma(\wedge^{p+2-r-s}S^\circ)$ since $Y \in \Gamma(S \wedge (\wedge^{r-1}T))$ and $\alpha_i \in \Gamma(\wedge^{p+1-s}S^\circ)$. $\square$

Finally, we comment on how our definition of higher Dirac structure differs from Hagiwara's Nambu–Dirac structures [**12**], which also are an extension of Courant's notion of Dirac structure.

Remark 4.6. A *Nambu–Dirac structure* on a manifold M [**12,** Def. 3.1, Definition 3.7] is an involutive subbundle $L \subset E^p$ satisfying

$$\langle X_1 + \alpha_1, X_2 + \alpha_2\rangle|_{\wedge^{p-1}(pr_{TM}(L))} = 0, \tag{4.5}$$

$$\wedge^p(pr_{TM}(L)) = pr_{\wedge^p TM} L^{\perp,p}, \tag{4.6}$$

where $L^{\perp,p} \subset \wedge^p TM \oplus T^*M$ is defined as in equation (4.1). When $p = 1$, Nambu–Dirac structures agree with Dirac structures. Graphs of closed forms and of Nambu–Poisson multivector fields are Nambu–Dirac structures.

Our isotropicity condition (3.1) is clearly stronger than (4.5). Nevertheless, higher Dirac structures are usually *not* Nambu–Dirac structures, for the former satisfy

$$pr_{TM}(L) \wedge (\wedge^{p-1}TM) = pr_{\wedge^p TM} L^{\perp,p}$$

by Lemma 4.3, and hence usually do not satisfy (4.6). A concrete instance is given by the 3-dimensional Lagrangian subspace $L \subset T \oplus \wedge^2 T^*$ given as in Corollary 3.9 by $T = \mathbb{R}^4$, S equal to the plane $\{x_3 = x_4 = 0\}$ and $\omega = dx_1 \wedge dx_2 \wedge dx_3$.

5. Review: L_∞-algebras

In this section, we review briefly the notion of L_∞-algebra, which generalizes Lie algebras and was introduced by Lada and Stasheff [**6**] in the 1990s. We will follow the conventions of Lada–Markl[2] [**22,** Sections 2 and 5].

[2]Except that on graded vector spaces we take the grading inverse to theirs.

Recall that a *graded vector space* is just a (finite dimensional, real) vector space $V = \oplus_{i\in\mathbb{Z}} V_i$ with a direct sum decomposition into subspaces. An element of V_i is said to have degree i, and we denote its degree by $|\cdot|$.

For any $n \geq 1$, $V^{\otimes n}$ is a graded vector space, and the symmetric group acts on it by the so-called odd representation: the transposition of the k-th and $(k+1)$-th element acts by

$$v_1 \otimes \cdots \otimes v_n \mapsto -(-1)^{|v_k||v_{k+1}|} v_1 \otimes \cdots \otimes v_{k+1} \otimes v_k \otimes \cdots \otimes v_n.$$

The *nth graded exterior product of* V is the graded vector space $\wedge^n V$, consisting of elements of $V^{\otimes n}$ which are fixed by the odd representation of the symmetric group.

Definition 5.1. An L_∞-*algebra* is a graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$ endowed with a sequence of multi-brackets ($n \geq 1$)

$$l_n \colon \wedge^n V \to V$$

of degree $2 - n$, satisfying the following quadratic relations for each $n \geq 1$:

(5.1)
$$\sum_{i+j=n+1} \sum_{\sigma\in Sh(i,n-i)} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(v_{\sigma(1)}, \dots, v_{\sigma(i)}), v_{\sigma(i+1)}, \dots, v_{\sigma(n)}) = 0.$$

Here, $Sh(i, n-i)$ denotes the set of $(i, n-i)$-unshuffles, i.e., permutations preserving the order of the first i elements and the order of the last $n - i$ elements. The sign $\chi(\sigma)$ is given by the action of σ on $v_1 \otimes \cdots \otimes v_n$ in the odd representation.

Remark 5.2. (1) The quadratic relations imply that the unary bracket l_1 squares to zero, so (V, l_1) is a chain complex of vector spaces. Hence, L_∞-algebras can be viewed as chain complexes with the extra data given by the multi-brackets l_n for $n \geq 2$.

(2) When V is concentrated in degree 0, (i.e., only V_0 is non-trivial) then $\wedge^n V$ is the usual n-th exterior product of V, and is concentrated in degree zero. Hence, by degree reasons only the binary bracket $[\cdot,\cdot]_2$ is non-zero, and the quadratic relations are simply the Jacobi identity, so we recover the notion of Lie algebra.

For any $p \geq 1$, we use the term *Lie p-algebra* to denote an L_∞-algebra whose underlying graded vector space is concentrated in degrees $-p + 1, \dots, 0$. Note that by degree reasons only the multi-brackets

$l_1, \dots, l_{p+1}$ can be non-zero. In particular, a Lie 2-algebra consists of a graded vector space V concentrated in degrees -1 and 0, together with maps

$$\begin{aligned} d &:= l_1 \colon V \to V, \\ [\cdot,\cdot] &:= l_2 \colon \wedge^2 V \to V, \\ J &:= l_3 \colon \wedge^3 V \to V, \end{aligned}$$

of degrees 1,0 and -1, respectively, subject to the quadratic relations.

An L_∞-*morphism* $\phi\colon V \rightsquigarrow V'$ between L_∞-algebras is a sequence of maps $(n \geq 1)$

$$\phi_n \colon \wedge^n V \to V'$$

of degree $1-n$, satisfying certain relations, which can be found in [**22,** Def. 5.2] in the case when V' has only the unary and binary bracket. The first of these relations says that $\phi_1\colon V \to V'$ must preserve the differentials (unary brackets). We spell out the definition when V and V' are Lie 2-algebras.

Definition 5.3. Let $(V, d, [\cdot,\cdot], J)$ and $(V', d', [\cdot,\cdot]', J')$ be Lie 2-algebras. A *morphism* $\phi\colon V \rightsquigarrow V'$ consists of linear maps

$$\begin{aligned} &\phi_0 \colon V_0 \to V_0, \\ &\phi_1 \colon V_{-1} \to V_{-1}, \\ &\phi_2 \colon \wedge^2 V_0 \to V_{-1}, \end{aligned}$$

such that

$$d' \circ \phi_1 = \phi_0 \circ d, \tag{5.2}$$

$$d'(\phi_2(x,y)) = \phi_0[x,y] - [\phi_0(x), \phi_0(y)]' \quad \text{for all } x, y \in V_0, \tag{5.3}$$

$$\phi_2(df, y) = \phi_1[f,y] - [\phi_1(f), \phi_0(y)]' \quad \text{for all } f \in V_{-1}, y \in V_0, \tag{5.4}$$

and for all $x, y, z \in V_0$:

$$\begin{aligned} &\phi_0(J(x,y,z)) - J'(\phi_0(x), \phi_0(y), \phi_0(z)) \\ &\quad = \phi_2(x, [y,z]) - \phi_2(y, [x,z]) + \phi_2(z, [x,y]) \\ &\qquad + [\phi_0(x), \phi_2(y,z)]' - [\phi_0(y), \phi_2(x,z)]' + [\phi_0(z), \phi_2(x,y)]'. \end{aligned} \tag{5.5}$$

6. L_∞-algebras from higher analogues of Dirac structures

Courant [**1,** Section 2.5] associated to every Dirac structure on M a subset of $C^\infty(M)$, which we refer to as Hamiltonian functions or observables. Usually the Hamiltonian vector field associated to such a function is not unique. Nevertheless, the set of Hamiltonian functions is endowed with a Poisson algebra structure (a Lie bracket compatible with the product of functions). Baez *et al.* associate to a p-plectic form a set of Hamiltonian $p-1$-forms and endow it with a bracket [**4,** Section 3]. Rogers shows that the bracket can be extended to obtain a Lie p-algebra [**5,** Thm. 5.2]. In this section, we mimic

Courant's definition of the bracket and extend Roger's results to arbitrary isotropic involutive subbundles.

Let $p \geq 1$ and let L be an isotropic, involutive subbundle of $E^p = TM \oplus \wedge^p T^*M$.

Definition 6.1. A $(p-1)$-form $\alpha \in \Omega^{p-1}(M)$ is called *Hamiltonian* if there exists a smooth vector field X_α such that $X_\alpha + d\alpha \in \Gamma(L)$. We denote the set of Hamiltonian forms by $\Omega^{p-1}_{\mathrm{ham}}(M, L)$. We refer to X_α as *a Hamiltonian vector field* of α.

Remark 6.2. (a) Hamiltonian vector fields are unique only up to smooth sections of $L \cap (TM \oplus 0)$.

(b) For all $X \in L_x \cap (T_x M \oplus 0)$ and for all $\eta \in pr_{\wedge^p T^*M} L_x$,

$$\iota_X \eta = 0.$$

Here, $x \in M$ and $pr_{\wedge^p T^*M}$ denotes the projection of E^p_x onto the second component. The above property follows from the fact that there exists $Y \in T_x M$ with $Y + \eta \in L_x$, so $\iota_X \eta = \langle X + 0, Y + \eta \rangle = 0$ by the isotropicity of L.

Definition 6.3. We define a bracket $\{\cdot, \cdot\}$ on $\Omega^{p-1}_{\mathrm{ham}}(M, L)$ by

$$\{\alpha, \beta\} := \iota_{X_\alpha} d\beta,$$

where X_α is any Hamiltonian vector field for α.

Lemma 6.4. *The bracket $\{\cdot, \cdot\}$ is well-defined and skew-symmetric. It does not satisfy the Jacobi identity, but rather*

$$\{\alpha, \{\beta, \gamma\}\} + c.p. = -d(\iota_{X_\alpha}\{\beta, \gamma\})$$

where "c.p." denotes cyclic permutations.

Proof. The bracket is well-defined: by Remark 6.2 the ambiguity in the choice of X_α is a section X of $L \cap (TM \oplus 0)$ and $\iota_X d\beta = 0$. Using $\mathcal{L}_Y = \iota_Y d + d\iota_Y$ one computes

$$[\![X_\alpha + d\alpha, X_\beta + d\beta]\!] = [X_\alpha, X_\beta] + d\{\alpha, \beta\}. \tag{6.1}$$

Hence, $[X_\alpha, X_\beta]$ is a Hamiltonian vector field for $\{\alpha, \beta\}$, showing that $\Omega^{p-1}_{\mathrm{ham}}(M, L)$ is closed under $\{\cdot, \cdot\}$. The bracket is skew symmetric because

$$0 = \langle X_\alpha + d\alpha, X_\beta + d\beta \rangle = \{\alpha, \beta\} + \{\beta, \alpha\}.$$

To compute the Jacobiator of $\{\cdot,\cdot\}$ we proceed as in[3] [**1,** Proposition 2.5.3]. Since L is isotropic and involutive we have

$$\begin{aligned}0 &= \langle [\![X_\alpha + d\alpha\ ,\ X_\beta + d\beta]\!], X_\gamma + d\gamma\rangle\\ &= \langle [X_\alpha, X_\beta] + d\{\alpha,\beta\}\ ,\ X_\gamma + d\gamma\rangle\\ &= \iota_{[X_\alpha,X_\beta]}d\gamma + \iota_{X_\gamma} d\{\alpha,\beta\}\\ &= (\{\alpha,\{\beta,\gamma\}\} + c.p.) + d(\iota_{X_\alpha}\{\beta,\gamma\}).\end{aligned}$$

Here the second equality uses equation (6.1) and the last equality uses $\iota_{[Y,Z]} = [\mathcal{L}_Y, \iota_Z]$. $\square$

Remark 6.5. Given a p-plectic form ω, Cantrijn, Ibort and de León [**3,** Section 4] define the space of Hamiltonian $(p-1)$-forms α by the requirement that $d\alpha = -\iota_{X_\alpha}\omega$ for a (necessarily unique) vector field X_α on M, and define the *semi-bracket* $\{\alpha,\beta\}_s$ by $\iota_{X_\beta}\iota_{X_\alpha}\omega$. These notions coincide with our Definition 6.1 and Definition 6.3 applied to $\mathrm{graph}(\omega) := \{X - \iota_X\omega : X \in TM\} \subset E^p$.

Remark 6.6. Given an p-plectic form, in [**4,** Def. 3.3] the *hemi-bracket* of $\alpha,\beta \in \Omega^{p-1}_{\mathrm{ham}}(M, \mathrm{graph}(\omega))$ is also defined, by the formula $\mathcal{L}_{X_\alpha}\beta$. This notion does not extend to the setting of arbitrary isotropic subbundles of E^p, since in that setting the Hamiltonian vector field X_α is not longer unique and the above expression depends on it.

For instance, take $M = \mathbb{R}^4$, consider the closed 3-form $\theta = dx_1\wedge dx_2\wedge dx_3$. By Proposition 3.2, $L = \{X - \iota_X\theta : X \in TM\}$ is a isotropic, involutive subbundle of E^2. Both $\frac{\partial}{\partial x_4} \in \Gamma(L\cap TM)$ and the zero vector field are Hamiltonian vector fields for $\alpha = 0$, and the hemi-bracket of α with $\beta = x_1dx_4 + x_4dx_1$ is not well-defined since

$$\mathcal{L}_{\frac{\partial}{\partial x_4}}\beta = dx_1 \neq 0 = \mathcal{L}_0\beta.$$

Rogers [**5,** Thm. 5.2] shows that for every p-plectic manifold there is an associated L_∞-algebra of observables. The statement and the proof generalize in a straightforward way to arbitrary isotropic, involutive subbundle of $E^p = TM \oplus \wedge^p T^*M$.

Theorem 6.7. *Let $p \geq 1$ and L be a isotropic, involutive subbundle of $E^p = TM \oplus \wedge^p T^*M$. Then the complex concentrated in degrees $-p+1,\dots,0$*

$$C^\infty(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p-2}(M) \xrightarrow{d} \Omega^{p-1}_{\mathrm{ham}}(M,L)$$

has a Lie p-algebra structure. The only non-vanishing multibrackets are given by the de Rham differential on $\Omega^{\leq p-2}(M)$ and, for $k = 2,\dots,p+1$, by

$$l_k(\alpha_1,\dots,\alpha_k) = \epsilon(k)\iota_{X_{\alpha_k}}\dots\iota_{X_{\alpha_3}}\{\alpha_1,\alpha_2\}$$

[3] There the case $p = 1$ is treated, and the term $\iota_{X_\alpha}\{\beta,\gamma\}$ vanishes by degree reasons.

where $\alpha_1, \dots, \alpha_k \in \Omega^{p-1}_{\mathrm{ham}}(M, L)$ *and* $\epsilon(k) = (-1)^{\frac{k}{2}+1}$ *if* k *is even,* $\epsilon(k) = (-1)^{\frac{k-1}{2}}$ *if* k *is odd.*

Proof. The expressions for the multibrackets are totally skew-symmetric, as a consequence of the fact that $\{\cdot, \cdot\}$ is skew-symmetric. This and the fact that $\{\cdot, \cdot\}$ is independent of the choice of Hamiltonian vector fields imply that the multibrackets are well-defined. Clearly, l_k has degree $2 - k$.

Now we check the L_∞ relations (5.1). For $n = 1$, the relation holds due to $d^2 = 0$. Now consider the relation (5.1) for a fixed $n \geq 2$, and let $\alpha_1, \dots, \alpha_n$ be homogeneous elements of the above complex. We will use repeatedly the fact that, for $k \geq 2$, the k-multibracket vanishes when one of its entries is of negative degree. For $j \in \{2, \dots, n-2\}$ (so $i \geq 3$), we have

$$l_j(l_i(\alpha_1, \dots, \alpha_i), \alpha_{i+1}, \dots, \alpha_n) = 0,$$

as a consequence of the fact that k-multibrackets for $k \geq 3$ take values in negative degrees. For $j = n$, we have

$$l_n(l_1(\alpha_1), \alpha_2, \dots, \alpha_n) = 0 :$$

if $|\alpha_1| = 0$ then $l_1(\alpha_1)$ vanishes, otherwise $l_1(\alpha_1) = d\alpha_1$ and its Hamiltonian vector field vanishes.

We are left with the summands of (5.1) with $j = 1$ and $j = n - 1$. When $n = 2$ we have just one summand $l_1(l_2(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}))$, which vanishes by degree reasons. For $n \geq 3$ it is enough to assume that all the α_i's have degree zero. We have

$$d(l_n(\alpha_1, \dots, \alpha_n)) + \sum_{\sigma \in Sh(2,n-2)} \chi(\sigma) l_{n-1}(\{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}\}, \alpha_{\sigma(3)} \dots, \alpha_{\sigma(n)}).$$

Writing out explicitly the unshuffles in $Sh(2, n-2)$ and the multibrackets we obtain

$$\begin{aligned}
&\epsilon(n) d(\iota_{X_{\alpha_n}} \dots \iota_{X_{\alpha_3}} \{\alpha_1, \alpha_2\}) \\
&\quad + \epsilon(n-1) \left[\sum_{2 \leq i < j \leq n} (-1)^{i+j-1} \iota_{X_{\alpha_n}} \dots \widehat{\iota_{X_{\alpha_j}}} \dots \widehat{\iota_{X_{\alpha_i}}} \dots \iota_{X_{\alpha_2}} \{\{\alpha_i, \alpha_j\}, \alpha_1\} \right. \\
&\quad + \sum_{3 \leq j \leq n} (-1)^j \iota_{X_{\alpha_n}} \dots \widehat{\iota_{X_{\alpha_j}}} \dots \iota_{X_{\alpha_3}} \{\{\alpha_1, \alpha_j\}, \alpha_2\} \\
&\quad \left. + \iota_{X_{\alpha_n}} \dots \dots \iota_{X_{\alpha_4}} \{\{\alpha_1, \alpha_2\}, \alpha_3\} \right].
\end{aligned}$$

By Lemma 6.8 we conclude that the above expression vanishes. $\square$

The following Lemma, needed in the proof of Theorem 6.7, extends [**5,** Lemma 3.7].

Lemma 6.8. *Let $p \geq 1$ and L be a isotropic, involutive subbundle of $E^p = TM \oplus \wedge^p T^*M$. Then for any $n \geq 3$, and for all $\alpha_1, \ldots, \alpha_n \in \Omega^{p-1}_{\mathrm{ham}}(M, L)$ we have*

$$\begin{aligned} &d(\iota_{X_{\alpha_n}} \ldots \iota_{X_{\alpha_3}} \{\alpha_1, \alpha_2\}) \\ &\quad = (-1)^{n+1} \Bigg[\sum_{2 \leq i < j \leq n} (-1)^{i+j-1} \iota_{X_{\alpha_n}} \ldots \widehat{\iota_{X_{\alpha_j}}} \ldots \widehat{\iota_{X_{\alpha_i}}} \ldots \iota_{X_{\alpha_2}} \{\{\alpha_i, \alpha_j\}, \alpha_1\} \\ &\qquad + \sum_{3 \leq j \leq n} (-1)^j \iota_{X_{\alpha_n}} \ldots \widehat{\iota_{X_{\alpha_j}}} \ldots \iota_{X_{\alpha_3}} \{\{\alpha_1, \alpha_j\}, \alpha_2\} \\ &\qquad + \iota_{X_{\alpha_n}} \ldots \ldots \iota_{X_{\alpha_4}} \{\{\alpha_1, \alpha_2\}, \alpha_3\} \Bigg]. \end{aligned}$$

Proof. We proceed by induction on n. For $n = 3$ the statement holds by Lemma 6.4. So let $n > 3$. To shorten the notation, denote $A := \iota_{X_{\alpha_{n-1}}} \ldots \iota_{X_{\alpha_3}} \{\alpha_1, \alpha_2\}$. Then we have

$$d(\iota_{X_{\alpha_n}} \ldots \iota_{X_{\alpha_3}} \{\alpha_1, \alpha_2\}) = d(\iota_{X_{\alpha_n}} A) = \mathcal{L}_{X_{\alpha_n}} A - \iota_{X_{\alpha_n}} dA. \tag{6.2}$$

The first term on the r.h.s. of (6.2) becomes

$$\begin{aligned} &\mathcal{L}_{X_{\alpha_n}} (\iota_{X_{\alpha_3} \wedge \cdots \wedge X_{\alpha_{n-1}}} \{\alpha_1, \alpha_2\}) \\ &\quad = \sum_{i=3}^{n-1} (-1)^{i+1} \iota_{X_{\alpha_{n-1}}} \ldots \widehat{\iota_{X_{\alpha_i}}} \ldots \iota_{X_{\alpha_3}} \iota_{[X_{\alpha_n}, X_{\alpha_i}]} \{\alpha_1, \alpha_2\} \\ &\qquad + \iota_{X_{\alpha_{n-1}}} \ldots \iota_{X_{\alpha_3}} \mathcal{L}_{X_{\alpha_n}} \{\alpha_1, \alpha_2\} \\ &\quad = \sum_{i=3}^{n-1} (-1)^{i+1} \iota_{X_{\alpha_{n-1}}} \ldots \widehat{\iota_{X_{\alpha_i}}} \ldots \iota_{X_{\alpha_2}} \{\{\alpha_n, \alpha_i\}, \alpha_1\} \\ &\qquad + \iota_{X_{\alpha_{n-1}}} \ldots \iota_{X_{\alpha_3}} (\{\{\alpha_2, \alpha_n\}, \alpha_1\} - \{\{\alpha_1, \alpha_n\}, \alpha_2\}) \\ &\quad = \sum_{i=2}^{n-1} (-1)^{i} \iota_{X_{\alpha_{n-1}}} \ldots \widehat{\iota_{X_{\alpha_i}}} \ldots \iota_{X_{\alpha_2}} \{\{\alpha_i, \alpha_n\}, \alpha_1\} \\ &\qquad - \iota_{X_{\alpha_{n-1}}} \ldots \iota_{X_{\alpha_3}} \{\{\alpha_1, \alpha_n\}, \alpha_2\}. \end{aligned}$$

Here in the second equality, we used $[X_{\alpha_n}, X_{\alpha_i}] = X_{\{\alpha_n, \alpha_i\}}$ (see the proof of Lemma 6.4) and

$$\iota_{X_{\{\alpha_n, \alpha_i\}}} \{\alpha_1, \alpha_2\} = -\iota_{X_{\{\alpha_n, \alpha_i\}}} \iota_{X_{\alpha_2}} d\alpha_1 = \iota_{X_{\alpha_2}} \{\{\alpha_n, \alpha_i\}, \alpha_1\},$$

as well as Cartan's formula for the Lie derivative and Lemma 6.4.

The second term on the r.h.s. of (6.2) can be developed using the induction hypothesis. The resulting expression for the l.h.s. of equation (6.2) is easily seen to agree with the one in the statement of this lemma. $\square$

Remark 6.9. The observables associated by Theorem 6.7 to the zero $p+1$-form on M are given by the abelian Lie algebra $\mathbb{R}$ for $p = 1$ and to the complex $C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{closed}}(M)$ (with vanishing higher brackets) for $p = 2$. It is a curious coincidence that they agree with the central extensions of observables of p-plectic structures given in [**23,** Proposition 9.4] for $p = 1$ and 2, respectively.

A closed 2-form B on M induce an automorphism of the Courant algebroid $TM \oplus T^*M$ by gauge transformations (see Section 1), and therefore acts on the set of Dirac structures. For instance, the Dirac structure $TM \oplus \{0\}$ is mapped to the graph of B. The Poisson algebras of observables of these two Dirac structures are not isomorphic (unless $B = 0$).

Similarly, for $p \geq 1$, gauge transformations of E^p by closed $p+1$-forms usually do not induce an isomorphism of the Lie p-algebra of observables. We display a quite trivial operation which, on the other hand, does have this property.

Lemma 6.10. *Let $\lambda \in \mathbb{R} - \{0\}$ and consider*

$$\begin{aligned} m_\lambda \colon\ & E^p \to E^p \\ & X + \eta \mapsto X + \lambda\eta \end{aligned}$$

Let $L \subset E^p$ be an involutive isotropic subbundle. Then $m_\lambda(L)$ is also an involutive isotropic subbundle, and the Lie p-algebras of observables of L and $m_\lambda(L)$ are isomorphic.

Proof. m_λ is an automorphism of the Dorfman bracket $[\![\cdot,\cdot]\!]$ and $\langle m_\lambda \cdot, m_\lambda \cdot\rangle = \lambda\langle\cdot,\cdot\rangle$. Hence, $m_\lambda(L)$ is also involutive and isotropic.

We consider the Lie p-algebras of observables associated to L and $m_\lambda(L)$, respectively, as in Theorem 6.7. We denote them by $\mathcal{O}^L$ and $\mathcal{O}^{m_\lambda(L)}$, respectively. The underlying complexes coincide, both being

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p-1}_{\mathrm{ham}}(M, L).$$

Note that if $\alpha \in \Omega^{p-1}_{\mathrm{ham}}(M, L)$ has Hamiltonian vector field X^L_α, then $\lambda\alpha$ is a Hamiltonian $(p-1)$-form for $m_\lambda(L)$, and X^L_α itself is a Hamiltonian vector field for it. Hence, from Theorem 6.7 it is clear that the unary map given by multiplication by λ

$$\phi\colon (\beta_0, \dots, \beta_{p-1}) \mapsto (\lambda\beta_0, \dots, \lambda\beta_{p-1})$$

intertwines the multibrackets of $\mathcal{O}^L$ and $\mathcal{O}^{m_\lambda(L)}$, where $\beta_i \in \Omega^i(M)$ for $i < p-1$ and $\beta_{p-1} \in \Omega^{p-1}_{\mathrm{ham}}(M, L)$. Therefore, setting the higher maps to zero, we obtain a strict morphism [**24,** Section 7] of Lie p-algebras, which clearly is an isomorphism. $\square$

As an application of Lemma 6.10 we show that to any compact, connected, orientable $p+1$-dimensional manifold ($p \geq 1$) there is an associated Lie p-algebra. A dual version of this Lie p-algebra appeared in [**25,** Thm. 6.1].

Corollary 6.11. *Let M be a compact, connected, orientable $p+1$-dimensional manifold. For any volume form ω consider the Lie p-algebra associated to graph(ω) by Theorem* 6.7*, whose underlying complex is*

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p-1}(M).$$

(Note that all $p-1$-forms are Hamiltonian). Its isomorphism class is independent of the choice of ω, and therefore depends only on the manifold M.

Proof. Let ω_0 and ω_1 be two volume forms on M. They define non-zero cohomology classes in $H^{p+1}(M,\mathbb{R}) = \mathbb{R}$, so there is a (unique) $\lambda \in \mathbb{R} - \{0\}$ such that $[\omega_1] = \lambda[\omega_0]$. By Moser's theorem [**26**] there is a diffeomorphism ψ of M such that $\psi^*(\omega_1) = \lambda\omega_0$. This explains the first isomorphism in

$$\text{Lie } p\text{-algebra of } \omega_1 \;\cong\; \text{Lie } p\text{-algebra of } \lambda\omega_0 \;\cong\; \text{Lie } p\text{-algebra of } \omega_0,$$

whereas the second one holds by Lemma 6.10. □

7. Relations to L_∞-algebras arising from split Courant algebroids

In this section, we construct an L_∞-morphism from a Lie algebra associated to E^0 with the σ-twisted bracket, where σ is a closed 2-form, to a Lie 2-algebra associated to E^1 with the untwisted Courant bracket (in other words, the Courant bracket twisted by $d\sigma = 0$).

We consider again $E^p := TM \oplus \wedge^p T^*M$. For $p = 0$ we have $E^0 = TM \oplus \mathbb{R}$. Fix a closed 2-form $\sigma \in \Omega^2_{\text{closed}}(M)$. Then $\Gamma(E^0)$ with the σ-twisted Dorfman bracket

$$[X+f, Y+g]_\sigma = [X,Y] + (X(g) - Y(f)) + \sigma(X,Y)$$

is an honest Lie algebra. (See [**14,** Section 3.8], where a geometric interpretation in terms of circle bundles is given too.)

For $p = 1$ we have the (untwisted) Courant algebroid $E^1 = TM \oplus T^*M$. Roytenberg and Weinstein [**9**] associated to it an L_∞-algebra. In the version given in [**11,** Thm. 4.4], the underlying complex is

$$C^\infty(M) \xrightarrow{d} \Gamma(E^1), \tag{7.1}$$

where d is the de Rham differential. The binary bracket $[\cdot,\cdot]'$ is given by the *Courant* bracket $[\![\cdot,\cdot]\!]_{Cou}$ on $\Gamma(E^1)$ and by

$$[e,f]' = -[f,e]' := \frac{1}{2}\langle e, df\rangle$$

for $e \in \Gamma(E^1)$ and $f \in C^\infty(M)$. The trinary bracket J' is given by

$$J'(e_1,e_2,e_3) = -\frac{1}{6}(\langle [\![e_1,e_2]\!]_{Cou}, e_3\rangle + c.p.)$$

for elements of $\Gamma(E^1)$, where "c.p." denotes cyclic permutation. All other brackets vanish.

We show that there is a canonical morphism between these two Lie 2-algebras:

Theorem 7.1. *Let M be a manifold and $\sigma \in \Omega^2_{\text{closed}}(M)$. There is a canonical morphism of Lie 2-algebras*

$$\phi\colon (\Gamma(E^0), [\cdot,\cdot]_\sigma) \rightsquigarrow (C^\infty(M) \xrightarrow{d} \Gamma(E^1), [\cdot,\cdot]', J') \tag{7.2}$$

given by

$$\phi_0\colon \Gamma(E^0) \to \Gamma(E^1), \quad (X,f) \mapsto (X, df)$$
$$\phi_2\colon \wedge^2 \Gamma(E^0) \to C^\infty(M), \quad (X,f),(Y,g) \mapsto \frac{1}{2}\big(X(g) - Y(f)\big) + \sigma(X,Y).$$

Proof. We check that the conditions of Definition 5.3 are satisfied. Equation (5.2) is satisfied because $\Gamma(E^0)$ is concentrated in degree zero.

Equation (5.3) is satisfied because for any $X+f, Y+g \in \Gamma(E^0)$ we have

$$\begin{aligned}
&\phi_0[X+f, Y+g]_\sigma - [\![\phi_0(X+f), \phi_0(Y+g)]\!]_{Cou} \\
&\quad = ([X,Y] + d(X(g) - Y(f) + \sigma(X,Y))) - \left([X,Y] + \frac{1}{2} d(X(g) - Y(f))\right) \\
&\quad = d(\phi_2(X+f, Y+g)).
\end{aligned}$$

Equation (5.4) is satisfied because $\Gamma(E^\circ)$ is concentrated in degree zero.

We are left with checking equation (5.5). Let $X+f, Y+g, Z+h \in \Gamma(E^1)$. We want to show that

$$\begin{aligned}
&-J'(X+df, Y+dg, Z+dh) \\
&\quad \overset{!}{=} \phi_2(X+f, [Y,Z] + Y(h) - Z(g) + \sigma(Y,Z)) + c.p. \\
&\qquad + [X+df, \phi_2(Y+g, Z+h)]' + c.p.
\end{aligned} \tag{7.3}$$

where as usual "*c.p.*" denotes cyclic permutation. The l.h.s. of equation (7.3) is equal to

$$\begin{aligned}
&\frac{1}{6}(\langle [\![X+df, Y+dg]\!]_{Cou}, Z+dh\rangle) + c.p. \\
&\quad = \frac{1}{6}\left([X,Y](h) + \frac{1}{2}Z(X(g)) - \frac{1}{2}Z(Y(f))\right) + c.p. \\
&\quad = \frac{1}{4}[X,Y](h) + c.p.
\end{aligned}$$

The r.h.s. is equal to

$$\begin{aligned}
&\frac{1}{2}(X(Y(h)-Z(g)+\sigma(Y,Z))-[Y,Z](f))+\sigma(X,[Y,Z])+c.p.\\
&\qquad+\frac{1}{2}X\left(\frac{1}{2}(Y(h)-Z(g))+\sigma(Y,Z)\right)+c.p.\\
&\quad=\frac{3}{4}X(Y(h)-Z(g))-\frac{1}{2}[Y,Z](f)+c.p.\\
&\qquad+\sigma(X,[Y,Z])+X(\sigma(Y,Z))+c.p.\\
&\quad=\frac{1}{4}[X,Y](h)+c.p.\\
&\qquad+d\sigma(X,Y,Z).
\end{aligned}$$

Since σ is a closed form, we conclude that equation (7.3) is satisfied. □

8. L_∞-algebras from higher analogues of split Courant algebroids

In this section we apply Getzler's recent construction [**10**] to obtain an L_∞ structure on the complex concentrated in degrees $-r+1,\dots,0$

$$C^\infty(M)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{r-2}(M)\xrightarrow{d}\Gamma(E^{r-1})=\chi(M)\oplus\Omega^{r-1}(M), \tag{8.1}$$

for any manifold M and integer $r\geq 2$. When $r=2$ we obtain exactly the Lie 2-algebra given just before Theorem 7.1.

Let us first recall Getlzer's recent theorem [**10,** Thm. 3]. Let $(V,\delta,\{\ ,\ \})$ be a differential graded Lie algebra (DGLA). Getlzer endows the graded[4] vector space $V^-:=\oplus_{i<0}V_i$ with multibrackets satisfying the relations [**10,** Def. 1], which after a degree shift provide $V^-[-1]$ with a L_∞-algebra structure in the sense of our Definition 5.1. Note that, $V^-[-1]$ is concentrated in non-positive degrees: its degree 0 component is V_{-1}, its degree -1 component is V_{-2}, and so on. The multibrackets are built out of a derived bracket construction using the restriction of the operator δ to V_0, and the Bernoulli numbers appear as coefficients.

Now let M be a manifold, fix an integer $r\geq 2$, and consider the graded manifold

$$T^*[r]T[1]M$$

(see [**27, 29**] [**28,** Section 2] for background material on graded manifolds). $T^*[r]T[1]M$ is endowed with a canonical Poisson structure of degree $-r$: there is a bracket $\{\ ,\ \}$ of degree $-r$ on the graded commutative algebra of functions $\mathcal{C}:=C(T^*[r]T[1]M)$ such that

$$(\mathcal{C}\ ,\ \cdot\ ,\ \{\ ,\ \})$$

[4]We take the opposite grading as in [**10**] so that our differential δ has degree 1.

is a *Poisson algebra of degree* r [**30,** Def. 1.1]. This means that $\{\ ,\ \}$ defines a (degree zero) graded Lie algebra structure on $\mathcal{C}[r]$, the graded vector space defined by the degree shift $(\mathcal{C}[r])_i := \mathcal{C}_{r+i}$, and that $\{a, \cdot\}$ is a degree $|a| - r$ derivation of the product for any homogeneous element $a \in \mathcal{C}$.

More concretely, choose coordinates x_i on M, inducing fiber coordinates v_i on $T[1]M$, and conjugate coordinates P_i and p_i on the fibers of $T^*[r]T[1]M \to T[1]M$. The degrees of these generators of $\mathcal{C}$ are

$$|x_i| = 0, \ \ |v_i| = 1, \ \ |P_i| = r, \ \ |p_i| = r - 1.$$

Then

$$\begin{aligned} \{P_i, x_i\} &= 1 = -\{x_i, P_i\}, \\ \{p_i, v_i\} &= 1 = -(-1)^{r-1}\{v_i, p_i\} \end{aligned}$$

for all i, and all the other brackets between generators vanish. Note that the coordinate v_i corresponds canonically to $dx_i \in \Omega^1(M)$ and that p_i corresponds canonically to $\frac{\partial}{\partial x_i} \in \chi(M)$. Also, note that $\mathcal{C}$ is concentrated in non-negative degrees, and that there are canonical identifications

$$\mathcal{C}_i = \Omega^i(M) \text{ for } 0 \leq i < r-1, \quad \mathcal{C}_{r-1} = \Omega^{r-1}(M) \oplus \chi(M). \tag{8.2}$$

Indeed, for $i < r - 1$ the elements of degree i are sums of expressions of the form $f(x)v_{j_1} \dots v_{j_i}$, while for $i = r - 1$ they are sums of expressions $f(x)v_{j_1} \dots v_{j_{r-1}} + g(x)p_j$.

The degree $r+1$ function $\mathcal{S} := \sum v_i P_i$, given by the De Rham differential on M, satisfies $\{\mathcal{S}, \mathcal{S}\} = 0$, hence $\{\mathcal{S},\ \}$ squares to zero. This and the fact that $(\mathcal{C}[r], \{\ ,\ \})$ is a graded Lie algebra imply that

$$(\mathcal{C}[r], \delta := \{\mathcal{S},\ \}, \{\ ,\ \}). \tag{8.3}$$

is a DGLA. Hence, Getlzer's construction can be applied to (8.3), endowing $(\mathcal{C}[r])^-[-1] = (\oplus_{0 \leq i \leq r-1} \mathcal{C}_i)[r-1]$ (the complex displayed in (8.1)) with an L_∞-algebra structure.

We write out explicitly the multibrackets. The twisted case will be considered in Proposition 8.4 below.

Proposition 8.1. *Let M be a manifold, $r \geq 2$ an integer. There exists a Lie r-algebra structure on the complex* (8.1) *concentrated in degrees $-r+1, \dots, 0$, i.e.,*

$$C^\infty(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{r-2}(M) \xrightarrow{d} \Gamma(E^{r-1}) = \chi(M) \oplus \Omega^{r-1}(M),$$

whose only non-vanishing brackets (up to permutations of the entries) are

- *unary bracket: the de Rham differential in negative degrees.*
- *binary bracket:*
  for $e_i \in \Gamma(E^{r-1})$ the Courant bracket as in equation (2.6),

$$[e_1, e_2] = [\![e_1, e_2]\!]_{Cou};$$

for $e=(X,\alpha)\in\Gamma(E^{r-1})$ *and* $\xi\in\Omega^{\bullet<r-1}(M)$,

$$[e,\xi]=\frac{1}{2}\mathcal{L}_X\xi.$$

- *trinary bracket:*
  for $e_i\in\Gamma(E^{r-1})$,

$$[e_0,e_1,e_2]=-\frac{1}{6}\left(\langle[\![e_0,e_1]\!]_{Cou},e_2\rangle+c.p.\right);$$

  for $\xi\in\Omega^{\bullet<r-1}(M)$ *and* $e_i=(X_i,\alpha_i)\in\Gamma(E^{r-1})$,

$$[\xi,e_1,e_2]=-\frac{1}{6}\left(\frac{1}{2}(\iota_{X_1}\mathcal{L}_{X_2}-\iota_{X_2}\mathcal{L}_{X_1})+\iota_{[X_1,X_2]}\right)\xi.$$

- *n-ary bracket for $n\geq 3$ with n an* odd *integer:*
  for $e_i=(X_i,\alpha_i)\in\Gamma(E^{r-1})$, $[e_0,\dots,e_{n-1}]=\sum_i[X_0,\dots,\alpha_i,\dots,X_{n-1}]$, *with*

$$\begin{aligned}&[\alpha,X_1,\dots,X_{n-1}]\\ &\quad=\frac{(-1)^{\frac{n+1}{2}}12B_{n-1}}{(n-1)(n-2)}\\ &\qquad\times\sum_{1\leq i<j\leq n-1}(-1)^{i+j+1}\iota_{X_{n-1}}\dots\widehat{\iota_{X_j}}\dots\widehat{\iota_{X_i}}\dots\iota_{X_1}[\alpha,X_i,X_j];\end{aligned}$$

  for $\xi\in\Omega^{\bullet<r-1}(M)$ *and* $e_i=(X_i,\alpha_i)\in\Gamma(E^{r-1})$,

$$\begin{aligned}&[\xi,e_1,\dots,e_{n-1}]\\ &\quad=\frac{(-1)^{\frac{n+1}{2}}12B_{n-1}}{(n-1)(n-2)}\\ &\qquad\times\sum_{1\leq i<j\leq n-1}(-1)^{i+j+1}\iota_{X_{n-1}}\dots\widehat{\iota_{X_j}}\dots\widehat{\iota_{X_i}}\dots\iota_{X_1}[\xi,X_i,X_j].\end{aligned}$$

Here the B's denote the Bernoulli numbers.

Remark 8.2. Bering [**31,** Section 5.6] shows that the vector fields and differential forms on a manifold M are naturally endowed with multibrackets forming an algebraic structure which generalizes L_∞-algebras: the quadratic relations satisfied by Bering's multibrackets have Bernoulli numbers as coefficients. The multibrackets appearing in Proposition 8.1 are similar to Bering's, and they differ not only in the coefficients, but also in that the expression for $[\xi,e_1,\dots,e_{n-1}]$ (for $n\geq 3$) does not appear among Bering's brackets. This is a consequence of the fact that Getzler's multibracket are constructed not out of δ, but out of its restriction to V_0.

Remark 8.3. We write more explicitly the trinary bracket of elements $e_i = (X_i, \alpha_i) \in \Gamma(E^{r-1})$: we have $[e_0, e_1, e_2] = [\alpha_0, X_1, X_2] - [\alpha_1, X_0, X_2] + [\alpha_2, X_0, X_1]$ with

$$[\alpha_0, X_1, X_2] = -\frac{1}{6}\left(\frac{1}{2}(\iota_{X_1}\mathcal{L}_{X_2} - \iota_{X_2}\mathcal{L}_{X_1}) + \iota_{[X_1,X_2]} + \iota_{X_1}\iota_{X_2} d\right)\alpha_0.$$

Proof. Let $X_1, X_2, \dots \in \chi(M)$ and $\xi_1, \xi_2, \dots$ be differential forms on M. In the following we identify them with elements of $\mathcal{C}$ as indicated in equation (8.2), and we adopt the notation introduced in the text before Proposition 8.1. The following holds:

(a) If $\xi_i \in \Omega^{k_i}(M)$ for k_1, k_2 arbitrary, we have

$$\{X_1 + \xi_1, X_2 + \xi_2\} = \iota_{X_1}\xi_2 + (-1)^{r-1-k_1}\iota_{X_2}\xi_1.$$

In particular, when $\xi_1, \xi_2 \in \Omega^{r-1}(M)$, we obtain the pairing $\langle \cdot, \cdot \rangle$ as in equation (2.1).

(b) For any differential form ξ_1, the identity

$$\{\mathcal{S}, \xi_1\} = d\xi_1$$

is immediate in coordinates.

(c) If $\xi_1, \xi_2 \in \Omega^{r-1}(M)$, we have

$$\{\{\mathcal{S}, X_1 + \xi_1\}, X_2 + \xi_2\} = [\![X_1 + \xi_1, X_2 + \xi_2]\!],$$

the Dorfman bracket as in equation (2.2). This holds by the following identities, which we write for $\xi_i \in \Omega^{k_i}(M)$ for arbitrary k_1, k_2:

$$\{\{\mathcal{S}, X_1\}, X_2\} = [X_1, X_2] \text{ and } \{\{\mathcal{S}, \xi_1\}, \xi_2\} = 0$$

are checked in coordinates, and

$$\begin{aligned}
\{\{\mathcal{S}, X_1\}, \xi_2\} &= \{\mathcal{S}, \{X_1, \xi_2\}\} + \{X_1, \{\mathcal{S}, \xi_2\}\} \\
&= d(\iota_{X_1}\xi_2) + \iota_{X_1} d\xi_2 = \mathcal{L}_{X_1}\xi_2, \\
\{\{\mathcal{S}, \xi_1\}, X_2\} &= -(-1)^{r-1-k_1}\{X_2, \{\mathcal{S}, \xi_1\}\} = -(-1)^{r-1-k_1}\iota_{X_2} d\xi_1.
\end{aligned}$$

(d) For $n \geq 3$, and letting a_i be either a vector field X_i or a differential form ξ_i of arbitrary degree (not a sum of both),

$$\{\{\dots\{\mathcal{S}, a_1\}, \dots\}, a_n\} = 0$$

except when exactly one of a_1, a_2, a_3 is a differential form and all the remaining a_i's are vector fields.

Using this it is straighforward to write out the (graded symmetric) multibrackets of [**10,** Thm. 3], which we denote by $(\cdot,\dots,\cdot)$. More precisely, (b) gives the unary bracket, (c) gives the binary bracket, (c) and (d) give the trinary bracket. For the higher brackets ($n \geq 3$ odd) one uses (d) and then (a) to compute

$$\begin{aligned}
&(\alpha, X_1, \dots, X_{n-1})\\
&\quad = \frac{c_{n-1}}{c_2} \sum_{\sigma \in \text{Sectionigma}_{n-1},\, \sigma_1<\sigma_2}\\
&\qquad \times (-1)^{\sigma} \{\{\dots\{(\alpha, X_{\sigma_1}, X_{\sigma_2}), X_{\sigma_3}\},\dots\}, X_{\sigma_{n-1}}\}\\
&\quad = (-1)^{\binom{n-2}{2}}(n-3)!\frac{c_{n-1}}{c_2} \sum_{1\leq i<j\leq n-1}\\
&\qquad \times (-1)^{i+j+1} \iota_{X_{n-1}} \dots \widehat{\iota_{X_j}} \dots \widehat{\iota_{X_i}} \dots \iota_{X_1}(\xi, X_i, X_j),
\end{aligned}$$

where we abbreviate $c_{n-1} := \frac{(-1)^{\binom{n+1}{2}}}{(n-1)!} B_{n-1}$. The computation for $[\xi, e_1, \dots, e_{n-1}]$ with $\xi \in \Omega^{\bullet < r-1}(M)$ delivers the same expression and uses the fact that n is odd. The coefficient can be simplified:

$$(-1)^{\binom{n-2}{2}}(n-3)!\frac{c_{n-1}}{c_2} = \frac{12}{(n-1)(n-2)} B_{n-1}$$

since n is odd and $c_2 = \frac{1}{12}$.

This gives us the (graded symmetric) multibrackets $(\cdot,\dots,\cdot)$ of [**10**]. As pointed out in [**10**], multiplying the n-ary bracket by $(-1)^{\binom{n-1}{2}}$ delivers (graded symmetric) multibrackets that satisfy the Jacobi rules given just before [**32,** Def. 4.2].

These Jacobi rules coincide with Voronov's [**33,** Def. 1], and according to [**33,** Rem. 2.1], the passage from these (graded symmetric) multibrackets to the (graded skew-symmetric) multibrackets satisfying our Definition 5.1 is given as follows: multiply the multibracket of elements $x_1, \dots, x_n$ by

$$(-1)^{\tilde{x}_1(n-1)+\tilde{x}_2(n-2)+\cdots+\tilde{x}_{n-1}}, \tag{8.4}$$

where $\tilde{x}_i$ denotes the degree of x_i as an element of (8.1), a complex concentrated in degrees $-r+1, \dots, 0$. One easily checks that in all the cases relevant to us (8.4) does not introduce any sign.

In conclusion, to pass from the conventions of [**10**] to the conventions of our Definition 5.1 we just have to multiply the n-ary bracket $(\cdot,\dots,\cdot)$ by $(-1)^{\binom{n-1}{2}}$, which for $n = 1, 2$ equals 1 and for n odd equals $(-1)^{\frac{n-1}{2}}$. $\square$

Now, let $H \in \Omega^{r+1}_{\text{closed}}(M)$ be a closed $r+1$-form. H can be viewed as an element $\mathcal{H}$ of $\mathcal{C}_{r+1}$, and $\boldsymbol{\{}\mathcal{S}-\mathcal{H}, \mathcal{S}-\mathcal{H}\boldsymbol{\}} = -2\boldsymbol{\{}\mathcal{S}, \mathcal{H}\boldsymbol{\}} = -2dH = 0$. Hence

$$\big(\mathcal{C}[r], \delta := \boldsymbol{\{}\mathcal{S}-\mathcal{H},\ \boldsymbol{\}}, \boldsymbol{\{}\ ,\ \boldsymbol{\}}\big) \tag{8.5}$$

is a DGLA, and again we can apply Getzler's construction. We obtain an L_∞-algebra structure that extends the H-twisted Courant bracket:

Proposition 8.4. *Let M be a manifold, $r \geq 2$ an integer and $H \in \Omega^{r+1}_{\text{closed}}(M)$. There exists a Lie r-algebra structure on the complex* (8.1) *concentrated in degrees $-r+1, \ldots, 0$, whose only non-vanishing brackets (up to permutations of the entries) are those given in Proposition* 8.1 *and additionally for $e_i = (X_i, \alpha_i) \in \Gamma(E^{r-1})$:*

- *binary bracket:*
$$[e_1, e_2] = \iota_{X_2}\iota_{X_1}H$$
- *n-ary bracket for $n \geq 3$ with n an* odd *integer:*
$$[e_1, \ldots, e_n] = (-1)^{\frac{n-1}{2}} \cdot n \cdot B_{n-1} \cdot \iota_{X_n} \ldots \iota_{X_1} H.$$

Proof. It is easy to see (in coordinates, or using that $T[1]M \subset T^*[r]T[1]M$ is Lagrangian) that for any $n \geq 1$, letting a_i be either a vector field X_i or a differential form ξ_i of arbitrary degree (not a sum of both), one has:

$$\boldsymbol{\{}\boldsymbol{\{}\ldots\boldsymbol{\{}\mathcal{H}, a_1\boldsymbol{\}}, \ldots\boldsymbol{\}}, a_n\boldsymbol{\}} = 0$$

except when all of the a_i's are vector fields X_i's. In this case one obtains

$$(-1)^{\binom{n}{2}} \iota_{X_n} \ldots \iota_{X_1} H \tag{8.6}$$

using (a) in the proof of Proposition 8.1. Denoting by $(\cdot, \ldots, \cdot)$ the (graded symmetric) multibrackets as in [**10**] from the DGLA (8.5), we see that $(X_1, \ldots, X_n)$ is equal to (8.6) multiplied by $-n! \cdot c_{n-1}$. In order to pass from the conventions of [**10**] to those of our Definition 5.1 we multiply by $(-1)^{\binom{n-1}{2}}$ and obtain the formulae in the statement. $\square$

For any $B \in \Omega^r(M)$, the gauge transformation of E^{r-1} given by $e^{-B}\colon X + \alpha \mapsto X + \alpha - \iota_X B$ maps the H-twisted Courant bracket to the $(H+dB)$-twisted Courant bracket. Defining properly the notion of higher Courant algebroid — of which the E^r's should be the main examples – and extending to this general setting Proposition 8.1, will presumably imply that the L_∞-algebras defined by cohomologous differential forms are isomorphic. We show this directly:

Proposition 8.5. *Let M be a manifold, $r \geq 2$ an integer and $H \in \Omega^{r+1}_{\text{closed}}(M)$. For any $B \in \Omega^r(M)$, there is a strict isomorphism*

$$(\textit{the Lie-}r\textit{ algebra defined by } H) \to (\textit{the Lie-}r\textit{ algebra defined by } H + dB)$$

between the Lie-r algebra structures defined as in Proposition 8.4 *on the complex* (8.1). *Explicitly, the isomorphism is given by* e^{-B} *on* $\Gamma(E^{r-1})$ *and is the identity elsewhere.*

Proof. View B as an element $\mathcal{B} \in \mathcal{C}_r$. As $\{\mathcal{B}, \;\}$ is a degree zero derivation of the graded Lie algebra $(\mathcal{C}[r], \{\;,\;\})$ and is nilpotent, it follows that the exponential $\Phi := e^{\{\mathcal{B}, \;\}}$ is an automorphism. Therefore, it is an isomorphism of DGLAs

$$\Phi\colon (\mathcal{C}[r], \delta := \{\mathcal{S} - \mathcal{H}, \;\}, \{\;,\;\}) \to (\mathcal{C}[r], \Phi\delta\Phi^{-1}, \{\;,\;\}).$$

From the formulas for the multibrackets in Getzler's [**10,** Thm. 3] it is then clear that $\Phi|_{(\oplus_{0\leq i\leq r-1}\mathcal{C}_i)[r-1]}$ is a strict isomorphism between the L_∞-algebras induced by these two DGLAs.

The differential $\Phi\delta\Phi^{-1}$ on $\mathcal{C}$ is not equal to $\{\mathcal{S} - (\mathcal{H} + \{\mathcal{S}, \mathcal{B}\}), \;\}$, which is the differential associated to $H + dB \in \Omega^{r+1}_{\text{closed}}(M)$ as in (8.5). However, on $\oplus_{0\leq i\leq r-1}\mathcal{C}_i$ the two differentials do agree. (This follows from the fact that on $\oplus_{0\leq i\leq r-1}\mathcal{C}_i$ we have $\Phi(y) = y + \{\mathcal{B}, y\}$.) This assures that the L_∞-algebras induced by the two differentials agree. $\square$

9. Open questions: the relation between the L_∞-algebras of Sections 6–8

In this section, we speculate about the relations among the L_∞-algebras that appeared in Sections 6–8 and their higher analogues, and relate them to prequantization.

Let M be a manifold. Given an integer $n \geq 0$ and $H \in \Omega^{n+2}_{\text{closed}}(M)$, we use the notation E^n_H to denote the vector bundle $E^n = TM \oplus \wedge^n T^*M$ with the H-twisted Dorfman bracket $[\cdot,\cdot]_H$. In particular, E^n_0 denotes $TM \oplus \wedge^n T^*M$ with the untwisted Dorfman bracket (2.2).

9.1. Relations between L_∞-algebras. To any $n \geq 0$ and $H \in \Omega^{n+2}_{\text{closed}}(M)$, we associated in Proposition 8.4 a Lie $n+1$-algebra $\mathcal{S}^{E^n_H}$. We ask:

Is there a natural L_∞-morphism D from $\mathcal{S}^{E^n_H}$ to $\mathcal{S}^{E^{n+1}_0}$?

When $n = 0$ the answer is affirmative by Theorem 7.1.

Let $p \geq 1$ and $L \subset E^p_0$ an involutive isotropic subbundle. Denote by $\mathcal{O}^{L\subset E^p_0}$ the Lie p-algebra associated in Theorem 6.7. Since L is an involutive subbundle of E^p_0 it is natural to ask:

What is the relation between $\mathcal{O}^{L\subset E^p_0}$ and $\mathcal{S}^{E^p_0}$?

When L is equal to graph(H) for a p-plectic form H, we expect the relation to be given by an L_∞-morphism

$$P\colon \mathcal{O}^{\mathrm{graph}(H)\subset E_0^p} \rightsquigarrow \mathcal{S}^{E_H^{p-1}}$$

with the property that the unary map of the L_∞-morphism $D\circ P$, restricted to the degree zero component, coincide with

$$\Omega^{p-1}_{\mathrm{ham}}(M, \mathrm{graph}(H)) \to \Gamma(E_0^p), \quad \alpha \mapsto X_\alpha - d\alpha. \tag{9.1}$$

We summarize the situation in this diagram:

$$\begin{tikzcd} \mathcal{S}^{E_H^{p-1}} \arrow[r, rightsquigarrow, "D"] & \mathcal{S}^{E_0^p} \\ \mathcal{O}^{\mathrm{graph}(H)\subset E_0^p} \arrow[u, rightsquigarrow, "P"] \arrow[ur, rightsquigarrow, "D\circ P"'] & \end{tikzcd}$$

Remark 9.1. In the case $p=1$ (so H is a symplectic form) the embedding P exists and is given as follows. We have two honest Lie algebras

$$\mathcal{O}^{\mathrm{graph}(H)\subset E_0^1} = (C^\infty(M), \{\cdot,\cdot\}), \qquad \mathcal{S}^{E_H^0} = (\Gamma(TM\oplus\mathbb{R}), [\cdot,\cdot]_H),$$

where $\{\cdot,\cdot\}$ is the usual Poisson bracket defined by H. The map

$$P\colon C^\infty(M) \to \Gamma(TM\oplus\mathbb{R}), \quad f\mapsto (X_f, -f)$$

is a Lie algebra morphism. Lie 2-algebra morphism D is given by Theorem 7.1. One computes that the composition consists only of a unary map, given by the Lie algebra morphism (9.1).

Remark 9.2. We interpret P as a prequantization map. Indeed, for $p=1$ and integral form H, the Lie algebra $\mathcal{S}^{E_H^0}$ can be identified with the space of S^1-invariant vector fields on a circle bundle over M [**14,** Section 3.8]. The composition of P with the action of vector fields on the S^1-equivariant complex valued functions is then a faithful representation of the Lie algebra $\mathcal{O}^{\mathrm{graph}(H)\subset E_0^1} = C^\infty(M)$, i.e., a prequantization representation. For $p=2$ the morphism P is described by Rogers in [**11,** Thm. 5.2] and [**23,** Thm. 7.1], to which we refer for the interpretation as a prequantization map.

9.2. The twisted case. We pose three questions about higher analogues of twisted Dirac structures. Let H be a closed $p+1$-form for $p\geq 2$. Let $L'\subset E_H^{p-1}$ be an isotropic subbundle, involutive w.r.t. the H-twisted Dorfman bracket.

Can one associate to L' an L_∞-algebra of observables $\mathcal{O}^{L'\subset E_H^{p-1}}$?

To the author's knowledge, this is not known even in the simplest case, i.e., when $p=2$ and L' is the graph of an H-twisted Poisson structure [**34**]. In that case, one defines in the usual manner a skew-symmetric bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$. It does not satisfy the Jacobi identity but rather [**34,** equation

(4)] $\{\{f,g\},h\}+c.p. = -H(X_f,X_g,X_h)$, hence it is natural to wonder if one can extend this bracket to an L_∞-structure.

> Is there a natural L_∞-morphism D' from $\mathcal{O}^{L'\subset E_H^{p-1}}$ to $\mathcal{O}^{\mathrm{graph}(H)\subset E_0^p}$?

This question is motivated by the fact that L' plays the role of a primitive of H. In the simple case that L' is the graph of a symplectic form the answer is affirmative, by the morphism from $(C^\infty(M),\{\cdot,\cdot\})$ to $C^\infty(M)\xrightarrow{d}\Omega^1_{\mathrm{closed}}(M)$ (a complex with no higher brackets) with vanishing unary map and binary map $\phi_2(f,g)=\{f,g\}$.

> Is there an L_∞-morphism from $\mathcal{O}^{L'\subset E_H^{p-1}}$ to $\mathcal{S}^{E_H^{p-1}}$, assuming that L' is the graph of a non-degenerate differential form?

Such a morphism would be interesting because it could be interpreted as a weaker (because not injective) version of a prequantization map for (M,L').

We summarize the discussion of this whole section in the following diagram, in which for the sake of concreteness and simplicity we take $H\in\Omega^3_{\mathrm{closed}}(M)$ to be a 2-plectic form and $L'\subset TM\oplus T^*M$ to be a H-twisted Dirac structure. The arrows denote L_∞-morphisms.

$$
\begin{tikzcd}
 & \mathcal{S}^{E_H^1} \arrow[r, rightsquigarrow, "D"] & \mathcal{S}^{E_0^2} \\
\mathcal{O}^{L'\subset E_H^1} \arrow[ur, rightsquigarrow] \arrow[r, rightsquigarrow, "D'"] & \mathcal{O}^{\mathrm{graph}(H)\subset E_0^2} \arrow[u, rightsquigarrow, "P"] \arrow[ur, rightsquigarrow, "D\circ P"'] &
\end{tikzcd}
$$

We conclude presenting an interesting example in which the geometric set-up described above applies.

Example 9.3. Let G be a Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a non-degenerate bi-invariant quadratic form $(\cdot,\cdot)_{\mathfrak{g}}$. There is a well-defined closed Cartan 3-form H, which on $\mathfrak{g}=T_eG$ is given by $H(u,v,w)=\frac{1}{2}(u,[v,w])_{\mathfrak{g}}$ [**35,** Section 2.3]. There is also a canonical H-twisted Dirac structure $L'\subset TG\oplus T^*G$: it is given by $L'=\{(v_{\mathrm{r}}-v_{\mathrm{l}})+\frac{1}{2}(v_{\mathrm{r}}+v_{\mathrm{l}})^*: v\in\mathfrak{g}\}$ where $v_{\mathrm{r}},v_{\mathrm{l}}$ denote the right and left translations of $v\in\mathfrak{g}$ and the quadratic form is used to identify a tangent vector $X\in TG$ with a covector $X^*\in T^*G$ [**34**] [**35,** Ex. 3.4].

Appendix A. The proof of Proposition 3.8

In this appendix, we present the proof of Proposition 3.8. We start giving an alternative characterization of Lagrangian subspaces.

Lemma A.1. *Let T be a vector space and $p \geq 1$. For all subspaces $L \subset T \oplus \wedge^p T^*$, denoting $S := pr_T L$, the following holds:*

$$L \text{ is Lagrangian } \Leftrightarrow \begin{cases} L \text{ is isotropic}, \\ L \cap \wedge^p T^* = \wedge^p S^\circ, \\ dim(S) \leq (dim(T) - p) \text{ or } S = T. \end{cases}$$

Proof. "$\Rightarrow$:" Assume first that L is Lagrangian. It is straightforward to check that for any subspace $F \subset T \oplus \wedge^p T^*$ we have

$$F^\perp \cap \wedge^p T^* = \wedge^p (pr_T(F))^\circ. \tag{A.1}$$

We apply this to $F = L = L^\perp$ and derive $L \cap \wedge^p T^* = \wedge^p S^\circ$.

Hence, we are left with showing that S satisfies $\dim(S) \leq \dim(T) - p$ or $S = T$. We argue by contradiction: we assume that $\wedge^p S^\circ = \{0\}$ and S is strictly included in T, and deduce from this that $pr_T(L^\perp) \not\subset S$, which contradicts $L = L^\perp$. Let $\{X_j\}_{j \leq \dim(T)}$ be a basis of T whose first $\dim(S)$ elements form a basis of S. Let Y be a basis element *not* lying in S (it exists since $S \neq T$). It is enough to prove the following *claim*:

$$Y + \beta \in L^\perp, \quad \text{where } \beta = -\sum_{j=1}^{\dim(S)} \left(X_j^* \wedge \left(\sum_{q=0}^{p} \frac{1}{q+1} \iota_Y \alpha_j^q \right) \right),$$

because it implies that $Y \in pr_T(L^\perp)$. Here $\{X_j^*\}_{j \leq \dim(T)}$ denotes the basis of T^* dual to $\{X_j\}_{j \leq \dim(T)}$, and for all $j \leq \dim(S)$, $\alpha_j \in \wedge^p T^*$ is such that $X_j + \alpha_j \in L$. Further, we adopt the following notation: for any $\alpha \in \wedge^p T^*$, α^q denotes the component of α, written in the basis of $\wedge^p T^*$ induced by $\{X_j^*\}_{j \leq \dim(T)}$, for which the number of X_j^*'s with $j \leq \dim(S)$ is exactly q.

To prove the claim fix $j_0 \leq \dim(S)$. We have

$$\begin{aligned} \iota_{X_{j_0}} \beta &= -\sum_{q=0}^{p} \frac{1}{q+1} \iota_Y \alpha_{j_0}^q + \sum_{j=1}^{\dim(S)} X_j^* \wedge \left(\sum_{q=0}^{p} \frac{1}{q+1} \iota_{X_{j_0}} \iota_Y \alpha_j^q \right) \\ &= -\sum_{q=0}^{p} \frac{1}{q+1} \iota_Y \alpha_{j_0}^q - \iota_Y \sum_{j=1}^{\dim(S)} X_j^* \wedge \left(\sum_{q=0}^{p} \frac{1}{q+1} \iota_{X_j} \alpha_{j_0}^q \right) \\ &= -\sum_{q=0}^{p} \frac{1}{q+1} \iota_Y \alpha_{j_0}^q - \sum_{q=0}^{p} \frac{q}{q+1} \iota_Y \alpha_{j_0}^q \\ &= -\iota_Y \alpha_{j_0}, \end{aligned}$$

where in the second equality we used $\iota_{X_{j_0}} \alpha_j^q = -\iota_{X_j} \alpha_{j_0}^q$ and in the third $\sum_{j=1}^{\dim(S)} X_j^* \wedge (\iota_{X_j} \alpha_{j_0}^q) = q \alpha_{j_0}^q$. Hence, $\langle Y + \beta, X_j + \alpha_j \rangle = 0$ for all $j \leq \dim(S)$. Since $L \cap \wedge^p T^* = \wedge^p S^\circ = \{0\}$, we have $L = span\{X_j + \alpha_j\}_{j \leq \dim(S)}$, and we conclude that $Y + \beta \in L^\perp$, proving the claim.

"$\Leftarrow$:" We need to show that L is Lagrangian, i.e., $L = L^{\perp}$. We claim that $pr_T(L^{\perp}) = S$. If $S = T$ this is clear, so we prove the claim in the case $\dim(S) \leq \dim(T) - p$, for which we have $\wedge^p S^{\circ} \neq \{0\}$. Since $\wedge^p S^{\circ} \subset L$, this implies that $pr_T(L^{\perp}) \subset S$. By the isotropicity of L we therefore have $pr_T(L^{\perp}) = S$, as claimed.

Hence, if $X + \beta \in L^{\perp}$ there exists $\alpha \in \wedge^p T^*$ such that $X + \alpha \in L \subset L^{\perp}$. So $\beta - \alpha \in L^{\perp} \cap \wedge^p T^* = \wedge^p S^{\circ} \subset L$, where the equality holds by equation (A.1). Therefore $X + \beta = (X + \alpha) + (\beta - \alpha)$ is the sum of two elements of L, showing $L^{\perp} \subset L$. $\square$

Lemma A.2. *Let $S \subset T$ a subspace and $p \geq 1$. Let $\Omega \in \wedge^2 S^* \otimes \wedge^{p-1} T^*$. Then Ω admits an extension to $S^* \otimes \wedge^p T^*$ iff it admits an extension to $\wedge^{p+1} T^*$.*

Proof. If there exists $\alpha \in \wedge^{p+1} T^*$ with $\alpha|_{S \otimes S \otimes \bigotimes^{p-1} T} = \Omega$, the clearly $\alpha|_{S \otimes \bigotimes^p T}$ is an element of $S^* \otimes \wedge^p T^*$ with the required property.

Conversely, let $\beta' \in S^* \otimes \wedge^p T^*$ be an extension of Ω. We choose a complement C to S in T, and by the identification $S^* \cong C^{\circ}$ from β' we obtain an element $\beta \in T^* \otimes \wedge^p T^*$. The skew-symmetrization $\bar{\beta} \in \wedge^{p+1} T^*$ of β is given as follows:

$$\bar{\beta}(x_0, \ldots, x_p) = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j \beta(x_j, x_0, \ldots, \hat{x}_j, \ldots, x_p)$$

for all $x_i \in T$. In general $\bar{\beta}$ does not restrict to Ω, but a weighted sum of its component does, as we now show. We have $\bar{\beta} = \sum_{q=0}^{p+1} \bar{\beta}^q$. Here, for any basis $\{X_j\}_{j \leq \dim(T)}$ of T whose first $\dim(S)$ elements span S and whose remaining elements span C, taking $\{X_j^*\}_{j \leq \dim(T)}$ to be the dual basis of T^*, we denote by $\bar{\beta}^q$ the component of $\bar{\beta}$ for which, in the basis of $\wedge^p T^*$ induced by $\{X_j^*\}_{j \leq \dim(T)}$, the number of X_j^*'s with $j \leq \dim(S)$ is exactly q. We have $\bar{\beta}^0 = 0$, since β is an extension of β'. For $q = 1, \ldots, p+1$, vectors $x_0, \ldots, x_{q-1} \in S$ and $x_q, \ldots, x_p \in C$ we have[5]

$$\bar{\beta}^q(x_0, \ldots, x_p) = \bar{\beta}(x_0, \ldots, x_p) = \frac{q}{p+1} \beta(x_0, \ldots, x_p).$$

Therefore, $\sum_{q=1}^{p+1} \frac{p+1}{q} \bar{\beta}^q$ is an element of $\wedge^{p+1} T^*$ whose restriction to $S \otimes \bigotimes^p T$ agrees with β', and in particular its restriction to $S \otimes S \otimes \bigotimes^{p-1} T$ agrees with Ω. $\square$

Proof of Proposition 3.8. We make use of the characterization of Lagrangian subspaces given in Lemma A.1.

[5]This of course does not imply that β is totally skew, as the element x_0 of S is plugged in the first slot of β.

We first show that the correspondence "$L \mapsto (S,\Omega)$" is well-defined. Let L be a Lagrangian subspace. The dimension restriction on S follows from Lemma A.1. Since $L \cap \wedge^p T^* = \wedge^p S^\circ$, for any $X \in S$, the definition of $\iota_X \Omega$ in Proposition 3.8 is independent of the choice of α with $X + \alpha \in L$, and determines a unique $\Omega \in \otimes^2 S^* \otimes \wedge^{p-1} T^*$. Clearly, Ω is skew in the first two components: if $X+\alpha, Y+\beta \in L$ then the isotropicity of L implies $\iota_Y \iota_X \Omega = \iota_Y \alpha = -\iota_X \beta = -\iota_X \iota_Y \Omega$. By construction, Ω is the restriction of an element of $S^* \otimes \wedge^p T^*$, hence by Lemma A.2 it is the restriction of an element of $\wedge^{p+1} T^*$

Next, we show that the correspondence "$(S,\Omega) \mapsto L$" is well-defined. Let (S,Ω) a pair as in the statement of Proposition 3.8. This pair maps to a subspace L which is isotropic, due to the skew-symmetry of Ω in its first 2 components. By inspection we have $L \cap \wedge^p T^* = \wedge^p S^\circ$, and further S agrees with $pr_T(L)$ because Ω is the restriction of an element of $S^* \otimes \wedge^p T^*$. Hence, L is Lagrangian by Lemma A.1.

The maps "$L \mapsto (S,\Omega)$" and "$(S,\Omega) \mapsto L$" are inverses of each other. $\square$

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