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SYMPLECTIC MICROGEOMETRY I: MICROMORPHISMS

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We introduce the notion of symplectic microfolds and symplectic micromorphisms between them. They form a symmetric monoidal category, which is a version of the "category" of symplectic manifolds and canonical relations obtained by localizing them around lagrangian submanifolds in the spirit of Milnor's microbundles.

Contents

1. Introduction

There is a category **Sympl** whose objects are finite-dimensional symplectic manifolds (M, ω) and whose morphisms are symplectomorphisms $\Psi : (M, \omega_M) \to (N, \omega_N)$. In attempting to understand the quantization procedure of physicists from a mathematical perspective, one may think of it as a functor from this symplectic category, where classical mechanics takes place, into the category of Hilbert spaces and unitary operators, which is the realm of quantum mechanics.

It is well known that the category **Sympl** is too large, since there are "no-go" theorems that show that the group of all symplectomorphisms on (M, ω) does not act in a physically meaningful way on a corresponding Hilbert space. One standard remedy for this is to replace **Sympl** with a smaller category, replacing the symplectomorphism groups with certain finite-dimensional subgroups. Another is to replace the Hilbert spaces and operators by objects depending on a formal parameter.

But there is also a sense in which the category **Sympl** is too *small*, since it does not contain morphisms corresponding to operators such as projectors and the self-adjoint (or skew-adjoint) operators that play the role of observables in quantum mechanics, nor can it encode the algebra structure itself on the space of observables. (This collection of observables is not actually a Hilbert space, but certain sets of operators do carry a vector space structure, with the inner product associated to the Hilbert–Schmidt norm.)

To enlarge the symplectic category, we look at the "dictionary" of quantization, following, for example, [**1**]. In this dictionary, the cartesian product of symplectic manifolds corresponds to the tensor product of Hilbert spaces, and replacing a symplectic manifold (M, ω) with $(M, -\omega)$ (which we denote by $\overline{M}$ when we omit the symplectic structure from the notation for a given symplectic manifold) corresponds to replacing a Hilbert space $\mathcal{H}$ by its conjugate, or dual, space $\mathcal{H}^*$. Thus, if symplectic manifolds M_1 and M_2 correspond to Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the product $\overline{M} \times N$ corresponds to $\mathcal{H}_1^* \otimes \mathcal{H}_2$, which, with a suitable definition of the tensor product, is a space $L(\mathcal{H}_1, \mathcal{H}_2)$ of linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Another entry in the dictionary says that lagrangian submanifolds (perhaps carrying half-densities) in symplectic manifolds correspond to vectors or lines in Hilbert space. Combining this idea with the one in the previous paragraph, we conclude that lagrangian submanifolds in $\overline{M} \times N$ should correspond to linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. This suggests that, if the space of observables $\mathcal{H}$ for a quantum system corresponds to a symplectic manifold M, then the algebra structure on $\mathcal{H}$ should be given by a lagrangian submanifold μ in $\overline{M} \times \overline{M} \times M$. The algebra axioms of unitality and associativity should be encoded by monoidal properties of μ in an extended symplectic category, $\mathbf{Sympl^{ext}}$, where the morphisms from M to N are the canonical relations, i.e., all the lagrangian submanifolds of $\overline{M} \times N$ (not just those that are the graphs of symplectomorphisms) and where the morphism composition is the usual composition of relations. However, a problem immediately occurs: the composition of canonical relations may yield relations that are not submanifolds any more, and thus are not canonical relations! $\mathbf{Sympl^{ext}}$ is then not a true category, as the morphisms cannot always be composed. It is rather awkward to speak about a quantization functor in this context.

There have already been several approaches to remedy this defect. One approach, developed by Guillemin and Sternberg in [**5**] (see [**4**] for a recent version), is to consider only symplectic vector spaces and linear canonical relations. Another, suggested by Wehrheim and Woodward in [**8**], is to enlarge the category still further by allowing arbitrary "formal" products of canonical relations and equating them to actual products when the latter exist as manifolds.

In this paper, we take yet another approach. We construct a version of the extended symplectic "category," which is a true category, by localizing it around lagrangian submanifolds. Its objects, called symplectic microfolds in the spirit of Milnor's microbundles [**7**], are equivalence classes $[M, A]$ of pairs consisting of a symplectic manifold M and a lagrangian submanifold $A \subset M$, called the core. The equivalence reflects the fact that these objects really describe the geometry of a neighborhood — or a "micro" neighborhood — of A in M.

In this "micro" setting, there is also a notion of canonical "micro" relations between two symplectic microfolds: They are lagrangian submicrofolds $[L, C]$ of the symplectic microfold product $[\overline{M}, A] \times [N, B]$. Their composition is generally as ill behaved as it is for regular canonical relations.

One of the main points of this paper is to identify a certain subset of canonical "micro" relations satisfying a new transversality condition which ensures that their composition is always well defined. We consider these transverse "micro" canonical relations as morphisms between symplectic microfolds; in this way, we obtain a new symmetric monoidal category: the extended microsymplectic category.

The extended symplectic "category" has been used as a sort of heuristic guideline in an attempt to quantize Poisson manifolds (see [**10**]) in a geometric way. These attempts have been only partially successful due in part to the existence of nonintegrable Poisson manifolds (hence restricting the class of Poisson manifold one can quantize), as well as to the ill-defined composition of canonical relations (limiting thus the functorial properties of these geometric quantization methods). The replacement of the extended symplectic "category" by its "micro" version provides new ways of dealing with both issues.

This paper lays the foundation for a series of work that revolves around two main themes: the categorification of Poisson geometry and its functorial quantization as explained below.

1.1. Categorification. Since $\mathbf{Sympl}^{\mathbf{ext}}_{\mathbf{mic}}$ is a monoidal category, it is natural to consider its category of monoids. The main statement we are aiming at is the equivalence between this latter category and the category of Poisson manifolds and Poisson maps. Future research directions will include the study of a weakened version of the monoid category whose maps are "bimodules". This should correspond to a "micro" Morita theory for Poisson manifolds.

1.2. Quantization. Our second line of work will focus on constructing a monoidal functor from the extended microsymplectic category (enhanced with half-density germs on the morphisms) to the category of vector spaces. Since monoidal functors between monoidal categories induce functors between their respective categories of monoid objects, we obtain in this way a "quantization" functor from the category of Poisson manifolds to the category of monoids in **Vect**.

2. Symplectic microfolds

A *local manifold pair* (M, A) consists of a manifold M and a submanifold $A \subset M$, called the *core*. Two manifold pairs (M, A) and (N, B) are said to be equivalent if $A = B$ and if there is a third manifold pair (U, A) such that U is an open subset in both M and N simultaneously. A map between local pairs is a smooth map from M to N that sends A to B. Note that we require equality of neighborhoods and not merely diffeomorphism.

Definition 2.1. A *microfold* is an equivalence class of a local pairs (M, A). We denote these equivalence classes either by $[M, A]$ or by $([M], A)$. Sometimes, $[M]$ will be referred to as a *manifold germ* around A.

We define an equivalence relation on the maps of local pairs that send a representative of $[M, A]$ to a representative of $[N, B]$ by declaring two such maps equivalent if there is a common neighborhood of A where they coincide. The equivalence classes, written as

$$[\Psi] : [M, A] \longrightarrow [N, B],$$

are the maps between microfolds. We say that $[\Psi]$ is a germ above $\phi : A \to B$ if, for $\Psi \in [\Psi]$, we have that $\Psi_{|A} = \phi$.

A *submicrofold* of a microfold $[M, A]$ is a microfold $[N, B]$ such that $N \subset M$ and $B \subset A$. We define the graph of a microfold map $[\Psi]$ as the submicrofold

$$\mathrm{gr}[\Psi] := \big([\mathrm{gr}\,\Psi], \mathrm{gr}\,\Psi_{|A}\big)$$

of the product microfold

$$[M, A] \times [N, B] := [M \times N, A \times B].$$

Microfolds and microfold maps form a category. Fibered products of microfolds are defined in the obvious way.

There are micro counterparts of symplectic manifolds.

Definition 2.2. A *symplectic microfold* is a microfold $[M, L]$ where M is a symplectic manifold and $L \subset M$ a lagrangian submanifold. We call *cotangent microbundles* the symplectic microfolds of the form $[T^*M, M]$.

Note. We will write Z_E to denote the zero section of a vector bundle $E \to M$. In Definition 2.2, we abused notation by writing $[T^*M, M]$ instead of $[T^*M, Z_{T^*M}]$.

A symplectomorphism between symplectic microfolds is a microfold map for which there is a representative that is a symplectomorphism.

Note. To avoid an explosion in the use of the prefix "micro," we will keep the usual "manifold terminology" when available and assume that we are talking about the "micro" version when microfolds are around and no confusion is possible. For examples, we choose to use "symplectomorphism" instead of "microsymplectomorphism," and so on.

Symplectic microfolds and their symplectomorphisms form a category, which we denote by $\mathbf{Sympl_{mic}}$.

Many special submanifolds of symplectic geometry have their corresponding "micro" versions.

Definition 2.3. A submicrofold $[S, X]$ of a symplectic microfold $[M, L]$ will be called *isotropic, lagrangian*, or *coisotropic* if there are representatives S and M such that S is isotropic, lagrangian or coisotropic in M.

The language of microfolds is useful to express local geometric properties; that is, properties that are true for all neighborhoods of some submanifold. For instance, the lagrangian embedding theorem can be phrased as follows.

Lagrangian Embedding Theorem. *For any symplectic microfold* $[M, L]$, *there exists a symplectomorphism*

$$[\Psi_{M,L}] : [M, L] \longrightarrow [T^*L, L], \tag{2.1}$$

above the identity.

Actually, this theorem was first stated and proved using the language of local manifold pairs and their equivalences (see [**9**]).

An important notion in symplectic geometry is that of a *canonical relation*, i.e., a lagrangian submanifold $V \subset \overline{M} \times N$, where (M, ω_M) and (N, ω_N) are symplectic manifolds and $\overline{M}$ is the symplectic manifold $(M, -\omega_M)$. Canonical relations are usually thought of as "generalized symplectomorphisms" and written as $V : M \to N$ instead of $V \subset \overline{M} \times N$. The rationale behind this is two-fold: the graph of a symplectomorphism is a canonical relation, and it is formally possible to extend the composition of symplectomorphisms to canonical relations. Namely, the composition of $V \subset \overline{M} \times N$

and $W \subset \overline{N} \times P$ is the subset of $\overline{M} \times P$ defined by

$$W \circ V = \mathrm{Red}\Big(\big(V \times W\big) \cap \big(M \times \Delta_N \times P\big)\Big) \\ = \mathrm{Red}\,\big(V \times_N W\big),$$

where Red is the "reduction" map that projects (m, n, n, p) to (m, p). The major issue here is that composition of canonical relations is generally ill defined: $W \circ V$ may fail to be a submanifold, although, when it is, it is a lagrangian one.[1] There is a well-known criterion which limits the wildness of the composition and which we will need later.

Theorem 2.1. *The composition $W \circ V$ of the canonical relations V and W as above is an immersed lagrangian submanifold of $\overline{M} \times P$ if the submanifolds $V \times W$ and $M \times \Delta_N \times P$ intersect cleanly.*

Nevertheless, it is standard to think of symplectic manifolds and canonical relations as a category. It is called the *extended symplectic "category"*[2] and will be denoted by $\mathbf{Sympl}^{\mathbf{ext}}$. Many constructions in **Sympl** extend to $\mathbf{Sympl}^{\mathbf{ext}}$. For instance, we define the image of a point $x \in M$ by a canonical relation $V \subset \overline{M} \times N$ as the subset of N given by

$$V(x) := \pi_N\Big(V \cap \big(\{x\} \times N\big)\Big), \tag{2.2}$$

where π_N is the projection on the second factor of $M \times N$. The tangent relation $TV : TM \to TN$ to a canonical relation $V : M \to N$ is the subset TV of $TM \times TN$ given by the set of tangent vectors to V.

The notion of canonical relation between symplectic manifolds can be transported to symplectic microfolds.

Definition 2.4. A *canonical relation* $([V], K)$ between the symplectic microfolds $[M, A]$ and $[N, B]$ is a lagrangian submicrofold $([V], K)$ of $[\overline{M} \times N, A \times B]$.

Note. We will often prefer the notation $([V], K)$ for canonical relations and reserve the notation $[M, A]$ for symplectic microfolds in order to distinguish between objects and morphisms. We will also use the arrow notation

$$\big([V], K\big) : [M, A] \longrightarrow [N, B]$$

to represent canonical relations between symplectic microfolds. We consider the core K as a submanifold of $B \times A$ and not as a submanifold of $A \times B$. In other words, we decide to regard the core as a generalized morphism from

[1] This is a special instance of symplectic reduction. Namely, the submanifolds $M \times \Delta_N \times P$ and $V \times W$ are, respectively, coisotropic and lagrangian in $\overline{M} \times N \times \overline{N} \times P$. The composition $W \circ V$ is exactly the quotient of $V \times W$ by the characteristic foliation of $M \times \Delta_N \times P$. This ensures that $W \circ V$ is a lagrangian submanifold whenever it is a submanifold.

[2] The quotes are there as a reminder that it is not really a category.

B to A. The reason for this contraintuitive convention will become apparent later on.

The composition of canonical relations in the microworld

$$[M, A] \overset{([V],K)}{\longrightarrow} [N, B] \overset{([W],L)}{\longrightarrow} [P, C]$$

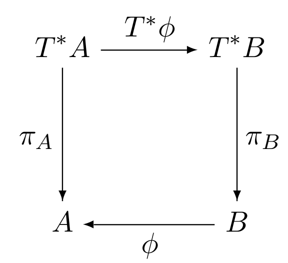
is given by the binary relation composition of their "components"

$$([W], L) \circ ([V], K) := ([W \circ V], K \circ L). \tag{2.3}$$

At this point, we still have the same kind of ill-defined composition for canonical relations between symplectic microfolds as we had for symplectic manifolds. However, we may consider a special type of canonical relations between symplectic microfolds that always compose well. This is what we do next.

3. Symplectic micromorphisms

3.1. Definitions. Our starting point is the cotangent lift $T^*\phi$ of a diffeomorphism ϕ:



It induces a canonical relation of the form

$$\big([\mathrm{gr}\, T^*\phi], \mathrm{gr}\, \phi\big) : [T^*A, A] \longrightarrow [T^*B, B].$$

The fact that the canonical relation $\big([\mathrm{gr}\, T^*\phi], \mathrm{gr}\, \phi\big)$ comes from a map implies the identities[3]

$$\Big(\mathrm{gr}\, T^*\phi\Big)(x) = \phi^{-1}(x), \tag{3.1}$$

$$\Big(T\, \mathrm{gr}\, T^*\phi\Big)(v) = \big(T\phi\big)^{-1}(v), \tag{3.2}$$

for all x in the zero section of T^*A and all tangent vectors v to the zero section of T^*A. Obviously, canonical relations coming from cotangent lifts compose well. It turns out that identities (3.1) and (3.2) are the key to this nice composability. We therefore make the following definition.

[3]Note that, in general, the graph of a map $f : X \to Y$, seen as a generalized morphism from Y to X, satisfies $\big(\mathrm{gr}\, f\big)(y) = f^{-1}(y)$, for $y \in Y$.

Definition 3.1. A *symplectic micromorphism* is a canonical relation of the form

$$\big([V], \operatorname{gr}\phi\big) : [M, A] \longrightarrow [N, B],$$

where ϕ is a smooth map from B to A and such that there exists a representative $V \in [V]$ for which

$$V(a) = \phi^{-1}(a), \quad \text{for all } a \in A, \tag{3.3}$$

$$TV(v) = \big(T\phi\big)^{-1}(v), \quad \text{for all } v \in T_a A, \tag{3.4}$$

where $V(a)$ is the image of a under the relation V as defined by (2.2) and $TV(v)$ is the image of v under the tangent relation TV. We will usually write $\big([V], \phi\big)$ instead of $\big([V], \operatorname{gr}\phi\big)$.

The next proposition gives various characterizations of symplectic micromorphisms. Recall that a submanifold X of a manifold M is transverse to a subbundle $\lambda \to Y$ of $TM \to M$ along a submanifold $Z \subset X \cap Y$ if

$$T_z X + \lambda_z = T_z M, \quad z \in Z.$$

In this case we write $X \pitchfork_Z \lambda$. In particular, a submanifold X is transverse to a submanifold Y along $Z \subset X \cap Y$ if X is transverse to TY along Z; we write this $X \pitchfork_Z Y$. A *splitting* of a symplectic microfold $[M, A]$ is a lagrangian subbundle K of the tangent bundle $TM\big|_A$ of M restricted to A such that, for all $x \in A$, we have[4] $T_x M = T_x A \oplus K_x$ (i.e., A is transverse to K along A).

Definition 3.2. A canonical relation of the form

$$\big([V], \phi\big) : [M, A] \longrightarrow [N, B]$$

is said to be *transverse to a splitting* K of $[N, B]$ if there is a $V \in [V]$ such that

$$V \cap (A \times N) = \operatorname{gr}\phi \quad \text{and} \quad V \pitchfork_{\operatorname{gr}\phi} \Big(\big(TA \oplus 0\big) \times \big(0 \oplus K\big)\Big).$$

In this case, we will abuse notation slightly and write

$$[V] \pitchfork_{\operatorname{gr}\phi} \big(TA \times K\big).$$

If $([V], \phi)$ is transverse to all splittings, we will say that it is *strongly transverse.*

The following proposition offers alternative descriptions of symplectic micromorphisms in terms of transverse intersections as pictured below:

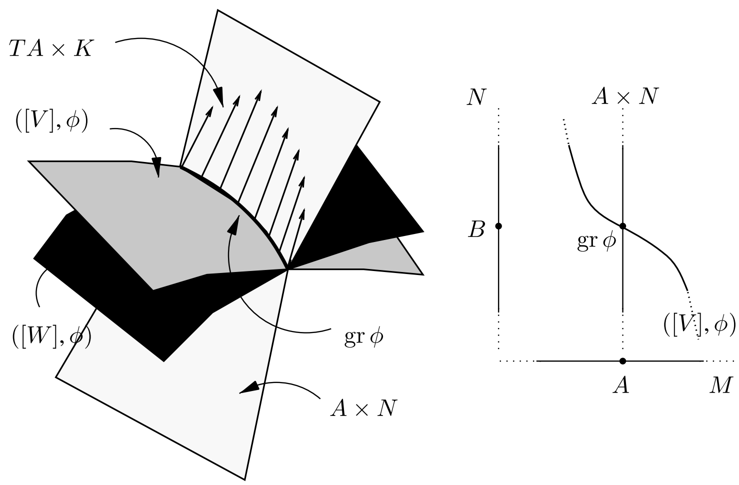
[4]Note that we will sometimes abuse notation slightly by writing $T_x L$ instead of $T_x L \oplus 0$ and L_x instead of $0 \oplus L_x$.

Proposition 3.1. *Consider a canonical relation of the form*

$$\big([V],\phi\big) : [M,A] \longrightarrow [N,B].$$

Then, the following statements are equivalent:

1. $([V],\phi)$ *is a symplectic micromorphism,*
2. $([V],\phi)$ *is strongly transverse,*
3. *there is a* $V\in[V]$ *such that* $V\cap(A\times N)=\operatorname{gr}\phi$ *is transverse.*



Proof. First step : $1\Rightarrow 2$. In general, we have that

$$V\cap(A\times N)=\bigcup_{a\in A}\{a\}\times V(a),$$

which yields $V\cap(A\times N)=\operatorname{gr}\phi$, because of (3.3). Similarly, one obtains

$$TV\cap(TA\times TN)=\operatorname{gr}T\phi$$

from (3.4). Now, for any splitting K of $[N,B]$, we have that

$$TA\times K\subset TA\times TN,$$

and, therefore, that

$$\begin{aligned}TV\cap(TA\times K)&=\Big(TV\cap(TA\times TN)\Big)\cap(TA\times K)\\&=T\operatorname{gr}\phi\cap(TA\times K),\end{aligned}$$

along $\operatorname{gr}\phi$. Since $T\operatorname{gr}\phi\subset TA\times TB$ and since $TB\cap K=0$, we have that

$$(v,w)\in T\operatorname{gr}\phi\cap(TA\times K)$$

if and only if $w=0$. By definition, we have that

$$T_{(\phi(b),b)}\operatorname{gr}\phi=\big\{(T_b\phi(w),w):\ w\in T_bB\big\},$$

and, therefore, we can conclude that

$$TV \cap (TA \times K) = 0$$

along $\mathrm{gr}\,\phi$. A dimension count yields

$$\dim T_{(\phi(b),b)}V + \dim(T_{\phi(b)}A \times K_b) = \dim T_{(\phi(b),b)}(M \times N),$$

which completes the proof that TV intersects $TA \times K$ transversally along $\mathrm{gr}\,\phi$.

Second step : $2 \Rightarrow 3$. First, note that the condition $V \cap (A \times N) = \mathrm{gr}\,\phi$ is part of both statements. Let K be a splitting of $[N, B]$. By hypothesis, we have that

$$TV + (TA \times K) = T(M \times N)$$

along $\mathrm{gr}\,\phi$, which implies in particular that

$$TV + (TA \times TN) = T(M \times N),$$

meaning that $A \cap (A \times N) = \mathrm{gr}\,\phi$ is a transverse intersection.

Third step : $3 \Rightarrow 1$. The fact that there exists $V \in [V]$ such that

$$V \cap (A \times N) = \mathrm{gr}\,\phi$$

implies (3.3). Namely, this gives immediately that

$$V \cap \big(\{a\} \times N\big) = \{a\} \times \phi^{-1}(a),$$

and, by definition of $V(a)$ (see (2.2)), we obtain that $V(a) = \phi^{-1}(a)$. Since the intersection of V with $A \times N$ is transverse, it is also clean, i.e., we have that

$$TV \cap (TA \times TN) = \mathrm{gr}\,T\phi.$$

Using this equation and the same argument as above, one obtains (3.4). □

In Step 1 of the proof of Proposition 3.1, we showed that if a canonical relation $([V], \phi)$ is a symplectic micromorphism then there exists $V \in [V]$, such that V intersects $A \times N$ cleanly in $\mathrm{gr}\,\phi$:

$$V \cap (A \times N) = \mathrm{gr}\,\phi, \tag{3.5}$$

$$TV \cap (TA \times TN) = \mathrm{gr}\,T\phi. \tag{3.6}$$

In turn, we proved that conditions (3.5) and (3.6) imply that $([V], \phi)$ is strongly transverse. This makes conditions (3.5) and (3.6) equivalent to $([V], \phi)$ being a symplectic micromorphism. Therefore, if we compare this with Statement 3 of Proposition 3.1, we see that clean intersection is enough. In Step 2, we see that it is enough to have a single splitting K of $[N, B]$ transverse to $([V], \phi)$ to show that it is a symplectic micromorphism. We obtain thus the following, apparently weaker, version of Proposition 3.1.

Corollary 3.1. *Consider a canonical relation of the form*

$$\big([V], \phi\big) : [M, A] \longrightarrow [N, B].$$

Then, the following statements are equivalent:

1. $([V], \phi)$ *is a symplectic micromorphism,*
2. $([V], \phi)$ *is transverse to a splitting of* $[N, B]$.
3. *there is a* $V \in [V]$ *such that* $V \cap (A \times N) = \operatorname{gr}\phi$ *is clean.*

3.2. Examples.

3.2.1. The unit symplectic microfold E. Let us denote by $\mathbf{E}$ the cotangent bundle of the one point manifold[5] $\{\star\}$, which we regard as a symplectic microfold, i.e.,

$$\mathbf{E} := \Big[T^*\{\star\}, \big\{(0, \star)\big\}\Big].$$

For any microsymplectic manifold $[M, L]$, there is a unique symplectic micromorphism

$$\mathbf{e}_{[M,L]} : \mathbf{E} \longrightarrow [M, L]$$

given by

$$\mathbf{e}_{[M,L]} := \Big(\big[\{(0, \star)\} \times L\big], \operatorname{pr}_L\Big),$$

where pr_L is the unique map from L to $\{\star\}$. On the other hand, symplectic micromorphisms

$$\nu : [M, L] \longrightarrow \mathbf{E}$$

are in bijection with lagrangian submanifold germs $[V_x]$ around a point $x \in L$ that are transverse to L at x. Namely, the core map

$$\mathbf{core}(\nu) : \{\star\} \longrightarrow L$$

is specified by the image $x \in M$ of the unique point $\star$; hence

$$\nu = \big([V_x], \{x\}\big).$$

Conditions (3.3) and (3.4) read $V \cap L = \{x\}$ and $T_xV \cap T_xL = 0$, i.e., V_x and L are transverse.

3.2.2. Cotangent lifts. Recall the standard identification

$$\overline{T^*M} \times T^*N \simeq T^*(M \times N)$$

via the "Schwartz transform" (see [**1**])

$$\mathcal{S}\big((p_1, x_1), (p_2, x_2)\big) = \big((-p_1, p_2), (x_1, x_2)\big).$$

For any smooth map $\phi : N \to M$, the transform of the conormal bundle

$$N^*(\operatorname{gr}\phi) \subset T^*(M \times N)$$

[5] We may define $\{\star\}$ as the manifold containing only the singleton $\star = \{\emptyset\}$.

is a symplectic micromorphism from $[T^*M, M]$ to $[T^*N, N]$ given by

$$T^*\phi := \Big(\big[\mathcal{S}^{-1}\big(N^*(\mathrm{gr}\,\phi)\big)\big], \phi\Big).$$

We call it the *cotangent lift* of ϕ and denote it again by $T^*\phi$. Note that, whenever ϕ is a diffeomorphism, we slightly abuse notation since in this case

$$T^*\phi = \Big(\big[\,\mathrm{gr}\,T^*\phi], \phi\Big).$$

3.2.3. Symplectomorphism germs. As in the macroworld, the graph

$$\mathrm{gr}[\Psi] = \big([\mathrm{gr}\,\Psi], \Psi^{-1}_{|B}\big)$$

of symplectomorphism germ $[\Psi] : [M, A] \to [N, B]$ is a symplectic micromorphism: obviously, we have that

$$(\mathrm{gr}\,\Psi) \cap (A \times N) = \mathrm{gr}\,\Psi^{-1}_B,$$
$$(\mathrm{gr}\,T\Psi) \cap (TA \times TN) = \mathrm{gr}\,T\Psi^{-1}_B.$$

The following proposition tells us that every symplectic micromorphisms whose core map is a diffeomorphism is the graph of a symplectomorphism germ.

Proposition 3.2. *If the core ϕ of a symplectic micromorphism*

$$([V], \phi) : [M, A] \longrightarrow [N, B]$$

is a diffeomorphism, then there exists a symplectomorphism germ

$$[\Psi] : [M, A] \longrightarrow [N, B]$$

such that $\mathrm{gr}[\Psi] = ([V], \phi)$.

Proof. For each $a \in A$, we have that the intersection

$$V \cap \big(\{a\} \times N\big) = \big(a, \phi^{-1}(a)\big)$$

is transverse. We see this by counting the dimensions and by remarking that the tangent space intersection

$$T_{(a,\phi^{-1}(a))}V \cap \big(\{0\} \times T_{\phi^{-1}(a)}N\big)$$

is contained in $\mathrm{gr}\,T\phi^{-1}$, which implies that this intersection must be zero since

$$\big(\{0\} \times T_{\phi^{-1}(a)}N\big) \cap \mathrm{gr}\,T\phi^{-1} = \{0\}.$$

This transverse intersection guarantees that, for each $a \in A$, there is a neighborhood K_a of $\big(a, \phi^{-1}(a)\big)$ in $M \times N$ and a neighborhood U_a of a in M such that the restriction of the first factor projection,

$$M \times N \supset V \cap K_a \longrightarrow U_a \subset M,$$

is a diffeomorphism. Let us denote by ρ_a its inverse and set $\Psi_a := \pi_N \circ \rho_a$. By construction, we have that

$$\mathrm{gr}\,\Psi_a = V \cap K_a.$$

Since, for two overlapping neighborhoods U_a and $U_{a'}$ as above, the corresponding maps Ψ_a and $\Psi_{a'}$ coincide on $U_a \cap U_{a'}$, we obtain a germ

$$[\Psi] : [M, A] \longrightarrow [N, B]$$

such that $\mathrm{gr}[\Psi] = ([V], \phi)$. Because $[V]$ is a lagrangian submanifold germ, $[\Psi]$ is a symplectomorphism germ. $\square$

3.3. Composition. We now prove that the composition of symplectic micromorphisms

$$[M, A] \xrightarrow{\big([V], \phi\big)} [N, B] \xrightarrow{\big([W], \psi\big)} [P, C].$$

is always well defined and that

$$\big([W \circ V], \phi \circ \psi\big) : [M, A] \longrightarrow [P, C]$$

is again a symplectic micromorphism. We first prove a cleanness result in order to apply Theorem 2.1 to our micro setting.

Lemma 3.1. *Let* $\big([V], \phi\big)$ *and* $\big([W], \psi\big)$ *be symplectic micromorphisms as above. For all* $V \in [V]$ *and* $W \in [W]$*,* $V \times W$ *intersects* $M \times \Delta_N \times N$ *transversally along* $\mathrm{gr}\,\phi \times_B \mathrm{gr}\,\psi$*.*

Proof. We need to show that

$$T(V \times W) + T(M \times \Delta_M \times P) = T(M \times N \times N \times P)$$

at all points

$$K(p) := \Big(\phi \circ \psi(p), \psi(p), \psi(p), p\Big)$$

in $\mathrm{gr}\,\phi \times_B \mathrm{gr}\,\psi$. In the symplectic vector space

$$T_{\phi\circ\psi(p)}\overline{M} \times T_{\psi(p)}N \times T_{\psi(p)}\overline{N} \times T_p P$$

we have that

$$T_{K(p)}(V \times W)^{\perp} = T_{K(p)}(V \times W),$$
$$(T_{\phi\circ\psi(p)}M \times T_{(\psi(p),\psi(p))}\Delta_N \times T_p P)^{\perp} = \{0\} \times T_{(\psi(p),\psi(p))}\Delta_N \times \{0\}.$$

Using the relation $(A + B)^{\perp} = A^{\perp} \cap B^{\perp}$, which holds for any subspaces A and B of a symplectic vector space, one sees that the transversality equation

$$T_{K(p)}(V \times W) + T_{K(p)}(M \times \Delta_M \times P) = T_{K(p)}(\overline{M} \times N \times \overline{N} \times P)$$

becomes equivalent to

$$\underbrace{T_{K(p)}(V\times W)\cap\Big(\{0\}\times T_{(\psi(p),\psi(p))}\Delta_N\times\{0\}\Big)}_{U}=\{(0,0,0,0)\}.$$

We shall now prove that this last equation holds. By assumption, we have that

$$T_{(\phi\circ\psi(p),\psi(p))}V\cap\big(T_{\phi\circ\psi(p)}A\times T_{\psi(p)}N\big)=T_{(\phi\circ\psi(p),\psi(p))}\operatorname{gr}\phi, \tag{3.7}$$

$$T_{(\psi(p),p)}W\cap\big(T_{\psi(p)}B\times T_pP\big)=T_{(\psi(p),p)}\operatorname{gr}\psi. \tag{3.8}$$

Moreover, we may rewrite U as

$$U=\underbrace{\Big(T_{(\phi\circ\psi(p),\psi(p))}V\cap\big(\{0\}\times T_{\psi(p)}N\big)\Big)}_{J}\times_{T_{\psi(p)}N}$$
$$\underbrace{\times_{T_{\psi(p)}N}\Big(T_{(\psi(p),p)}W\cap\big(T_{\psi(p)}N\times\{0\}\big)\Big)}_{L}.$$

Equation (3.7) tells us that $J\subset T_{\phi\circ\psi(p)}A\times T_{\psi(p)}B$. Using this and Equation (3.8), we can then write

$$U=\Big(T_{(\phi\circ\psi(p),\psi(p))}\operatorname{gr}\phi\cap\Big(\{0\}\times T_{\psi(p)}B\Big)\Big)\times_{T_{\psi(p)}N}$$
$$\times_{T_{\psi(p)}N}\Big(T_{(\psi(p),p)}\operatorname{gr}\psi\cap\Big(T_{\psi(p)}B\times\{0\}\Big)\Big).$$

Since

$$T_{(\psi(p),p)}\operatorname{gr}\psi=\Big\{(T_p\psi(v),v):\ v\in T_pC\Big\},$$

we see that

$$T_{(\psi(p),p)}\operatorname{gr}\psi\cap\Big(T_{\psi(p)}B\times\{0\}\Big)=\{(0,0)\}$$

and finally, that $U=\{(0,0,0,0)\}$, as desired. □

Proposition 3.3. *The composition of two symplectic micromorphisms* $([V],\phi)$ *and* $([W],\psi)$ *via* (2.3) *is well defined and yields a symplectic micromorphism again.*

Proof. Lemma 3.1 together with a continuity argument yield that there is a neighborhood U of $\operatorname{gr}\phi\times_B\operatorname{gr}\psi$ where $V\times W$ and $M\times\Delta_N\times P$ still intersect transversally. Therefore, the map

$$\mathrm{Red}:(V\times_N W)\cap U\longrightarrow M\times P,$$

restricted to this neighborhood, is a immersion according to Theorem 2.1. At this point, recall that a proper immersion $i:X\to Y$ that is injective on a closed submanifold $A \subset X$ is a embedding on a neighborhood of A. Since the maps ϕ and ψ are smooth, $\operatorname{gr}\phi \times_B \operatorname{gr}\psi$ is closed. Moreover, on this submanifold, we have that

$$\operatorname{Red}\Big(\big((\phi\circ\psi)(p),\psi(p),\psi(p),p\big)\Big) = \big((\phi\circ\psi)(p),p\big),$$

meaning that Red maps $\operatorname{gr}\phi\times_B\operatorname{gr}\psi$ diffeomorphically to $\operatorname{gr}(\phi\circ\psi)$. Therefore, there is a neighborhood $\overline{U}$ of $\operatorname{gr}\phi \times_B \operatorname{gr}\psi$ such that $\operatorname{Red}(\overline{U})$ is a lagrangian submanifold containing $\operatorname{gr}(\phi\circ\psi)$. This proves that the lagrangian submanifold germ $([W\circ V],\phi\circ\psi)$ is well defined. We need to show that it is a symplectic micromorphism, i.e., conditions (3.5) and (3.6) hold. To begin with, note that

$$W\circ V \cap (A\times P) = \operatorname{Red}\Big(\big(V\cap(A\times N)\big)\times_N W\Big).$$

Now, since, by assumption, $V\cap(A\times N) = \operatorname{gr}\phi \subset A\times B$, we see that

$$\begin{aligned}\big(V\cap(A\times N)\big)\times_N W &= \big(V\cap(A\times N)\big)\times_N \big(W\cap(B\times P)\big)\\ &= (\operatorname{gr}\phi)\times_B(\operatorname{gr}\psi).\end{aligned}$$

Therefore, we obtain (3.5) for $W\circ V$, namely

$$\begin{aligned}W\circ V\cap(A\times P) &= \operatorname{Red}\big((\operatorname{gr}\phi)\times_N(\operatorname{gr}\psi)\big)\\ &= \operatorname{gr}(\phi\circ\psi).\end{aligned}$$

Set $K(p) = \big(\phi\circ\psi(p),\psi(p),\psi(p),p\big)$ with $p\in P$. Realizing that

$$T_{\left(\phi\circ\psi(p),p\right)}(W\circ V) = T_{K(p)}\operatorname{Red}\Big(\Big(T_{\left(\phi\circ\psi(p),\psi(p)\right)}V\Big)\times_{T_{\psi(p)}N}\Big(T_{\left(\psi(p),p\right)}W\Big)\Big),$$

a similar computation on the tangent space level yields (3.6) for $W\circ V$. □

4. Symplectic categories

In this section, we reinterpret the results obtained so far in the language of monoidal categories. We refer the reader to [**6**] for an exposition on monoidal categories.

Notation. We will sometimes write $\mathcal{C}_0$ for the objects and $\mathcal{C}_1$ for the morphisms of a category $\mathcal{C}$. Accordingly, given a functor $F:\mathcal{C}\to\mathcal{D}$, we denote by $F_0:\mathcal{C}_0\to\mathcal{D}_0$ the object component of F and by $F_1:\mathcal{C}_1\to\mathcal{D}_1$ its morphism component.

So far, we have seen that "symplectic categories" come in four flavors. In the macroworld of symplectic manifolds we have:

- **Sympl**, the usual symplectic category of symplectic manifolds and symplectomorphisms,

- $\mathbf{Sympl^{ext}}$, the extended symplectic "category", where symplectomorphisms are replaced by canonical relations, and which is not a category.

In the microworld of symplectic microfolds, we have:

- $\mathbf{Sympl_{mic}}$, the *microsymplectic category*, i.e., the category of symplectic microfolds and symplectomorphism germs,
- $\mathbf{Sympl^{ext}_{mic}}$, the *extended microsymplectic category*, i.e., the category of symplectic microfolds and symplectic micromorphisms.

A major improvement in the microworld is that, this time, symplectic micromorphisms always compose. Hence, $\mathbf{Sympl^{ext}_{mic}}$ is a category, which enlarges the category $\mathbf{Sympl_{mic}}$ in the following precise sense:

Definition 4.1. A category $\mathcal{D}$ is said to be an *enlargement* of a category $\mathcal{C}$ if there is a functor $F : \mathcal{C} \to \mathcal{D}$ such that F_0 is a bijection and such that F_1 is injective and bijective on the isomorphisms, i.e.,

$$\mathrm{Iso}(x, y) \simeq \mathrm{Iso}\big(F_0(x), F_0(y)\big),$$

for all objects $x, y \in \mathcal{C}_0$. In this case, we also call the functor an *enlargement*.

Intuitively, enlarging a category means keeping the same objects while adding morphisms that are not isomorphisms.

Clearly, the functor

$$\mathbf{Gr} : \mathbf{Sympl_{mic}} \longrightarrow \mathbf{Sympl^{ext}_{mic}}$$

that is the identity on objects and that takes a symplectomorphism germ to its graph is an enlargement of categories.

The extended microsymplectic category is a symmetric monoidal category. The tensor product of symplectic microfolds is simply given by

$$[M, A] \otimes [N, B] := [M \times N, A \times B].$$

Given two symplectic micromorphisms

$$([V_i], \phi_i) : [M_i, A_i] \longrightarrow [N_i, B_i], \quad i = 1, 2,$$

we define their tensor product as

$$([V_1], \phi_1) \otimes ([V_2], \phi_2) := \Big(\Big[\big(\,\mathrm{id}_{M_1} \times \epsilon_{N_1, M_2} \times \mathrm{id}_{N_2}\,\big)(V_1 \times V_2)\Big], \phi_1 \times \phi_2\Big),$$

where $\epsilon_{X,Y}(x, y) = (y, x)$ is the usual factor permutation. The unit object $\mathbf{E}$ is the cotangent bundle of the one-point manifold $\{\star\}$. As shown in Section 3.2.1, $\mathbf{E}$ is initial. The symmetry isomorphisms are given by

$$\sigma_{[M,A],[N,B]} := \Big(\big[\,\mathrm{gr}\,\epsilon_{M,N}\big], \epsilon_{B,A}\Big).$$

Note that the opposite symplectic manifold $\overline{(M, \omega)} = (M, -\omega)$ has its natural micro version $\overline{[M, L]} := [\overline{M}, L]$. It is straightforward, although cumbersome, to verify the following:

Theorem 4.1. $(\mathbf{Sympl}^{\mathbf{ext}}_{\mathbf{mic}}, \otimes, E, \sigma)$ *is a symmetric monoidal category with initial unit* E.

We conclude this section by commenting on the relationship between the extended microsymplectic category and the lagrangian operads introduced in [**2**, **3**].

Definition 4.2. An *operad* is a collection $\{A(n)\}_{n\geq 0}$ of sets together with composition laws

$$\begin{aligned} A(n) \times A(k_1) \times \cdots \times A(k_n) &\longrightarrow A(k_1 + \cdots + k_n), \\ (F, G_1, \ldots, G_n) &\longmapsto F(G_1, \ldots, G_n) \end{aligned}$$

for each $n, k_1, \ldots, k_n \in \mathbb{N}$, satisfying the associativity equations

$$\begin{aligned} &\Big(F(G_1, \ldots, G_n)\Big)(H_{11}, \ldots, H_{1k_1}, \ldots H_{n1}, \ldots, H_{nk_n}) \\ &\quad = F\Big(G_1(H_{11}, \ldots, H_{1k_1}), \ldots, G_n(H_{n1}, \ldots, H_{nk_n})\Big), \end{aligned}$$

and unit $I \in A(1)$ such that $F(I, \ldots, I) = F$ for all $F \in A$.

For any object X in a monoidal category $(\mathcal{C}, \otimes, \mathbf{E})$, one defines the endomorphism operad $\mathcal{END}(X)$ of X to be the collection

$$\mathcal{END}(X)(n) = \hom(X^{\otimes n}, X)$$

with the usual convention that $X^{\otimes n} = \mathbf{E}$ for $n = 0$. The composition laws are given by the tensor product and the usual composition in the category:

$$F(G_1, \cdots, G_n) := F \circ (G_1 \otimes \cdots \otimes G_n).$$

The unit is the identity morphism $\mathrm{id}_X \in \hom(X, X)$.

Since $\mathbf{Sympl}^{\mathbf{ext}}_{\mathbf{mic}}$ is a monoidal category, it makes sense to consider the endomorphism operad $\mathcal{END}([M, A])$ of a symplectic microfold $[M, A]$. There are two special operads sitting inside of it. First, the cotangent lifts of the n-diagonal maps $\Delta^n : A \to A^n$, $n \geq 1$, form an operad

$$\begin{aligned} \mathcal{L}_\Delta([M, A])(n) &:= \big\{T^*\Delta^n\big\}, \quad n \geq 1, \\ \mathcal{L}_\Delta([M, A])(0) &:= \big\{\mathbf{e}_{[M,A]}\big\}, \end{aligned}$$

thanks to the properties

$$\begin{aligned} T^*\Delta^1 &= \mathrm{id}_{[M,A]}, \\ \Delta^{k_1 + \cdots + k_n} &= \Delta^n \circ (\Delta^{k_1} \times \cdots \times \Delta^{k_n}), \\ T^*\Delta^{n-1} &= T^*\Delta^n \circ (\mathrm{id}_{[M,A]} \otimes \cdots \otimes \mathbf{e}_{[M,A]} \otimes \cdots \otimes \mathrm{id}_{[M,A]}). \end{aligned}$$

Now, $\mathcal{L}_\Delta([M, A])$ sits in the suboperad $\mathcal{L}([M, A])$ of $\mathcal{END}([M, A])$ defined as follows. For $n \geq 1$, $\mathcal{L}([M, A])(n)$ is the set of symplectic micromorphisms $[M, A]^{\otimes n} \to [M, A]$ whose core map is the n-diagonal Δ^n. For $n = 0$, we set $\mathcal{L}([M, A])(0) = \{\mathbf{e}_{[M,A]}\}$. Note that the first degree of this suboperad is interesting: $\mathcal{L}([M,A])(1)$ is the group of symplectorphism germs $[\psi] : [M,A] \to [M,A]$ fixing A.

In [**3**], $\mathcal{L}_\Delta([T^*\mathbb{R}^n, \mathbb{R}^n])$ was called the cotangent lagrangian operad over $T^*\mathbb{R}^n$ and $\mathcal{L}([T^*\mathbb{R}^n, \mathbb{R}^n])$ the local lagrangian operad over $T^*\mathbb{R}^n$. They were introduced *ad hoc* in terms of generating functions of lagrangian submanifold germs.

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