COVERING SPACES AND ℚ-GRADINGS ON HEEGAARD FLOER HOMOLOGY

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Heegaard Floer homology, first introduced by P. Ozsváth and Z. Szabó in [**OS04b**], associates to a 3-manifold Y, a family of relatively graded abelian groups $HF(Y, \mathfrak{t})$, indexed by Spin^c structures $\mathfrak{t}$ on Y. In the case that Y is a rational homology sphere, Ozsváth and Szabó lift the relative $\mathbb{Z}$-grading to an absolute $\mathbb{Q}$-grading [**OS06**]. This induces a relative $\mathbb{Q}$-grading on $\bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} HF(Y, \mathfrak{t})$. In this paper, we describe an alternate construction of this relative $\mathbb{Q}$-grading by studying the Heegaard Floer homology of covering spaces.

1. Introduction

In [**OS04b**], P. Oszváth and Z. Szabó associated to a 3-manifold Y families $\widehat{HF}(Y, \mathfrak{t})$, $HF^+(Y, \mathfrak{t})$, $HF^-(Y, \mathfrak{t})$, and $HF^\infty(Y, \mathfrak{t})$ of abelian groups, indexed by Spin^c structures $\mathfrak{t}$ on Y, collectively known as Heegaard Floer homology. (Below, we shall use $HF(Y, \mathfrak{t})$ to refer to any of these groups.) These groups arise as the homology groups of certain Lagrangian intersection Floer chain complexes $\widehat{CF}(Y, \mathfrak{t})$, $CF^+(Y, \mathfrak{t})$, $CF^-(Y, \mathfrak{t})$, and $CF^\infty(Y, \mathfrak{t})$, and as such inherit relative $\mathbb{Z}$-gradings or, if $c_1(\mathfrak{t})$ is non-torsion, relative $\mathbb{Z}/n$ gradings; we denote all of these gradings by gr.

In [**OS06**], for $c_1(\mathfrak{t})$ torsion, Ozsváth and Szabó used a bordism construction to lift gr to an absolute $\mathbb{Q}$-grading on $HF(Y, \mathfrak{t})$. In other words, they found an absolute $\mathbb{Q}$-grading $\widetilde{\mathrm{gr}}$ on $HF(Y, \mathfrak{t})$ satisfying $\mathrm{gr}(\xi, \eta) = \widetilde{\mathrm{gr}}(\xi) - \widetilde{\mathrm{gr}}(\eta)$ for all homogeneous elements $\xi, \eta \in HF(Y, \mathfrak{t})$. This defines an absolute $\mathbb{Q}$-grading on the group

$$HF(Y, \text{torsion}) := \bigoplus_{c_1(\mathfrak{t}) \text{ is torsion}} HF(Y, \mathfrak{t}).$$

In [**OS03**], they used this absolute $\mathbb{Q}$-grading to give restrictions on which knots can, under surgery, give rise to lens spaces, and to give restrictions

on intersection forms of 4-manifolds bounding a given 3-manifold; in [**MO**], C. Manolescu and B. Owens used the absolute $\mathbb{Q}$-grading on the branched double cover of a knot to produce a concordance invariant of knots.

In this paper, we use covering spaces to give an alternate construction of the relative $\mathbb{Q}$-grading on $HF(Y, \text{torsion})$ induced by the absolute $\mathbb{Q}$-grading $\widetilde{\text{gr}}$. Specifically, given homogeneous elements $\xi, \eta \in HF(Y, \text{torsion})$ not necessarily lying in the same $HF(Y, \mathfrak{t})$, we are able to reconstruct $\widetilde{\text{gr}}(\xi) - \widetilde{\text{gr}}(\eta)$. See Theorems 2.6 and 4.1. Of course, the relative $\mathbb{Q}$-grading contains less information about Y than the absolute $\mathbb{Q}$-grading, and therefore our covering space construction is less powerful than the bordism construction of Ozsváth and Szabó; on the other hand, our construction offers a new perspective on $\widetilde{\text{gr}}$ and leads to a simple algorithm for computing the relative $\mathbb{Q}$-grading at the chain level.

Recall ([**OS04b**, Section 4]) that there are short exact sequences of chain complexes

$$0 \to \widehat{CF}(Y, \mathfrak{t}) \to CF^+(Y, \mathfrak{t}) \to CF^+(Y, \mathfrak{t}) \to 0,$$
$$0 \to CF^-(Y, \mathfrak{t}) \to CF^\infty(Y, \mathfrak{t}) \to CF^+(Y, \mathfrak{t}) \to 0.$$

We specify that the relative $\mathbb{Q}$-gradings on these chain complexes be preserved by the maps in these short exact sequences. Consequently, it suffices to define the relative $\mathbb{Q}$-grading on $\widehat{CF}(Y, \text{torsion})$, and we restrict our attention to this chain complex for the rest of the paper.

We now describe the structure of this paper. In Section 2, we explain our covering space construction of the relative $\mathbb{Q}$-grading. This section requires no prior knowledge of Heegaard Floer homology. We also explicitly describe how to compute the relative $\mathbb{Q}$-grading at the chain level. In Section 3, we briefly review Ozsváth and Szabó's construction of $\widetilde{\text{gr}}$, and in Section 4 we prove that our definition of the relative $\mathbb{Q}$-grading agrees with the Ozsváth–Szabó definition. We defer a necessary computation for lens spaces until Section 5. Finally, we mention some directions for future research in Section 6.

2. Gradings and covering spaces

2.1. Review of the relative $\mathbb{Z}$-grading. We begin by defining our use of the term "grading."

Definition. Let G and S be abelian groups.

We say that f is an *absolute S-grading on G with homogeneous elements* $\mathcal{H}$ if $\mathcal{H} \subset A$ generates A and $f : \mathcal{H} \to S$ such that for each $s \in S$, $f^{-1}(s) \cup \{0\}$ is a subgroup of A.

A *relative S-grading on G with homogeneous elements* $\mathcal{H}$ is an equivalence class of absolute S-gradings of G with homogeneous elements $\mathcal{H}$, where two absolute S-gradings are equivalent if they differ by a constant in S.

In this paper, S will always be $\mathbb{Z}$ or $\mathbb{Q}$. If G is free abelian, one can specify an absolute S-grading on G with homogeneous elements $\mathcal{H}$ by declaring a basis $\mathcal{U}$ of G to be contained in $\mathcal{H}$ and specifying a function $f : \mathcal{U} \to S$. One can specify a relative S-grading of G by instead specifying a function $F : \mathcal{U} \times \mathcal{U} \to S$ that is *additive* in the sense that $F(x,y) + F(y,z) = F(x,z)$.

As mentioned in the introduction, for any Spin^c structure $\mathfrak{t}$ on Y with $c_1(\mathfrak{t})$ torsion, there is a relative $\mathbb{Z}$-grading, gr, on $\widehat{HF}(Y, \mathfrak{t})$. We will describe gr at the chain level. That is, given a pointed Heegaard diagram $\mathcal{S}$ for Y, we will grade the group $\widehat{CF}(\mathcal{S}, \mathfrak{t})$. More generally, we will define gr on $\widehat{CF}(\mathcal{S}, \mathfrak{t})$ when $\mathcal{S}$ is an ℓ-pointed Heegaard diagram (defined below). The material in this subsection has been extracted from Section 3 of [**OS**], Section 4 of [**Lip06**], and the first half of [**OS04b**]; we gather it here for the reader's convenience.

Definition. An *ℓ-pointed Heegaard diagram $\mathcal{S}$* is a 4-tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ where:

- Σ is an oriented surface of genus $g > 0$.
- $\boldsymbol{\alpha}$ is a union of disjoint simple closed curves $\alpha_1, \ldots, \alpha_{g+\ell-1}$, which span a rank g sublattice of $H_1(\Sigma)$.
- $\boldsymbol{\beta}$ is a union of disjoint simple closed curves $\beta_1, \ldots, \beta_{g+\ell-1}$, which span a rank g sublattice of $H_1(\Sigma)$.
- $\boldsymbol{\alpha}$ intersects $\boldsymbol{\beta}$ transversely.
- $\mathbf{z}$ is a collection of points $\{z_1, \ldots, z_\ell\} \subset \Sigma \smallsetminus \boldsymbol{\alpha} \smallsetminus \boldsymbol{\beta}$.
- Each component of $\Sigma \smallsetminus \boldsymbol{\alpha}$ contains exactly one z_i.
- Each component of $\Sigma \smallsetminus \boldsymbol{\beta}$ contains exactly one z_i.

The α and β circles specify two handlebodies, U_α and U_β, with boundary Σ. We say that $\mathcal{S}$ is an *ℓ-pointed Heegaard diagram for $Y_{\alpha,\beta} = U_\alpha \cup_\Sigma U_\beta$.*

When $\ell = 1$, we say that $\mathcal{S}$ is a *pointed Heegaard diagram.* (The concept of an ℓ-pointed Heegaard diagram was introduced in [**OS**, Section 3]; earlier papers restricted attention to the $\ell = 1$ case. Many theorems that were originally proved in the $\ell = 1$ case can be trivially generalized to the case of arbitrary ℓ.)

Fix an oriented 3-manifold Y and a metric on Y. Let f be a self-indexing Morse–Smale function on Y with ℓ index zero and ℓ index three critical points, and choose ℓ flowlines of ∇f connecting the index zero and index three critical points in pairs. One can then construct an ℓ-pointed Heegaard diagram $\mathcal{S}$ for Y as follows:

- Take $\Sigma = f^{-1}(3/2)$.
- Take $\boldsymbol{\alpha}$ to be the intersection of Σ with the flowlines leaving index one critical points.

- Take $\boldsymbol{\beta}$ to be the intersection of Σ with the flowlines entering index two critical points.
- Take $\mathbf{z}$ to be the intersection of Σ with the ℓ flowlines chosen above.

In this case we say that f is *compatible* with $\mathcal{S}$. Observe that, given $\mathcal{S}$, one can always construct a compatible Morse function f.

Let $\mathcal{S}$ be an ℓ-pointed Heegaard diagram for Y. Define $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to be the set of all $(g + \ell - 1)$-element subsets $\mathbf{x} \subset \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ such that each α_i contains exactly one element of $\mathbf{x}$, and each β_i contains exactly one element of $\mathbf{x}$.[1]

We now define a map

$$s : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y).$$

Fix a metric on Y, and choose a Morse function f compatible with $\mathcal{S}$. Then $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ determines $g + \ell - 1$ flowlines connecting the $g + \ell - 1$ index one critical points to the $g + \ell - 1$ index two critical points in pairs, and $\mathbf{z}$ determines ℓ flowlines connecting the ℓ index zero critical points to the ℓ index three critical points in pairs. Consider small neighborhoods of these flowlines containing the critical points. Notice that ∇f is nonvanishing outside these neighborhoods, and that, since each neighborhood contains two critical points of opposite parity, we can extend ∇f to a nonvanishing vector field $V(\mathbf{x})$ on all of Y.

The vector field $V(\mathbf{x})$ reduces the structure group of TY from $\mathrm{SO}(3)$ to $\mathrm{SO}(2)$, and since $\mathrm{SO}(2) = \mathrm{U}(1) \subset \mathrm{U}(2) = \mathrm{Spin}^c(3)$, it follows that $V(\mathbf{x})$ determines a Spin^c structure $s(\mathbf{x})$ on Y. One can check that $s(\mathbf{x})$ does not depend on the choices of metric, compatible Morse function, and extension of ∇f. (Specifically, while different choices will determine a different vector field $V(\mathbf{x})$, the new vector field will be *homologous* to the original one in the sense of [**Tur97**]. Consequently $s(\mathbf{x})$ is unchanged, as shown in [**Tur97**].)

Now let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ where $\mathbf{x} = \{x_1, \ldots, x_{g+\ell-1}\}$ and $\mathbf{y} = \{y_1, \ldots, y_{g+\ell-1}\}$. Choose 1-chains a in $\boldsymbol{\alpha}$ and b in $\boldsymbol{\beta}$ such that

$$\partial a = \partial b = (y_1 + \cdots + y_{g+\ell-1}) - (x_1 + \cdots + x_{g+\ell-1}).$$

Then $\partial(a - b) = 0$, so $a - b$ descends to an element of $H_1(\Sigma)$, which in turn defines an element $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(Y)$ via the isomorphism

$$H_1(Y) \cong \frac{H_1(\Sigma)}{[\alpha_1], \ldots, [\alpha_{g+\ell-1}], [\beta_1], \ldots, [\beta_{g+\ell-1}]}.$$

It is clear that $\epsilon(\mathbf{x}, \mathbf{y})$ is independent of choices a and b.

There is a nice relationship between ϵ and Spin^c structures. Recall that the set of (homotopy classes of) Spin^c structures on Y forms an affine copy of

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[1] The notation comes from thinking of $\mathbb{T}_\alpha$ as the quotient of the torus $\alpha_1 \times \cdots \times \alpha_{g+\ell-1}$ by permutations (and similar for $\mathbb{T}_\beta$) so that the intersection $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ takes place in the $(g + \ell - 1)$-fold symmetric product of Σ.

$H^2(Y)$. The following lemma is a generalization of Lemma 2.19 of [**OS04b**] to the ℓ-pointed case, and is closely related to Lemma 3.11 in [**OS**].

Lemma 2.1. *For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the difference class $(s(\mathbf{x}) - s(\mathbf{y})) \in H^2(Y)$ is the Poincaré dual of $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(Y)$.*

Let $D_1, \ldots, D_N$ be the closures of the connected components of $\Sigma \smallsetminus \boldsymbol{\alpha} \smallsetminus \boldsymbol{\beta}$, thought of as 2-chains, labeled so that $z_i \in D_i$ for $i \leq \ell$. Define $C_2(\mathcal{S})$ to be the group of 2-chains in Σ generated by the D_i's, and define $\hat{C}_2(\mathcal{S})$ to be the subgroup of $C_2(\mathcal{S})$ generated by the D_i's with $i > \ell$, or in other words, $\hat{C}_2(\mathcal{S})$ is generated by the closures of connected components of $\Sigma \smallsetminus \boldsymbol{\alpha} \smallsetminus \boldsymbol{\beta}$ not containing any element of $\mathbf{z}$.

For any $A \in C_2(\mathcal{S})$, define $\partial_\alpha A$ to be the intersection of ∂A with $\boldsymbol{\alpha}$. For all $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ where $\mathbf{x} = \{x_1, \ldots, x_{g+\ell-1}\}$ and $\mathbf{y} = \{y_1, \ldots, y_{g+\ell-1}\}$, we define

$$\pi_2(\mathbf{x}, \mathbf{y}) = \{A \in C_2(\mathcal{S}) \mid \partial \partial_\alpha A = (y_1 + \cdots + y_{g+\ell-1}) - (x_1 + \cdots + x_{g+\ell-1})\}$$

$$\hat{\pi}_2(\mathbf{x}, \mathbf{y}) = \pi_2(\mathbf{x}, \mathbf{y}) \cap \hat{C}_2(\mathcal{S}).$$

Lemma 2.2. *For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, if $s(\mathbf{x}) = s(\mathbf{y})$, then $\hat{\pi}_2(\mathbf{x}, \mathbf{y})$ is nonempty.*

Proof. Construct a and b as in our definition of $\epsilon(\mathbf{x}, \mathbf{y})$. By the previous lemma, $\epsilon(\mathbf{x}, \mathbf{y}) = 0$. By the definition of ϵ, this means that $a - b$ plus some α and β circles is zero in $H_1(\Sigma)$ and hence equal to the boundary of some $C = \sum_{i=1}^{N} c_i D_i \in C_2(\mathcal{S})$. Observe that $C \in \pi_2(\mathbf{x}, \mathbf{y})$. For $i \leq \ell$, let A_i be the closure of the connected component of $\Sigma \smallsetminus \boldsymbol{\alpha}$ containing z_i, thought of as a 2-chain in $C_2(\mathcal{S})$. By the definition of an ℓ-pointed Heegaard diagram, the coefficient of D_i in the expression for A_j must be δ_{ij}, so one can check that $C - \sum_{i=1}^{\ell} c_i A_i \in \hat{\pi}_2(\mathbf{x}, \mathbf{y})$. $\square$

Let $\mathcal{U}_{\mathfrak{t}} = \{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid s(\mathbf{x}) = \mathfrak{t}\}$, and define $\widehat{CF}(\mathcal{S}, \mathfrak{t})$ to be the free abelian group generated by $\mathcal{U}_{\mathfrak{t}}$. We also define $\widehat{CF}(\mathcal{S}) = \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} \widehat{CF}(\mathcal{S}, \mathfrak{t})$. Our goal in this subsection is to define a relative $\mathbb{Z}$-grading, gr, on $\widehat{CF}(\mathcal{S}, \mathfrak{t})$ when $\mathfrak{t}$ is torsion. We declare $\mathcal{U}_{\mathfrak{t}}$ to be homogeneous so that we just need to define an additive function $\mathrm{gr} : \mathcal{U}_{\mathfrak{t}} \times \mathcal{U}_{\mathfrak{t}} \to \mathbb{Z}$. Before doing so, we introduce some notation. For any $A = \sum_{i=1}^{N} a_i D_i \in C_2(\mathcal{S})$, we define the *Euler measure of A*, $e(A)$, as follows. If D_i has p_i (not necessarily distinct) "vertices" in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$, define $e(D_i) = \chi(D_i) - p_i/4$ where $\chi(D_i)$ is the Euler characteristic. Extend this definition linearly to $C_2(\mathcal{S})$, so that

$$e(A) = \sum_{i=1}^{N} a_i(\chi(D_i) - p_i/4).$$

For any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $A \in C_2(\mathcal{S})$, we define $n_\mathbf{x}(A)$ as follows. Each $x \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ "touches" four of the D_is (with possible repetition); define $n_x(A)$

to be the average of the coefficients of these four D_is in the expression for A. Finally, we define $n_{\mathbf{x}}(A) = \sum_{x \in \mathbf{x}} n_x(A)$.

Definition. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s(\mathbf{x}) = s(\mathbf{y})$ is torsion. By the previous lemma there exists $A \in \hat{\pi}_2(\mathbf{x}, \mathbf{y})$. Define $\mathrm{gr}(\mathbf{x}, \mathbf{y})$ by the formula

$$(2.1) \qquad \mathrm{gr}(\mathbf{x}, \mathbf{y}) = e(A) + n_{\mathbf{x}}(A) + n_{\mathbf{y}}(A).$$

The formula (2.1), first suggested by Ozsváth and Szabó, comes from [**Lip06**, Section 4], where it was proved to agree with the standard definition in [**OS04b**] in terms of the Maslov index.[2] It follows that gr is additive. For completeness, we include a proof that gr is well-defined.

Proposition 2.3. *In the definition before,* $\mathrm{gr}(\mathbf{x}, \mathbf{y})$ *does not depend on choice of* $A \in \hat{\pi}_2(\mathbf{x}, \mathbf{y})$.

Proof. Observe that A is unique up to addition of elements in $\hat{\pi}_2(\mathbf{x}, \mathbf{x})$. By [**OS04a**, Proposition 7.5], for any $P \in \hat{\pi}_2(\mathbf{x}, \mathbf{x})$,

$$\langle c_1(s(\mathbf{x})), P \rangle = e(P) + 2n_{\mathbf{x}}(P)$$
$$\langle c_1(s(\mathbf{y})), P \rangle = e(P) + 2n_{\mathbf{y}}(P).$$

Since $s(\mathbf{x})$ and $s(\mathbf{y})$ are torsion, the left sides must vanish and we are left with

$$n_{\mathbf{x}}(P) = n_{\mathbf{y}}(P) = -\frac{1}{2}e(P).$$

(Indeed, the reason we deal only with torsion Spin^c structures $\mathfrak{t}$ throughout this paper is that we need $\langle c_1(\mathfrak{t}), P \rangle$ to be zero.) Thus, calculating $\mathrm{gr}(\mathbf{x}, \mathbf{y})$ using $A + P$ instead of A, we have

$$\begin{aligned} \mathrm{gr}(\mathbf{x}, \mathbf{y}) &= e(A + P) + n_{\mathbf{x}}(A + P) + n_{\mathbf{y}}(A + P) \\ &= e(A) + e(P) + n_{\mathbf{x}}(A) + n_{\mathbf{x}}(P) + n_{\mathbf{y}}(A) + n_{\mathbf{y}}(P) \\ &= e(A) + n_{\mathbf{x}}(A) + n_{\mathbf{y}}(A) \end{aligned}$$

$\square$

We have now defined a relative $\mathbb{Z}$-grading on $\widehat{CF}(\mathcal{S}, \mathfrak{t})$ when $\mathfrak{t}$ is torsion. If $\mathcal{S}$ is weakly admissible in the sense of [**OS04b**, Section 5], one can make $\widehat{CF}(\mathcal{S})$ into a chain complex whose homology is an invariant of (Y, ℓ), independent of the choice of $\mathcal{S}$ [**OS**].[3] When $\ell = 1$, we can define the *Heegaard Floer homology group*

$$\widehat{HF}(Y) := H(\widehat{CF}(\mathcal{S})).$$

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[2]The proofs in [**Lip06**] all deal with the case $\ell = 1$ but can easily be extended to the general case using the proof of Theorem 2.4.

[3]We omit the definition of *weakly admissible* and the definition of the differential because both are peripheral to the content of this paper. In later sections of this paper, all Heegaard diagrams will be implicitly assumed to be weakly admissible. Arranging this is never difficult; see [**OS04b**, Section 5].

Turning our attention back to the case of general ℓ, it follows from [**Lip06**, Corollary 4.3] and the definition of the differential that gr descends to a relative $\mathbb{Z}$-grading on $H(\widehat{CF}(\mathcal{S}, \mathfrak{t}))$ when $\mathfrak{t}$ is torsion.

We recall the relationship between the $\ell = 1$ case and the general case. The following is [**OS**, Theorem 4.5].

Theorem 2.4. *Let $\mathcal{S}$ be a weakly admissible ℓ-pointed Heegaard diagram for Y, and let $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. Then*

$$H(\widehat{CF}(\mathcal{S}, \mathfrak{t})) \cong \widehat{HF}(Y \# (\#^{\ell-1}(S^1 \times S^2)), \mathfrak{t} \# \mathfrak{s}_0),$$

where $\mathfrak{s}_0$ is the unique Spin^c structure on $S^1 \times S^2$ with $c_1 = 0$.

Furthermore, when $\mathfrak{t}$ is torsion, the relative $\mathbb{Z}$-gradings on the two sides are the same.

Proof. Given $\mathcal{S} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$, define a pointed Heegaard diagram $\mathcal{S}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z')$ as follows. Define Σ' by taking Σ and attaching $\ell - 1$ tubes connecting $D_i \ni z_i$ to $D_{i+1} \ni z_{i+1}$ for $1 \leq i \leq \ell - 1$. Let $\boldsymbol{\alpha}' = \boldsymbol{\alpha}$, $\boldsymbol{\beta}' = \boldsymbol{\beta}$, and $z' = z_1$. Now observe that $\mathcal{S}'$ is a pointed Heegaard diagram for $Y \#(\#^{\ell-1}(S^1 \times S^2))$, and that for each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta = \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$, $s'(\mathbf{x}) = s(\mathbf{x}) \# \mathfrak{s}_0$, where $s'(\mathbf{x})$ is computed with respect to $\mathcal{S}'$ and $s(\mathbf{x})$ is computed with respect to $\mathcal{S}$. Therefore $\widehat{CF}(\mathcal{S}) = \widehat{CF}(\mathcal{S}')$, and $\widehat{CF}(\mathcal{S}, \mathfrak{t}) = \widehat{CF}(\mathcal{S}', \mathfrak{t} \# \mathfrak{s}_0)$ for each $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. It is immediate from its definition, which we have omitted, that the differential is the same on both sides. Finally, it is clear that when $\mathfrak{t}$ is torsion, the relative $\mathbb{Z}$-gradings on $\widehat{CF}(\mathcal{S}, \mathfrak{t})$ and $\widehat{CF}(\mathcal{S}', \mathfrak{t} \# \mathfrak{s}_0)$ are the same. The result follows. $\square$

To summarize, we have constructed a map from ℓ-pointed Heegaard diagrams $\mathcal{S}$ and torsion Spin^c structures $\mathfrak{t} \in \mathrm{Spin}^c_{\mathrm{tor}}(Y_{\alpha,\beta})$ to relative $\mathbb{Z}$-gradings on $\widehat{CF}(\mathcal{S}, \mathfrak{t})$,

$$(\mathcal{S}, \mathfrak{t}) \mapsto (\widehat{CF}(\mathcal{S}, \mathfrak{t}), \mathrm{gr}).$$

Moreover, for each ℓ (after restricting to weakly admissible diagrams), this map descends to a map from oriented 3-manifolds Y with torsion Spin^c structures $\mathfrak{t} \in \mathrm{Spin}^c_{\mathrm{tor}}(Y)$ to relative $\mathbb{Z}$-gradings on $\widehat{HF}(Y \# (\#^{\ell-1}(S^1 \times S^2)), \mathfrak{t} \# \mathfrak{s}_0)$,

$$(Y, \mathfrak{t}, \ell) \mapsto \left(\widehat{HF}(Y \# (\#^{\ell-1}(S^1 \times S^2)), \mathfrak{t} \# \mathfrak{s}_0), \mathrm{gr} \right)$$

with the property that $(Y, \mathfrak{t}, \ell)$ and $(Y \# (\#^{\ell-1}(S^1 \times S^2)), \mathfrak{t} \# \mathfrak{s}_0, 1)$ produce the same gr.

2.2. Gradings and covering spaces. Let $p : \tilde{Y} \to Y$ be an n-fold, connected covering map.[4] Given an ℓ-pointed Heegaard diagram $\mathcal{S} =$

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[4]All covers of 3-manifolds will be assumed to be connected, unless explicitly stated otherwise.

$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for Y and the covering map p, consider the preimage of $\mathcal{S}$ under p, which we denote by $\tilde{\mathcal{S}} = (\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\mathbf{z}})$. We claim that $\tilde{\mathcal{S}}$ is an $n\ell$-pointed Heegaard diagram for $\tilde{Y}$.

To verify the claim, fix a metric on Y and a Morse function f on Y compatible with $\mathcal{S}$. Pull both of them back to $\tilde{Y}$, so that we have a self-indexing Morse–Smale function $\tilde{f}$ on $\tilde{Y}$ that has $n\ell$ index zero and $n\ell$ index three critical points. Also, $\mathbf{z}$ determines ℓ flowlines for ∇f, which lift to $n\ell$ flowlines for $\tilde{\nabla}\tilde{f}$. As described in Section 2.1, $\tilde{f}$ and the $n\ell$ flowlines determine an $n\ell$-pointed Heegaard diagram for $\tilde{Y}$. It is clear that this $n\ell$-pointed Heegaard diagram is exactly $\tilde{\mathcal{S}}$.

We say that $\tilde{\mathcal{S}}$ is a *covering Heegaard diagram* of $\mathcal{S}$. Note that $\tilde{\Sigma}$ is a connected n-fold cover of Σ and hence a surface of genus $ng - n + 1$. Also observe that $\tilde{\boldsymbol{\alpha}}$ is a disjoint union of $n(g + \ell - 1)$ circles, as is $\tilde{\boldsymbol{\beta}}$.

In the situation described above, we have the following useful fact.

Lemma 2.5. *For any* $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

$$p^*(s(\mathbf{x})) = s(\tilde{\mathbf{x}}),$$

where $\tilde{\mathbf{x}}$ *is the inverse image of* $\mathbf{x}$ *under* p.

Proof. As before, consider a metric g on Y, a Morse function f on Y compatible with $\mathcal{S}$, and their pullbacks $\tilde{g}$ and $\tilde{f}$ under p. Observe that $\mathbf{x}$ determines $g + \ell - 1$ flowlines of ∇f connecting the index one and index two critical points of f in pairs, which must be covered by the $n(g + \ell - 1)$ flowlines of $\tilde{\nabla}\tilde{f}$ determined by $\tilde{\mathbf{x}}$. Recalling the definition of s from Section 2.1, it is now evident that one can choose the vector fields $V(\mathbf{x})$ and $V(\tilde{\mathbf{x}})$ so that $V(\tilde{\mathbf{x}})$ is the pullback of $V(\mathbf{x})$. It follows that the diagram

commutes. This is the desired result. □

The preimage map $p^{-1} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{T}_{\tilde{\alpha}} \cap \mathbb{T}_{\tilde{\beta}}$ induces a map of groups $\widehat{CF}(\mathcal{S}) \to \widehat{CF}(\tilde{\mathcal{S}})$, which we denote $\xi \mapsto \tilde{\xi}$. (Note that in general this is not a chain map.) Let us define

$$\widehat{CF}(\mathcal{S}, \text{torsion}) := \bigoplus_{\mathfrak{t} \in \text{Spin}^c_{\text{tor}}(Y)} \widehat{CF}(\mathcal{S}, \mathfrak{t}).$$

Our goal in this subsection is to prove the following theorem, which completely characterizes our relative $\mathbb{Q}$-grading.

Theorem 2.6. *There is a unique map from ℓ-pointed Heegaard diagrams to relative $\mathbb{Q}$-gradings on $\widehat{CF}(\mathcal{S}, \text{torsion})$,*

$$\mathcal{S} \mapsto (\widehat{CF}(\mathcal{S}, \text{torsion}), \text{Gr})$$

such that the relative $\mathbb{Q}$-grading Gr *has the following properties:*

- *For each $\mathfrak{t} \in \text{Spin}^c_{\text{tor}}(Y)$, the restriction of* Gr *to $\widehat{CF}(\mathcal{S}, \mathfrak{t})$ is equal to* gr, *where* gr *is the relative $\mathbb{Z}$-grading described in the previous subsection. In particular, they have the same homogeneous elements.*
- *If $\tilde{Y} \to Y$ is an n-fold cover, then for all homogeneous elements $\xi, \eta \in \widehat{CF}(\mathcal{S}, \text{torsion})$,*

$$\text{Gr}(\xi, \eta) = \frac{1}{n} \text{Gr}(\tilde{\xi}, \tilde{\eta}).$$

Moreover, for each ℓ (after restricting to weakly admissible diagrams), this map descends to a map from oriented 3-manifolds Y to relative $\mathbb{Q}$-gradings on $\widehat{HF}(Y \# (\#^{\ell-1}(S^1 \times S^2)), \text{torsion})$,

$$(Y, \ell) \mapsto \left(\widehat{HF}(Y \# (\#^{\ell-1}(S^1 \times S^2)), \text{torsion}), \text{Gr} \right)$$

with the property that (Y, ℓ) and $(Y \# (\#^{\ell-1}(S^1 \times S^2)), 1)$ produce the same Gr.

We are now ready to describe our construction of the relative $\mathbb{Q}$-grading on $\widehat{CF}(\mathcal{S}, \text{torsion})$. As in the previous subsection, we simply need to define Gr on the homogeneous generators in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Definition. Let $p : \tilde{Y} \to Y$ be an n-fold covering map, and let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s(\mathbf{x})$ and $s(\mathbf{y})$ are torsion. If it happens that $s(\tilde{\mathbf{x}}) = s(\tilde{\mathbf{y}})$, then we define

$$\text{Gr}(\mathbf{x}, \mathbf{y}) = \frac{1}{n} \text{gr}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}).$$

In order for this definition to make sense, we need to prove two things. First, we must show that $\text{Gr}(\mathbf{x}, \mathbf{y})$ is independent of the choice of cover. Second, we must show that given any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s(\mathbf{x})$ and $s(\mathbf{y})$ torsion, there always exists a cover such that $s(\tilde{\mathbf{x}}) = s(\tilde{\mathbf{y}})$.

Lemma 2.7. *Let $p : \tilde{Y} \to Y$ be an n-fold covering map, and let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. If $s(\mathbf{x}) = s(\mathbf{y})$ is torsion, then $s(\tilde{\mathbf{x}}) = s(\tilde{\mathbf{y}})$ and*

$$\text{gr}(\mathbf{x}, \mathbf{y}) = \frac{1}{n} \text{gr}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}).$$

In other words, computing Gr using the trivial cover, when possible, is consistent with computing Gr using any other cover.

Proof. The previous lemma shows that $s(\tilde{\mathbf{x}}) = s(\tilde{\mathbf{y}})$. To prove the formula, choose $A \in \hat{\pi}_2(\mathbf{x}, \mathbf{y})$. Now consider the total preimage $\tilde{A} \in C_2(\tilde{\mathcal{S}})$ and observe that $\tilde{A} \in \hat{\pi}_2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. It is clear from the definitions that $e(\tilde{A}) = n \cdot e(A)$, $n_{\tilde{\mathbf{x}}}(\tilde{A}) = n \cdot n_{\mathbf{x}}(A)$, and $n_{\tilde{\mathbf{y}}}(\tilde{A}) = n \cdot n_{\mathbf{y}}(A)$. The result now follows from the definition of gr in equation (2.1). □

Lemma 2.8. *Let $p_1 : \tilde{Y}_1 \to Y$ be an n_1-fold covering map, and let $p_2 : \tilde{Y}_2 \to Y$ be an n_2-fold covering map. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s(\mathbf{x})$ and $s(\mathbf{y})$ are torsion, and suppose that $s(p_1^{-1}\mathbf{x}) = s(p_1^{-1}\mathbf{y})$ and $s(p_2^{-1}\mathbf{x}) = s(p_2^{-1}\mathbf{y})$. Then*

$$\frac{1}{n_1} \operatorname{gr}(p_1^{-1}\mathbf{x}, p_1^{-1}\mathbf{y}) = \frac{1}{n_2} \operatorname{gr}(p_2^{-1}\mathbf{x}, p_2^{-1}\mathbf{y}).$$

Proof. Let Y' be a connected component of the fibered product $\tilde{Y}_1 \times_Y \tilde{Y}_2$ so that the projections $p_1' : Y' \to \tilde{Y}_1$ and $p_2' : Y' \to \tilde{Y}_2$ are n_1'- and n_2'-fold covering maps, respectively, and thus $n_1 n_1' = n_2 n_2'$. Note that since $p_1 p_1' = p_2 p_2'$, given our ℓ-pointed Heegaard diagram for Y, we are led to the same covering Heegaard diagram for Y', whether we go through $\tilde{Y}_1$ or $\tilde{Y}_2$. Therefore, the previous lemma shows that

$$\frac{1}{n_1} \operatorname{gr}(p_1^{-1}\mathbf{x}, p_1^{-1}\mathbf{y}) = \frac{1}{n_1 n_1'} \operatorname{gr}(p_1'^{-1}p_1^{-1}\mathbf{x}, p_1'^{-1}p_1^{-1}\mathbf{y})$$

$$= \frac{1}{n_2 n_2'} \operatorname{gr}(p_2'^{-1}p_2^{-1}\mathbf{x}, p_2'^{-1}p_2^{-1}\mathbf{y})$$

$$= \frac{1}{n_2} \operatorname{gr}(p_2^{-1}\mathbf{x}, p_2^{-1}\mathbf{y}).$$

□

Lemma 2.8 implies that for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s(\mathbf{x})$ and $s(\mathbf{y})$ torsion, $\operatorname{Gr}(\mathbf{x}, \mathbf{y})$ is uniquely defined whenever it is defined at all. We must now prove that $\operatorname{Gr}(\mathbf{x}, \mathbf{y})$ can always be defined. For this we will use the following lemma from algebraic topology.

Lemma 2.9. *Let Y be a connected topological space with the homotopy-type of a CW-complex. Suppose that $a \in H^2(Y; \mathbb{Z})$ is n-torsion. Then there exists a $\mathbb{Z}/n$-covering map $p : \tilde{Y} \to Y$ such that $p^*a = 0$.*

Proof. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \overset{n}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

induces the exact sequence

$$H^1(Y; \mathbb{Z}/n) \overset{\beta}{\longrightarrow} H^2(Y; \mathbb{Z}) \overset{n}{\longrightarrow} H^2(Y; \mathbb{Z}) .$$

Since a is n-torsion, $a = \beta(q)$ for some $q \in H^1(Y; \mathbb{Z}/n)$. Under the isomorphism $H^1(Y; \mathbb{Z}/n) \cong [Y, K(\mathbb{Z}/n, 1)]$, we can represent q by a map

$f : Y \to K(\mathbb{Z}/n, 1) = B\mathbb{Z}/n$ such that $q = f^*\iota$, where $\iota \in H^1(B\mathbb{Z}/n; \mathbb{Z}/n)$ is the image of the identity via the canonical isomorphism $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/n) \cong H^1(B\mathbb{Z}/n; \mathbb{Z}/n)$. Let $\pi : E\mathbb{Z}/n \to B\mathbb{Z}/n$ denote the contractible principal $\mathbb{Z}/n$-bundle over $B\mathbb{Z}/n$. Let $\tilde{Y}$ be the pullback $f^*(E\mathbb{Z}/n)$, so we have the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\ g\ } & E\mathbb{Z}/n \ . \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\
Y & \xrightarrow{\ f\ } & B\mathbb{Z}/n
\end{array}
$$

We claim that $p : \tilde{Y} \to Y$ is the desired n-fold covering map. Indeed, we have $a \in \mathrm{im}(\beta f^*) = \mathrm{im}(f^*\beta)$, and thus $p^*a \in \mathrm{im}(p^*f^*) = \mathrm{im}(g^*\pi^*) \subset \mathrm{im}(g^*) = 0$, since $H^2(E\mathbb{Z}/n; \mathbb{Z}) = 0$. $\qquad\square$

Corollary 2.10. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *such that* $s(\mathbf{x})$ *and* $s(\mathbf{y})$ *are torsion. Then for some* n, $s(\mathbf{x}) - s(\mathbf{y})$ *is* n-*torsion and there is a* $\mathbb{Z}/n$-*cover* $\tilde{Y} \to Y$ *such that* $s(\tilde{\mathbf{x}}) = s(\tilde{\mathbf{y}})$. *Consequently,* $\mathrm{Gr}(\mathbf{x}, \mathbf{y})$ *is well defined.*

Proof. Since $c_1(s(\mathbf{x})) - c_1(s(\mathbf{y})) = 2(s(\mathbf{x}) - s(\mathbf{y}))$, the hypotheses imply that $s(\mathbf{x}) - s(\mathbf{y})$ is n-torsion for some n. By the previous theorem, there is a $\mathbb{Z}/n$-covering map $p : \tilde{Y} \to Y$ such that $p^*(s(\mathbf{x}) - s(\mathbf{y})) = 0$. But $p^*(s(\mathbf{x}) - s(\mathbf{y})) = s(\tilde{\mathbf{x}}) - s(\tilde{\mathbf{y}})$. $\qquad\square$

We have shown that we can always find a cover that allows us to compute $\mathrm{Gr}(\mathbf{x}, \mathbf{y})$, but for a rational homology sphere, there is one cover that always works. Recall that the maximal abelian cover of Y is the cover corresponding to the commutator subgroup of $\pi_1(Y)$.

Corollary 2.11. *Let* Y *be a rational homology sphere and* $\tilde{Y}$ *its maximal abelian cover. Then* $\tilde{Y}$ *is a finite cover of* Y, *and for all* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, *we have* $s(\tilde{\mathbf{x}}) = s(\tilde{\mathbf{y}})$. *Thus* $\mathrm{Gr}(\mathbf{x}, \mathbf{y})$ *can be computed using this cover.*

Proof. Since $H_1(Y) \cong \pi_1(Y)/[\pi_1(Y), \pi_1(Y)]$ is finite, $\tilde{Y}$ is a finite cover of Y. The rest of the conclusion follows from the previous corollary, Lemma 2.5, and the fact that every cyclic cover is covered by the maximal abelian cover (since every homomorphism from a group G to an abelian group factors through the abelianization of G). $\qquad\square$

We now turn to additivity.

Corollary 2.12. *For* $\mathbf{x}, \mathbf{y}, \mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *such that* $s(\mathbf{x})$, $s(\mathbf{y})$, *and* $s(\mathbf{w})$ *are torsion,*

$$\mathrm{Gr}(\mathbf{x}, \mathbf{y}) + \mathrm{Gr}(\mathbf{y}, \mathbf{w}) = \mathrm{Gr}(\mathbf{x}, \mathbf{w}).$$

Proof. Lemma 2.9 and the proof of Corollary 2.10 allow us to find a cover where $s(\tilde{\mathbf{x}}) = s(\tilde{\mathbf{y}}) = s(\tilde{\mathbf{w}})$. The additivity of gr then implies the result. $\qquad\square$

This completes the proof of Theorem 2.6, except for the last part regarding the invariance of Gr and the statement that Gr is essentially independent of ℓ. The invariance of Gr is a direct consequence of the invariance of gr. The proof that Gr is independent of ℓ follows from the proof that gr is independent of ℓ.

2.3. How to compute. From the previous subsection, it would appear that every time one wants to compute $\mathrm{Gr}(\mathbf{x}, \mathbf{y})$, one must find an appropriate cover and perform computations in the covering Heegaard diagram $\tilde{\mathcal{S}}$. However, it turns out that the mere existence of an appropriate cover allows us to perform all of the computations in the original ℓ-pointed Heegaard diagram $\mathcal{S}$, without ever thinking about covers at all.

Given $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s(\mathbf{x})$ and $s(\mathbf{y})$ are torsion, Corollary 2.10 gives us an n-fold cover for which we can find $A \in \hat{\pi}_2(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. Consider the projection $\bar{A} \in C_2(\mathcal{S})$, and observe that $\bar{A} \in \hat{\pi}_2(n\mathbf{x}, n\mathbf{y})$ in the sense that $\bar{A} \in \hat{C}_2(\mathcal{S})$ and

$$\partial \partial_\alpha \bar{A} = n(y_1 + \cdots + y_{g+\ell-1}) - n(x_1 + \cdots + x_{g+\ell-1}).$$

Also observe that

$$\begin{aligned}
\mathrm{Gr}(\mathbf{x}, \mathbf{y}) &= \frac{1}{n} \left[e(A) + n_{\tilde{\mathbf{x}}}(A) + n_{\tilde{\mathbf{y}}}(A) \right] \\
&= \frac{1}{n} \left[e(\bar{A}) + n_{\mathbf{x}}(\bar{A}) + n_{\mathbf{y}}(\bar{A}) \right].
\end{aligned}$$

The moral here is that we do not need to find A in order to compute $\mathrm{Gr}(\mathbf{x}, \mathbf{y})$; it is sufficient to find $\bar{A}$.

Proposition 2.13. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *such that* $s(\mathbf{x})$ *and* $s(\mathbf{y})$ *are torsion. Then* $s(\mathbf{x}) - s(\mathbf{y})$ *is n-torsion for some n, and there exists some* $A \in \hat{\pi}_2(n\mathbf{x}, n\mathbf{y})$. *For any such* A,

$$\mathrm{Gr}(\mathbf{x}, \mathbf{y}) = \frac{1}{n} \left[e(A) + n_{\mathbf{x}}(A) + n_{\mathbf{y}}(A) \right].$$

Proof. The argument above shows that there exists some A for which the result holds. We just have to prove that the right side of the formula is independent of the choice of A. Observe that A is unique up to addition of elements in $\hat{\pi}_2(\mathbf{x}, \mathbf{x})$. The rest of the proof is identical to the proof of Proposition 2.3. $\square$

Therefore, computing $\mathrm{Gr}(\mathbf{x}, \mathbf{y})$ is reduced to an elementary exercise in linear algebra: for each $i > \ell$, compute $\partial \partial_\alpha D_i$. Then find integers a_i so that $\sum_{i=\ell+1}^{N} a_i \partial \partial_\alpha D_i$ equals some multiple n of $(y_1 + \cdots + y_{g+\ell-1}) - (x_1 + \cdots + x_{g+\ell-1})$. Finally, plug $A = \sum_{i=\ell+1}^{N} a_i D_i$ into the formula in the previous proposition.

3. Bordisms and Ozsváth–Szabó's construction

The material in this section is all essentially contained in Ozsváth–Szabó's paper [**OS06**]. However, our purposes require us to articulate more precise chain-level statements in Section 3.3.

3.1. Review of Heegaard triples.

Definition. An ℓ-*pointed Heegaard triple* is a 5-tuple $\mathcal{T}_{\alpha,\beta,\gamma} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$ such that each of $\mathcal{S}_{\alpha,\beta} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$, $\mathcal{S}_{\beta,\gamma} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$, and $\mathcal{S}_{\alpha,\gamma} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z})$ is an ℓ-pointed Heegaard diagram and $\boldsymbol{\alpha} \cap \boldsymbol{\beta} \cap \boldsymbol{\gamma} = \emptyset$. Let $Y_{\alpha,\beta}$ be the 3-manifold specified by $\mathcal{S}_{\alpha,\beta}$ (and similarly for $Y_{\beta,\gamma}$ $Y_{\alpha,\gamma}$). When $\ell = 1$, $\mathcal{T}_{\alpha,\beta,\gamma}$ is called a *pointed Heegaard triple*.

An ℓ-pointed Heegaard triple specifies a 4-manifold with boundary $W_{\alpha,\beta,\gamma}$ as follows. Let T denote a triangle. The $\boldsymbol{\alpha}$ circles (respectively β, γ) specify a handlebody U_α (respectively U_β, U_γ) with boundary Σ. Glue each of $U_\alpha \times [0,1]$, $U_\beta \times [0,1]$, and $U_\gamma \times [0,1]$ to an edge of $\Sigma \times T$ (clockwise). The result is a 4-manifold $W_{\alpha,\beta,\gamma}$ with boundary $-Y_{\alpha,\beta} \cup -Y_{\beta,\gamma} \cup Y_{\alpha,\gamma}$. See [**OS04b**, Section 8.1] for details.

Let $\mathcal{T}_{\alpha,\beta,\gamma}$ be an ℓ-pointed Heegaard triple. Let $D_1, \dots, D_N$ be the closures of the connected components of $\Sigma \smallsetminus \boldsymbol{\alpha} \smallsetminus \boldsymbol{\beta} \smallsetminus \boldsymbol{\gamma}$, thought of as 2-chains, labeled so that $z_i \in D_i$ for $i \leq \ell$. Define $C_2(\mathcal{T}_{\alpha,\beta,\gamma})$ to be the group of 2-chains in Σ generated by the D_i's, and define $\hat{C}_2(\mathcal{T}_{\alpha,\beta,\gamma})$ to be the subgroup of $C_2(\mathcal{T}_{\alpha,\beta,\gamma})$ generated by the D_i's with $i > \ell$, or in other words, $\hat{C}_2(\mathcal{T}_{\alpha,\beta,\gamma})$ is generated by the closures of connected components of $\Sigma \smallsetminus \boldsymbol{\alpha} \smallsetminus \boldsymbol{\beta} \smallsetminus \boldsymbol{\gamma}$ not containing any element of $\mathbf{z}$.

For all $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{x} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$, define

$$\pi_2(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \left\{ B \in C_2(\mathcal{T}_{\alpha,\beta,\gamma}) \; \middle| \; \begin{array}{l} \partial\partial_\alpha B = (y_1 + \cdots + y_{g+\ell-1}) - (w_1 + \cdots + w_{g+\ell-1}) \\ \partial\partial_\beta B = (w_1 + \cdots + w_{g+\ell-1}) - (x_1 + \cdots + x_{g+\ell-1}) \end{array} \right\}$$

$$\hat{\pi}_2(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \pi_2(\mathbf{w}, \mathbf{x}, \mathbf{y}) \cap \hat{C}_2(\mathcal{T}_{\alpha,\beta,\gamma}).$$

Observe that addition gives a map

$$\pi_2(\mathbf{w}', \mathbf{w}) \times \pi_2(\mathbf{x}', \mathbf{x}) \times \pi_2(\mathbf{y}', \mathbf{y}) \times \pi_2(\mathbf{w}, \mathbf{x}, \mathbf{y}) \to \pi_2(\mathbf{w}', \mathbf{x}', \mathbf{y}').$$

For $B \in \pi_2(\mathbf{w}, \mathbf{x}, \mathbf{y})$ and $B' \in \pi_2(\mathbf{w}', \mathbf{x}', \mathbf{y}')$, say that B and B' are Spin^c-*equivalent* if there are $A_{\alpha,\beta} \in \pi_2(\mathbf{w}', \mathbf{w})$, $A_{\beta,\gamma} \in \pi_2(\mathbf{x}', \mathbf{x})$, and $A_{\alpha,\gamma} \in \pi_2(\mathbf{y}, \mathbf{y}')$ such that $B' = A_{\alpha,\beta} + A_{\beta,\gamma} + A_{\alpha,\gamma} + B$.

Lemma 3.1. *There is a map* $s : \pi_2(\mathbf{w}, \mathbf{x}, \mathbf{y}) \to \mathrm{Spin}^c(W_{\alpha,\beta,\gamma})$ *with the following properties:*

- $\mathfrak{s} \in \mathrm{im}(s)$ *if and only if* $\mathfrak{s}|_{Y_{\alpha,\beta}} = s(\mathbf{w})$, $\mathfrak{s}|_{Y_{\beta,\gamma}} = s(\mathbf{x})$, *and* $\mathfrak{s}|_{Y_{\alpha,\gamma}} = s(\mathbf{y})$.

- *For $B \in \pi_2(\mathbf{w}, \mathbf{x}, \mathbf{y})$ and $B' \in \pi_2(\mathbf{w}', \mathbf{x}', \mathbf{y}')$, $s(B) = s(B')$ if and only if B and B' are Spin^c-equivalent.*

For a definition of the map s and the proof of this lemma, see [**OS04b**, Section 8.1].

Let $\hat{\pi}_2^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \{B \in \hat{\pi}_2(\mathbf{w}, \mathbf{x}, \mathbf{y}) \mid s(B) = \mathfrak{s}\}$. For any $\mathfrak{s} \in \mathrm{Spin}^c(W_{\alpha,\beta,\gamma})$, we say that an ℓ-pointed Heegaard triple $\mathcal{T}_{\alpha,\beta,\gamma}$ *realizes* $\mathfrak{s}$ when there exist $\mathbf{w}$, $\mathbf{x}$, and $\mathbf{y}$ such that $\hat{\pi}_2^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) \neq \emptyset$. The previous lemma implies that $\hat{\pi}_2^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) \neq \emptyset$ if and only if $\mathfrak{s}|_{Y_{\alpha,\beta}} = s(\mathbf{w})$, $\mathfrak{s}|_{Y_{\beta,\gamma}} = s(\mathbf{x})$, and $\mathfrak{s}|_{Y_{\alpha,\gamma}} = s(\mathbf{y})$.

Given an element $B \in \pi_2(\mathbf{w}, \mathbf{x}, \mathbf{y})$, one can assign an index $\mathrm{ind}(B) \in \mathbb{Z}$ to B; this is the index of the $\bar{\partial}$-operator on some space of holomorphic curves.[5] This index has the property that for $B \in \pi_2(\mathbf{w}, \mathbf{x}, \mathbf{y})$, $A_{\alpha,\beta} \in \pi_2(\mathbf{w}', \mathbf{w})$, $A_{\beta,\gamma} \in \pi_2(\mathbf{x}', \mathbf{x})$, and $A_{\alpha,\gamma} \in \pi_2(\mathbf{y}', \mathbf{y})$,

$$\mathrm{ind}(A_{\alpha,\beta} + A_{\beta,\gamma} + A_{\alpha,\gamma} + B) = \mathrm{ind}(A_{\alpha,\beta}) + \mathrm{ind}(A_{\beta,\gamma}) + \mathrm{ind}(A_{\alpha,\gamma}) + \mathrm{ind}(B),$$

where $\mathrm{ind}(A_{\alpha,\beta}) = e(A_{\alpha,\beta}) + n_{\mathbf{w}'}(A_{\alpha,\beta}) + n_{\mathbf{w}}(A_{\alpha,\beta})$, and similarly for $\mathrm{ind}(A_{\beta,\gamma})$ and $\mathrm{ind}(A_{\alpha,\gamma})$.

Definition. Let $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{x} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ such that $s(\mathbf{w})$, $s(\mathbf{x})$, and $s(\mathbf{y})$ are torsion, and let $\mathfrak{s} \in \mathrm{Spin}^c(W_{\alpha,\beta,\gamma})$ such that $\hat{\pi}_2^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) \neq \emptyset$. Then for any $B \in \hat{\pi}_2^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y})$, define

$$\mathrm{gr}^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \mathrm{ind}(B).$$

The definition is independent of choice of B by the same reasoning as in Proposition 2.3, but note that $\mathrm{gr}^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y})$ is only defined when $\hat{\pi}_2^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) \neq \emptyset$. This "relative grading" gr is additive in the sense that

$$\mathrm{gr}^{\mathfrak{s}}(\mathbf{w}', \mathbf{x}', \mathbf{y}') = \mathrm{gr}(\mathbf{w}', \mathbf{w}) + \mathrm{gr}(\mathbf{x}', \mathbf{x}) + \mathrm{gr}(\mathbf{y}, \mathbf{y}') + \mathrm{gr}^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y}).$$

In order to prove that Gr agrees with the relative grading induced by $\widetilde{\mathrm{gr}}$, we will need to use covering Heegaard triples; we collect a few basic facts about these here. Given an ℓ-pointed Heegaard triple $\mathcal{T}_{\alpha,\beta,\gamma} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$ and an n-fold covering $\pi : \tilde{W} \to W_{\alpha,\beta,\gamma}$, we can define a new $n\ell$-pointed Heegaard triple $\tilde{\mathcal{T}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}} = (\tilde{\Sigma}, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}, \tilde{\mathbf{z}})$ for $\tilde{W}$ by taking $\tilde{\mathcal{T}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$ to be the preimage of $\mathcal{T}_{\alpha,\beta,\gamma}$ under π, as we did in Section 2.2 for ordinary Heegaard diagrams. That $\tilde{\mathcal{T}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$ is a Heegaard triple for $\tilde{W}$ follows by exactly the same argument as in Section 2.2. We say that $\tilde{\mathcal{T}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$ is a *covering Heegaard triple* of $\mathcal{T}_{\alpha,\beta,\gamma}$.

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[5]For the definition of ind, see [**OS04b**, Section 8], where the notation μ is used instead of ind.

Lemma 3.2. *Fix* $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{x} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, *and* $\mathfrak{s} \in$ $\mathrm{Spin}^c(W_{\alpha,\beta,\gamma})$. *For any* $B \in \hat{\pi}_2^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y})$ *and its total preimage* $\tilde{B}$,

(1) $s(\tilde{B}) = \pi^* s(B)$.

(2) $\mathrm{ind}(\tilde{B}) = n \cdot \mathrm{ind}(B)$.

In particular, if the restriction of $\mathfrak{s}$ *to* $\partial W_{\alpha,\beta,\gamma}$ *is torsion and* $\hat{\pi}_2^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) \neq \emptyset$, *then* $\mathrm{gr}^{\pi^* \mathfrak{s}}(\tilde{\mathbf{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = n \cdot \mathrm{gr}^{\mathfrak{s}}(\mathbf{w}, \mathbf{x}, \mathbf{y})$.

Proof. The proof of statement 1 follows from the definition of $s(B)$ (which we have not given) in an exactly analogous way to Lemma 2.5.

To prove statement 2, recall from [**OS04b**] that ind denotes the index of the linearized $\overline{\partial}$-operator at an appropriate map ϕ of a triangle Δ to $\mathrm{Sym}^{g+\ell-1}(\Sigma)$. There is a map $u : S \to \Sigma \times \Delta$ for some Riemann surface S which *tautologically corresponds* to ϕ (see, e.g., [**Lip06**, Section 13]). The index of the $\overline{\partial}$-operators at u and ϕ agree. (Indeed, there is even an identification of index bundles – see [**Lip06**, Section 13] – though we do not need this stronger result.)

In our situation, the covering map $\tilde{\Sigma} \to \Sigma$ induces an *inclusion* $\mathrm{Sym}^{g+\ell-1}(\Sigma) \to \mathrm{Sym}^{ng+n\ell-n}(\tilde{\Sigma})$ and a covering map $\tilde{\Sigma} \times \Delta \to \Sigma \times \Delta$. The maps ϕ and u then induce maps $\tilde{\phi} : \Delta \to \mathrm{Sym}^{ng+n\ell-n}(\tilde{\Sigma})$ and $\tilde{u} : \tilde{S} \to \tilde{\Sigma} \times \Delta$. (Here, $\tilde{S}$ is an n-fold covering of S.) It is straightforward to check that the maps $\tilde{u}$ and $\tilde{\phi}$ again tautologically correspond.

Now, it follows from the Atiyah–Singer index theorem that the index of the $\overline{\partial}$-operator at $\tilde{u}$ is n times the index of the $\overline{\partial}$-operator at u. Alternately, this is immediate from the index formula on page 1018 of [**Lip06**], $\mathrm{ind}(u) = (g + \ell - 1)/2 - \chi(S) + 2e(D(u))$. $\square$

Remark. Since the first version of this paper, S. Sarkar has proved a combinatorial formula for the index of triangles ([**Sar**]). It should be possible to use his formula to prove part (2) of Lemma 3.2. We leave this to the interested reader.

3.2. Bordisms and Heegaard triples. Recall that any bordism W^4 from Y_1^3 to Y_2^3 can be decomposed as a collection of 1-handle attachments, followed by 2-handle attachments, followed by 3-handle attachments. (See [**GS99**] for an efficient exposition of handle decompositions and Kirby calculus.) Attaching a 1-handle to W has the effect of either changing ∂W to $\partial W \# (S^1 \times S^2)$ or connect summing two connected components of ∂W. Attaching 2-handles to W has the effect on ∂W of doing framed surgery along the attaching circles of the 2-handles. Attaching a 3-handle to W has the opposite effect of attaching a 1-handle: it either removes an $S^1 \times S^2$ summand or it disconnects a connected sum.

In [**OS06**], Ozsváth and Szabó associate maps to 1-handle, 2-handle, and 3-handle attachings, and show that the composition of these maps depends

only on the bordism. The absolute $\mathbb{Q}$-grading, however, is defined using bordisms composed entirely of 2-handles. We will restrict our attention here to such bordisms, which we call for convenience *link surgery bordisms*.

Fix a framed n-component link $\mathbb{L} = \{L_i\}$ in Y. By a *bouquet for* $\mathbb{L}$ we mean the union of $\mathbb{L}$ and a path from each component L_i of $\mathbb{L}$ to a fixed reference point. Fix a bouquet $B(\mathbb{L})$ for $\mathbb{L}$. A neighborhood V of $B(\mathbb{L})$ is a handlebody of genus n. In general, $Y \smallsetminus V$ will not be a handlebody; however, it is possible to annex tubes to V to produce a new handlebody V' so that $Y \smallsetminus V'$ is a handlebody. Suppose $\partial V'$ has genus g. We can choose V' so that a small neighborhood of L_i intersects $\partial V'$ in a punctured torus F_i, with $F_i \cap F_j = \emptyset$ for $i \neq j$.

A Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is *subordinate to* $\mathbb{L}$ if, for some choice of bouquet and V' as before, there is an identification of Σ with $\partial V'$ such that

- Each α_i bounds a disk in $Y \smallsetminus V'$.
- Each β_i bounds a disk in V'.
- For $1 \leq i \leq n$, β_i lies in F_i and is a meridian of L_i.
- For $n < i \leq g$, β_i is disjoint from each F_j.
- For $1 \leq i \leq n$, γ_i lies in F_i and the homology class of γ_i corresponds to the framing of L_i.
- For $n < i \leq g$, γ_i is a small perturbation of β_i.

This definition comes from [**OS06**, Section 4.1].

Observe that for such a Heegaard triple, $(\Sigma, \boldsymbol{\alpha}, \beta_{n+1}, \ldots, \beta_g)$ and $(\Sigma, \boldsymbol{\alpha}, \gamma_{n+1}, \ldots, \gamma_g)$ both specify $Y \smallsetminus \mathbb{L} = Y(\mathbb{L}) \smallsetminus \mathbb{L}$, where $Y(\mathbb{L})$ denotes the 3-manifold obtained by surgery along $\mathbb{L}$. Filling in the boundary of $Y \smallsetminus \mathbb{L}$ according to the β_i, $i \leq n$, gives back Y; filling in the boundary according to the γ_i, $i \leq n$, gives $Y(\mathbb{L})$. That is, $Y_{\alpha,\beta} = Y$ and $Y_{\alpha,\gamma} = Y(\mathbb{L})$.

For any $B(\mathbb{L})$, there always exists a pointed Heegaard triple subordinate to it. Note that if we start with a link surgery bordism W and find a Heegaard triple $\mathcal{T}_{\alpha,\beta,\gamma}$ subordinate to the corresponding link, then W is obtained from $W_{\alpha,\beta,\gamma}$ by filling in $Y_{\beta,\gamma} = \#^{g-n}(S^1 \times S^2)$ with $\natural^{g-n}(S^1 \times \mathbb{D}^3)$.

3.3. The absolute $\mathbb{Q}$-grading. We will now define Ozsváth and Szabó's absolute $\mathbb{Q}$-grading, $\widetilde{\mathrm{gr}}$, on $\widehat{HF}(Y, \mathrm{torsion})$, following the treatment in [**OS06**, Section 7]. First, for $\mathfrak{t}_i \in \mathrm{Spin}^c(Y_i)$, we define a Spin^c *bordism from* $(Y_1, \mathfrak{t}_1)$ *to* $(Y_2, \mathfrak{t}_2)$ to be a pair $(W, \mathfrak{s})$ such that W is a bordism from Y_1 to Y_2, and $\mathfrak{s} \in \mathrm{Spin}^c(W)$ satisfies $\mathfrak{s}|_{Y_i} = \mathfrak{t}_i$. (We distinguish this concept from that of *stable* Spin^c bordism, which will be relevant in Section 4.) For any $\mathfrak{t} \in \mathrm{Spin}^c_{\mathrm{tor}}(Y)$, there exists a link surgery Spin^c bordism $(W, \mathfrak{s})$ from $(S^3, \mathfrak{s}_0)$ to $(Y, \mathfrak{t})$, where $\mathfrak{s}_0$ is the unique Spin^c structure on S^3. Let $\mathcal{T}_{\alpha,\beta,\gamma}$ be a pointed Heegaard triple realizing $\mathfrak{s}$ and subordinate to the link. Then $Y_{\alpha,\beta} = S^3$, $Y_{\beta,\gamma} = \#^k(S^1 \times S^2)$ for some k, and $Y_{\alpha,\gamma} = Y$. Choose

$\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ so that they have the same grading as the highest graded elements of $\widehat{HF}(S^3, \mathfrak{s}_0)$ and $\widehat{HF}(\#^k(S^1 \times S^2), \mathfrak{s}_0)$, respectively. We say that $\mathbf{x}_0$ and θ lie *in the canonical degree.*

Definition. With notation as before, define $\widetilde{\mathrm{gr}}$ on $\widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t})$ by

$$\widetilde{\mathrm{gr}}(\mathbf{y}) = -\mathrm{gr}^{\mathfrak{s}}(\mathbf{x}_0, \theta, \mathbf{y}) + \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}$$

for $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ with $s(\mathbf{y}) = \mathfrak{t}$.

In [**OS06**], Ozsváth and Szabó proved that this definition gives a well-defined absolute $\mathbb{Q}$-grading on $\widehat{HF}(Y, \text{torsion})$. Observe that $\widetilde{\mathrm{gr}}$ is not obviously defined for a general pointed Heegaard diagram, but only those $\mathcal{S}_{\alpha,\gamma}$ that arise from the earlier construction. Call such a diagram a $\widetilde{\mathrm{gr}}$-*admissible pointed Heegaard diagram for* Y.

For our purposes, we need to be able to work with $\widetilde{\mathrm{gr}}$ at the chain level. The following cumbersome lemma may be thought of as a generalization of the previous definition. This argument is essentially the same as in the proofs of [**OS06**, Proposition 4.9, Lemma 7.5, and Theorem 7.1]. Since we want a chain-level statement, rather than a homology-level statement, we need to be slightly more precise.

Lemma 3.3. *Let* $(W, \mathfrak{s})$ *be a link surgery* Spin^c *bordism from* $(Y_1, \mathfrak{t}_1)$ *to* $(Y_2, \mathfrak{t}_2)$. *Then there is a pointed Heegaard triple* $\mathcal{T}_{\alpha,\beta,\gamma}$ *subordinate to a link inducing* W, *realizing* $\mathfrak{s}$, *such that* $\mathcal{S}_{\alpha,\beta}$ *and* $\mathcal{S}_{\alpha,\gamma}$ *are* $\widetilde{\mathrm{gr}}$-*admissible for* $Y_1 = Y_{\alpha,\beta}$ *and* $Y_2 = Y_{\alpha,\gamma}$, *respectively.*

In this case, for all $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *with* $s(\mathbf{x}) = \mathfrak{t}_1$ *and* $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ *with* $s(\mathbf{y}) = \mathfrak{t}_2$,

$$(3.1) \qquad \widetilde{\mathrm{gr}}(\mathbf{y}) = \widetilde{\mathrm{gr}}(\mathbf{x}) - \mathrm{gr}^{\mathfrak{s}}(\mathbf{x}, \theta, \mathbf{y}) + \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

As before, $\theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ *lies in the canonical degree of* $\widehat{HF}(Y_{\beta,\gamma} = \#^k(S^1 \times S^2), \mathfrak{s}_0)$ *for some* k.

(We abuse notation here and elsewhere by identifying $\mathfrak{s}$ with $\mathfrak{s}|_{W_{\alpha,\beta,\gamma}}$.)

Proof. Suppose that $\mathbb{L}'$ is a framed link in S^3 so that surgery on $\mathbb{L}'$ produces Y_1 and $\mathfrak{t}_1$ extends over the induced bordism from S^3 to Y_1. Let $\mathbb{L}$ be any framed link in Y_1 inducing the link surgery bordism W. By a small perturbation we can choose $\mathbb{L}$ to be disjoint from image of $\mathbb{L}'$ in Y_1, so that $\mathbb{L}$ is the image of some framed link in S^3. Let $\mathbb{L}\mathbb{L}'$ be the union of $\mathbb{L}'$ and the preimage, in S^3, of $\mathbb{L}$.

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\gamma})$ be a genus g Heegaard triple subordinate to $\mathbb{L}\mathbb{L}'$, with the δ circles ordered so that $\delta_1, \ldots, \delta_m$ are meridians for components of $\mathbb{L}'$ and $\delta_{m+1}, \ldots, \delta_n$ are meridians for components of the preimage of $\mathbb{L}$. Let β_i

be a small isotopic translate of γ_i for $1 \leq i \leq m$; let β_i be a small isotopic translate of δ_i for $m < i \leq n$; and let β_i be a small isotopic translate of both δ_i and γ_i for $n < i \leq g$, all chosen so that each β_i is transverse to $\boldsymbol{\alpha} \cup \boldsymbol{\delta} \cup \boldsymbol{\gamma}$. Set $\boldsymbol{\beta} = \beta_1 \cup \cdots \cup \beta_g$. Then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\beta})$ is a Heegaard triple subordinate to $\mathbb{L}'$, and $\mathcal{T}_{\alpha,\beta,\gamma} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is a Heegaard triple subordinate to $\mathbb{L}$. We will now verify that $\mathcal{T}_{\alpha,\beta,\gamma}$ satisfies the requirements in the statement of the lemma.

Let W' denote the link surgery bordism induced by $\mathbb{L}'$, and WW' the link surgery bordism induced by $\mathbb{L}\mathbb{L}'$, so that WW' is the result of gluing W to W' along Y_1. Let $\mathfrak{s}'$ be a Spin^c structure on W' extending $\mathfrak{t}_1$, and $\mathfrak{s}\mathfrak{s}'$ the Spin^c structure on WW' induced by $\mathfrak{s}$ and $\mathfrak{s}'$. (Note that it is easy to make choices above so that $\mathfrak{s}$, $\mathfrak{s}'$, and $\mathfrak{s}\mathfrak{s}'$ are realized by the corresponding Heegaard triples.)

It is classical that

$$c_1(\mathfrak{s}\mathfrak{s}')^2 - 2\chi(WW') - 3\sigma(WW') = c_1(\mathfrak{s})^2 + c_1(\mathfrak{s}')^2 - 2\chi(W) - 2\chi(W')$$
$$- 3\sigma(W) - 3\sigma(W').$$

(In particular, for additivity of the signature, see [**AS68**, Section 7.1].) To prove the result, then, it remains to check that the gr-terms add.

Note that $Y_{\beta,\gamma} = \#^{g-n+m}(S^1 \times S^2)$, $Y_{\delta,\beta} = \#^{g-m}(S^1 \times S^2)$, and $Y_{\delta,\gamma} = \#^{g-n}(S^1 \times S^2)$. Choose $\theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\theta' \in \mathbb{T}_\delta \cap \mathbb{T}_\beta$, and $\Theta \in \mathbb{T}_\delta \cap \mathbb{T}_\gamma$ lying in the canonical degree.

It is easy to check directly that in $(\Sigma, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, $\mathrm{gr}(\theta', \theta, \Theta) = 0$. Finally, additivity properties of the index imply that for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ with $s(\mathbf{x}) = \mathfrak{t}_1$ and $s(\mathbf{y}) = \mathfrak{t}_2$, and $\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ in the canonical degree,

$$\mathrm{gr}(\mathbf{x}_0, \theta\theta', \mathbf{y}) + \mathrm{gr}(\theta', \theta, \Theta) = \mathrm{gr}(\mathbf{x}_0, \theta', \mathbf{x}) + \mathrm{gr}(\mathbf{x}, \theta, \mathbf{y}).$$

where the suppressed superscripts are understood. The result follows. $\qquad \square$

It is convenient to also understand how $\widetilde{\mathrm{gr}}$ behaves under connected sums.

Lemma 3.4. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be $\widetilde{\mathrm{gr}}$-admissible pointed Heegaard diagrams for Y_1 and Y_2. Then $\mathcal{S}_1 \# \mathcal{S}_2$ is a $\widetilde{\mathrm{gr}}$-admissible pointed Heegaard diagram for $Y_1 \# Y_2$, and*

$$\widehat{CF}(\mathcal{S}_1 \# \mathcal{S}_2) = \widehat{CF}(\mathcal{S}_1) \otimes \widehat{CF}(\mathcal{S}_2).$$

Furthermore, for any homogeneous elements $\xi_1 \in \widehat{CF}(\mathcal{S}_1)$ and $\xi_2 \in \widehat{CF}(\mathcal{S}_2)$ with $s(\xi_1)$ and $s(\xi_2)$ torsion,

$$\widetilde{\mathrm{gr}}(\xi_1 \otimes \xi_2) = \widetilde{\mathrm{gr}}(\xi_1) + \widetilde{\mathrm{gr}}(\xi_2).$$

This lemma can be proved by considering a link surgery bordism from Y_1 to $Y_1 \# Y_2$ and applying the same reasoning used in the proof of the previous lemma. Here, it is convenient to choose the $\mathbb{L}$ and $\mathbb{L}'$ so that $\mathbb{L}\mathbb{L}' \subset S^3$ is a split link. Then the corresponding Heegaard triple subordinate to $\mathbb{L}\mathbb{L}'$ can be chosen to be a connected sum of a triple for Y_1 and a triple for Y_2.

Finally, given a pointed Heegaard diagram $\mathcal{S}$ for Y, observe that if $\mathfrak{t} \in \mathrm{Spin}^c_{\mathrm{tor}}(Y)$ and $\widehat{HF}(Y, \mathfrak{t}) \neq 0$, then $\widetilde{\mathrm{gr}}$ on $\widehat{HF}(Y, \mathfrak{t})$ determines $\widetilde{\mathrm{gr}}$ on $\widehat{CF}(\mathcal{S}, \mathfrak{t})$.

4. Proof that the two constructions agree

The purpose of this section is to prove the following theorem.

Theorem 4.1. $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ *as relative* $\mathbb{Q}$-*gradings on* $\widehat{HF}(Y, \text{torsion})$.

The idea of the proof is the following. The grading $\widetilde{\mathrm{gr}}$ is well behaved under Spin^c bordisms. In particular, if we know $\widetilde{\mathrm{gr}}$ on $\widehat{CF}(Y, \mathfrak{t})$, then we can determine $\widetilde{\mathrm{gr}}$ on any pair that is Spin^c bordant to $(Y, \mathfrak{t})$. The definition of $\widetilde{\mathrm{gr}}$ in Section 3.3 works because there is only one Spin^c bordism class in dimension three. The relative grading Gr has similar good behavior under Spin^c bordism, but only those bordisms admitting $\mathbb{Z}/n$-covers. In particular if we know Gr on $\widehat{CF}(Y, \mathfrak{t}) \oplus \widehat{CF}(Y, \mathfrak{t}')$, then we can determine Gr on something that is "suitably" Spin^c bordant to $(Y, \mathfrak{t}, \mathfrak{t}')$. Therefore, if we can show that Gr is consistent with $\widetilde{\mathrm{gr}}$ on a representative of each "suitable" bordism class (it turns out that lens spaces will suffice), then the result will follow.

We begin by showing that Gr has the desired behavior under connected sums.

Lemma 4.2. *Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be pointed Heegaard diagrams for* Y_1 *and* Y_2. *Then* $\mathcal{S}_1 \# \mathcal{S}_2$ *is a pointed Heegaard diagram for* $Y_1 \# Y_2$, *and*

$$\widehat{CF}(\mathcal{S}_1 \# \mathcal{S}_2) = \widehat{CF}(\mathcal{S}_1) \otimes \widehat{CF}(\mathcal{S}_2).$$

Furthermore, for any homogeneous elements $\xi_1, \xi_1' \in \widehat{CF}(\mathcal{S}_1)$ *and* $\xi_2, \xi_2' \in \widehat{CF}(\mathcal{S}_2)$ *with* $s(\xi_1)$, $s(\xi_1')$, $s(\xi_2)$, *and* $s(\xi_2')$ *torsion,*

$$\mathrm{Gr}(\xi_1 \otimes \xi_2, \xi_1' \otimes \xi_2') = \mathrm{Gr}(\xi_1, \xi_1') + \mathrm{Gr}(\xi_2, \xi_2').$$

Proof. Consider a cover of Y_1 for which $\mathfrak{s}(\tilde{\xi}_1) = \mathfrak{s}(\tilde{\xi}_1')$ and a cover of Y_2 for which $\mathfrak{s}(\tilde{\xi}_2) = \mathfrak{s}(\tilde{\xi}_2')$. From these two covers, we can construct a cover of $Y_1 \# Y_2$ for which the computation becomes obvious. $\square$

Combining this lemma with Lemma 3.4, we have the following.

Corollary 4.3. *If* $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ *as relative* $\mathbb{Q}$-*gradings on* $\widehat{HF}(Y_1, \text{torsion})$ *and* $\widehat{HF}(Y_2, \text{torsion})$, *then they are equal on* $\widehat{HF}(Y_1 \# Y_2, \text{torsion})$.

Now observe that trivially, $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ as relative $\mathbb{Q}$-gradings on $\widehat{HF}(S^1 \times S^2, \mathfrak{s}_0) \cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore, Lemmas 3.4 and 4.2 give us the following.

Corollary 4.4. *Let* $\mathfrak{t}, \mathfrak{t}' \in \mathrm{Spin}^c_{\mathrm{tor}}(Y)$. *Then* $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ *as relative* $\mathbb{Q}$-*gradings on* $\widehat{HF}(Y, \mathfrak{t}) \oplus \widehat{HF}(Y, \mathfrak{t}')$ *if and only if they are equal on* $\widehat{HF}(Y \# (S^1 \times S^2), \mathfrak{t} \# \mathfrak{s}_0) \oplus \widehat{HF}(Y \# (S^1 \times S^2), \mathfrak{t}' \# \mathfrak{s}_0)$.

The following lemma will be useful for producing equivariant Spin^c bordisms. When $\mathfrak{t}$ is a Spin structure we will denote the induced Spin^c structure by $\mathfrak{t}$ as well.

Lemma 4.5. *Given $f : Y \to B\mathbb{Z}/n$ and a Spin structure $\mathfrak{t}$ on Y, there exists a bordism $F : W \to B\mathbb{Z}/n$ from f to a disjoint union of maps $f_1 : L(n, 1) \to B\mathbb{Z}/n$, and there exists $\mathfrak{s} \in \mathrm{Spin}^c(W)$ such that $\mathfrak{s}|_Y = \mathfrak{t}$.*

Proof. First, recall the following (stable) Spin bordism groups of a point.

$$\Omega_0^{\mathrm{Spin}} = \mathbb{Z}$$
$$\Omega_1^{\mathrm{Spin}} = \mathbb{Z}/2$$
$$\Omega_2^{\mathrm{Spin}} = \mathbb{Z}/2$$
$$\Omega_3^{\mathrm{Spin}} = 0.$$

We use the Atiyah–Hirzebruch spectral sequence in order to understand the third (stable) Spin bordism group of $B\mathbb{Z}/n$, $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/n)$. We know that each class in $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/n)$ can be described by an element of

$$\bigoplus_{p+q=3} E_{p,q}^2 \cong H_0(B\mathbb{Z}/n; \Omega_3^{\mathrm{Spin}}) \oplus H_1(B\mathbb{Z}/n; \Omega_2^{\mathrm{Spin}}) \oplus H_2(B\mathbb{Z}/n; \Omega_1^{\mathrm{Spin}})$$

$$\oplus H_3(B\mathbb{Z}/n; \Omega_0^{\mathrm{Spin}}).$$

Consider the last summand $H_3(B\mathbb{Z}/n; \Omega_0^{\mathrm{Spin}}) \cong \mathbb{Z}/n$. Choose a model of $B\mathbb{Z}/n$ with $L(n, 1)$ as its 3-skeleton; this map $f_1 : L(n, 1) \to B\mathbb{Z}/n$, together with a Spin structure on $L(n, 1)$ gives us a generator of $H_3(B\mathbb{Z}/n; \Omega_0^{\mathrm{Spin}})$.

We now consider the other three summands. The first is obviously zero, since $\Omega_3^{\mathrm{Spin}} = 0$. Since $\Omega_1^{\mathrm{Spin}^c} = 0$ and $\Omega_2^{\mathrm{Spin}^c} = \mathbb{Z}$, the maps $\Omega_1^{\mathrm{Spin}} \to \Omega_1^{\mathrm{Spin}^c}$ and $\Omega_2^{\mathrm{Spin}} \to \Omega_2^{\mathrm{Spin}^c}$ are both zero. By naturality of the Atiyah–Hirzebruch spectral sequence, it follows that any element of $H_1(B\mathbb{Z}/n; \Omega_2^{\mathrm{Spin}}) \oplus H_2(B\mathbb{Z}/n; \Omega_1^{\mathrm{Spin}})$ is *stably* Spin^c bordant to 0.

We would be done now, except that the stable Spin bordisms and stable Spin^c bordisms are manifolds with Spin or Spin^c structures on the stable normal bundle, not the tangent bundle. For Spin bordism this is no additional complication: there is a one-to-one correspondence between Spin structures on the tangent bundle and Spin structures on the stable normal bundle (see [**LM89**, Proposition 2.15]). In particular, $(f, \mathfrak{t})$ must be Spin bordant to something in $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/n)$

For Spin^c structures on 3-manifolds, there is no such correspondence. However, for manifolds of dimension less than three, there is such a correspondence: in these dimensions, Spin^c structures on the tangent bundle and Spin^c structures on the stable normal bundle are both classified

by their first Chern classes. Therefore, any element of $H_1(B\mathbb{Z}/n; \Omega_2^{\mathrm{Spin}}) \oplus H_2(B\mathbb{Z}/n; \Omega_1^{\mathrm{Spin}})$ is honestly Spin^c bordant to 0. The result follows.
$\square$

Corollary 4.6. *Fix a 3-manifold Y, a Spin structure $\mathfrak{t}$ on Y, and a torsion Spin^c structure $\mathfrak{t}'$ on Y. Suppose that $\langle \mathfrak{t} - \mathfrak{t}' \rangle = \mathbb{Z}/n \subset H^2(Y; \mathbb{Z})$. Let $\tilde{Y} \to Y$ be a $\mathbb{Z}/n$-covering space as in the statement of Corollary 2.10. Then for some $\mathfrak{u}, \mathfrak{u}' \in \mathrm{Spin}^c(\coprod^m(L(n,1)))$, there are Spin^c bordisms $(W, \mathfrak{s})$ from $(\coprod^m(L(n,1)), \mathfrak{u})$ to $(Y, \mathfrak{t})$ and $(W, \mathfrak{s}')$ from $(\coprod^m(L(n,1)), \mathfrak{u}')$ to $(Y, \mathfrak{t}')$, and another Spin^c bordism $(\tilde{W}, \tilde{\mathfrak{s}})$ from S^3 to $\tilde{Y}$ covering $(W, \mathfrak{s})$ and $(W, \mathfrak{s}')$. That is,*

$$
\begin{array}{ccc}
(\coprod^m(L(n,1)), \mathfrak{u}) \!—\!(W,\mathfrak{s})\!—\! & (Y, \mathfrak{t}) \\
\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \\
(S^3, \mathfrak{s}_0) \!———\!(\tilde{W},\check{\mathfrak{s}})\!—\! & (\tilde{Y}, \tilde{\mathfrak{t}}) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
(\coprod^m(L(n,1)), \mathfrak{u}') \!—\!(W,\mathfrak{s}')\!—\! & (Y, \mathfrak{t}')
\end{array}
$$

Further, $\mathfrak{s} - \mathfrak{s}'$ is a torsion element of $H^2(W; \mathbb{Z})$.

Proof. Let $\tilde{Y} \to Y$ be the covering map given by Corollary 2.10, and let $f : Y \to B\mathbb{Z}/n$ be its classifying map. Then by Lemma 4.5 there is a Spin^c-bordism $F : (W, \mathfrak{s}) \to B\mathbb{Z}/n$ from $f : (Y, \mathfrak{t}) \to B\mathbb{Z}/n$ to a disjoint union of m maps of the form $f_1 : (L(n,1), \mathfrak{u}) \to B\mathbb{Z}/n$. (Here, $\mathfrak{u}$ is some fixed Spin structure on $L(n,1)$.) Recall from the proof of Lemma 2.9 that $\mathfrak{t} - \mathfrak{t}' = f^*b$ for some $b \in H^2(B\mathbb{Z}/n; \mathbb{Z})$. Setting $\mathfrak{s}' = \mathfrak{s} - F^*b$, we see that $\mathfrak{s}'|_Y = \mathfrak{t}'$. Setting $\tilde{W} = F^*(E\mathbb{Z}/n)$, we see that $\mathfrak{s}$ and $\mathfrak{s}'$ pullback to the same $\tilde{\mathfrak{s}} \in \mathrm{Spin}^c(\tilde{W})$.
$\square$

As mentioned earlier, we will need to know that the two gradings agree for lens spaces.

Lemma 4.7. $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ *as relative $\mathbb{Q}$-gradings on $\widehat{HF}(L(n,1))$.*

We defer the proof of this lemma until Section 5, where it follows from Proposition 5.3 and Corollary 5.2.

Proof of Theorem 4.1. For the sake of clarity, assume for now that there exists a Spin structure $\mathfrak{t}$ on Y such that $\widehat{HF}(Y, \mathfrak{t}) \neq 0$; this is necessarily the case, for instance, if Y is a rational homology sphere. At the end of the proof we will explain how to remove this assumption. With this assumption, it is sufficient to show that for all $\mathfrak{t}' \in \mathrm{Spin}_{\mathrm{tor}}^c(Y)$, $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ as relative $\mathbb{Q}$-gradings on $\widehat{HF}(Y, \mathfrak{t}) \oplus \widehat{HF}(Y, \mathfrak{t}')$.

Let W, $\tilde{W}$, $\mathfrak{s}$, $\mathfrak{s}'$, $\mathfrak{u}$, $\mathfrak{u}'$, and m be as in the statement of Corollary 4.6. As described in Section 3.2, we can decompose W into a bordism from $\coprod^m (L(n, 1))$ to $Y_1 = (\#^m(L(n, 1)))\#(\#^{k_1}(S^1 \times S^2))$, a link surgery bordism W_1 from Y_1 to $Y_2 = Y\#(\#^{k_2}(S^1 \times S^2))$, and a bordism from Y_2 to Y. By the previous lemma, we know that $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ as relative $\mathbb{Q}$-gradings on $\widehat{HF}(L(n, 1))$. Then by Corollaries 4.3 and 4.4, we know that $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ as relative $\mathbb{Q}$-gradings on $\widehat{HF}(Y_1, \text{torsion})$.

Now choose a pointed Heegaard triple $\mathcal{T}_{\alpha,\beta,\gamma}$ corresponding to W_1 as in the statement of Lemma 3.3 such that $\mathcal{T}_{\alpha,\beta,\gamma}$ realizes both $\mathfrak{s}|_{W_1}$ and $\mathfrak{s}'|_{W_1}$. Let $\mathbf{x}, \mathbf{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s(\mathbf{x}) = (\#\mathfrak{u})\#\mathfrak{s}_0$ and $s(\mathbf{x}') = (\#\mathfrak{u}')\#\mathfrak{s}_0$, and let $\mathbf{y}, \mathbf{y}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ with $s(\mathbf{y}) = \mathfrak{t}\#\mathfrak{s}_0$ and $s(\mathbf{y}') = \mathfrak{t}'\#\mathfrak{s}_0$. Then

$$\widetilde{\mathrm{gr}}(\mathbf{y}) - \widetilde{\mathrm{gr}}(\mathbf{y}') = \widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{x}') - (\mathrm{gr}^{\mathfrak{s}}(\mathbf{x}, \theta, \mathbf{y}) - \mathrm{gr}^{\mathfrak{s}'}(\mathbf{x}', \theta, \mathbf{y}'))$$

$$+ \frac{c_1(\mathfrak{s}|_{W_1})^2 - c_1(\mathfrak{s}'|_{W_1})^2}{4}$$

(4.1) $$= \widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{x}') - \mathrm{gr}^{\mathfrak{s}}(\mathbf{x}, \theta, \mathbf{y}) + \mathrm{gr}^{\mathfrak{s}'}(\mathbf{x}', \theta, \mathbf{y}')$$

where the last line follows because $\mathfrak{s} - \mathfrak{s}'$ is torsion.

Claim 4.8. $\mathrm{Gr}(\mathbf{y}, \mathbf{y}') = \mathrm{Gr}(\mathbf{x}, \mathbf{x}') - \mathrm{gr}^{\mathfrak{s}}(\mathbf{x}, \theta, \mathbf{y}) + \mathrm{gr}^{\mathfrak{s}'}(\mathbf{x}', \theta, \mathbf{y}')$.

Using the cover $\tilde{W} \to W$, construct a covering Heegaard triple $\tilde{\mathcal{T}}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$ and compute:

$$\mathrm{Gr}(\mathbf{y}, \mathbf{y}') = \frac{1}{n}\mathrm{gr}(\tilde{\mathbf{y}}, \tilde{\mathbf{y}}')$$

$$= \frac{1}{n}(\mathrm{gr}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') - \mathrm{gr}^{\pi^*\mathfrak{s}}(\tilde{\mathbf{x}}, \tilde{\theta}, \tilde{\mathbf{y}}) + \mathrm{gr}^{\pi^*\mathfrak{s}'}(\tilde{\mathbf{x}}', \tilde{\theta}, \tilde{\mathbf{y}}')) \text{ by additivity of gr}$$

$$= \mathrm{Gr}(\mathbf{x}, \mathbf{x}') - \mathrm{gr}^{\mathfrak{s}}(\mathbf{x}, \theta, \mathbf{y}) + \mathrm{gr}^{\mathfrak{s}'}(\mathbf{x}', \theta, \mathbf{y}') \text{ by Lemma 3.2.}$$

Since $\widehat{HF}(Y_1, (\#\mathfrak{u})\#\mathfrak{s}_0)$ and $\widehat{HF}(Y_1, (\#\mathfrak{u}')\#\mathfrak{s}_0)$ are nonzero and $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ as relative $\mathbb{Q}$-gradings on $\widehat{HF}(Y_1, \text{torsion})$, it follows that $\mathrm{Gr}(\mathbf{x}, \mathbf{x}') = \widetilde{\mathrm{gr}}(\mathbf{x}') - \widetilde{\mathrm{gr}}(\mathbf{x})$. Equation (4.1) and Claim 4.8 now show that $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ as relative $\mathbb{Q}$-gradings on $\widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}\#\mathfrak{s}_0) \oplus \widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}'\#\mathfrak{s}_0)$. Consequently they are equal on $\widehat{HF}(Y_2, \mathfrak{t}\#\mathfrak{s}_0) \oplus \widehat{HF}(Y_2, \mathfrak{t}'\#\mathfrak{s}_0)$, and Corollary 4.4 then shows that they are equal on $\widehat{HF}(Y, \mathfrak{t}) \oplus \widehat{HF}(Y, \mathfrak{t}')$.

We now deal with the general case when the simplifying assumption fails. Choose a Spin structure $\mathfrak{t}$ on Y. To prove the theorem, it is sufficient to show that for all $\mathfrak{t}', \mathfrak{t}^\circ \in \mathrm{Spin}^c_{\mathrm{tor}}(Y)$, $\mathrm{Gr} = \widetilde{\mathrm{gr}}$ as relative $\mathbb{Q}$-gradings on $\widehat{HF}(Y, \mathfrak{t}') \oplus \widehat{HF}(Y, \mathfrak{t}^\circ)$. We will do this by constructing a Heegaard diagram $\mathcal{S}_{\alpha,\gamma}$ for $Y\#(\#^{k_2}S^1 \times S^2)$ for which we know the relative gradings Gr and $\widetilde{\mathrm{gr}}$ agree on $\widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}\#\mathfrak{s}_0) \oplus \widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}'\#\mathfrak{s}_0)$ and also agree on $\widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}\#\mathfrak{s}_0) \oplus \widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}^\circ\#\mathfrak{s}_0)$. By Corollary 4.4, this is sufficient to prove that Gr and $\widetilde{\mathrm{gr}}$ agree as relative gradings on $\widehat{HF}(Y, \mathfrak{t}') \oplus \widehat{HF}(Y, \mathfrak{t}^\circ)$.

Let W, $\tilde{W}$, $\mathfrak{s}$, $\mathfrak{s}'$, $\mathfrak{u}$, $\mathfrak{u}'$, m, Y_1, and Y_2 be the objects constructed before using $\mathfrak{t}$ and $\mathfrak{t}'$. Let W°, $\tilde{W}^\circ$, $\mathfrak{s}^\circ$, $\mathfrak{s}'^\circ$, $\mathfrak{u}^\circ$, $(\mathfrak{u}^\circ)'$, m°, Y_1°, and Y_2° be the corresponding objects constructed using $\mathfrak{t}$ and $\mathfrak{t}^\circ$. By adding some canceling 1- and 3-handles to the decompositions of W or W° if necessary, we may assume that $Y_2 = Y_2^\circ$.

Now, choose Heegaard triples $\mathcal{T}_{\alpha,\beta,\gamma}$ and $\mathcal{T}_{\alpha^\circ,\beta^\circ,\gamma^\circ}^\circ$ for W_2 and W_2° respectively, with the property that $\mathcal{S}_{\alpha,\gamma} = \mathcal{S}_{\alpha^\circ,\gamma^\circ}^\circ$. This is possible by the argument used in Lemma 3.3. That is, we view W_2 and W_2° as given by surgery on disjoint links $\mathbb{L}$ and $\mathbb{L}'$ in Y_2. Then it is easy to construct the desired triple diagrams from a diagram subordinate to $\mathbb{L} \cup \mathbb{L}'$. Further, arrange that $\widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}\#\mathfrak{s}_0)$ is nonzero, i.e., that there is some generator in the Spin^c-structure $\mathfrak{t}\#\mathfrak{s}_0$.

As before, it follows from Corollaries 4.3 and 4.4 that Gr $= \widetilde{\mathrm{gr}}$ as relative Q-gradings on both $\widehat{CF}(\mathcal{S}_{\alpha,\beta}, \mathfrak{t}\#\mathfrak{s}_0) \oplus \widehat{CF}(\mathcal{S}_{\alpha,\beta}, \mathfrak{t}'\#\mathfrak{s}_0)$ and $\widehat{CF}(\mathcal{S}_{\alpha,\beta^\circ}^\circ, \mathfrak{t}\#\mathfrak{s}_0) \oplus \widehat{CF}(\mathcal{S}_{\alpha,\beta^\circ}^\circ, \mathfrak{t}^\circ\#\mathfrak{s}_0)$. Then, by (4.1) and Claim 4.8, it follows that Gr $= \widetilde{\mathrm{gr}}$ as relative Q-gradings $\widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}\#\mathfrak{s}_0) \oplus \widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}'\#\mathfrak{s}_0)$ and $\widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}\#\mathfrak{s}_0) \oplus \widehat{CF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}^\circ\#\mathfrak{s}_0)$. It then follows that Gr $= \widetilde{\mathrm{gr}}$ as relative Q-gradings on $\widehat{HF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}'\#\mathfrak{s}_0) \oplus \widehat{HF}(\mathcal{S}_{\alpha,\gamma}, \mathfrak{t}^\circ\#\mathfrak{s}_0)$. The result now follows from Corollary 4.4. □

As discussed in Section 2.1, there is a group $\widehat{HF}(Y, \ell)$ that is independent of choice of ℓ-pointed Heegaard diagram. The invariant absolute grading $\widetilde{\mathrm{gr}}$ was originally only defined on $\widehat{HF}(Y, 1)$ [**OS06**]. In light of the two main theorems of this paper, Theorems 2.6 and 4.1, one might hope that some generalization of $\widetilde{\mathrm{gr}}$ to an invariant absolute grading of $\widehat{HF}(Y, \ell)$ would behave well with respect to coverings. Specifically, given a one-pointed Heegaard diagram $\mathcal{S}$ for Y and an n-fold cover of Y, one might hope that $\widetilde{\mathrm{gr}}(\mathbf{x}) = \frac{1}{n}\widetilde{\mathrm{gr}}(\tilde{\mathbf{x}})$ (whenever $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are both nontrivial in homology). Unfortunately, the example of lens spaces shows that this is impossible.

Explicitly, consider the standard one-pointed Heegaard diagrams $\mathcal{S}_1$ and $\mathcal{S}_2$ for $L(5, 1)$ and $L(5, 2)$, respectively, together with the covering maps from S^3 to $L(5, 1)$ and $L(5, 2)$. One can find $\mathbf{x}_1 \in \widehat{CF}(\mathcal{S}_1)$ and $\mathbf{x}_2 \in \widehat{CF}(\mathcal{S}_2)$ with the properties that:

- $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ define the same nontrivial element of $\widehat{HF}(S^3, 5)$.
- $\mathbf{x}_1$ and $\mathbf{x}_2$ are both nontrivial in homology, and $\widetilde{\mathrm{gr}}(\mathbf{x}_1) \neq \widetilde{\mathrm{gr}}(\mathbf{x}_2)$.

5. Lens spaces and gradings

Let p and q be relatively prime positive integers with $p > q$. In this section we will compute Gr on $L(p, q)$ and show that the answer agrees with the relative Q-grading induced by $\widetilde{\mathrm{gr}}$, as computed by Ozsváth and Szabó in [**OS03**, Proposition 4.8]. We first describe a pointed Heegaard diagram
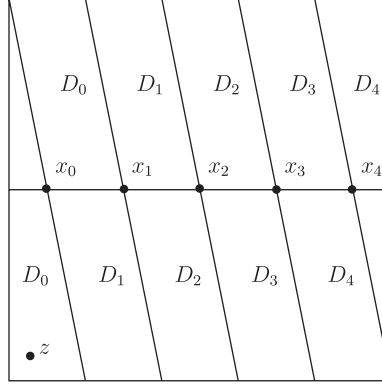
Figure 1. The lens space $-L(5,1)$.

$\mathcal{S} = (\Sigma, \alpha, \beta, z)$ for $-L(p,q)$. Let Σ be a torus, realized as the square $[0,1] \times [0,1]$ with edges identified. Let α be the horizontal circle $y = 1/2$, and let β be any circle with slope $-p/q$. Choose z to be any point in $\Sigma \smallsetminus \alpha \smallsetminus \beta$ lying below $y = 1/2$. There are p connected components of $\Sigma \smallsetminus \alpha \smallsetminus \beta$. We label their closures $D_0, \ldots, D_{p-1}$, going from left to right, starting with $z \in D_0$. (Note that this is not the same notational convention used in Section 2.) There are p elements of $\alpha \cap \beta$. We label them $x_0, \ldots, x_{p-1}$, going from left to right along α, such that x_0 is the upper right vertex of D_0. See Figure 1.

It is easy to see that $\epsilon(\mathbf{x}_i, \mathbf{x}_j) \neq 0$ if $i \neq j$, so each $\mathfrak{s}_i = s(x_i)$ is distinct, and these are all of the Spin^c structures on $-L(p,q)$. Consequently, the differential on $\widehat{CF}(\mathcal{S})$ is trivial, and we can identify $\widehat{CF}(\mathcal{S})$ with $\widehat{HF}(-L(p,q))$. Ozsváth and Szabó proved the following inductive formula for their absolute $\mathbb{Q}$-grading on $\widehat{HF}(-L(p,q))$ in [**OS03**, Proposition 4.8]. Since their formula relates $\widetilde{\mathrm{gr}}$ on different lens spaces, we will use the notation $\widetilde{\mathrm{gr}}_{p,q}$ for the absolute $\mathbb{Q}$-grading on $\widehat{HF}(-L(p,q))$.

Proposition 5.1. *In the setup described above, for $0 \leq i < p + q$,*

$$\widetilde{\mathrm{gr}}_{p,q}(x_{i \bmod p}) = \left(\frac{pq - (2i + 1 - p - q)^2}{4pq} \right) - \widetilde{\mathrm{gr}}_{q,p \bmod q}(x_{i \bmod q}).$$

We can use this formula to derive a noninductive formula for the relative $\mathbb{Q}$-grading induced by $\widetilde{\mathrm{gr}}$.

Corollary 5.2. *Fix p and q, and let $\widetilde{\mathrm{gr}}$ refer to the absolute $\mathbb{Q}$-grading on $\widehat{HF}(-L(p,q))$. For $0 \leq i < p$,*

$$\widetilde{\mathrm{gr}}(x_{i+q \bmod p}) - \widetilde{\mathrm{gr}}(x_i) = \frac{1}{p}(p - 1 - 2i).$$

Note that this formula completely determines the relative $\mathbb{Q}$-grading on $\widehat{HF}(-L(p,q))$.

Proof. By the previous proposition,

$$
\begin{aligned}
\widetilde{\mathrm{gr}}_{p,q}&(x_{i+q \bmod p}) - \widetilde{\mathrm{gr}}_{p,q}(x_{i \bmod p}) \\
&= \left(\frac{pq - (2(i+q) + 1 - p - q)^2}{4pq} \right) - \widetilde{\mathrm{gr}}_{q,p \bmod q}(x_{i+q \bmod q}) \\
&\quad - \left(\frac{pq - (2i + 1 - p - q)^2}{4pq} \right) + \widetilde{\mathrm{gr}}_{q,p \bmod q}(x_{i \bmod q}) \\
&= \left(\frac{-(2i+1-p+q)^2}{4pq} \right) + \left(\frac{(2i+1-p-q)^2}{4pq} \right) \\
&= \frac{2(2i+1-p)(-2q)}{4pq} \\
&= \frac{1}{p}(p - 1 - 2i).
\end{aligned}
$$

$\square$

We now derive the same formula for Gr.

Proposition 5.3. *Let* Gr *be the relative $\mathbb{Q}$-grading on $\widehat{HF}(-L(p,q))$ defined in Section 2. For $0 \le i < p$,*

$$
\mathrm{Gr}(x_{i+q \bmod p}, x_i) = \frac{1}{p}(p - 1 - 2i).
$$

Proof. Since p and q are now fixed, we can interpret the indices for x and D as being defined mod p. In order to compute $\mathrm{Gr}(x_{i+q}, x_i)$ using Proposition 2.13, we need an element of $\hat{\pi}_2(px_{i+q}, px_i)$.

Claim 5.4. *For $0 \le i < p$, let $A_i = \sum_{j=1}^{p} (p - (i+j)) D_{i+j}$. Then $A_i \in \hat{\pi}_2(px_{i+q}, px_i)$.*

To see that $A \in \hat{C}_2(\mathcal{S})$, note that the coefficient of $D_0 = D_p$ is zero. The proof that $A \in \pi_2(px_{i+q}, px_i)$ is a simple computation using the fact that

$$
\partial \partial_\alpha D_i = x_{i+q} - x_{i+q-1} + x_{i-1} - x_i.
$$

We can now plug A_i into the formula in Proposition 2.13. Noting that $e(D_i) = 0$, x_i touches D_{p+i}, D_{i+1}, D_{p+i-q}, and $D_{p+i-q+1}$, and x_{i+q} touches

D_{i+q}, D_{i+q+1}, D_{p+i}, and D_{i+1}, we compute

$$
\begin{aligned}
\mathrm{Gr}(x_{i+q}, x_i) &= \frac{1}{p}[e(A_i) + n_{x_{i+q}}(A_i) + n_{x_i}(A_i)] \\
&= \frac{1}{p}\left(0 + \frac{1}{4}[(p - i - q) + (p - i - q - 1) + (-i) + (p - i - 1)]\right. \\
&\quad \left. + \frac{1}{4}[(-i) + (p - i - 1) + (-i + q) + (-i + q - 1)]\right) \\
&= \frac{1}{p}(p - 1 - 2i).
\end{aligned}
$$

$\square$

6. Directions for future research

This paper is an offshoot of an ongoing attempt by the authors to understand the relationship between the Heegaard Floer homology of a space and the Heegaard Floer homology of its finite covering spaces, the main technical goal of which is a localization theorem. This is one direction for further study.

More directly related to gradings, it would be interesting to extend our definition to non-torsion Spin^c structures. This has two meanings. The simpler is that if $\mathfrak{t}_1 - \mathfrak{t}_2 \in H^2(Y)$ is torsion then one can find a covering space $p : \tilde{Y} \to Y$ such that $p^*\mathfrak{t}_1 = p^*\mathfrak{t}_2$; one could try to use this covering space to define a relative grading between generators of $\widehat{CF}(Y, \mathfrak{t}_1)$ and $\widehat{CF}(Y, \mathfrak{t}_2)$. The difficulty arises from the fact that when $\mathfrak{t}$ is nontorsion, the relative grading on $\widehat{CF}(Y, \mathfrak{t})$ is only defined modulo $\gcd_{A \in H_2(Y)}\{\langle c_1(\mathfrak{t}), A\rangle\}$. Hopefully, this difficulty could either be overcome or exploited.

A larger generalization would be to obtain a relative grading between Spin^c structures with non-torsion difference by considering infinite covering spaces. In light of [**APS75**], it seems likely that here the real numbers, rather than the rationals, would come into play. Conceivably, attempts to generalize the index could lead to some kind of "ℓ^2 Heegaard Floer homology" for infinite covering spaces.

Finally, it would be interesting to explore how covering spaces might be used to construct the absolute $\mathbb{Q}$-grading $\widetilde{\mathrm{gr}}$, as well as the induced relative $\mathbb{Q}$-grading. In view of the example of lens spaces, such a construction might require significant new ideas.

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