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# SYMPLECTIC FORMS AND SURFACES OF NEGATIVE SQUARE

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We introduce an analogue of the inflation technique of Lalonde–McDuff, allowing us to obtain new symplectic forms from symplectic surfaces of negative self-intersection in symplectic 4-manifolds. We consider the implications of this construction for the symplectic cones of Kähler surfaces, proving along the way a result which can be used to simplify the intersections of distinct pseudo-holomorphic curves via a perturbation.

## 1. Introduction

Given an embedded symplectic surface $C$ of non-negative self-intersection in a symplectic 4-manifold $(M, \omega)$, the inflation process in [**9**] gives rise to new symplectic forms in the class $[\omega] + tPD[C]$ for arbitrary $t > 0$. In this paper, we show that there is an analogous construction in the case of an embedded symplectic surface of negative self-intersection.

**Theorem 1.1.** *Suppose $C$ is an embedded connected symplectic surface representing a class $e$ with $e \cdot e = -k < 0$ and $a = \omega(e)$. Let $h = k$ if $C$ has positive genus or $C$ is a sphere with $k$ even, and $h = k + 1$ if $C$ is a sphere with $k$ odd. Then there are symplectic forms $\omega_t$ representing the classes $[\omega] + tPD(e)$ for any $t \in [0, \frac{2a}{h})$.*

This is achieved by the normal connected sum construction [**4**, **13**]. In fact, the inflation process can be viewed this way as well. However, there are two distinct features from the inflation process. The first is the upper bound on $t$. The second is that the surface $C$ is not symplectic with respect to the forms $\omega_t$ when $t > \frac{a}{k}$ (such values of $t$ occur as long as $C$ is not a $(-1)$-sphere). Indeed, the symplectic area of $C$ is non-positive for these values of $t$.

From the known characterization of the symplectic cones of $S^2$-bundles [**11**], for any triple $(g, k, a)$ with $g \geq 0$ and $k, a > 0$ and any $\epsilon > 0$, it is a routine exercise to find a symplectic 4-manifold $(M, \omega)$ containing a symplectic surface $\Sigma_\epsilon$ of genus $g$, square $-k$, and area $a$ such that $[\omega] + (2a/h + \epsilon)PD[\Sigma_\epsilon]$ does *not* admit symplectic forms, where $h$ is as in the statement of Theorem 1.1. In this regard, Theorem 1.1 may be considered a best possible result for the generality in which it is stated.

Using the pairwise normal connected sum construction, we will also show how to apply the construction of Theorem 1.1 to a configuration of surfaces intersecting each other positively and transversally.

To apply such a construction, we need to locate configurations of surfaces. Such configurations sometimes appear as pseudo-holomorphic curves. It is shown in [**12**] that any irreducible simple pseudo-holomorphic curve can be perturbed to a pseudo-holomorphic immersion, possibly after a $C^1$-small change in the almost complex structure. We show how to further perturb such an immersion to an embedding. In fact, we are able to show that any configuration of simple $J$-holomorphic curves can be perturbed to a configuration of symplectic surfaces which intersect each other positively and transversally and which are $J'$-holomorphic for an almost complex structure arbitrarily $C^1$-close to $J$.

Holomorphic curves of negative self-intersection actually characterize the Kähler cone by (the extension of) the Nakai–Moishezon criterion. Thus, it is interesting to apply this construction to Kähler surfaces. Let $(M, J)$ be a Kähler surface and $H_J^{1,1}$ denote the real part of the $(1,1)$-subspace of $H^2(M; \mathbb{C})$ determined by $J$. The classical Hodge index theorem then asserts that the restriction of the intersection form to $H_J^{1,1}$ is a bilinear form of type $(1, h^{1,1} - 1)$. The *positive cone* of $H_J^{1,1}$ is then by definition the set of classes in $H_J^{1,1}$ which have positive square and pair positively with the class of the given Kähler form. Buchdahl and Lamari have recently independently proven the following result:

**Theorem 1.2.** [**2**, **7**] *For a Kähler surface $(M, J)$, any class in the positive cone of $H_J^{1,1}$ is represented by a Kähler form if it is positive on each holomorphic curve with negative self-intersection.*

(Note that the Hodge index theorem implies that any class in the positive cone of $H_J^{1,1}$ is automatically positive on each curve of *non*-negative self-intersection.) Applying Theorem 1.1 to a curve $C$ of negative self-intersection, the Kähler cone can be enlarged across the "wall" consisting of cohomology classes which vanish on $[C]$ unless $C$ is a $(-1)$-sphere. This suggests the following symplectic Nakai–Moishezon criterion:

**Question 1.3.** *For a Kähler surface $(M, J)$, is every class in the positive cone of $H_J^{1,1}$ which is positive on each $(-1)$-sphere (possibly reducible) represented by a symplectic form?*

To motivate this, note that by the Riemann–Roch theorem and the adjunction formula, the expected dimension of the space of embedded pseudo-holomorphic genus $g$ curves in the class $[C]$ is

$$d([C]) = 2(g-1+\langle c_1(M),[C]\rangle) = [C]\cap[C]-\langle c_1(M),[C]\rangle = [C]\cap[C]+1-g,$$

which as the last expression above demonstrates is negative if $[C]$ is the class of any negatively self-intersecting curve other than a $(-1)$-sphere. Thus for generic almost complex structures $J'$ close to $J$, there will be no $J'$-holomorphic curves in the class $C$. The theory of pseudo-holomorphic curves hence does not provide any obstruction to deforming the symplectic form to one which pairs negatively with $C$. If $C$ is the class of a $(-1)$-sphere, on the other hand, Gromov–Taubes theory shows that any symplectic form deformation equivalent to the Kähler form must pair positively with $C$.

When $p_g = 0$, Question 1.3 has an affirmative answer. In this case, we have $b^+ = 1$, so every class of positive square which is positive on $-1$ symplectic spheres is realized by a symplectic form [**8**]. In addition, for a minimal surface of general type, the canonical class $K$ has been shown to be in the symplectic cone [**3**, **15**].

In a more general setting, the answer to Question 1.3 seems elusive. Our methods do, however, enable us to progress somewhat farther on the following related question:

**Question 1.4.** *Let $\{C_1,\dots,C_n\}$ be reduced irreducible holomorphic curves of negative square, none of which is a $(-1)$-sphere, such that there exist classes $\alpha$ in the positive cone of $H_J^{1,1}$ satisfying $\langle\alpha,[C_i]\rangle < 0$ for each $i$. Do some of these classes $\alpha$ admit symplectic forms?*

In Section 4, we outline methods for using Theorem 1.1 to answer this question in certain situations and we illustrate these methods by applying them in detail to all of the subsets of a particular set of 21 negative-square curves in a rigid surface $K$ that was introduced in [**6**]. We choose a rigid surface as our primary example in order to ensure that the curves in question cannot be made to disappear by an integrable variation in the complex structure; as such, we may state with certainty that the new symplectic forms that we construct are not directly obtainable by considerations of Kähler geometry.

The methods of Section 4 can be applied to a wide variety of configurations of the curves $C_1,\dots,C_n$ in Question 1.4, but there are also many configurations to which these methods do not apply. It seems unlikely that there is any necessary and sufficient condition on the configuration that can be expressed at all concisely, but we provide an example of a moderately general sufficient condition in Theorem 4.14.

We would like to thank McDuff for her valuable suggestions on how to extend her result in [**12**] to our situation. The first author is also grateful

to Ruan for discussions on the 6-dimensional symplectic minimal model program which inspired Theorem 1.1.

## 2. The construction

Theorem 1.1 is an application of the normal connected sum construction with symplectic $S^2$-bundles. So let us collect some facts about symplectic structures on such manifolds and embedded symplectic surfaces in them.

Up to diffeomorphisms, there are two orientable $S^2$-bundles over a Riemann surface $\Sigma$: the trivial one $\Sigma \times S^2$ and the non-trivial one $M_\Sigma$. By [**9**], symplectic forms on $S^2$-bundles are determined by their cohomology classes up to isotopy. Thus we can pick any convenient symplectic form in a fixed cohomology class.

We begin with the easier case: the product bundle. In this case, we use split forms as our model forms. Clearly, every class of the positive cone is represented by a split symplectic form. And for a split symplectic form, the vertical fibers and horizontal sections are symplectic. The class of any section with (even) positive square is then represented by an immersed symplectic surface with only positive transverse self-intersections, which can then be smoothed to an embedded symplectic surface.

Now let us deal with the non-trivial bundle $M_\Sigma$ over a positive genus surface. We use Kähler forms as our model forms. The following result is essentially contained in [**5**, **11**] (we present it here since it may not be very well known).

**Proposition 2.1.** *Let $\mathcal{E}$ be a holomorphic rank 2 bundle over $\Sigma$ with $g(\Sigma) > 0$ and $c_1(\mathcal{E}) = -1$. Let $(M, J_{\mathcal{E}})$ be the complex ruled surface $P(\mathcal{E})$. Then the Kähler cone is the positive cone if and only if $\mathcal{E}$ is stable. Furthermore, for appropriately chosen holomorphic structures on $\mathcal{E}$, the class of any section with (odd) positive square can be represented by an embedded surface which is symplectic with respect to any Kähler form.*

*Proof.* Notice that the slope of $\mathcal{E}$ is $-\frac{1}{2}$. Therefore, the stability of $\mathcal{E}$ is equivalent to the statement that every holomorphic line subbundle $\mathcal{L}$ of $\mathcal{E}$ has $c_1(\mathcal{L}) \leq -1$. Observe that any holomorphic line bundle $\mathcal{L} \subset \mathcal{E}$ gives rise to a holomorphic section $Z(\mathcal{L})$ of $P(\mathcal{E})$, and vice versa. Since the normal bundle to $Z(\mathcal{L})$ is $\mathcal{L}^* \otimes \mathcal{E}/\mathcal{L}$, all sections of $P(\mathcal{E})$ have positive self-intersection if and only if $\mathcal{E}$ is stable. The statement about Kähler cone now follows from the arguments in Proposition 3.1 in [**5**, **11**].

For the second statement, it suffices to show that the class $[s^+]$ of a section with square $+1$ is symplectic. As all the fibers are holomorphic and hence symplectic and the classes of sections with higher squares have form $[s^+] + m[fiber]$ for $m > 0$, these classes are represented by positively immersed symplectic surfaces, which can be smoothed to embedded ones. We may take the holomorphic structure on $\mathcal{E}$ to be that on a non-trivial extension of $\mathcal{L}$ by the trivial line bundle $\mathcal{O}$, where $\mathcal{L}$ is a degree $-1$ holomorphic line bundle. The section $Z(\mathcal{L})$ is then a holomorphic, and so in particular symplectic, $+1$ section. $\square$

Finally, the non-trivial bundle over a sphere is diffeomorphic to the blow up of $\mathbb{C}P^2$ at a point, and the exceptional divisor is a section with square $-1$. As is well known, either using the standard symplectic reduction picture or algebraic geometry, we can construct symplectic forms in every class in the positive cone which is positive on the class of a section with square $-1$, such that, for every odd $k \geq -1$, there are symplectic sections with square $k$.

Now we are ready to prove Theorem 1.1.

*Proof.* Let $R$ be the trivial sphere bundle over the surface of genus $g(C)$ if $k$ is even, and the non-trivial one if $k$ is odd. Let $s^{\pm k}$ be the class of a section with square $\pm k$. Then, $s^{+k}$ and $s^{-k}$ form a basis for $H_2(R;\mathbb{Z})$. Since $s^{+k} \cdot s^{-k} = 0$, a cohomology class of the form

$$c^+ PD(s^{+k}) + c^- PD(s^{-k})$$

has positive square if and only if $c^+ > |c^-|$.

Suppose first that $C$ is not a sphere with $k$ odd. By Proposition 2.1 and the discussions preceding it, there exists a symplectic form $\tau_t$ on $R$ in the class

$$\frac{a}{k} PD(s^{+k}) + \left(t - \frac{a}{k}\right) PD(s^{-k})$$

for any $t \in (0, \frac{2a}{k})$. By Proposition 2.1, there is a $\tau_t$-symplectic section $S^{+k}$ in the class $s^{+k}$. Notice that the symplectic surfaces $C$ and $S^{+k}$ have opposite self-intersection and equal symplectic area $a$. Thus we can perform the symplectic sum construction to $(M, C, \omega)$ and $(R, S^{+k}, \tau_t)$ to obtain a new symplectic manifold $(X, \omega_t)$. As observed in [**4**], $X$ and $M$ are diffeomorphic. Moreover, because the surface $S^{-k}$ is disjoint from the surface $S^{+k}$ in $R$, it becomes a surface in $M$ which is homologous to $C$. Thus we have

$$\omega_t(e) = \tau_t(s^{-k}) = a - tk.$$

Therefore, $[\omega_t] = [\omega] + tPD(e)$.

In the case that $C$ is a sphere with $k$ odd, by the discussions after Proposition 2.1, there exists a symplectic form in the class $c^+PD(s^{+1}) + c^-PD(s^{-1})$ if and only if $c^+ > -c^- > 0$. We would like to express the condition in terms of the basis $s^{+k}$ and $s^{-k}$. Since $s^m \cdot s^n = \frac{m+n}{2}$, we see that a class $\alpha$ contains a symplectic form if and only if

$$\frac{\alpha(s^{-k})}{\alpha(s^{+k})} > \frac{1-k}{k+1}.$$

Thus the allowed values of $t$ are those between 0 and $\frac{2a}{k+1}$, as claimed. $\square$

**Remark 2.2.** Notice that

$$\omega_t \cdot \omega_t = \omega^2 + 2t\omega(e) + t^2 e \cdot e = -t^2 k + 2ta + \omega^2 = \omega^2 + k\left[\frac{a^2}{k^2} - \left(t - \frac{a}{k}\right)^2\right].$$

So the volume of the symplectic manifold $(M, \omega_t)$ is greater than that of $(M, \omega)$ for each $t \in (0, 2a/k)$.

We can generalize Theorem 1.1 to a configuration of transversally intersecting symplectic surfaces.

**Theorem 2.3.** *Suppose $C_1, \ldots, C_l$ is a set of embedded symplectic surfaces with self-intersection $C_i \cdot C_i = -k_i < 0$ and intersecting positively and transversally. Let $e_i$ be the class of $C_i$. Then there are symplectic forms in the class $[\omega] + \sum_i t_i PD(e_i)$ for any $0 < t_i < 2\omega(e_i)/h_i$, where $h_i$ is as in Theorem* 1.1*.*

*Proof.* We prove the theorem in the case that there are only two curves $C_1$ and $C_2$. The idea for the general case is the same.

The key point is that the symplectic sum construction in Theorem 1.1, when applied to $C_1$, can be done in a way such that $C_2$, possibly after an isotopy, is still symplectic with respect to the new symplectic structures $\omega_1$, which is in the class $[\omega]+t_1PD(e_1)$ with $t_1 \in (0, 2\omega(e_1)/h_1)$. This is possible due to the pairwise sum feature in [**4**].

First, by applying Lemma 2.3 of [**4**], perturb $C_2$ such that $C_2$ intersects $C_1$ orthogonally with respect to $\omega$. Since the fiber spheres in $R$ are symplectic and intersect the symplectic section $S^{+k_1}$ transversally, we can likewise assume that the symplectic section $S^{+k_1}$ intersects a total of $k_{12} = C_1 \cdot C_2$ fibers, all orthogonally. Denote this union of the fibers by $F$. Now apply pairwise sum to $(M, C_1, C_2)$ to $(R, S^{+k_1}, F)$ to get a symplectic surface $C_2'$. Finally, apply the symplectic sum construction to $C_2'$ as in the proof of Theorem 1.1. □

**Remark 2.4.** Notice that since $e_1 \cdot e_i \geq 0$ for $i \geq 2$, $[\omega]+t_iPD(e_1)$ is positive on $e_2$ for $t_1$ positive. One has $S_i^+ = S_i^- + k_i f$, where $f$ is the homology class of the fiber in $R$, so

$$\int_{S_i^-} \tau = \int_{C_i} \omega + t_i PD(e_i) = a_i - t_i k_i = \int_{S_i^+} \tau - k_i \tau(f).$$

Thus $\tau(f) = t_i$. This is consistent with the normal connected sum picture. The area of the surface $C_j$ increases by $(e_j \cdot e_i)\tau(f)$, which is indeed equal to $t_i(e_i \cdot e_j)$.

**Remark 2.5.** If these surfaces actually intersect, then some of the values of $t_i$ can be taken larger than in the statement of the theorem.

## 3. Configurations of embedded symplectic surfaces and pseudo-holomorphic curves

In attempting to answer questions such as Question 1.3, we might wish to apply Theorem 2.3 to some finite set of holomorphic curves. However, the proof of Theorem 2.3 depends on the assumption that the symplectic submanifolds being considered intersect positively and transversely, which is a property that our set of holomorphic curves might not be known to have. Assume that we are given a collection of distinct $J$-holomorphic curves $C_1, \ldots, C_k$ in the symplectic 4-manifold $M$ (we adopt the convention that a $J$-holomorphic curve is the image of a generically injective $J$-holomorphic map from some irreducible compact Riemann surface). Corollary 4.2.1 of [**12**] asserts that, at the possible cost of $C^1$ slightly changing the almost complex structure $J$, we may perturb any one of these curves to a pseudo-holomorphic immersion. We first give a simple modification of McDuff's argument to show that, in fact, we may perturb all of the curves and the almost complex structure simultaneously so that $C_1, \ldots, C_k$ become immersed.

**Lemma 3.1.** *Let $u_i\colon \Sigma_i \to M$ be $J$-holomorphic maps with images $C_i$. Then given $\epsilon > 0$, there are an almost complex structure $\tilde{J}$ and $\tilde{J}$-holomorphic immersions $\tilde{u}_i\colon \Sigma_i \to M$ such that $\|\tilde{u}_i - u_i\|_{C^2} < \epsilon$ and $\|\tilde{J} - J\|_{C^1} < \epsilon$.*

*Proof.* Let $p \in M$ be a critical value for one or more of the $u_i$. It is shown in [**12**] that the various $u_i$ each have just finitely many critical points, so denote the various critical points in $\cup\Sigma_i$ having image $p$ by $z_1, \ldots, z_m$. For $j = 1, \ldots, m$, if $z_j \in \Sigma_i$, let $D_j \subset \Sigma_i$ be a disc around $z_j$, and let $v_j = u_i|_{D_j}$. By shrinking the various $D_j$, we assume that the $D_j$ are disjoint and that $z_j$ is the only critical point of the restriction $v_j$. Since the intersections (and self-intersections) of the various $C_i$ are isolated, let $U \subset M$ be a coordinate neighborhood of $p$ in which the $C_i$ meet each other and themselves only at $p$ and such that for each $j$ $v_j^{-1}(U) \subset D_j$ is a connected component of $\cup u_i^{-1}(U)$. Shrinking $U$ if necessary, assume also that $U$ contains no critical values of the various $u_i$ other than $p$. Now fix neighborhoods $W_m \subset U_m \subset \cdots \subset W_1 \subset U_1 \subset U$ of $p$. By Theorem 4.1.1 of [**12**], there is a family $v_1^\delta$ ($\delta > 0$) of $J$-holomorphic immersions $D_1 \to M$, converging in $C^2$ norm to $v_1$ as $\delta \to 0$. For $\delta$ small, define $\tilde{v}_1(z) = \chi(z)v_1^\delta(z) + (1-\chi)(z)v_1(z)$, where $\chi$ is a smooth cutoff function which is 1 on a neighborhood of $v_1^{-1}(W_1)$ and 0 on a neighborhood of the complement of $v_1^{-1}(U_1)$. $\tilde{v}_1$ is then $C^2$-close to $v_1$, so there is an almost complex structure $J_1'$ which agrees with $J$ away from $U_1 \setminus W_1$, makes $\tilde{v}_1$ $J_1'$ holomorphic, and is $C^1$-close to $J$ everywhere (see the proof of Corollary 4.2.1 of [**12**] or the proof of Proposition 3.3). Furthermore, if $U \cap \mathrm{Im} v_1$ is a distance at least $K$ from $\cup C_i \setminus \mathrm{Im} v_1$, then for $\delta$ small enough, $U \cap \mathrm{Im}\tilde{v}_1$ will be a distance $K/2$ from $\cup C_i \setminus \mathrm{Im} v_1$, and so using a cutoff function supported in a $(K/3)$-neighborhood of $\mathrm{Im}(\tilde{v}_1) \cap (U_1 \setminus W_1)$, we can patch together $J$ and $J_1'$ to obtain an almost complex structure $\tilde{J}_1$ which is $C^1$-close to $J$, agrees with $J$ outside $U_1 \setminus W_1$ and on a neighborhood of $\cup C_i \setminus \mathrm{Im}(\tilde{v}_1)$, and makes $\tilde{v}_1$ pseudo-holomorphic.

With this done, we now apply the same procedure sequentially to $v_2, \ldots, v_m$, obtaining almost complex structures $\tilde{J}_j$ which are $C^1$-close to $J$ globally and which agree with $\tilde{J}_{j-1}$ both near $\cup C_i \setminus \mathrm{Im}(v_j)$ and outside $U_j \setminus W_j$, and $\tilde{J}_j$-holomorphic immersions $\tilde{v}_j$ which are $C^2$-close to $v_j$. Modifying the original maps $u_i \colon \Sigma_i \to M$ by replacing the restrictions $v_j \colon D_j \to M$ by $\tilde{v}_j$, we get $\tilde{J}_m$-holomorphic maps $\tilde{u}_i$ which have no critical values inside $U$ and agree with the $u_i$ outside $U$. So we have reduced the number of critical values by 1, and repeating the process at each critical value gives the almost complex structure $\tilde{J}$ and the $\tilde{J}$-holomorphic immersions $\tilde{u}_i$ that we desire. □

Applying Lemma 3.1, we may assume that we now have a set of distinct immersed $J$-holomorphic curves $C_i$, and we now aim to show that these curves may be perturbed further to a set of symplectic surfaces $C_i'$ whose intersections are all transverse and positive with $C_i' \cap C_j' \cap C_k' = \varnothing$ when $i, j, k$ are all distinct. In fact, our perturbed curves $C_i'$ will agree with $C_i$ outside an arbitrarily small neighborhood of the initial intersection points; will be arbitrarily $C^1$-close to $C_i$ (from which it immediately follows that they are symplectic); and will be made simultaneously pseudo-holomorphic by an almost complex structure $J'$ arbitrarily $C^1$-close to $J$.

We start by finding a nice coordinate system near any given intersection point of our curves. In the case where $J$ is integrable, any given holomorphic coordinate chart can be modified by an element of $GL(2, \mathbb{C})$ to satisfy the conditions we need, so the arguments below are only needed in the non-integrable case.

**Lemma 3.2.** *Given immersed $J$-holomorphic curves $C_0, \ldots, C_m \subset M$ all having an isolated intersection at the point $p$, there is a coordinate chart $U$ around $p$ with coordinates $z, w$ such that:*

(i) $C_0 \cap U = \{(z, w) \in U | w = 0\}$,
(ii) *Each set* $\{(z, w) \in U | w = \mathrm{const}\}$ *is $J$-holomorphic, and*
(iii) *For $i \geq 1$, there are smooth functions $g_i$ of the form $g_i(z) = a_i z^{k_i} + O(|z|^{k_i+1})$ ($a_i \neq 0$, $k_i \geq 1$) such that $C_i \cap U = \{(z, g_i(z))\}$.*

*Proof.* A coordinate chart $U_0 = \{(z', w')\}$ satisfying (i) and (ii) may be constructed by using Lemmas 5.4 and 5.5(d) of [**16**]. To obtain condition (iii), first note that for a generic linear change of coordinates $(z', w') \mapsto (z' + cw', w')$, we retain properties (i) and (ii) and additionally ensure that $\{z' = 0\}$ is transverse to each of the $C_i$. Now condition (ii) implies that the antiholomorphic tangent space of our almost complex structure $J$ is given in these coordinates by

$$T_J^{0,1} = \langle \partial_{\bar{z}'} + \alpha(z', w')\partial_{z'}, v(z', w')\rangle$$

for a certain function $\alpha$ and complexified vector field $v$. By Ahlfors–Bers' Riemann mapping theorem with smooth dependence on parameters [**1**], the equation $u_{\bar{z}'} + \alpha(z', w')u_{z'} = 0$ can be solved for a smooth function $u(z', w')$ with $u(0, w') = 0$.

Changing coordinates to $(z, w) = (z' + u(z', w'), w')$, we have that $\{z = 0\} = \{z' = 0\}$ is transverse to each of the $C_i$, so that after possibly shrinking the coordinate chart $U$, we have $C_i \cap U = \{(z, g_i(z))\}$ for some smooth functions $g_i$. In terms of the coordinates $(z, w)$, we have for certain functions $a$ and $b$ both vanishing at the origin,

$$T_J^{0,1} = \langle \partial_{\bar{z}}, \partial_{\bar{w}} + a(z, w)\partial_w + b(z, w)\partial_z\rangle.$$

It is then a simple matter to check that a curve $\{(z, g(z))\} \subset U$ is $J$-holomorphic exactly if

$$b(z, g(z)) = \frac{g_{\bar{z}} - a(z, g(z))g_z}{|g_{\bar{z}}|^2 - |g_z|^2}.$$

But then the fact that $a(z, g(z))$ and $b(z, g(z))$ are smooth functions of $z$ and vanish at $z = 0$ implies that the lowest order terms in the Taylor expansion of $g$ are functions only of $z$ and not of $\bar{z}$. Of course, our functions $g_i$ cannot be constants (since the $C_i$ $(i \geq 1)$ have an isolated intersection point with $C_0 = \{w = 0\}$ at the origin), so it follows that the $g_i$ all have the form specified in condition (iii). ☐

**Proposition 3.3.** *In the situation of Lemma* 3.2*, given any sufficiently small* $\delta > 0$*, there is a surface* $C_0^\delta$ *such that, where* $B_\delta = \{(z, w) \in U||z| < \delta\}$*,* $C_0^\delta \cap (X \setminus B_\delta) = C_0 \cap (X \setminus B_\delta)$*, while all intersection points of* $C_0^\delta$ *with* $C_i$ $(i > 0)$ *that are contained in* $B_\delta$ *are in fact contained in* $B_{\delta^2}$ *and are transverse, positive, and distinct from* $p$ *and from each other as* $i$ *varies. Further, there is a constant* $A$ *depending on the curves* $C_i$ *but not on* $\delta$ *such that* $\operatorname{dist}_{C^2}(C_0^\delta, C_0) \leq A\delta^2$*, and there is an almost complex structure* $J'$ *agreeing with* $J$ *near* $C_i$ $(i > 0)$ *and making* $C_0^\delta$ *holomorphic with* $\|J' - J\|_{C^1} \leq A\delta^2$*.*

*Proof.* Work in coordinates provided by the conclusion of Lemma 3.2, and let $c_i$ be constants such that for $i > 0$

$$|g_i(z) - a_i z^{k_i}| < c_i |z^{k_i+1}|.$$

Given $\epsilon > 0$, write

$$R_\epsilon = \max_{i \geq 1} \left(\frac{2\,\epsilon}{|a_i|}\right)^{1/k_i};$$

we will only work with $\epsilon$ so small that

$$\sqrt{R_\epsilon} < \min_{i \geq 1} \frac{|a_i|}{2c_i}$$

Then for any such $\epsilon$, if $R_\epsilon \leq |z| \leq \sqrt{R_\epsilon}$, we have

$$|g_i(z)| \geq (|a_i| - c_i|z|)|z|^{k_i} > \frac{|a_i|}{2}|z|^{k_i} \geq \epsilon\,.$$

Write $\delta = \sqrt{R_\epsilon}$; note that we may alternatively express $\epsilon$ in terms of an arbitrary $\delta > 0$, and then for $\delta$ small enough, $\epsilon$ is bounded by a constant times $\delta^2$. Fix a cutoff function $\chi(z)$ with image $[0,1]$ equal to one for $|z| \leq \delta^2$ and zero for $|z| \geq \delta$, with $\|\chi\|_{C^2} < \frac{4}{\delta^2}$. Let $C_0^\delta = \{(z, \epsilon^2\,\chi(z))\}$; obviously $C_0^\delta$ agrees with $C_0$ outside $B_\delta$. Since $\epsilon^2\,\chi(z) \leq \epsilon^2 < \epsilon$ while each $|g_i(z)| > \epsilon$ for $|z| \in [\delta^2, \delta]$, evidently the intersection points of $C_i$ with $C_0^\delta$ contained in $B_\delta$ are just those points $(z, \epsilon^2)$ with $|z| < \delta^2 = R_\epsilon$ such that $g_i(z) = \epsilon^2$.

Write $\tilde{g}_i(z) = \epsilon^{-2}\, g_i((\frac{\epsilon^2}{a_i})^{1/k_i} z)$; then

$$\tilde{g}_i(z) = 1 \Leftrightarrow g_i\left(\left(\frac{\epsilon^2}{a_i}\right)^{1/k_i} z\right) = \epsilon^2,$$

so the intersections of $C_i$ with $C_0^\delta$ are of just the same type as the intersections of the graph of $\tilde{g}_i(z)$ with $\{w = 1\}$. Now we see that $\tilde{g}_i(z) = z^{k_i} + \tilde{r}_i(z)$, where $|\tilde{r}_i(z)| \leq \tilde{c}_i\,\epsilon^{2/k_i}\,|z|^{k_i+1}$. Hence the graph of $\tilde{g}_i(z)$ is $O(\epsilon^{2/k_i})$ away in $C^1$ norm from that of $z \mapsto z^{k_i}$, so since the latter's only intersections with $\{w = 1\}$ are positive and transverse at the $k_i$th roots of unity, for $\epsilon$ small enough the graph of $\tilde{g}_i$ will also have just $k_i$ distinct positive transverse intersections with $\{w = 1\}$, each at a point a distance $O(\epsilon^{2/k_i})$ from a different one of the $k_i$th roots of unity. Scaling back, we conclude that the intersections of $C_i$ with $C_0^\delta$ that are contained in $B_\delta$ are in fact contained in $B_{\delta^2}$ and are transverse, positive, and located at points a distance $O(\epsilon^{4/k_i})$ from the various $(\epsilon^2\,/a_i)^{1/k_i}\eta$ for $\eta$ a $k_i$th root of unity.

Obviously for any given $i$, the points of $C_i \cap C_0^\delta$ are all distinct for small enough $\epsilon$. For small enough $\epsilon$, these intersections vary continuously in $\epsilon > 0$, so if it were not the case that the sets $C_i \cap C_j \cap C_0^\delta$ were all eventually empty for $\epsilon$ small enough and $i, j$ distinct, we would then, by varying $\epsilon$, obtain a continuous family of points in $C_i \cap C_j$, which is impossible since $C_i$ and $C_j$ are distinct holomorphic curves and so have isolated intersections.

Finally, note that $\|\,\epsilon^2\,\chi\|_{C^2} \leq \epsilon^2(4/\delta^2) \leq A\delta^2$ for a certain constant $A$ and $\delta$ sufficiently small, so that $C_0^\delta$ is indeed less than $A\delta^2$ away from $C_0$ in $C^2$ norm. Letting $\beta(w)$ be a cutoff function, which is 1 for $|w| < 2\,\epsilon^2$ and 0 for $|w| \geq \epsilon$, if $J$ is defined by $T_J^{0,1} = \langle \partial_{\bar{z}}, v\rangle$, then setting

$$T_{J'}^{0,1} = \langle \partial_{\bar{z}} + \beta(w)\left((\epsilon^2\,\chi)_{\bar{z}}\partial_w + (\overline{\epsilon^2\,\chi})_{\bar{z}}\right)\partial_{\bar{w}}, v\rangle$$

defines an almost complex structure $J'$ which makes $C_0^\delta$ holomorphic and which (since $|g_i(z)| > \epsilon$ whenever $\nabla(\epsilon\chi) \neq 0$) agrees with $J$ near $C_i$ for $i > 0$. Further, one easily sees that $\|J' - J\|_{C^1} = O(\epsilon) \leq O(\delta^2)$. □

**Corollary 3.4.** *Any set of distinct $J$-holomorphic curves $C_0, \dots, C_m$ can be perturbed to symplectic surfaces $C_0', \dots, C_m'$ whose intersections are all transverse and positive, with $C_i' \cap C_j' \cap C_k' = \varnothing$ when $i$, $j$, $k$ are all distinct. Furthermore, there is an almost complex structure $J'$ arbitrarily $C^1$-close to $J$ such that the $C_i'$ are $J'$-holomorphic.*

*Proof.* Assume that the process used in the proof of the above proposition has been repeated to yield surfaces $C_0^{\delta_0}, \dots, C_i^{\delta_i}$, each missing $p$ and hitting the other $C_j$ transversely and positively. Let our neighborhood $U$ and the parameter $\delta_{i+1}$ be so small that each $C_j^{\delta_j}$ $(j \leq i)$ misses $B_{\delta_{i+1}}$ (this is possible since the $C_j^{\delta_j}$ all miss $p$); then since $C_{i+1}^{\delta_{i+1}} \cap (X \setminus B_{\delta_{i+1}}) = C_{i+1} \cap (X \setminus B_{\delta_{i+1}})$, the intersection points of $C_{i+1}^{\delta_{i+1}}$ with $C_j^{\delta_j}$ $(j \leq i)$ are the same as those of $C_{i+1}$ and $C_j^{\delta_j}$, and so are transverse, positive, and away from $p$. By the proposition, we have the same conclusion for the intersection points of $C_{i+1}^{\delta_{i+1}}$ with $C_j$ $(j > i+1)$. So by induction, we may perturb all of the $C_i$ to $C_i' = C_i^{\delta_i}$ with the desired intersection configuration. Moreover, by choosing $\delta_0 > \delta_1 > \cdots \delta_{m-1} > 0$ small enough, the $C_i^{\delta_i}$ can be made arbitrarily $C^2$-close to the $C_i$, so since the property of being a symplectic submanifold persists under $C^1$-small perturbations, the $C_i^{\delta_i}$ can be taken to be all symplectic. Repeating this local construction at all of the intersection points of two or more of the $C_i$ gives the global result. □

## 4. Towards a symplectic Nakai–Moishezon criterion

In this subsection, let $(M, J)$ be a minimal Kähler surface and $H_J^{1,1}$ denote the real part of the $(1,1)$-subspace of $H^2(M;\mathbb{C})$ determined by $J$. We apply Theorem 1.1 to study the symplectic classes in $H_J^{1,1}$.

Given a homology class $e$ with $e \cdot e < 0$, we define the reflection along $e$ to be

$$R_e(\alpha) = \alpha - 2\frac{\alpha(e)}{e \cdot e} PD(e).$$

Notice that this is an automorphism of $H^2(M;\mathbb{Q})$ preserving the intersection form. But it is an automorphism of $H^2(M;\mathbb{Z})$ only if $e \cdot e = -1$ or $-2$. Geometrically, the annihilator of $e$ is a hyperplane in $H^2(M;\mathbb{R})$, which we call the "$e$-wall," and $R_e$ is the reflection across this hyperplane.

**Definition 4.1.** A homology class $e$ is called small and effective if it is represented by a reduced irreducible holomorphic curve with negative self-intersection.

Notice that there is only one holomorphic curve $C$ representing a small and effective class.

**Proposition 4.2.** *Let $e$ be a small and effective class which is not represented by a curve of zero arithmetic genus and odd self-intersection. Then the reflection of the Kähler chamber along the $e$-wall is contained in the symplectic cone.*

*Proof.* Let $x$ be a point in the Kähler cone. The Kähler cone is open in $H_J^{1,1}$, since the sum of a small closed real $(1,1)$ form and a Kähler form on a closed manifold is still a closed positive $(1,1)$ form, hence a Kähler form. Thus, for small $\epsilon$, $x-\epsilon e$ is also in the Kähler cone, and hence represented by a Kähler form $\omega$. By Proposition 3.3, we can perturb $C$ to get an embedded $\omega$-symplectic surface, still denoted by $C$. Applying Theorem 1.1 to $\omega$ and $C$, we see that $R_e(x) = [\omega_t]$ for some $t$. □

**Remark 4.3.** For an embedded $-2$ rational curve $C$, there is a diffeomorphism whose induced action on cohomology is $R_{[C]}$. Pulling the Kähler form back by this diffeomorphism gives an alternative way of enlarging the Kähler cone by reflection. However, this latter method, unlike Theorem 1.1, does not allow us to obtain symplectic forms in classes which vanish on the $(-2)$-curve.

We mention a simple case where the symplectic Nakai–Moishezon criterion can be established.

**Proposition 4.4.** *Suppose that $H_2(M;\mathbb{Z})$ contains only one small and effective class, $e$, and that $e$ is not represented by a sphere of odd square. Then every class $\alpha$ in the positive cone which is negative on $e$ lies in the image of the Kähler chamber under $R_e$. Therefore, the symplectic Nakai–Moishezon criterion holds in this case.*

*Proof.* Suppose $e \cdot e = -k$ and $\alpha$ is as in the statement of the proposition. Choose $s > 0$ such that

$$\alpha^2 + 2s|\alpha(e)| > s^2 k > 2s|\alpha(e)|.$$

Let $\beta = \alpha - sPD(e)$. Then

$$\beta(e) = \alpha(e) + sk > |\alpha(e)|, \quad \beta^2 = \alpha^2 + 2s|\alpha(e)| - s^2 k > 0,$$
$$\beta \cdot \alpha = \alpha^2 + |\alpha(e)| > 0.$$

Therefore, $\beta$ is in the Kähler cone by Theorem 1.2. Now apply Theorem 1.1 to $\beta$. □

The much more common situation in which $M$ contains more than one small and effective class is more difficult to analyze. We begin by establishing the following finiteness result, which might be known to experts.

**Lemma 4.5.** *For any* $(1,1)$ *class* $\alpha$ *with positive square and in the positive cone, there are only finitely many classes which are represented by reduced irreducible holomorphic curves and pair non-positively with* $\alpha$. *Further, the intersection form on* $M$ *is negative definite on the subspace of* $H_J^{1,1}$ *spanned by the Poincaré duals of these classes.*

*Proof.* Suppose $e_i$ are distinct such classes with negative square, which are represented by reduced irreducible holomorphic curves. Notice that $e_i \cdot e_j \geq 0$ if $i \neq j$.

Then if a finite positive linear combination of $e_i$, say $\sum_i a_i e_i$, has non-negative square, it must be in the positive cone or its boundary, as $\omega$ is positive on each $e_i$, $a_i \geq 0$, and $\omega$ itself in the positive cone. By the Hodge index theorem, as $\alpha$ is also in the positive cone, $\alpha$ is strictly positive on $\sum_i a_i e_i$. But $\alpha$ is non-positive on each $e_i$, so $\alpha$ is non-positive on $\sum_i a_i e_i$ as $a_i \geq 0$.

This contradiction shows that any positive linear combination of the $e_i$ has negative square. But this implies that for *any* $a_i \in \mathbb{R}$ not all zero, we have, using positivity of intersections between the distinct $e_i$,

$$\begin{aligned}\Big(\sum a_i e_i\Big)^2 &= \sum_i a_i^2 e_i^2 + 2\sum_{i<j} a_i a_j e_i \cdot e_j \leq \sum_i |a_i|^2 e_i^2 + 2\sum_{i<j} |a_i||a_j| e_i \cdot e_j \\ &\leq \Big(\sum |a_i| e_i\Big)^2 < 0.\end{aligned}$$

Thus, the $e_i$ are linearly independent, and they span a negative definite subspace of $H_2(X;\mathbb{Z})$. In particular, there are at most $h^{1,1} - 1 = b^-$ many $e_i$. □

In view of the lemma above, we make the following definition. Here $\mathcal{P}$ denotes the positive cone in $H_J^{1,1}(X;\mathbb{R})$.

**Definition 4.6.** A finite set of small and effective classes $G = \{e_1, \ldots, e_l\}$ is called admissible if they are linearly independent and the intersection form on the subspace of $H_2(M;\mathbb{Z})$ generated by these $e_i$ is negative definite. Given an admissible set $G$, the $G$-chamber is

$$\mathcal{C}(G) = \{\alpha \in \mathcal{P} | \alpha(e_i) \leq 0 \text{ if } e_i \in G,\ \alpha(e) > 0 \text{ if } e \notin G\}.$$

The $G$-corner is

$$\mathcal{C}^c(G) = \{\alpha \in \mathcal{P} | \alpha(e_i) = 0 \text{ if } e_i \in G,\ \alpha(e) > 0 \text{ if } e \notin G\}.$$

The following simple observation will be useful.

**Proposition 4.7.** *Let* $M$ *be a symmetric negative definite matrix such that* $M_{ij} \geq 0$ *if* $i \neq j$. *Then every entry of* $-M^{-1}$ *is non-negative.*

*Proof.* By multiplying $M$ by a scalar, assume without loss of generality that all diagonal entries and all eigenvalues of $M$ are greater than $-1$.

Then, where $I$ is the identity, $I+M$ has all its entries non-negative and all its eigenvalues between 0 and 1. The latter condition implies that we have a convergent Taylor series expansion

$$-M^{-1} = (I-(I+M))^{-1} = \sum_{n=0}^{\infty}(I+M)^n,$$

and the proposition follows from the fact that the set of matrices with all entries non-negative is closed under addition and multiplication. □

**Lemma 4.8.** *The chambers $\mathcal{C}(G)$ for admissible sets $G$ form a partition of the positive cone and are all non-empty, as are the $G$-corners $\mathcal{C}^c(G)$. Each $G$-chamber and each $G$-corner is convex and hence connected.*

*Proof.* That the $\mathcal{C}(G)$ form a partition of the positive cone follows directly from Lemma 4.5. Convexity is obvious from the definitions.

To see that each $\mathcal{C}(G) \neq \varnothing$, let $G = \{e_1, \dots, e_n\}$ be an admissible set and denote by $M$ the matrix representing the restriction of the intersection form to the span of $G$, so that $M$ is negative definite. Pick an arbitrary $\alpha$ in the Kähler cone, and let $v_i = \langle \alpha, e_i \rangle$, so that each $v_i > 0$. Then where $\vec{t} = -M^{-1}\vec{v}$ and $\alpha' = \alpha + \sum t_i PD(e_i)$, we have $\langle \alpha', e_j \rangle = v_j - v_j = 0$ for each $j$, and

$$(\alpha')^2 = \alpha^2 + 2\vec{v}\cdot\vec{t} + (M\vec{t})\cdot\vec{t} = \alpha^2 - (M\vec{t})\cdot\vec{t} \geq \alpha^2 > 0$$

since $M$ is negative definite, so $\alpha'$ is in the positive cone. Also, by Proposition 4.7, we have each $t_i > 0$ since each $v_i > 0$, so if $e$ is small and effective with $e \notin G$, then by positivity of intersections $\langle \alpha', e \rangle \geq \langle \alpha, e \rangle > 0$. Thus $\alpha' \in \mathcal{C}^c(G)$, and $\mathcal{C}^c(G)$ is non-empty. Where $s_i = -\sum (M^{-1})_{ij}$, $\alpha' + \epsilon \sum s_i PD(e_i)$ will evaluate as $-\epsilon$ on each $e_i$, will be positive on each $e \notin G$ (noting that each $s_i > 0$), and will remain in the positive cone for small $\epsilon > 0$, so $\mathcal{C}(\{e_1, \dots, e_n\})$ is also non-empty. □

**Remark 4.9.** By Theorem 1.2, the Kähler cone is just $\mathcal{C}(\emptyset)$. Within the positive cone, the *boundary* of the Kähler cone is the disjoint union of the $\mathcal{C}^c(G)$ over the admissible sets $G$.

Applying Theorem 1.1 with $\omega$ equal to a Kähler form, $e = [C]$, and $t$ between $a/k$ and $2a/h$ shows that each chamber $\mathcal{C}(e)$ contains symplectic classes. Iterating Theorem 1.1, the same can be said for any $G$-chamber $\mathcal{C}(e_1, \dots, e_n)$ with $e_i \cdot e_j = 0$ for $i \neq j$.

We can apply Theorem 2.3 to show that more general $G$-chambers contain symplectic classes. To do this, it suffices to show that the corresponding $G$-corner contains symplectic classes, since as in the proof of Lemma 4.8 suitably chosen arbitrarily small perturbations of these will lie in $\mathcal{C}(G)$ and will remain symplectic. Under suitable hypotheses on the set $G$, we shall see that every class in the $G$-corner $\mathcal{C}^c(G)$ contains symplectic forms.

Accordingly, let $\alpha \in H^{1,1}_J(M;\mathbb{R})$ be an arbitrary class in the boundary of the Kähler cone and have positive square, so that $\alpha$ satisfies $\langle \alpha, D\rangle \geq 0$ for every effective divisor $D$. $\alpha$ is then in some $G$-corner; say $G = \{e_1, \ldots, e_n\}$, so that $\alpha$ vanishes only on the $e_i$ and the $PD(e_i)$ span a negative definite subspace of $H^{1,1}_J(M;\mathbb{R})$. Our strategy for attempting to show that $\alpha$ contains symplectic forms consists of two steps:

(i) Find $t_i > 0$ such that $\alpha - \sum t_i PD(e_i)$ lies in the Kähler cone.

(ii) Beginning with a Kähler form in the class $\alpha - \sum t_i PD(e_i)$, apply the inflation procedure sequentially to the $e_i$ (and/or smoothings of unions thereof) to obtain a symplectic form in class $\alpha$.

We shall show presently that step (i) can always be completed.

**Lemma 4.10.** *If $\alpha$ and $e_i$ are as above, and if $s_i > 0$ are such that $\sum_i s_i e_i \cdot e_j < 0$ for every $j$, then for $r > 0$ sufficiently small, $\alpha - \sum r s_i PD(e_i)$ admits Kähler forms.*

*Proof.* Multiplying the $s_i$ by a small constant if necessary, assume that $\beta := \alpha - \sum s_i PD(e_i)$ is in the positive cone. By Lemma 4.5, there are then just finitely many curves on which $\beta$ is non-positive; denote them by $f_1, \ldots, f_m$ (note that the assumption on the $s_i$ implies that none of the $f_j$ is among the $e_i$). Now for each $f_j$ we have $\langle \alpha, f_j\rangle > 0$, so since there are only finitely many $f_j$, for $r > 0$ small enough, $\alpha - \sum r s_i e_i = (1-r)c + rd$ will be positive on each $f_j$. Meanwhile $\langle \alpha, e_i\rangle = 0$ and $\langle \beta, e_i\rangle > 0$, and if $C$ is any curve not among the $e_i$ and $f_j$, both $\alpha$ and $\beta$ are positive on $[C]$, so for $r > 0$, $(1-r)\alpha + r\beta$ is also positive on all curves other than those represented by the $f_j$. Hence by Theorem 1.2, $(1-r)\alpha + r\beta$ admits Kähler forms if $r > 0$ is small enough. □

**Corollary 4.11.** *If $\alpha \in H^{1,1}_J$ has positive square and lies in the boundary of the Kähler cone, and if $e_1, \ldots, e_n$ are the homology classes of the finitely many curves on which $\alpha$ vanishes, then there are $t_i > 0$ such that $\alpha - \sum t_i PD(e_i)$ contains Kähler forms.*

*Proof.* By Lemma 4.10, it suffices to find $s_i > 0$ such that $\sum_i s_i e_i \cdot e_j < 0$ for every $j$; we then set $t_i = r s_i$ for $r$ small. Define the $n \times n$ matrix $M$ by $M_{ij} = e_i \cdot e_j$. $M$ is negative definite by Lemma 4.5, and its off-diagonal entries are non-negative by positivity of intersections, so $-M^{-1}$ has all non-negative entries by Proposition 4.7. Then for any $v_i > 0$ ($i = 1, \ldots, n$), the $s_i = \sum -M^{-1}_{ik} v_k$ will each be positive, and we have $\sum_i s_i e_i \cdot e_j = -\sum_{i,k} M_{ji} M^{-1}_{ik} v_k = -v_j < 0$, as desired. □

Carrying out step (ii) of our strategy is more difficult (and often impossible). As we allude to above, instead of applying inflation sequentially to curves $C_i$ representing the $e_i$, we will sometimes wish to smooth the union of the $C_i$ into an embedded symplectic submanifold $C$ (as is always possible since the $C_i$ may be assumed to meet positively and transversely by Corollary 3.4) and then apply the inflation procedure to $C$. Now $C$ will no longer be symplectic after we do this, and in the smoothing construction $C$ will contain all but a small subset of each $C_i$, so the $C_i$ would not be symplectic either. As such, it will not be possible to apply inflation to $C_i$ after we apply inflation to $C$. The following trick allows us to evade this issue in certain circumstances.

**Proposition 4.12.** *Let $C_0, \dots, C_k$ be symplectic surfaces such that $C_0$ has only positive transverse intersections with the $C_i$ ($i > 0$). Assume that*

$$\#\left(C_0 \cap \left(\bigcup_{i\geq 1} C_i\right)\right) \geq -[C_0]^2$$

*Then there exist symplectic surfaces $\tilde{C_0}$ and $C$, homologous to $C_0$ and $\cup_{r\geq 0} C_r$, respectively, such that all intersections between $\tilde{C_0}$ and $C$ are positive and transverse.*

*Proof.* Where $m = -[C_0]^2$, assume that, for some points $p_1, \dots, p_m$, $C_0$ meets the surface $C_{i_j}$ at $p_j$; in complex coordinates $(z, w)$ in a neighborhood $U_j$ around $p_j$, we may assume $C_0 \cap U_j = \{z = 0\}$ and $C_{i_j} \cap U_j = \{w = 0\}$. By exponentiating a small scalar multiple of a smooth section of the normal bundle to $C_0$ which vanishes negatively precisely at the $m = -[C_0]^2$ points $p_j$, we take for $\tilde{C_0}$ a surface such that $\tilde{C_0} \cap C_0 = \{p_1, \dots, p_m\}$ and, for each of the above neighborhoods $U_j$, $\tilde{C_0} \cap U_j = \{(z, \epsilon\bar{z})\}$. For $\epsilon$ small enough, $\tilde{C_0}$ will be sufficiently $C^1$-close to $C_0$ as to guarantee that $\tilde{C_0}$ is symplectic and (like $C_0$) only meets the $C_i$ ($i > 0$) positively and transversely. For $C$, we take a surface which coincides with $\cup_{r\geq 0} C_r$ outside the $U_j$ and whose intersection with $U_j$ is given by

$$C \cap U_j = \{(z, w) | zw = \delta f_j(z, w)\}$$

where $f_j$ is a *real-valued* function supported on $U_j$ with $f(p_j) \neq 0$ and $\delta$ is a complex constant chosen small enough as to guarantee that $C$ is symplectic. Now for any $(z, w) \in \tilde{C_0} \cap U_j$, we have $zw \in \mathbb{R}\,\epsilon$, while for any $(z, w) \in C \cap U_j$ we have $zw \in \mathbb{R}\delta$, so as long as we choose $\epsilon, \delta \in \mathbb{C}$ to have different phases, we ensure that $C$ and $\tilde{C_0}$ have no intersections within $\cup_{j\geq 1} U_j$. By construction, any intersections of $\tilde{C_0}$ with $C$ outside $\cup_{j\geq 1} U_j$ are positive and transverse, proving the result. □

There are many examples of intersection patterns of curves $C_1, \dots, C_n$ for which our methods give rise to symplectic classes on the Kähler cone, but it does not seem possible at this juncture to give a concise yet anywhere-near-exhaustive list of the assumptions on the $[C_i]$ which are sufficient. Instead, we shall demonstrate the techniques on a particular complex surface, which we believe illustrates nicely both the subtleties involved and the fact that our construction gives rise to symplectic forms that cannot be obtained by classical methods.

**4.1. The Kharlamov–Kulikov surface.** If $(M, J)$ is a complex surface admitting Kähler structures and $\mathcal{C}_J \subset H^{1,1}_J$ is the Kähler cone as given by the Buchdahl–Lamari theorem, then every class in $\mathcal{C}_J + Re\, H^{2,0}_J$ is of course represented by symplectic forms. Although our method gives seemingly new symplectic forms in classes $c$ outside $\mathcal{C}_J$ in the presence of ($J$-holomorphic) curves of negative square, a skeptic might imagine that if we were to vary the complex structure on $M$ to some other (integrable) $J'$, then the negative-square curves might disappear, and so these classes $c$ might lie in $\mathcal{C}_{J'} + Re\, H^{2,0}_{J'}$, in which case our method would not have been necessary to obtain the new forms.

Now the list of underlying manifolds $M$ of complex surfaces for which the effective cone is known for *every* complex structure on $M$ is rather short, so for most complex surfaces, it is difficult to tell whether our new forms could have been obtained by algebro-geometric considerations. In the case that $M$ is *rigid*, though, there is no room to vary $J$, and so we can confidently assert that our main theorems give genuinely new forms as soon as we know that there are curves of negative square in the surface. We present here an example of a rigid surface $K$, borrowed from [**6**], which contains several (21) curves of negative square intersecting each other in a non-trivial fashion and on which we can find symplectic forms in all classes in the positive cone which are non-negative on each of these 21 curves. It seems likely (though we shall not attempt to prove this) that all curves of negative square in $K$ lie in the cone generated by these 21 special curves; if this is indeed the case, then it would follow that the entire boundary of the Kähler cone of $K$ is contained in the symplectic cone. In any event, our results show that at least a rather substantial portion of the boundary of the Kähler cone of $K$ is contained in the symplectic cone, even though the standard methods of Kähler geometry alone seem to provide no reason to expect this to be the case.

We now recall the construction of $K$ from Section 2 of [**6**]. Begin with an arbitrary smooth cubic curve in $\mathbb{C}P^2$ and consider its nine inflection points. Since these inflection points are each 3-torsion under the group law of the cubic, any line through two of them also passes through a third which is distinct from the first two; as such, we obtain 12 lines each passing through precisely three of the inflection points. The dual arrangement provides us with nine lines $L_1, \dots, L_9$ and 12 points $p_{\{i,j,k\}}$ ($\{i,j,k\} \in \{\{1,2,3\}, \{1,4,7\}, \{1,5,9\}, \{1,6,8\}, \{2,4,9\}, \{2,5,8\}, \{2,6,7\}, \{3,4,8\}, \{3,5,7\}, \{3,6,9\}, \{4,5,6\}, \{7,8,9\}\}$) in (the dual plane) $\mathbb{C}P^2$, with $p_{\{i,j,k\}} \in L_l$ iff $l \in \{i,j,k\}$. Let $\sigma \colon \tilde{\mathbb{P}}^2 \to \mathbb{C}P^2$ denote the blowup at the various $p_{\{i,j,k\}}$; let $E_{\{i,j,k\}}$ denote the corresponding exceptional divisors, and let $L_i'$ denote the strict transform of $L_i$. As is seen in [**6**], for suitable choices of a homomorphism

$\phi\colon H_1(\tilde{\mathbb{P}}^2 \setminus \sigma^{-1}(\cup_{i=1}^9 L_i);\mathbb{Z}) \to (\mathbb{Z}/5\mathbb{Z})^2$, the total space of the Galois cover branched over $\cup_{i=1}^9 L_i$ associated to $\phi$ will be smooth. Call this total space $K$ and the covering map $g\colon K \to \tilde{\mathbb{P}}^2$.

Write $C_i = g^{-1}(L_i')$, $D_{\{i,j,k\}} = g^{-1}(E_{\{i,j,k\}})$. Lemma 2.1 of [**6**] shows that each $C_i$ is a square-$(-3)$ curve of genus 4 and each $D_{\{i,j,k\}}$ is a square-$(-1)$ curve of genus 2. Further, the canonical class of $K$ is ample and is given by

$$K_K = \frac{1}{3}PD\left(7\sum[C_i] + 12\sum[D_{\{i,j,k\}}]\right);$$

we have $K_K^2 = 333$ and $e(K) = 111$, so $K$ is the quotient of the unit ball in $\mathbb{C}^2$ by a famous result of Miyaoka [**10**] and Yau [**17**]; a theorem of Siu [**14**] then shows that $K$ is rigid as promised.

**Theorem 4.13.** *Let $\alpha$ be any class in the positive cone of $H^{1,1}$ which is non-negative on all holomorphic curves in $K$ and positive on all curves whose homology classes are not in the cone spanned by the $[C_i]$ and $[D_{\{i,j,k\}}]$. Then $\alpha$ is represented by symplectic forms.*

*Proof.* First, note that the intersections of the distinct $C_i$ and $D_{\{i,j,k\}}$ are given by

$$[C_i]\cdot[C_j] = 0; \qquad [D_{\{i,j,k\}}]\cdot[D_{\{l,m,n\}}] = 0;$$
$$[C_l]\cdot[D_{\{i,j,k\}}] = \begin{cases} 1 & l \in \{i,j,k\} \\ 0 & l \notin \{i,j,k\} \end{cases}.$$

Let $\Gamma$ denote the dual graph to the subset of $\{[C_i],[D_{\{i,j,k\}}]\}$ on which $\alpha$ vanishes (in other words, $\Gamma$ has a vertex for each element of this set, and the number of edges connecting two distinct vertices of $\Gamma$ is the intersection number of the corresponding pair of classes). If $\Gamma$ were to contain a loop, then by virtue of the intersection pattern of the $C_i$ and $D_{\{i,j,k\}}$, that loop would consist of some number (say $a$) of curves $A_0 = C_{i_0}, \dots, A_{a-1} = C_{i_{a-1}}$ and an equal number of curves $B_0 = D_{\{i_0,j_0,k_0\}}, \dots, B_{a-1} = D_{\{i_{a-1},j_{a-1},k_{a-1}\}}$ such that $[A_m]\cdot[B_m] = [A_m]\cdot[B_{m+1}] = 1$ for each $m$ (where $m \in \mathbb{Z}/a\mathbb{Z}$). Hence, since $[A_m]^2 = -3$ and $[B_m]^2 = -1$,

$$\left(\sum_{m=0}^{a-1}[A_m] + \sum_{m=0}^{a-1}[B_m]\right)^2 \geq -3a - a + 2(2a) = 0,$$

which is impossible since $\alpha$ lies in the positive cone and vanishes on $\sum[A_m] + \sum[B_m]$.

In general, if $\Gamma$ contains a connected component with at least three distinct $[B_m] = [D_{\{i_m,j_m,k_m\}}]$ $(1 \leq m \leq 3)$, then it contains a subgraph consisting of vertices $\{[B_1],[C_{i_1}],[B_2],[C_{i_2}],[B_3]\}$, where $[B_1]\cdot[C_{i_1}] = [B_2]\cdot[C_{i_1}] = [B_2]\cdot[C_{i_2}] = [B_3]\cdot[C_{i_2}] = 1$. But then

$$([B_1] + [C_{i_1}] + 2[B_2] + [C_{i_2}] + [B_3])^2 = -1 - 3 - 4 - 3 - 1 + 2 + 4 + 4 + 2 = 0,$$

which is again a contradiction since $\alpha$ is in the positive cone. Likewise, if $\Gamma$ contains a connected component with three distinct $[C_i]$ (say $[C_i]$, $[C_j]$, $[C_k]$), then it must also contain some $[D_{\{i,j,l\}}]$ and $[D_{\{j,k,m\}}]$ and we see

$$\begin{aligned}&([C_i] + 3[D_{\{i,j,l\}}] + 2[C_j] + 3[D_{\{j,k,m\}}] + [C_k])^2\\&\quad = -3 - 9 - 12 - 9 - 3 + 6 + 12 + 12 + 6 = 0,\end{aligned}$$

again a contradiction.

Now it will suffice to consider the case in which $\Gamma$ is connected, since if it is not, we can apply our argument successively to each component. Assuming $\Gamma$ is connected, then the above shows that it contains at most two $[C_i]$ and at most two $[D_{\{i,j,k\}}]$, so after relabeling, it is a subgraph of the graph $\Gamma_0$ with vertices $[C_1]$, $[B_1] := [D_{\{1,2,3\}}]$, $[C_2]$, and $[B_2] := [D_{\{2,4,9\}}]$, with just one edge each connecting $[C_1]$ to $[B_1]$, $[B_1]$ to $[C_2]$, and $[C_2]$ to $[B_2]$. Suppose that $\Gamma = \Gamma_0$. Since $\alpha$ is positive on all curves represented by classes which are not in the span of $[C_1]$, $[B_1]$, $[C_2]$, and $[B_2]$, by taking $t > 0$ small enough, we ensure that

$$\alpha_0 = \alpha - tPD(8[C_1] + 21[B_1] + 12[C_2] + 14[B_2])$$

will have the same property; we calculate

$$\langle\alpha_0, [C_1]\rangle = 3t, \quad \langle\alpha_0, [B_1]\rangle = t, \quad \langle\alpha_0, [C_2]\rangle = t, \quad \text{and} \quad \langle\alpha_0, [B_2]\rangle = 2t,$$

so $\alpha_0$ is represented by Kähler forms. Apply Proposition 4.12 twice: first to get a symplectic surface $\tilde{C}$ representing $[C_1] + [B_1]$ and disjoint from a symplectic representative of $[B_1]$ and then to get a symplectic surface $S$ representing $[C_2] + [\tilde{C}] + [B_1] + [B_2] = [C_1] + 2[B_1] + [C_2] + [B_2]$ which is disjoint from $C_2$, $B_1$, and $B_2$. $S$ then has positive genus and square $-1$, so we can apply inflation to $S$ to get a symplectic form in the class $\alpha_0 + sPD[S]$ for any parameter $s$ less than $2\langle\alpha_0, [S]\rangle = 16t$. Take $s = 8t$ to get a symplectic form $\omega_1$ representing

$$\alpha_1 = \alpha - tPD(5[B_1] + 4[C_2] + 6[B_2])$$

with respect to which $[B_1]$, $[C_2]$, and $[B_2]$ are symplectic. Now use Proposition 4.12 to obtain a positive-genus $\omega_1$-symplectic surface $S'$ representing $[B_1] + [C_2] + [B_2]$ and meeting $[B_1]$ and $[B_2]$ transversely and positively. $[S']^2 = -1$, and $\langle\alpha_1, [S']\rangle = 4t$, so inflation using $S'$ gives a symplectic form $\omega_2$ in the class

$$\alpha_1 + 4tPD[S] = \alpha - tPD([B_1] + 2[B_2]).$$

Since $B_1 \cdot B_2 = 0$, we can now apply Theorem 1.1 rather directly to get the desired symplectic form in $\alpha$, by first inflating using (say) $B_1$ and then inflating using $B_2$.

In each case that $\Gamma$ is a *proper* subgraph of $\Gamma_0$, the desired symplectic representative of $\alpha$ can be obtained by similar (but easier) arguments, which we leave to the reader. $\square$

**4.2. A more general criterion.** As a more general example of the circumstances in which our methods can be used to show that a class in the boundary of the Kähler cone admits symplectic representatives, we present the following theorem. Note that while condition (b) below is rather subtle, condition (a) is occasionally easy to check; for instance, it holds for the canonical class in a minimal surface of general type and for any class in the positive cone of a minimal surface of Kodaira dimension 0 (though in both of these cases, there exist other methods to prove that such a class is in the symplectic cone).

**Theorem 4.14.** *Let $(M, \omega, J)$ be a Kähler surface and $\alpha \in H^{1,1}_J$ any class in the positive cone such that*

(a) *If $e \in H_2(M; \mathbb{Z})$ is represented by a reduced, irreducible holomorphic curve of negative square, then $\langle \alpha, e \rangle \geq 0$, with equality only if $e^2 = -2$ or $e^2 = -1$ and $g(e) > 0$; and*
(b) *There are no $E_6$-trees of holomorphic curves of square $-2$ on which a vanishes.*

*Then $\alpha$ is represented by symplectic forms deformation equivalent to $\omega$.*

*Proof.* (Sketch) Using negative-definiteness as in the case of the Kharlamov–Kulikov surface, one first shows that each connected component of the dual graph of the curves on which $\alpha$ is negative either

- contains just one curve of square $-1$ and (say) $n-1$ curves of square $-2$, in which case the dual graph is the Dynkin diagram $A_n$, with the square-$(-1)$ curve as one of the univalent vertices; or
- consists entirely of square-$(-2)$ curves, in which case it is one of the ADE Dynkin diagrams.

Now assumption (b) in the statement of the theorem restricts the Dynkin diagrams that can appear to $A_n$ and $D_n$ and is imposed because our methods do not seem strong enough to apply to the cases of $E_6$, $E_7$, or $E_8$. In the cases of $A_n$ and $D_n$, an approach parallel to that used in the case of the Kharlamov–Kulikov surface provides the desired form; the details of this are left as a mildly amusing exercise to the interested reader. □

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