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REIDEMEISTER TORSION IN FLOER–NOVIKOV THEORY AND COUNTING PSEUDO-HOLOMORPHIC TORI, II

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This is the second part of an article in two parts, which builds the foundation of a Floer-theoretic invariant, $\boldsymbol{I}_{\mathbf{F}}$. (See [Y-J. Lee, *Reidemeister torsion in Floer–Novikov theory and counting pseudo-holomorphic tori, I*, J. Symplectic Geom. **3** (2005), no. 2, 221–311.] for Part I). Having constructed $\boldsymbol{I}_{\mathbf{F}}$ and outlined a proof of its invariance based on bifurcation analysis in Part I, in this part we prove a series of gluing theorems to confirm the bifurcation behavior predicted in Part I. These gluing theorems are different from (and much harder than) the more conventional versions in that they deal with broken trajectories or broken orbits connected at degenerate rest points which are not Morse–Bott. The issues of orientation and signs are also settled in the last section. This part is strongly *dependent* on Part I, and is meant only for readers familiar with the previous part of this article.

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1. Overview

This second part forms the main technical core of the present article.

We have not attempted to make this part independent of Part I and shall frequently make use of the definitions, results, notation, and convention from Part I without repetition. Thus, we urge the reader to familiarize him/herself with Part I before attempting this one, paying particular attention to the convention in I.1.3.

References in the form of I.* shall refer to section, theorem, or equation numbers from Part I.

1.1. A brief summary. The following summarizes the results contained in this part. Recall the definitions of (RHFS*), (NEP), and admissible (J, X)-homotopies from Sections 4.3, 4.4, and 6.2 of [**1**], respectively.

Theorem. *Let* $\Lambda = [1, 2]$, *and* (J^Λ, X^Λ) *be an admissible* (J, X)*-homotopy connecting two regular pairs*, (J_1, X_1), (J_2, X_2). *Then*:

(1) (*Corner structures of parameterized moduli spaces*):
The properties (*RHFS2c*) *and* (*RHFS3c*) *hold for the CHFS generated by* (J^Λ, X^Λ);

(2) (***Orientation***): *The parameterized moduli spaces* $\mathcal{M}_P^{\Lambda,+}$ *and* $\bar{\mathcal{M}}_O^{\Lambda,1,+}$ *may be, respectively, given coherent and grading compatible orientations such that* (*RHFS4*) *holds*;

(3) (***Existence of nonequivariant perturbations***): (*NEP*) *holds for all Type II handleslides in the CFHS generated by* (J^Λ, X^Λ).

Combining with Propositions I.4.4.6 and I.6.2.2, this completes the proof of the general invariance theorem stated in Part I, Theorem I.4.1.1.

Item (1) above follows from the gluing theorems proven in Sections 2–6 below. Section 7 contains the discussion on orientability of the moduli spaces, the definitions of coherent and grading-compatible orientations, and as a consequence, the proof of item (2) above. Item (3) is established in Sections 6.2–6.3. There we introduce a class of (possibly nonlocal) perturbations to the induced flow on the finite-cyclic covers in the statement of (NEP), establish the expected regularity and compactness properties of the moduli spaces of such perturbed flows, and show how the arguments in the proof of Theorem I.6.2.2 may be adapted to establish the R-regularity of parameterized moduli spaces in this context, as required by (NEP).

Gluing theory is the unifying theme of Part II. Not only is it used repeatedly to establish the bifurcation analysis, but it also appears in the definition of coherent orientations in Section 7. Linearized versions of the gluing theorems in Sections 2–6, which actually form part of the proofs of these gluing theorems, play a major role in the verification of signs for item 2 of Theorem 1.1 above. It is for this reason that we postpone all discussion of orientations until the gluing results have been fully treated.

Thus, we begin with a quick overview of the general features of gluing theory in next subsection, then give a more specific outline of the variants contained in this article in Section 1.3.

1.2. Basics of gluing theory. This subsection gives only a minimal outline of gluing theory and its applications in Floer theory. Rather than a general account, our aim is to set up the basic framework for the proofs of the gluing theorems contained in this article and to introduce some basic notions and terminologies frequently used, some of which are not conventional. The reader may find more details and better-balanced treatments in the vast literature on this subject, for example, [**2**, **3**] and Floer's original papers. Also, precision will sometimes be sacrificed here for the overall picture. We shall be precise in later sections, when we return to the specific context of this article.

1.2.1. The four steps of gluing theory. Gluing is useful for studying the local structure of a stratified moduli space, usually coming from compactification. Given a space of "gluing parameters" $\Xi(\mathbb{S})$ associated with a codimension > 0 stratum $\mathbb{S}$, a typical gluing theorem constructs a map from

$\Xi(\mathbb{S})$ to a neighborhood of $\mathbb{S}$ in the moduli space of solutions to a PDE

$$\mathcal{F}(w) = 0,$$

which is a local diffeomorphism.

The proof of a typical gluing theorem comprises the following four major steps:

Step 1. *Constructing the pregluing map and error estimates.* For each gluing parameter χ, one constructs an approximate solution w_χ to the PDE considered, which varies smoothly with χ. The *pregluing map* $\chi \mapsto w_\chi$ maps the space of gluing parameters into a set in the ambient configuration space that is close to the space of solutions. An explict estimate, referred to as the "error estimate," is required to show that $\mathcal{F}(w_\chi)$ is sufficiently small.

Step 2. *Kuranishi structure.* Let $\mathfrak{D}_w \colon E \to F$ denote the linearization of $\mathcal{F}$ at w (i.e., the deformation operator). This should be a Fredholm operator, and ideally, one wants to show that $\mathfrak{D}_{w_\chi}$ has a right inverse bounded uniformly in χ. Namely, there is a χ-independent constant $C_P > 0$, and operators P_χ depending continuously on χ, such that

$$\mathfrak{D}_{w_\chi} P_\chi = \mathrm{id}, \quad \|P_\chi\| \leq C_P.$$

For this to hold, judicious choices of normed spaces for E, F are often called for.

Step 3. *Obtaining a quadratic bound on the nonlinear part of* $\mathcal{F}$, namely, (3) below. In local coordinates, one may write

$$\mathcal{F}(w) = \mathcal{F}(w_\chi) + \mathfrak{D}_{w_\chi}\xi + N_{w_\chi}(\xi) \quad \text{for } w = w_\chi + \xi. \tag{1}$$

Setting $\xi = P_\chi \eta_\chi$, a solution to $\mathcal{F}(w) = 0$ is obtained by solving

$$\eta_\chi = -N_{w_\chi}(P_\chi \eta) - \mathcal{F}(w_\chi). \tag{2}$$

The contraction mapping theorem shows that

Lemma. *Let C_P be the upper bound on $\|P_\chi\|$ as above, and suppose that there is a χ-independent constant k such that*

$$\|\mathcal{F}(w_\chi)\| \leq \frac{1}{10kC_P^2},$$

$$\|N_{w_\chi}(\xi_1) - N_{w_\chi}(\xi_2)\| \leq k(\|\xi_1\| + \|\xi_2\|)\|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2. \tag{3}$$

Then there exists a unique η_χ with $\|\eta_\chi\| \leq 1/(5kC_P^2)$ solving (2). Moreover, the solution η_χ varies smoothly with χ, and $\|\eta_\chi\| \leq 2\|\mathcal{F}(w_\chi)\|$.

Thus, by assigning to each gluing parameter χ the corresponding $w_\chi + P_\chi \eta_\chi$, one obtains a smooth map from the space of gluing parameters to the moduli space. This is the *gluing map*.

Step 4. *Showing that the gluing map is a local diffeomorphism to a neighborhood of S.*

1.2.2. Typical pregluing constructions in Floer theory. In Floer theory, $\mathcal{F} = \partial_s + \mathcal{V}$, and $\mathbb{S}$ is a stratum in a moduli space of broken trajectories or broken orbits. Thus, it is a product of reduced moduli spaces

$$\begin{aligned} \mathbb{S} = & \hat{\mathcal{M}}_0 \times \hat{\mathcal{M}}_1 \times \cdots \times \hat{\mathcal{M}}_k, \quad \text{or} \\ & \hat{\mathcal{M}}_1 \times \hat{\mathcal{M}}_2 \times \cdots \times \hat{\mathcal{M}}_k/\mathbb{Z}/k\mathbb{Z}. \end{aligned}$$

In the case of a family of Floer theories parameterized by Λ, $\mathbb{S}$ is a fiber product of reduced, parameterized moduli spaces over Λ

$$\begin{aligned} \mathbb{S} = & \hat{\mathcal{M}}_0^\Lambda \times_\Lambda \hat{\mathcal{M}}_1^\Lambda \times_\Lambda \cdots \times_\Lambda \hat{\mathcal{M}}_k^\Lambda, \quad \text{or} \\ & \hat{\mathcal{M}}_1^\Lambda \times_\Lambda \hat{\mathcal{M}}_2^\Lambda \times_\Lambda \cdots \times_\Lambda \hat{\mathcal{M}}_k^\Lambda/\mathbb{Z}/k\mathbb{Z}. \end{aligned}$$

The space of gluing parameters in these cases is $\Xi(\mathbb{S}) = \mathbb{S} \times (\Re, \infty)^k$ for certain large $\Re$, and the gluing maps map into a reduced moduli space or a reduced, parameterized moduli space.

We now describe the typical pregluing construction in these situations.

Given a (unreduced) flow $u(s)$ from the critical point x to y, we define its truncation

$$u_{[-R_-,R+]}(s) := \begin{cases} u(s) & \text{when } -R_-/2 \le s \le R_+/2 \\ \exp(y, \beta(2 - 2s/R_+)\eta_y(s)) & \text{when } s \ge R_+/2 \\ \exp(x, \beta(2s/R_- + 2)\eta_x(s)) & \text{when } s \le -R_-/2, \end{cases}$$

where β is a smooth cutoff function with $\beta(s) = 0$ for $s \le 0$, and $\beta(s) = 1$ for $s \ge 1$, and η_y, η_x are defined such that

$$u(s) = \begin{cases} \exp(x, \eta_x(s)) & \text{for } s \ll -1, \\ \exp(y, \eta_y(s)) & \text{for } s \gg 1. \end{cases}$$

Let $u_{(-\infty,R+]}$, $u_{[-R_-,\infty)}$ be similarly defined, truncated only at the positive/negative end, respectively.

Let $\{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_k\}$ be a broken trajectory from x to y, and u_i be representatives in the respective unreduced moduli spaces. Given $(R_1, \ldots, R_k) \in \mathbb{R}_+^k$, we define the *glued trajectory*:

$$u_0\#_{R_1}u_1\#_{R_2}u_2\cdots\#_{R_k}u_k(s) :=$$

$$(4)\qquad \begin{cases} u_{0,(-\infty,R_1]}(s) & \text{if } s \leq R_1 \\ \tau_{2R_1}u_{1,[-R_1,R_2]}(s) & \text{if } s \in [R_1, 2R_1+R_2] \\ & \vdots \\ \tau_{2\sum i=1^{k-1}R_i}u_{k-1,[-R_{k-1},R_k]}(s) & \\ \qquad\qquad \text{if } s \in \left[2\sum_{i=1}^{k-2}R_i + R_{k-1}, \quad 2\sum_{i=1}^{k-1}R_i + R_k\right] & \\ \tau_{2\sum_{i=1}^{k}R_i}u_{k,[-R_k,\infty)}(s) & \text{if } s \in \left[2\sum_{i=1}^{k-1}R_i + R_k, \infty\right], \end{cases}$$

where τ_L denotes translation by L:

$$\tau_L w(s) := w(s-L).$$

When $\{\hat{u}_1, \ldots, \hat{u}_k\}$ is a broken orbit, we may also define the *glued orbit*

$$u_1\#_{R_1}u_2\#_{R_2}\cdots u_k\#_{R_k}(s) :=$$

$$(5)\qquad \begin{cases} \tau_{2R_1}u_{1,[-R_1,R_2]}(s) & \text{if } s \in [R_1, 2R_1+R_2] \\ & \vdots \\ \tau_{2\sum_{i=1}^{k-1}R_i}u_{k-1,[-R_{k-1},R_k]}(s) & \\ \qquad\qquad \text{if } s \in \left[2\sum_{i=1}^{k-2}R_i + R_{k-1},\, 2\sum_{i=1}^{k-1}R_i + R_k\right] & \\ \tau_{2\sum_{i=1}^{k}R_i}u_{k,[-R_k,R_1]}(s) & \\ \qquad\qquad \text{if } s \in \left[2\sum_{i=1}^{k-1}R_i + R_k,\, 2\sum_{i=1}^{k}R_i + R_1\right] & \end{cases}$$

$$\text{for } s \in \mathbb{R}/\left(2\sum_{i=1}^{k}R_i\right)\mathbb{Z}.$$

We shall sometimes suppress the subscript R_i from $\#$ when it is not important.

To define the pregluing map, in the case of broken trajectories, assign to each

$$\chi = \{\hat{u}_0\}\times\cdots\{\hat{u}_k\}\times(R_1,\ldots,R_k) \in \hat{\mathcal{M}}_0\times\cdots\times\hat{\mathcal{M}}_k\times\mathbb{R}_+^k$$

the $\mathbb{R}$-orbit $\hat{w}_\chi$ of the glued trajectory

$$w_\chi = u_0\#_{R_1}u_1\#_{R_2}u_2\cdots\#_{R_k}u_k$$

in the configuration space $\mathcal{B}_P(x,y)$, taking u_i to be *centered* representatives of $\hat{u}_i$. Similarly for the case of broken orbits or the parameterized case. Owing to the exponential decay of flows to nondegenerate critical points, these constructions typically give good approximation to flow lines when the connecting rest points in the broken trajectory are nondegenerate. In this article, they are used for handleslide bifurcations and in the discussion of coherent orientations.

Remark. Equivalently, there is an unreduced version of the above construction, where the gluing map maps products of unreduced moduli spaces to an unreduced moduli space. Namely, take the space of gluing parameters to be an appropriate open subset

$$\breve{\Xi}(\mathbb{S}) \subset \mathcal{M}_0 \times \cdots \times \mathcal{M}_k;$$

and let the pregluing be given by the same formulae above, for *fixed* large $(R_1, \dots, R_k)$, and *not necessarily centered* u_i. Notice that there is a free $\mathbb{R}^{k+1}$ action on $\breve{\Xi}(\mathbb{S})$, namely the product of translations on each factor moduli space $\mathcal{M}_i$, and the quotient $\breve{\Xi}(\mathbb{S})/\mathbb{R}^{k+1} = \mathbb{S}$.

The equivalence is easily seen by observing that, given $(L_0, \cdots, L_k) \in \mathbb{R}^{k+1}$, there is a unique $(L, R'_1, \dots, R'_k) \in \mathbb{R} \times \mathbb{R}^k_+$, so that

$$\begin{aligned}&\tau_{L_0} u_0 \#_{R_1} \tau_{L_1} u_1 \#_{R_2} \tau_{L_2} u_2 \cdots \#_{R_k} \tau_{L_k} u_k \quad \text{approximates}\\ &\tau_L\big(u_0 \#_{R'_1} u_1 \#_{R'_2} u_2 \cdots \#_{R'_k} u_k(s)\big).\end{aligned}$$

(They are equal if u_i are replaced by their truncations.) Furthermore, under this identification, a diagonal $\mathbb{R}$-translation $(L_0, \dots, L_k) \to (L_0 + l, \dots, L_k + l)$ corresponds to an $\mathbb{R}$ translation in the first factor $(L, R'_1, \dots, R'_k) \to (L + l, R'_1, \dots, R'_k)$. Thus, we have a diffeomorphism

$$\begin{gathered}\breve{\Xi}(\mathbb{S})/\mathbb{R} \xrightarrow{\sim} \Xi(\mathbb{S}),\\ (\tau_{L_0} u_0, \dots, \tau_{L_k} u_k) \bmod \mathbb{R} \mapsto (\{\hat{u}_0, \dots, \hat{u}_k\}, R'_1, \dots, R'_k),\end{gathered}$$

and a commutative diagram

$$\begin{array}{ccc}\breve{\Xi}(\mathbb{S}) & \xrightarrow{\text{pregluing map}} & \mathcal{B}\\ {\scriptstyle /\mathbb{R}}\big\downarrow & & {\scriptstyle /\mathbb{R}}\big\downarrow\\ \Xi(\mathbb{S}) & \xrightarrow{\text{pregluing map}} & \mathcal{B}/\mathbb{R}\end{array}$$

We prefer the reduced perspective in this article, because when the connecting rest points are degenerate, the (reduced) space of gluing paramaters Ξ can still be described in a way similar to the above discussion, while $\breve{\Xi}$ is no longer a product of unreduced moduli spaces.

1.2.3. K-models. In general, the deformation operator might not be surjective, and the gluing theory gives a local description of the moduli space as an analytic variety in the cokernel of the deformation operator. This is the “Kuranishi model.”

For our purpose, it is convenient to introduce a linear variant of Kuranishi models, which we call "K-models." This notion of K-model will be useful both for Step 2 of the gluing procedure and in discussing the orientation issue.

Definition. *A* K-model *for a Fredholm operator* $\mathfrak{D}\colon E \to F$, *denoted* $[\mathfrak{D}\colon K \to C]_B$, *or simply* $[K \to C]$ *when there is no danger of confusion, is a triple* K, C, B, *where* K, C *are finite-dimensional subspaces* $K \subset E$, $C \subset F$ *respectively, and* $B \subset E$ *is a closed subspace such that*

- $\mathfrak{D}|_B\colon B \to \mathfrak{D}(B)$ *is an isomorphism and*
- *there are decompositions* $E = K \oplus B$, $F = C \oplus \mathfrak{D}(B)$ *(possibly not orthogonally).*

An orientation *of a K-model is a choice of orientations for the spaces* K *and* C.

Example (**Standard K-models**). *In this article, the "cokernel"* $\operatorname{coker}\mathfrak{D}$ *refers to either the quotient space* $F/\operatorname{Image}(\mathfrak{D})$ *or an arbitrary subspace of* F *complementary to* $\operatorname{Image}(\mathfrak{D})$. *A trivial example of K-model is* $[\mathfrak{D}\colon \ker \mathfrak{D} \to \operatorname{coker}\mathfrak{D}]_B$, *for any subspace* $B \subset E$ *complementary to* $\ker \mathfrak{D}$. *Such will be called a* standard K-model *for* $\mathfrak{D}$.

We shall call K a *"generalized kernel"* of $\mathfrak{D}$, C a *"generalized cokernel"*, and B a *"B-space,"* for lack of better terminology. The honest kernel and cokernel of $\mathfrak{D}$ may be described in terms of K and C via the exact sequence:

$$0 \to \ker \mathfrak{D} \xrightarrow{\Pi_K} K \xrightarrow{\Pi_C \circ \mathfrak{D}} C \to \operatorname{coker} \mathfrak{D} \to 0, \tag{6}$$

where Π_K, Π_C are projections with respect to the above decompositions of E and F.

Here are some other simple examples of K-models frequently encountered in this article.

Example (**K-model of a stabilization**). *Let* $\hat{\mathfrak{D}}_\Psi\colon \mathbb{R}^k \oplus E \to F$ *denote a finite-dimensional extension of the Fredholm map* $\mathfrak{D}\colon E \to F$,

$$\hat{\mathfrak{D}}_\Psi(\vec{r}, \xi) = \Psi(\vec{r}) + \mathfrak{D}\xi,$$

where $\Psi\colon \mathbb{R}^k \to F$ *is a linear map. We call* $\hat{\mathfrak{D}}_\Psi$ *a (rank-k) stabilization of* $\mathfrak{D}$.

Let $[K \to C]_B$ *be a K-model for* $\mathfrak{D}$, *and*

$$\hat{K} := \mathbb{R}^k \oplus K \subset \mathbb{R}^k \oplus E, \quad \hat{B}\colon = * \oplus B \subset \mathbb{R}^k \oplus E,$$

where $*$ *denotes the trivial vector space. Then* $[\hat{K} \to C]_{\hat{B}}$ *is a K-model for* $\hat{\mathfrak{D}}_\Psi$, *called the* stabilization *of* $[K \to C]_B$.

Example (**Reductions of K-models**). *Let* $[\mathfrak{D}\colon K \to C]_B$ *be a K-model, and suppose that there are subspaces* $Q \subset K$, $K' \subset K$, $C' \subset C$ *such that* $\Pi_C \circ \mathfrak{D}|_Q$ *is injective, and* K, C *decompose as:*

$$K = K' \oplus Q; \quad C = C' \oplus \Pi_C(\mathfrak{D}(Q)).$$

Then $[K' \to C']_{B'}$ *is another K-model for* $\mathfrak{D}$*, where* $B' = Q + B$*. Such K-models will be called* reductions *(by* Q*) of* $[K \to C]$*.*

Notice that if two K-models for $\mathfrak{D}$, $[\mathfrak{D}\colon K_1 \to C_1]_{B_1}$, $[\mathfrak{D}\colon K_2 \to C_2]_{B_2}$ have the same B-space $B_1 = B_2$, then projections of K_1 to K_2 and C_1 to C_2 (with respect to the decompositions $E = K_1 \oplus B_1$, $F = C_1 \oplus \mathfrak{D}(B_1)$) are isomorphisms, and vice versa. In this case, we say that the two K-models are *equivalent*. Two oriented K-models are said to be *equivalent* if they are equivalent K-models in the above sense, and the projections involved are orientation-preserving.

K-models are particularly useful in family settings. We adopt the convention of denoting a Banach space bundle over Λ by V^Λ, with the fiber over $\lambda \in \Lambda$ denoted as V_λ. Let Λ be a connected manifold, and E^Λ, F^Λ be Banach space bundles over Λ. Let $\mathfrak{D}^\Lambda\colon = \{\mathfrak{D}_\lambda | \mathfrak{D}_\lambda\colon E_\lambda \to F_\lambda, \lambda \in \Lambda\}$ be a family of uniformly bounded Fredholm operators, continuous in operator norm. A (*family*) *K-model* for $\mathfrak{D}^\Lambda$, written as $[\mathfrak{D}^\Lambda\colon K^\Lambda \to C^\Lambda]_{B^\Lambda}$, is a triple of Banach space subbundles $K^\Lambda \subset E^\Lambda, C^\Lambda \subset F^\Lambda, B^\Lambda \subset E^\Lambda$, so that the fibers over each $\lambda \in \Lambda$, $[K_\lambda \to C_\lambda]_{B_\lambda}$ form a K-model for $\mathfrak{D}_\lambda$, and $\mathfrak{D}_\lambda|_{B_\lambda}$ has a uniformly bounded left inverse.

If Λ is finite-dimensional and compact, such K-models always exist by the Fredholmness of the family $\mathfrak{D}^\Lambda$. In contrast, $\bigcup_\lambda \ker \mathfrak{D}_\lambda$, $\bigcup_\lambda \operatorname{coker} \mathfrak{D}_\lambda$ may not form bundles as the dimensions of the kernels and cokernels may jump with λ.

Two K-models $[D_1\colon K_1 \to C_1]_{B_1}$, $[D_2\colon K_2 \to C_2]_{B_2}$ are said to be *correlated* via the family K-model $[D^\Lambda : K^\Lambda \to C^\Lambda]_{B^\Lambda}$ if they may be identified with two fibers $[D_{\lambda_1}\colon K_{\lambda_1} \to C_{\lambda_1}]_{B_{\lambda_1}}$, $[D_{\lambda_2}\colon K_{\lambda_2} \to C_{\lambda_2}]_{B_{\lambda_2}}$ over $\lambda_1, \lambda_2 \in \Lambda$. They are said to be *equivalent* via the family K-model if they are equivalent to two fibers. Finally, the notions of correlation and equivalence for *oriented* K-models are obtained by inserting the adjective "oriented" before every mention of K-model or family K-model in the above paragraph.

Example. If two Fredholm operators $\mathfrak{D}, \mathfrak{D}'$ are close in operator norm, one may always include them in a family

$$\mathfrak{D}^\Lambda = \{\mathfrak{D}_\lambda \,|\, \|\mathfrak{D} - \mathfrak{D}_\lambda\| < \varepsilon\} \quad \textit{for an } \varepsilon \ll 1.$$

Any K-model $[\mathfrak{D}\colon K \to C]_B$ may be extended into a family K-model for $\mathfrak{D}^\Lambda$, with trivial $B^\Lambda = B \times \Lambda$. In this case, we shall refer to the equivalence of K-models for $\mathfrak{D}$, $\mathfrak{D}'$ without specifying the family K-model — a family K-model of the above description will be implied. Moreover, if $\mathfrak{D}$ is surjective, and ε is sufficiently small, $[\bigcup_\lambda \ker \mathfrak{D}_\lambda \to *]$ form a K-model for $\mathfrak{D}^\Lambda$. Thus, in this case, we shall refer to correlated orientations of $\ker \mathfrak{D}$ and $\ker \mathfrak{D}'$ without further specifications.

1.2.4. Gluing operators and gluing K-models. A major motivation to introduce K-models is that, in gluing theory, generalized kernels and cokernels are typically easier to construct and work with than the honest kernels and cokernels. This subsubsection explains why.

We summarize the typical properties of the Fredholm operators appearring in Floer theories as follows. A *Floer-type operator* is a Fredholm operator of the form:

$$\mathfrak{D} = \partial_s + A(s) : E \to F, \quad \text{where}$$

- $E = W(\mathbb{R}_s \times Y, p_2^* V)$, $F = L(\mathbb{R}_s \times Y, p_2^* V)$ for suitable Sobolev norms W, L.
- V is an Euclidean or hermitian bundle over the manifold Y, $\mathbb{R}_s$ denotes the real line parameterized by s, $p_2 \colon \mathbb{R}_s \times Y \to Y$ denotes the projection.
- $A(s) \colon \Gamma(Y; V) \to \Gamma(Y; V)$ is a first-order linear differential operator, which is surjective and L^2-self-adjoint when $|s| \gg 1$.

A *stabilized Floer-type operator* is a stabilization of a Floer-type operator by multiplication with compactly supported functions.

Examples. In Morse theory, Y is a point. In the symplectic Floer theory considered in this article, $Y = S^1$, and $p_2^* V = \mathbb{R}^{2n}$ (obtained from trivializing some $u^* K$). Y is a 3-manifold in Seiberg–Witten or instanton Floer theories.

An ordered k-tuple of Floer-type operators

$$\mathfrak{D}_1 = \partial_s + A_1(s), \ldots, \mathfrak{D}_k = \partial_s + A_k(s) \colon E \to F$$

are said to be *glueable* if

- $A_1(s)$ is constant for large s, $A_k(s)$ constant for very negative s, and for $i = 2, \ldots, k-1$, $A_i(s)$ is constant in s for $|s| \gg 1$;
- $A_i(\infty) = A_{i+1}(-\infty)$ for $i = 1, \ldots, k-1$.

Given a glueable $k+1$-tuple of Floer-type operators $\mathfrak{D}_0, \ldots, \mathfrak{D}_k$, and $k+1$-tuple of functions $(f_0, \ldots, f_k) \in E^k$ or F^k, we may define the glued operator $\mathfrak{D}_0 \#_{R_1} \cdots \#_{R_k} \mathfrak{D}_k$ and glued function $f_0 \#_{R_1} \cdots \#_{R_k} f_k$ via the same formula (4), replacing $u_{i,[-R_i, R_{i+1}]}$ thereby $\mathfrak{D}_i$ and $f_{i,[-R_i, R_{i+1}]}$, respectively, where $f_{i,[-R_i, R_{i+1}]}$ is the truncation

$$f_{i,[-R_i,R_{i+1}]} = \beta_{[-R_i,R_{i+1}]}(s) f_i, \qquad \text{where } \beta_{[-R_i,R_{i+1}]}(s) = \beta(2s/R_i + 2)\beta(2 - 2s/R_{i+1}),$$

with R_0, R_{k+1} understood as $-\infty, \infty$, respectively, and $\beta_{(-\infty,R]}(s) := \beta(2 - 2s/R)$, $\beta_{[-R,\infty)}(s) := \beta(2s/R + 2)$.

Let K_i, $i = 0, \dots, k$ be subspaces in E or F. We denote by $K_0 \#_{R_1} \cdots \#_{R_k} K_k$ the subspace of E or F defined by

$$K_0 \#_{R_1} \cdots \#_{R_k} K_k := \{f_0 \#_{R_1} \cdots \#_{R_k} f_k \mid f_i \in K_i, i = 0, \dots, k\}.$$

In parallel, let $\mathfrak{D}_1 = \partial_s + A_1(s), \dots, \mathfrak{D}_k = \partial_s + A_k(s)$ be a k-tuple of glueable Floer-type operators, with $A_1(-\infty) = A_k(\infty)$. We call such operators *cyclically glueable*. In this case, we may define the cyclically glued operator and functions $\mathfrak{D}_1 \#_{R_1} \cdots \#_{R_{k-1}} \mathfrak{D}_k \#_{R_k}$, $f_1 \#_{R_1} \cdots \#_{R_{k-1}} f_k \#_{R_k} \in \Gamma(S^1_{2\sum_i R_i} \times Y; p_2^* V)$ via formula (5), with similar modifications. The subspace $K_1 \#_{R_1} \cdots \#_{R_{k-1}} K_k \#_{R_k} \subset \Gamma(S^1_{2\sum_i R_i} \times Y; p_2^* V)$ may also be similarly defined. Furthermore, gluing and cyclic-gluing extend in an obvious way to stabilized Floer-type operators.

We denote by $\iota^j_\# K_j$ the subspace

$$\begin{aligned} \iota^j_\# K_j = * \#_{R_1} \cdots * \#_{R_{j-1}} K_j \#_{R_j} * \cdots \#_{R_{k-1}} * &\subset K_1 \#_{R_1} \cdots \#_{R_{k-1}} K_k \\ \text{or } * \#_{R_1} \cdots * \#_{R_{j-1}} K_j \#_{R_j} * \cdots * \#_{R_k} &\subset K_1 \#_{R_1} \cdots \#_{R_{k-1}} K_k \#_{R_k} \end{aligned}$$

depending on the context. Notice that when $R_1, \dots, R_k$ are sufficiently large and the subspaces K_j are finite-dimensional, then $\iota^j_\#$ are injective for all j.

Given $f \in \Gamma(\mathbb{R} \times Y; p_2^* V)$ or $\Gamma(S^1 \times Y; p_2^* V)$ and $c \in \mathbb{R}$ or S^1, let

$$\operatorname{res}_c(f) := f|_{\{c\} \times Y}.$$

For a subspace $K \subset \Gamma(\mathbb{R} \times Y; p_2^* V)$ or $\Gamma(S^1 \times Y; p_2^* V)$, let $\operatorname{res}_c\, K$ denote the subspace $\{f|_{\{c\} \times Y} \mid f \in K\} \subset \Gamma(Y; V)$.

Lemma (Glued K-models). *Let $\mathfrak{D}_1, \dots, \mathfrak{D}_k$ be a k-tuple of glueable Floer-type operators, and $[\mathfrak{D}_i \colon K_i \to C_i]_{B_i}$ be K-models such that $\operatorname{res}_0|_{K_i} \colon K_i \to \operatorname{res}_0 K_i$ is an isomorphism, and let $\operatorname{res}_0\, B_i \subset \operatorname{res}_0\, E$ be a complementary subspace to $\operatorname{res}_0 K_i$. Set*

$$B_\# := \Big\{ f \;\Big|\; \operatorname{res}_0 (\tau_{-2\sum_{j=1}^{i-1} R_j} f) \in \operatorname{res}_0\, B_i, i = 1, \dots, k \Big\}.$$

(1a) *Suppose for $\Re \gg 1$ and $\vec{R} := (R_1, \dots, R_{k-1}) \in [\Re, \infty)^{k-1}$,*

$$\mathfrak{D}_{\#\vec{R}} := \mathfrak{D}_1 \#_{R_1} \cdots \#_{R_{k-1}} \mathfrak{D}_k \text{ is Fredholm of index } \sum_{i=1}^k \operatorname{ind} \mathfrak{D}_i, \; . \tag{7}$$

and $K_{\#\vec{R}} := K_1 \#_{R_1} \cdots \#_{R_{k-1}} K_k$, $C_{\#\vec{R}} := C_1 \#_{R_1} \cdots \#_{R_{k-1}} C_k$. Then $[\mathfrak{D}_{\#\vec{R}} : K_{\#\vec{R}} \to C_{\#\vec{R}}]_{B_\#}$ forms a K-model. In fact, these form a family K-model for the family of operators $\{\mathfrak{D}_{\#\vec{R}}\}_{\vec{R}}$. In particular, when $\mathfrak{D}_i$ are surjective $\forall i$, the glued operator has a right inverse bounded uniformly in $R_1, \dots, R_{k-1}$.

(1b) *The same holds for* $\vec{R} := (R_1, \ldots, R_k) \in [\Re, \infty)^k$,

$$\mathfrak{D}_{\#\vec{R}} = \mathfrak{D}_1 \#_{R_1} \cdots \#_{R_{k-1}} \mathfrak{D}_k \#_{R_k},$$
$$K_{\#\vec{R}} = K_1 \#_{R_1} \cdots \#_{R_{k-1}} K_k \#_{R_k},$$
$$C_{\#\vec{R}} = C_1 \#_{R_1} \cdots \#_{R_{k-1}} C_k \#_{R_k}$$

if, in addition, $\mathfrak{D}_1, \ldots, \mathfrak{D}_k$ *is cyclically glueable with*

$$\text{ind } \mathfrak{D}_{\#\vec{R}} = \sum_{i=1}^{k} \text{ind } \mathfrak{D}_i.$$

(2) *Furthermore, the projection* $\Pi_{\iota^j_\# C_j}$ (*with respect to the decomposition* $F = \bigoplus_{i=1}^k \iota^i_\# C_i \oplus \mathfrak{D}_{\#\vec{R}}(B_\#)$) *approximates* $\Pi_{C_j} \circ \beta_{[-R_{j-1}, R_j]} \circ \tau_{-2\sum_{i=1}^{j-1} R_i}$, *where the projection* Π_{C_j} *is with respect to the decomposition* $F = C_j \oplus \mathfrak{D}_j(B_j)$.

There are many other ways of choosing the B-space $B_\#$ for the statement of this lemma to hold [cf., e.g., [**4**], Proposition 9]; the one described above is that which we shall stick to for the gluing constructions in this article. Notice that with this choice of $B_\#$, the projection $\Pi_{\iota^j_\# K_j}$ (with respect to the decomposition $E = \bigoplus_i \iota^i_\# K_i \oplus B_\#$) is given by

$$\Pi_{\iota^j_\# K_j} = (\text{res}_0 |_{K_j})^{-1} \circ \Pi_{\text{res}_0 K_j} \circ \text{res}_0 \circ \tau_{-2\sum_{i=1}^{j-1} R_i},$$

where the projection $\Pi_{\text{res}_0\ K_j}$ is with respect to the decomposition $\text{res}_0 E = \text{res}_0\ K_j \oplus \text{res}_0\ B_j$.

This gluing procedure also generalizes to family situations to construct family K-models for glued family of operators from family K-models of the family of operators to be glued.

Example (**K-models of deformation operators at glued trajectories/orbits**)**.** Let $\{\hat{u}_0, \ldots, \hat{u}_k\}$ be a broken trajectory and u_i be centered representatives of $\hat{u}_i$. Then

$$E_{u_0 \#_{R_1} \cdots \#_{R_k} u_k} = E_{u_{(-\infty, -R_1]}} \#_{R_1} \cdots \#_{R_k} E_{v_{[R_k, \infty)}}.$$

When $R_0, \ldots, R_k$ are large enough, $[\text{ker } E_{u_i}, \text{coker } E_{u_i}]$ is a K-model for $E_{u_{i,[-R_i, R_{i+1}]}}$. Furthermore, viewing $\text{ker } E_{u_i}$ as the solution space of the first-order linear differential equation $E_{u_i}\xi = 0$, we see that $\text{res}_0 |_{\text{ker } E_{u_i}}$ is an isomorphism. Take $B_i = \{f | \text{res}_0(f) \in \text{ker } E_{u_i}^\perp\}$. By the above lemma,

$$[\text{ker } E_{u_0} \#_{R_1} \cdots \#_{R_k} \text{ ker } E_{u_k} \to \text{coker } E_{u_0} \#_{R_1} \cdots \#_{R_k} \text{ coker } E_{u_k}]_{B_\#}$$

is a K-model for $E_{u_0 \#_{R_1} \cdots \#_{R_k} u_k}$. Similarly, in the case of broken orbits, we obtain a K-model for the deformation operator at the glued orbit by cyclically gluing the standard K-models of the deformation operators at the component trajectories.

1.2.5. Proof by contradiction and excision for right-invertibility. Though Lemma 1.2.4 above is standard, we shall include a proof here, since it showcases the typical arguments for establishing the (uniform) right-invertibility of $\mathfrak{D}_{w_\chi}$ required by Step 2 of gluing. In simple situations, one may construct by excision a right inverse to $\mathfrak{D}_{w_\chi}$ from right inverses of the deformation operators $\mathfrak{D}_{u_i}$ associated to the gluing parameter χ [see, e.g., **2**, **3**,**5**]. In more intricate situations, such as those frequently encountered in this article, it is often convenient to use an indirect, nonconstructive method, which we refer to as "proof by contradiction." This method starts by choosing a codimension ind $\mathfrak{D}_{w_\chi}$ subspace $B_\chi \subset E$. By the Fredholmness of $\mathfrak{D}_{w_\chi}$, if $\mathfrak{D}_{w_\chi}|_{B_\chi}$ is injective, then $\mathfrak{D}_{w_\chi}$ has a bounded right inverse $P_\chi\colon F \to B_\chi$. Suppose otherwise that there is a sequence of unit length $\xi_\chi \in B_\chi$, such that $\mathfrak{D}_{w_\chi}\xi_\chi \to 0$. One then shows that this is impossible by estimating $\|\xi_\chi\|$ in terms of $\|\mathfrak{D}_{w_\chi}\xi_\chi\|$, showing that the former must go to 0 as the latter does so. This estimate is usually obtained by breaking ξ_χ into summands ξ_i supported in different regions and bounding the summands using the surjectivity of $\mathfrak{D}_{u_i}$.

Proof of Lemma 1.2.4. (1). The proofs of (1a) and (1b) are almost identical, so we shall focus on (1a). We follow the proof by contradiction framework. First, fix $\vec{R}$ and omit it from the subscripts to simplify notation. Let $\Psi_j\colon C_j \to F$ be the inclusion, and let $\mathfrak{D}_{\Psi_\#}\colon \bigoplus_j C_j \oplus B_\# \to F$ be defined by

$$\mathfrak{D}_{\Psi_\#}(v_1, \ldots, v_k, \xi) = \mathfrak{D}_\#\xi + \Psi_1(v_1)\#_{R_1} \cdots \#_{R_{k-1}}\Psi_k(v_k).$$

Choose $R_1, \ldots, R_{k-1}$ to be large enough so that dim $K_\# = \sum_i \dim K_i$ and dim $C_\# = \sum_i \dim C_i$. We have the decomposition $E = K_\# \oplus B_\#$ by construction. To show that $[\mathfrak{D}_\#\colon K_\# \to C_\#]_{B_\#}$ indeed forms a K-model, it suffices to show that $\mathfrak{D}_{\Psi_\#}$ is surjective, in view of the definition of $K_\#, C_\#, B_\#$ and the index constraint (7). Since $\mathfrak{D}_{\Psi_\#}$ is Fredholm of index 0, it is equivalent to show that it is injective.

Suppose the contrary that there exists $(v_1, \ldots, v_k, \xi) \in \bigoplus_j C_j \oplus B_\#$ with unit norm such that

$$\mathfrak{D}_{\Psi_\#}(v_1, \ldots, v_k, \xi) = 0. \tag{8}$$

The fact that $[\mathfrak{D}_j\colon K_j \to C_j]_{B_j}$ is a K-model implies that the operator

$$\mathfrak{D}_{\Psi_j}\colon C_j \oplus B_j \to F, \quad (v, \eta) \mapsto \mathfrak{D}_j\eta + \Psi_j(v)$$

has a bounded inverse, and hence

$$
\begin{aligned}
&\|(v_j, (\beta_{[-R_{j-1},R_j]} \circ \tau_{-2\sum_{i=1}^{j-1} R_i})\xi\| \\
&\le C\|\mathfrak{D}_j(\beta_{[-R_{j-1},R_j]} \circ \tau_{-2\sum_{i=1}^{j-1} R_i})\xi + \Psi_j(v)\| \\
&\le C\|\beta_{[-R_{j-1},R_j]} \circ \tau_{-2\sum_{i=1}^{j-1} R_i})\mathfrak{D}_{\Psi_\#}(v_1, \dots, v_k, \xi)\| \\
&\quad + C\|\beta'_{[-R_{j-1},R_j]}\tau_{-2\sum_{i=1}^{j-1} R_i})\xi\| + C\|(1-\beta_{[-R_{j-1},R_j]})\Psi_j(v_j)\| \\
&\ll 1,
\end{aligned} \tag{9}
$$

using the assumption (8) for the first term, the fact that $\beta'_{[-R_{j-1},R_j]} < C(\sum_i R_i^{-1}) \ll 1$ for the second term, and the facts that $\|\Psi_j(v_j)\|$ is bounded and R_i are large for the last term. Meanwhile, observe that the supports of $\beta_j := \tau_{2\sum_{i=1}^{j-1} R_i}\beta_{[-R_{j-1},R_j]}$ are disjoint for different j, and write

$$
1 - \sum_{j=1}^{k} \beta_j = \sum_{l=1}^{k-1} \varphi_l,
$$

where φ_l is a nonnegative function supported on

$$
\left(2\sum_{i=1}^{l-1} R_i - \frac{R_l}{2}, 2\sum_{i=1}^{l-1} R_i + \frac{R_l}{2}\right).
$$

Choose $R_1, \dots, R_k$ to be large enough so that over the support of φ_l, $\mathfrak{D}_l = \mathfrak{D}_{l+1} = \partial_s + A_l(\infty)$. Since by assumption $\partial_s + A_l(\infty)$ has a bounded inverse, we have

$$
\begin{aligned}
&\|\varphi_l \xi\| \\
&\le C'\|(\partial_s + A_l(\infty))(\varphi_l \xi)\| \\
&\le C'\|\varphi_l \mathfrak{D}_{\Psi_\#}(v_1, \dots, v_k, \xi)\| + C'\|\varphi_l' \xi\| \\
&\qquad + \left\|\varphi_l \sum_{j=l}^{l+1} \tau_{2\sum_{i=1}^{j-1} R_i}(\beta_{[-R_{j-1},R_j]}\Psi_j(v_j))\right\| \\
&\ll 1.
\end{aligned} \tag{10}
$$

Summing (9) and (10) for all j and l, we obtain the desired contradiction that

$$
\|(v_1, \dots, v_k, \xi)\| \le \sum_j \|(v_j, \beta_j \xi)\| + \sum_l \|\varphi_l \xi\| \ll 1.
$$

To see the assertion about family K-models, replace $(v_1, \dots, v_k, \xi)$ above by a sequence of unit vectors $\{(v_1^\nu, \dots, v_k^\nu, \xi^\nu)\}_\nu$ with $\|\mathfrak{D}_{\Psi_\# \vec{R}_\nu}(v_1^\nu, \dots, v_k^\nu, \xi^\nu)\| \to 0$. This is impossible by the same estimates, since the above estimates do not depend on the specific values of $R_1, \dots, R_{k-1}$.

(2) We now switch to the excision method. Let χ_j be a smooth cutoff function with value 1 on the support of β_j and vanishes outside the support of $\varphi_{j-1}+\varphi_j$ (with $\varphi_0 := 0 =: \varphi_k$). Let $\tilde\chi_l$ be a smooth cutoff function with value 1 on the support of φ_l and vanishes outside the support of $\sum_{j=l-1}^{l}\beta_j$. We choose these cutoff functions such that $|\chi_j'|$, $|\tilde\chi_l'|$ are both bounded by $\min_i R_i^{-1}/4$ for all j,l. Let $\mathfrak{G}_{\Psi_j} = (g_{\Psi_j}, G_{\Psi_j})\colon F \to C_j \oplus B_j$ and $\tilde G_l\colon F \to E$ be the inverses of $\mathfrak{D}_{\Psi_j}$ and $\partial_s + A_l(\infty)$, respectively. Let $\mathfrak{G}^\tau_{\Psi_j} = (g^\tau_{\Psi_j}, G^\tau_{\Psi_j}) := \Big(g_{\Psi_j}\tau_{-2\sum_{i=1}^{j-1}R_i}, \tau_{2\sum_{i=1}^{j-1}R_i}G_{\Psi_j}\tau_{-2\sum_{i=1}^{j-1}R_i}\Big)$ and set $\mathfrak{G}_{\Psi_\#}\colon F \to \bigoplus_j C_j \oplus B_\#$ to be

$$\mathfrak{G}_{\Psi_\#} = \Big(g^\tau_{\Psi_1}\beta_1, \ldots, g^\tau_{\Psi_k}\beta_k, \sum_j \chi_j G^\tau_{\Psi_j}\beta_j + \sum_l \tilde\chi_l \tilde G_l \varphi_l\Big).$$

A straightforward computation shows that $\mathfrak{D}_{\Psi_\#}\mathfrak{G}_{\Psi_\#} = 1 + \Xi$, where Ξ is small in operator norm, and so the inverse of $\mathfrak{D}_{\Psi_\#}$ is $\mathfrak{G}_{\Psi_\#}(1+\Xi)^{-1}$. Now, the projection from F to $\iota^j_\# C_j$ is given by $\Pi_{C_j}\mathfrak{G}_{\Psi_\#}(1+\Xi)^{-1}$ while the projection from F to C_j is given by $\Pi_{C_j}\mathfrak{G}_{\Psi_j}$. Claim (2) of Lemma 1.2.4 follows from comparing these two. □

1.2.6. Generalizing the gluing map. Suppose the deformation operator $\mathfrak{D}_{w_\chi}$ has a K-model $[K \to C]$ with nontrivial C, the construction of gluing map in Step 3 of Section 1.2.1 may be generalized as follows.

Write in local coordinates near w_χ as in Section 1.2.1, and project (1) to the subspaces $\mathfrak{D}(B)$, $C \subset F$, respectively, while decomposing

$$\xi = P_\chi\eta_\chi + \xi_K \quad \text{for } \xi_K \in K,\ P_\chi\eta_\chi \in B,$$

where $P_\chi\colon \mathfrak{D}(B) \to B$ being the left inverse of $\mathfrak{D}_{w_\chi}|_B$. We have

$$\eta_\chi + \Pi_{\mathfrak{D}(B)}(\mathcal{F}(w_\chi) + \mathfrak{D}_{w_\chi}\xi_K) + \Pi_{\mathfrak{D}(B)}N_{w_\chi}(\xi_K + P_\chi\eta_\chi) = 0,$$
$$\Pi_C(\mathcal{F}(w_\chi) + \mathfrak{D}_{w_\chi}\xi_K) + N_{w_\chi}(\xi_K + P_\chi\eta_\chi)) = 0.$$

If ξ_K is sufficiently small, the contraction mapping theorem (Lemma 1.2.1) applies to the first equation above to obtain a solution of η_χ depending on ξ_K. Substitute this into the second equation, we obtain a *finite rank* equation in ξ_K, which is itself in a finite-dimensional space. (The function on the LHS of this equation is the "Kuranishi map.") Thus, the solution space of ξ is now an analytic variety in C. If $[K \to C]$ is a fiber of a family K-model $[K^\Xi \to C^\Xi]$ for $\{\mathfrak{D}_{w_\chi}\}_{\chi\in\Xi}$, this describes the local structure of moduli space near the image of the pregluing map as an analytic variety in the finite-dimensional vector bundle C^Ξ. C^Ξ is the so-called "obstruction bundle," and this is essentially the "obstruction bundle technique" pioneered by Taubes.

In general, it is difficult to understand the structure of this analytic variety. An example from this article is the case of gluing a broken trajectory or orbit involving m Type II handleslides, where $m > 1$ (cf. Section 6). According to Lemma 1.2.4, in this case the glued K-model has an m-dimensional generalized cokernel. Our inability to describe the local structure of $\hat{\mathcal{M}}_P^{\Lambda,1,+}$ near the stratum $T_{P,\text{hs-m}}$ or that of $\hat{\mathcal{M}}_O^{\Lambda,1,+}$ near the stratum $T_{O,\text{hs-m}}$ is precisely due to the lack of understanding on the relevant analytic variety in this bundle of generalized cokernels.

1.2.7. Typical arguments for Step 4 in Floer theory. Typically, it follows directly from the discussion on Kuranishi structure in Step 2 that the gluing map is a local diffeomorphism. For example, let $\chi = \{\hat{u}_0, \dots, \hat{u}_k\} \times (R_1, \dots, R_{k-1})$, and $\check{\chi} = \{u_0\} \times \cdots \times \{u_k\}$ for corresponding representatives u_i of $\hat{u}_i$ given in the Remark of Section 1.2.2. When $\hat{u}_i$ are all nondegenerate, Lemma 1.2.4 asserts that $\ker \mathfrak{D}_{w_\chi}$ is isomorphic to

$$\ker \mathfrak{D}_{u_0} \# \cdots \# \ker \mathfrak{D}_{u_k} \simeq T_{\check{\chi}} \check{\Xi}(\mathbb{S}) \simeq T_\chi \Xi(\mathbb{S}) \times \mathbb{R} w'_\chi,$$

where the first isomorphism in the above expression is the differential of the pregluing map, and the second isomorphism is due to the remark of Section 1.2.2 and the fact that $\mathfrak{D}_L(\tau_L w_\chi) = w'_\chi$. On the other hand, the pregluing w_χ is close to the corresponding image of the gluing map, w. Thus, $\ker \mathfrak{D}_{w_\chi} \simeq \ker \mathfrak{D}_w = T_w \mathcal{M}_P$. These together imply that the differential of the gluing map is an isomorphism from $T_\chi \Xi$ to $T_w \hat{\mathcal{M}}_P$.

To show that the gluing map is actually surjective to a neighborhood of $\mathbb{S}$ in $\hat{\mathcal{M}}_P^+$, one starts with the following simple consequence of the implicit function theorem.

Lemma. *In the above situation, let $\mathcal{T}_\chi \subset T_{w_\chi} \mathcal{B}_P = E$ be the image of the differential of the pregluing map at χ. Suppose the following hold for all $\chi \in \Xi$:*

- *$\mathcal{T}_\chi$, and w'_χ vary smoothly with χ;*
- *$\exists$ subspaces $B_\chi \subset E$ forming fibers of a bundle $B^\Xi \to \Xi$, such that E decomposes as $E = B_\chi \oplus \mathcal{T}_\chi \oplus \mathbb{R} w'_\chi$, and the projections to the summands are bounded* uniformly *in χ.*

Let $\exp(w_\chi, b_\chi) \in \mathcal{B}_P$ denote the element of coordinates b_χ in the local chart centered at w_χ. Then there is a diffeomorphism from a small tubular neighborhood of $\{((\chi, 0), 0)\} \subset B^\Xi \times \mathbb{R}$ to a small tubular neighborhood, $U_\epsilon = \{\exp(\tau_L w_\chi, \xi) |\, \|\xi\|_E < \epsilon\} \subset \mathcal{B}_P$ defined by

$$((\chi, b_\chi), \tau) \mapsto \tau_L(\exp(w_\chi, b_\chi)).$$

In other words, B^Ξ gives a good coordinate system of a slice of the $\mathbb{R}$-action in U_ϵ. In our context, $\mathcal{B}_\chi$ is the B-space defined in Step 2, and the projection $\Pi_{B_\chi} = P_\chi \mathfrak{D}_{w_\chi}$. Proofs of analogous statements in the harder gauge-theoretic context, where the $\mathbb{R}$-action is replaced by the action of an infinite dimensional gauge group, may be found in [**2**, Section **7.3**], and [**3**, pp. 97–99].

Together with the contraction mapping theorem stated in Section 1.2.1, this lemma implies that the gluing map surjects to a tubular neighborhood (*in E-norm*) in the moduli space. However, the moduli space of broken trajectories is endowed with the coarser chain topology instead. Thus, a major task in Step 4 is to show that any flow line in a chain-topology neighborhood of $\mathbb{S}$ in fact lies in U_ϵ. This requires a decay estimate of the flow lines near the connecting rest points.

In the case where the connecting rest points are nondegenerate, the relevant exponential decay estimate is akin to the decay estimate for flows ending at y, which has been used to derive (global) compactness of $\mathcal{M}_P$ from Gromov (local) compactness. Proposition 4.4 of [**3**] is recommended for a well-written account of this estimate (in the gauge-theoretic context).

1.3. Gluing flow lines ending in degenerate critical points. To verify the prediction of (RHFS2c, 3c) on the corner structure of $\hat{\mathcal{M}}_P^{\Lambda,1,+}$ or $\hat{\mathcal{M}}_O^{\Lambda,1,+}$ near $\mathbb{T}_{P,\mathrm{db}}$, $\mathbb{J}_P$, or $\mathbb{T}_{O,\mathrm{db}}$, one needs to glue flow lines ending at a degenerate critical point.

Now let (J^Λ, X^Λ) be an admissible (J, X)-homotopy, and let $(0, y) \in \mathcal{P}^{\Lambda,deg}(J^\Lambda, X^\Lambda)$. In Sections 2–4, we set $\mathbb{S} = \mathbb{T}_{P,\mathrm{db}}$ or $\mathbb{T}_{O,\mathrm{db}}$, which consists of broken trajectories or orbits with all the connecting rest points being y. In Section 5, $\mathbb{S}$ is the subset in $\mathbb{J}_P$ consisting of connecting flow lines starting or ending in y. The space of gluing parameters in both cases will be $\Xi(\mathbb{S}) = \mathbb{S} \times S$, where S is an open interval in Λ with left or right end 0.

The gluing theory in these cases differ from the "standard" case outlined in Section 1.2 in many aspects. Much of the additional complication arises from the fact that, instead of the usual configuration space modeled locally on Sobolev spaces or exponentially weighted Sobolev spaces, the moduli spaces of flows to y now embed in configuration spaces modeled on the polynomially weighted W_u-norm, and the deformation operator is between the W_u and L_u spaces introduced in Section I.5. The main difference between working with these polynomially-weighted spaces and the more commonly seen exponentially weighted ones is that the range space L_u now has larger weights in the longitudinal direction than the domain space W_u. This often implies that all the estimates in the gluing theory need to be particularly precise in the longitudinal direction, especially near y, where the weight is large. Below is a quick outline of the strategies adopted in Sections 2–5.

1.3.1. Constructing pregluing. Let $\chi = (\{\hat{u}_1, \dots, \hat{u}_k\}, \lambda) \in \Xi(\mathbb{S})$. Due to the aforementioned problem with large weights in the longitudinal direction, one needs a more delicate pregluing construction instead of the typical one explained in Section 1.2.1.

Let $u_{\lambda,i}$ be a centered representative of $\hat{u}_i$ or a suitable cutoff version of it (to be specified later). Noticing that a variation in parameterization (by s) of an element in $\mathcal{B}_P$ or $\mathcal{B}_O$ gives rise to a variation of the element in the longitudinal direction, a natural solution to the above problem is to find (λ-dependent) diffeomorphisms $\gamma_{u_i}\colon I_i \to \mathbb{R}$, such that

- Setting the pregluing $w_\chi(s,t) = u_{\lambda,i}(\gamma_{u_i}(s),t)$ over $I_i \times S^1$, the error $\mathcal{F}(w_\chi)$ projects trivially to the longitudinal direction (i.e., the direction of w'_χ) where $w_\chi(s,\cdot)$ is close to y, and $\gamma'_{u_i} = 1$ elsewhere.
- $s_i < s_j$ if $s_i \in I_i$, $s_j \in I_j$, and $i < j$, and the closures of $I_i \times S^1$ cover the domain of w_χ, Θ.

The above condition gives an ODE which determines γ_{u_i}. Furthermore, from the ODE one may derive various behaviors of γ_{u_i}, which will be important for the estimates throughout the proof. For instance, the length of I_i is of order $|\lambda|^{-1/2}$ if u_i is not the first or last component of a broken trajectory.

1.3.2. λ-dependent *W*-norms and partitioning of Θ. In these settings, the gluing map to be constructed takes values in parameterized moduli spaces endowed with the ordinary L^p_1-topology. However, instead of the ordinary L^p_1-norms, we shall work with certain weighted norms W_χ, L_χ, because the right inverses of the deformation operator at w_χ is not bounded *uniformly* in the ordinary Sobolev norms. These weighted norms are defined similarly to the W_u-and L_u-norms in Section I.5.2, and are in some sense a combination of the W_{u_i}- or L_{u_i}-norms of the components u_i; thus, when u_i are all nondegenerate, the right inverse of the deformation operator at w_χ is expected to have a uniform bound in terms of the norms of the right inverses of the deformation operators at u_i. They are all commensurate with the usual Sobolev norms, though dependent on the gluing parameter χ.

When performing estimates, we typically partition Θ into several regions depending on whether γ'_{u_i} is close to 1 and estimate over each region separately. Over the region Θ_{u_i}, the values of γ'_{u_i} is close to 1, and hence w'_χ approximates $\partial_\gamma u(\gamma)$, the W_χ-norm approximates the W_{u_i}-norm, and the deformation operator at w_χ may be approximated by that at u_i. The length of these regions are typically of order $|\lambda|^{-1/2}$ or infinite, and the estimates over these regions are similar to those in Section I.5.

In the case considered in Sections 2–4, the other regions are Θ_{yj}. They have lengths of order $|\lambda|^{-1/2}$, and estimates over these regions often use the facts that on Θ_{yj}, $w_\chi(s,\cdot)$ is close to y (of distance $\leq C|\lambda|^{1/2}$ for some positive constant C) and that $\gamma_{u_i}(s)$ grows polynomially as positive multiples of $(|\lambda|(\mathfrak{l}-s))^{-1}$.

In the case considered in Section 5, the other regions are $\Theta_{y\pm}$. These are of infinite length, but γ'_u and hence also w'_χ decay exponentially in the form $C_\pm \exp(\mp\mu_\pm|\lambda|^{1/2}s)$, $C_\pm, \mu_\pm$ being positive constants of $O(1)$. In addition to this, we also often use the fact that over this region, $w_\chi(s,\cdot)$ is close to the new critical points $y_{\lambda\pm}$ (of distance $\leq C|\lambda|^{1/2}$ for some positive constant C) and the estimates about $y_{\lambda\pm}$ in Section I.5.3.

1.3.3. K-models. (A) *Choosing the triple* K, C, B. The deformation operators for parameterized moduli spaces are stabilizations of those for $\mathcal{M}_P$, $\mathcal{M}_O$, $\mathfrak{D}_u = E_u$ or $\tilde{D}_u$, respectively. Thus, it suffices to construct K-models for the latter. Similar to the case in Section 1.2.4, we shall always take the B-space to be W'_χ, the subspace of W_χ consisting of those ξ such that $\mathrm{res}_{\gamma_{u_i}^{-1}(0)}\,\xi$ is L^2_t-orthogonal to $\mathrm{res}_0\ \ker\, E_{u_i}\ \forall i$. The generalized kernel will be the sum of the subspaces $\gamma_{u_i}^*\ \ker\, E_{u_i} = \{\gamma_{u_i}^* f \mid f \in \ker\, E_{u_i}\}$. The generalized cokernel is trivial in the case of Section 5, but it is nontrivial in the case of Sections 2–4. In fact, by additivity of indices, its dimension is precisely the number of connecting rest points of the broken trajectory/orbit $\{\hat{u}_1, \ldots, \hat{u}_k\}$.

In this case, we choose the generalized cokernel to be spanned by $\{\mathfrak{f}_j\}$, where $\mathfrak{f}_j$ is a positive multiple of the product of the characteristic function of Θ_{yj} with a unit vector in the longitudinal direction. If one requires $\mathfrak{f}_j$ to be of unit L_χ-norm, the L^∞-norm of $\mathfrak{f}_j$ would be of order $|\lambda|^{1+1/(2p)}$. Heuristically, this choice is natural in the following sense.

(1) In this case, $\mathfrak{D}_{LL}$ is modeled on the operator

$$\frac{d}{ds}\colon L^p_1\left([\gamma_{u_1}^{-1}(0), \gamma_{u_k}^{-1}(0)]\right) \to L^p\left([\gamma_{u_1}^{-1}(0), \gamma_{u_k}^{-1}(0)]\right)$$

while $B = W'_\chi$ models on the subspace of functions vanishing at the points $\gamma_{u_i}^{-1}(0)$. Thus, $\mathfrak{D}(B)$ models on the space of functions integrating to 0 on all the intervals $[\gamma_{u_i}^{-1}(0), \gamma_{u_{i+1}}^{-1}(0)]$. A natural choice for the complementary space C is the subspace spanned by characteristic functions over these intervals.

(2) Let λ, λ' be, respectively, in a small death/birth neighborhood of 0, $\chi = (\{\hat{u}_1, \ldots, \hat{u}_k\}, \lambda)$, and $\tilde{u}_i \in \mathcal{M}_{P,\lambda'}$ be the flow line close to u_i, $\tilde{y} \in \mathcal{M}_{P,\lambda'}$ be the short flow line from $y_{\lambda'+}$ to $y_{\lambda'-}$ close to the constant flow line $\bar{y}(s) = y\ \forall s$. Let $\tilde{y}^{\mathrm{inv}}(s) := \tilde{y}(-s)$. There is a glued trajectory or orbit $w_\# = \tilde{u}_1\#\tilde{y}^{\mathrm{inv}}\#\tilde{u}_2\#\tilde{y}^{\mathrm{inv}}\#\cdots$ that approximates w_χ. Note that $\ker E_{\tilde{y}^{\mathrm{inv}}} \simeq \mathrm{coker}\ E_{\tilde{y}}$, $\mathrm{coker}\ E_{\tilde{y}^{\mathrm{inv}}} \simeq \ker\ E_{\tilde{y}}$; the former being trivial, while the latter approximates the 1-dimensional space of constant functions in the longitudinal direction (cf. Section 5.3.1). Thus, the glued K-model for $E_{w_\#}$ constructed in Example 1.2.4 also form a K-model for E_{w_χ}, in which the general cokernel is spanned by $\{*\#\cdots*\#\ker\ E_{\tilde{y}}\#*\cdots\}$, which approximates $\{\mathfrak{f}_j\}$.

(B) *Proving the isomorphism.* To verify that the above choices do give rise to a desired K-model, we need to show that the following operators are isomorphisms with uniformly bounded inverses:

- in the case of Section 5, $\mathfrak{D}_{w_\chi}|_{W'_\chi}\colon W'_\chi \to L_\chi$,
- in the case of Sections 2–4, the stabilization $\tilde{\mathfrak{D}}_{w_\chi}\colon \mathbb{R}^m \oplus W'_\chi \to L_\chi$, $\tilde{\mathfrak{D}}_{w_\chi}(\iota_1, \ldots, \iota_m, \xi) := \mathfrak{D}_{w_\chi}\xi + \sum_j \iota_j \mathfrak{f}_j$.

The general outline of the proofs follows the "proof by contradiction" framework sketched in Section 1.2.5, estimating over different regions in Θ separately according to the partition outlined in Section 1.3.2 and incoporating several extra ingredients including:

- variants of Floer's lemma (cf., e.g., Lemma 3.3.1), which gives a L^∞-bound on $|\lambda|^{-1/2}\xi$ over Θ_{yj}. This is useful for ensuring that, in spite of the potential problem with large weights, the extra term $\beta'\xi_T$ introduced by the cutoff function (as in (9)) when estimating the transversal component ξ_T is still sufficiently small. (A different method is needed for the longitudinal component, where the problem with large weights is worse.) This estimate is also useful for bounding the W_χ-norm of ξ_T over Θ_{yj}.
- estimates for ι_j and ξ_L over Θ_{yj}. In contrast to estimates over Θ_{u_i}, the estimates over Θ_{yj} differ substantially from the stereotype exemplified by the proof of Lemma 1.2.4, especially for the longitudinal direction, since $\partial_s + A_y$ is not surjective, or even Fredholm. Since $\mathfrak{D}_{LL}$ in this region is modeled on ∂_s, a basic tool of these estimates is a simple lemma (Lemma 3.3.3) bounding the L^p-norm of a real-valued function f over an interval I in terms of the $\|f'\|_{L^p(I)}$, the value of f at an end point of I, and the length of I. The latter are in turn bounded via $\|\tilde{\mathfrak{D}}_{w_\chi}(\iota_1, \ldots, \iota_m, \xi)\|_{W_\chi}$, the vanishing of ξ_L at the points $\gamma_{u_i}^{-1}(0)$, and the length estimate of Θ_{yj}.

(C) *Understanding the Kuranishi map.* As explained above, in the case of Sections 2–4, the Kuranishi model is more interesting, as the Kuranishi map is nontrivial. To understand the Kuranishi map, one needs a better description of the projection Π_C. In general, this is not easy to compute when the decomposition $C \oplus \mathfrak{D}(B)$ is not orthogonal. Fortunately, due to the special property of our $\mathfrak{D}$ and our choice of C, there is a relatively simple way of computing Π_C: very roughly speaking, modulo certain typically ignorable terms and multiplication by positive scalars, $\Pi_{\mathfrak{F}_j}$ is given by integrating the longitudinal component over the interval $[\gamma_{u_{j-1}}^{-1}(0)\gamma_{u_j}^{-1}(0)]$. (see Lemma 4.1.1 for the precise statement). Notice that this conforms with the heuristic picture sketched in item (1) of part (A) above.

1.3.4. Surjectivity of gluing map. As explained in Section 1.2.7, the main task of this step is a decay estimate for the flow line near y, which has to be particularly precise when y is degenerate, due to the polynomially weighted norms adopted. This will be done via various refinements of the decay estimate in Section I.5. Given $w = \exp(w_\chi, \xi) \in \mathcal{M}_P$ in a chain-topology neighborhood of the pregluing w_χ, we estimate the transversal and longitudinal components of ξ separately. First, reparameterize w_χ to get $\tilde{w}$, such that the difference between w and $\tilde{w}$ is transversal near y. This difference satisfies a differential equation which is used to obtain its pointwise estimate. On the other hand, comparing this parameterization with γ_{u_i}, which was used in the definition of w_χ, one may estimate the difference between the two parameterizations via an ODE, which in turn gives a pointwise bound on the difference between w_χ and $\tilde{w}$ (note that this is longitudinal). The desired bound on $\|\xi\|_{W_\chi}$ is obtained using the transversal and longitudal pointwise estimates above.

2. Gluing at deaths I: pregluing and estimates

The following three sections give a detailed proof of Proposition 2.1 below, following the outline in Section 1.

This section contains the pregluing construction, the definitions of the Banach spaces as the domain and range of the relevant deformation operator, the error estimates, and estimates for the nonlinear term. Namely, Steps 1 and 3 of the gluing construction sketched in Section 1.

2.1. Statement of the gluing theorem. The following Proposition describes the appearance of new trajectories and closed orbits near a death–birth bifurcation, by gluing broken trajectories and broken orbits at a death–birth. These trajectories all appear for λ in a death-neighborhood; for this reason, we call this a "gluing theorem at deaths," in contrast to the gluing theorems in Section 5, where the images of the gluing maps project via Π_Λ to birth-neighborhoods.

Proposition. *Let* (J^Λ, X^Λ) *be an admissible* (J, X)*-homotopy connecting two regular pairs, and* $\mathbf{x}, \mathbf{z}$ *be two path components of* $\mathcal{P}^\Lambda \backslash \mathcal{P}^{\Lambda,\mathrm{deg}}$. *Then*:

(a) *a chain-topology neighborhood of* $\mathbb{T}_{P,\mathrm{db}}(\mathbf{x}, \mathbf{z}; \Re)$ *in* $\hat{\mathcal{M}}_P^{\Lambda,1,+}(\mathbf{x}, \mathbf{z}; \mathrm{wt}_{-\langle \mathcal{Y} \rangle, e_{\mathcal{P}}} \leq \Re)$ *is l.m.b. along* $\mathbb{T}_{P,\mathrm{db}}(\mathbf{x}, \mathbf{z}; \Re)$;

(b) *a chain-topology neighborhood of* $\mathbb{T}_{O,\mathrm{db}}(\Re)$ *in* $\hat{\mathcal{M}}_O^{\Lambda,1,+}(\mathrm{wt}_{-\langle \mathcal{Y} \rangle, e_{\mathcal{P}}} \leq \Re)$ *is l.m.b. along* $\mathbb{T}_{O,\mathrm{db}}(\Re)$.

Furthermore, Π_Λ *maps these neighborhoods to death-neighborhoods.*

We shall focus on the proof of part (a), since the proof of part (b) is very similar: in fact, only the discussion in Section 4 on gluing maps needs slight modification. The necessary modification for part (b) will be briefly indicated in Section 4.3.

Recall that the admissibility of (J^Λ, X^Λ) implies that elements in $\mathcal{P}^{\Lambda,\mathrm{deg}}$ satisfy (RHFS1i), and lie in standard d-b neighborhoods, namely, satisfy the conditions described in Definition I.5.3.1. Thus, by possibly restricting to a sub-homotopy and/or reversing the orientation of Λ, we may assume without loss of generality that $\mathcal{P}^{\Lambda,\mathrm{deg}}$ contains exactly one point, y, which is a death. Namely, the constant

$$C'_y > 0$$

in Definition I.5.3.1 (2b). We may also assume without loss of generality that

$$\Pi_\Lambda y = 0.$$

We now begin the construction of a gluing map from $\Xi(\mathbb{S})$ to $\hat{\mathcal{M}}_P^{\Lambda,1}(\mathbf{x},\mathbf{z};\mathrm{wt}_{-\langle y\rangle,e_{\mathcal{P}}} \leq \Re)$, where in this case

$$\mathbb{S} = \mathbb{T}_{P,\mathrm{db}}(\mathbf{x},\mathbf{z};\Re); \quad \Xi(\mathbb{S}) = \mathbb{S} \times (0,\lambda_0) \quad \text{for a small } \lambda_0 > 0.$$

As $\mathbb{T}_{P,\mathrm{db}}(\mathbf{x},\mathbf{z};\Re)$ consists of finitely many isolated points, we may focus on a broken trajectory $\{\hat{u}_0,\dots,\hat{u}_{k+1}\}$ in $\mathbb{T}_{P,\mathrm{db}}(\mathbf{x},\mathbf{z};\Re)$. As usual, u_i will denote the centered representative of $\hat{u}_i$.

2.2. The pregluing. Let $\chi := (\{\hat{u}_0,\dots,\hat{u}_{k+1}\},\lambda) \in \mathbb{T}_{P,\mathrm{db}}(\mathbf{x},\mathbf{z};\Re)\times(0,\lambda_0)$. Choose the representatives u_i, $i=0,\dots,k+1$ such that $u_i(0)$ lies away from the neighborhood of y mentioned in Definition I.5.3.1 (2a) and (2d). Let

$$\delta_\lambda \mathcal{V} := \mathcal{V}_{X_\lambda} - \mathcal{V}_{X_0}.$$

By Definition I.5.3.1 (2a), this is given by $\check{\theta}_{X_\lambda} - \check{\theta}_{X_0}$ when $\lambda < \lambda_0$ is sufficiently small. That is, when λ_0 is so small such that J_λ is constant in λ for $\lambda \in (-\lambda_0,\lambda_0)$. We choose λ_0 so that this is the case and shall *simply write* $J_\lambda = J$ *for such* λ.

2.2.1. Lemma. *Let* $\mathfrak{l}_0 = -\infty$, $\mathfrak{l}_1 = 0$, *and* $\mathfrak{l}_{k+2} = \infty$. *Then there exist* $\mathfrak{l}_i \in \mathbb{R}$, $i = 2,\dots,k+1$, *and homeomorphisms*

$$\gamma_{u_i}\colon (\mathfrak{l}_i,\mathfrak{l}_{i+1}) \to \mathbb{R} \quad \forall i \in \{0,\dots,k+1\},$$

so that the configuration $\underline{w}_\chi \in \mathcal{B}_P(x_0,z_0)$ *defined by*

$$\underline{w}_\chi(s) := \begin{cases} u_i(\gamma_{u_i}(s)) & \textit{for } s \in (\mathfrak{l}_i,\mathfrak{l}_{i+1}),\ i=0,\dots,k+1; \\ y & \textit{for } s = \mathfrak{l}_j,\ \ j = 1,\dots,k+1 \end{cases} \tag{11}$$

satisfies

$$\begin{aligned} &\langle \underline{w}'_\chi(s), \bar{\partial}_{JX_\lambda}\underline{w}_\chi(s)\rangle_{2,t} = 0; \quad && \textit{on } \bigcup_{i=0}^{k+1} [\gamma_{u_i}^{-1}(0), \gamma_{u_{i+1}}^{-1}(0)] \\ &\gamma'_{u_i} = 1 && \textit{otherwise.} \end{aligned} \tag{12}$$

Furthermore,

$$C_0\lambda^{-1/2} \le -\gamma_{u_0}^{-1}(0) \le C_0'\lambda^{-1/2}$$
$$C_i\lambda^{-1/2} \le \mathfrak{l}_{i+1} - \mathfrak{l}_i \le C_i'\lambda^{-1/2} \quad \textit{for } i = 1, 2, \ldots, k.$$
$$C_{k+1}\lambda^{-1/2} \le \gamma_{u_{k+1}}^{-1}(0) - \mathfrak{l}_{k+1} \le C_{k+1}'\lambda^{-1/2}.$$

Notation. To avoid confusion, we write $u_\gamma = \partial_\gamma u$ and reserve $u' = u_s$ for $\partial_s u$.

γ_{u_i} will also be used to denote $\gamma_{u_i} \times \mathrm{id}\colon (\mathfrak{l}_i, \mathfrak{l}_{i+1}) \times S^1 \to \mathbb{R} \times S^1$.

Proof. We will focus on the case of $i = 0$, since the cases with other i's are similar. From the definition, $\gamma_{u_0}(s)$ satisfies

$$\frac{d}{ds}\gamma_{u_0}(s) = h_{u_0}(\gamma_{u_0}(s)),$$

where $h_{u_0}\colon \mathbb{R} \to \mathbb{R}$ is defined as:

(13)
$$h_{u_0}(\gamma) := \begin{cases} -\langle (u_0)_\gamma(\gamma), (\bar\partial_{J_\lambda X_\lambda} u_0)(\gamma)\rangle_{2,t} \|(u_0)_\gamma(\gamma)\|_{2,t}^{-2} + 1 & \text{when } \gamma \ge 0; \\ 1 & \text{when } \gamma < 0. \end{cases}$$

Our choice of the representatives u_i ensures that h_{u_i} is continuous. ¿From the decay estimates in Proposition I.5.1.3 and the fact that y is in standard d-b neighborhood, for large γ we have

$$(\bar\partial_{JX_\lambda})u_0(\gamma) = \delta_\lambda \mathcal{V}(u_0(\gamma)) = T_{y,u_0(\gamma)}\Big(\lambda C_y' \mathbf{e}_y\Big) + O(\lambda\gamma^{-1}) + O(\lambda^2).$$

On the other hand, $(u_0)_\gamma(\gamma)$ approaches the direction $-\mathbf{e}_y$ for large γ; therefore, there are λ-independent positive constants A, A', such that

$$A'\lambda\gamma^2 \ge h_{u_0}(\gamma) \ge A\lambda\gamma^2 \qquad \text{for } \gamma \gg 1. \tag{14}$$

We see that the inverse function of $\gamma_{u_0}(s)$, given by integration

$$\int_{\gamma_{u_0}}^{\infty} \frac{d\gamma}{h_{u_0}(\gamma)} = \int_{s(\gamma_{u_0})}^{0} ds' \tag{15}$$

is well defined where γ_{u_0} is large. On the other hand, h_{u_0} is always positive and goes to 1 when γ_{u_0} becomes negative; we see that $\gamma_{u_0}(s)$ defines a homeomorphism from $\mathbb{R}_-$ to $\mathbb{R}$. □

2.2.2. Definition. *The* pregluing *associated with the gluing data* χ *above is* $(\lambda, w_\chi) \in \mathcal{B}_P^\Lambda(\mathbf{x}, \mathbf{z})$, *where*

$$w_\chi := e_{R_-,R_+}(0, \underline{w}_\chi; \lambda, 0),$$

e_{R_-,R_+} *are defined in I.(62), and*

$$R_- = \gamma_{u_0}^{-1}(-C\lambda^{-1/2}); \quad R_+ = \gamma_{u_{k+1}}^{-1}(C'\lambda^{-1/2})$$

for fixed positive constants C, C'.

Remark. In general, more complicated pregluing constructions are needed if Definition I.5.3.1 (2b) is not assumed.

The following estimates for $R_\pm$ in terms of λ will be very useful.

2.2.3. Lemma. *$C'_\pm\lambda^{-1/2} \leq R_\pm \leq C_\pm\lambda^{-1/2}$ for some λ-independent positive constants $C_\pm$, $C'_\pm$.*

Proof. We shall only demonstrate the inequalities about R_-, since those for R_+ are similar.

Choose a large enough γ_0 such that when $s \geq \gamma_0$, the decay estimate in Proposition I.5.1.3 for $u_0(s)$ and $u'_0(s)$ holds, and

$$\|\delta_\lambda \mathcal{V}(u_0(s)) - T_{y,u_0(s)}(\lambda \mathbf{e}_y)\|_{2,t} \leq C_{\gamma_0}\lambda\|u_0(s)\|_{2,t}.$$

This implies that when $\gamma \geq \gamma_0\lambda^{-1/2}$, $A'\lambda\gamma^2 \geq h_{u_0}(\gamma) \geq A\lambda\gamma^2 \gg 1$ for some λ-independent constants A, A'. Thus

$$C_2\lambda^{-1/2} \leq -\gamma_{u_0}^{-1}(\gamma_0\lambda^{-1/2}) \leq \int_{\gamma_0\lambda^{-1/2}}^{\infty} \frac{d\gamma}{A\lambda\gamma^2} = C_1\lambda^{-1/2}. \tag{16}$$

On the other hand, $\frac{d\gamma_{u_0}(s)}{ds} = h_{u_0}(\gamma_{u_0}(s)) \geq 1$ always, so

$$\gamma_{u_0}^{-1}(\gamma_0\lambda^{-1/2}) - \gamma_{u_0}^{-1}(-C\lambda^{-1/2}) \leq C_2\lambda^{-1/2}.$$

Combining the above two inequalities, we get the claimed inequality for R_-. □

2.3. The weighted norms. Define the *weight function* $\sigma_\chi\colon \mathbb{R} \to \mathbb{R}^+$ by

$$\sigma_\chi(s) := \begin{cases} \|w'_\chi(s)\|_{2,t}^{-1} & \text{when } \gamma_{u_0}^{-1}(0) \leq s \leq \gamma_{u_{k+1}}^{-1}(0); \\ \|w'_\chi(\gamma_{u_0}^{-1}(0))\|_{2,t}^{-1} & \text{when } s \leq \gamma_{u_0}^{-1}(0); \\ \|w'_\chi(\gamma_{u_{k+1}}^{-1}(0))\|_{2,t}^{-1} & \text{when } s \geq \gamma_{u_{k+1}}^{-1}(0). \end{cases} \tag{17}$$

Let $\xi \in \Gamma(w_\chi^* K)$, define its "*longitudinal*" component as

$$\xi_L(s) := \beta(s - \gamma_{u_0}^{-1}(0))\beta(\gamma_{u_{k+1}}^{-1}(0) - s)\sigma_\chi(s)^2\langle w'_\chi(s), \xi(s)\rangle_{2,t} w'_\chi(s),$$

where $\beta\colon \mathbb{R} \to [0,1]$ is the smooth cutoff function supported on $\mathbb{R}^+$ such that $\beta(s) = 1\ \forall s \geq 1$ (cf. I.3.2.3).

The norms for the domain and range of E_{w_χ} are defined as follows.

2.3.1. Definition. *For $\xi \in \Gamma(w_\chi^* K)$,*

$$\|\xi\|_{L_\chi} := \|\sigma_\chi^{1/2}\xi\|_p + \|\sigma_\chi\xi_L\|_p;$$
$$\|\xi\|_{W_\chi} := \|\sigma_\chi^{1/2}\xi\|_{p,1} + \|\sigma_\chi\xi'_L\|_p.$$

As usual, we also use W_χ, L_χ to denote the Banach spaces which are C^∞-completion with respect to these norms.

We shall extend the norm W_χ to a norm on $T_{(\lambda,w_\chi)}B_P^\Lambda(\mathbf{x},\mathbf{z}) = \mathbb{R} \oplus W_\chi$ in a way such that $\hat{E}_{(\lambda,w_\chi)}$ is uniformly bounded (cf. Lemma 2.5.1).

2.3.2. Definition. *Define the following norm on* $\hat{W}_\chi := \mathbb{R} \oplus W_\chi$ *(denoted by the same notation):*

$$\|(\alpha, \xi)\|_{\hat{W}_\chi} := \|\xi\|_{W_\chi} + \lambda^{-1/(2p)-1}|\alpha|.$$

2.4. The error estimate. The main goal of this subsection is to obtain the following estimate.

2.4.1. Proposition. *In the notation of Sections 2.2 and 2.3*,

$$\|\bar{\partial}_{J_\lambda X_\lambda} w_\chi\|_{L_\chi} \le C\lambda^{1/2-1/(2p)}.$$

Proof. By direct computation, $\bar{\partial}_{J_\lambda X_\lambda} w_\chi$ is supported on $(-R_- - 1, R_+ + 1) \times S^1$, on which it is given by

$$T_{\underline{w}_\chi, w_\chi}\Big(\tilde{\Pi}^\perp_{\underline{w}'_\chi} \delta_\lambda \mathcal{V}(\underline{w}_\chi)\Big) + r_\lambda(x, z), \quad \text{where} \tag{18}$$

- $T_{\underline{w}_\chi, w_\chi}$ is as in Notation I.5.2.6;
- $r_\lambda(x, z)$ is a "remainder term" supported on
$$(-R_- - 1, R_+ + 1) \backslash (-R_-, R_+) \times S^1$$
which consists of terms involving $\beta(-R_- - s)\bar{x}_\lambda^{0,\underline{w}_\chi}$, $\beta(s - R_+)\bar{z}_\lambda^{0,\underline{w}_\chi}$ and their derivatives (cf. I.(62) for notation);
- letting $\Pi^\perp_{\underline{w}'_\chi(s)}$ denote the L^2_t-orthogonal projection to the orthogonal complement of $\mathbb{R}w'_\chi(s)$,
$$\tilde{\Pi}^\perp_{\underline{w}'_\chi(s)} = \begin{cases} \Pi^\perp_{\underline{w}'_\chi(s)} & \text{for } s \in [\gamma_{u_0}^{-1}(0), \gamma_{u_{k+1}}^{-1}(0)]; \\ \mathrm{Id} & \text{otherwise.} \end{cases}$$

To estimate the terms in (18), note:

2.4.2. Lemma. *When* $-R_- \le s \le R_+$, *there is a constant* C *independent of* λ *and* s, *such that*

$$\|\sigma_\chi(s)\tilde{\Pi}^\perp_{w'_\chi}\delta_\lambda\mathcal{V}(w_\chi(s))\|_{\infty,t} \le C\lambda^{1/2} \quad \forall \text{ sufficiently small } \lambda.$$

Combining this lemma with Lemma 2.2.3, one may bound the contribution to $\|\bar{\partial}_{J_\lambda X_\lambda} w_\chi\|_{L_\chi}$ from the first term in (18) by $C_1\lambda^{1/2-1/(2p)}$.

The contribution from the second term can be bounded by $C\lambda$, using the C^2 bound on J and X, and the following estimates:

$$\sum_{k=0}^{1} \sup_{s\in[-R_- -1, -R_-]} \Big\|\partial_s^k \bar{x}_\lambda^{0,\underline{w}_\chi}(s)\Big\|_{2,1,t} \le C_2\zeta_\lambda^x + C_3 \sum_{k=0}^{1} \sup_{s\in[-R_- -1, -R_-]} \|\partial_s^k \mu(s)\|_{2,1,t}$$

$$\leq C_2'\lambda + C_3' \mathrm{e}^{-C_4\lambda^{1/2}}$$
$$\leq C_x\lambda,$$

where $\mu(s), \zeta_\lambda^x$ are defined by $\exp(\underline{w}_\chi(s), \mu(s)) = x_0$, $\exp(x_0, x_\lambda) = \zeta_\lambda^x$, and the second inequality follows from the exponential decay of $\underline{w}_\chi$ to x_0, and the estimate for R_- in Lemma 2.2.3. Similarly,

$$\sum_{k=0}^{1} \sup_{s\in[R_+,R_++1]} \left\| \partial_s^k \bar{z}_\lambda^{0,\underline{w}_\chi}(s) \right\|_{2,1,t} \leq C_z\lambda.$$

These together imply the proposition. □

Proof of Lemma 2.4.2. By Sobolev embedding, it suffices to estimate the $L^2_{1,t}$ norm. Again we will estimate only the $s \leq 0$ part, since the other parts are entirely similar. Let γ_0 be as in Lemma 2.2.3. Consider the following two cases separately. Case 1: $-R_- \leq s \leq \gamma_{u_0}^{-1}(\gamma_0)$; Case 2: $\gamma_{u_0}^{-1}(\gamma_0) \leq s \leq 0$.
Case 1: In this region, $\|\tilde{\Pi}^\perp_{w'_\chi}(\delta_\lambda \mathcal{V}(w_\chi(s))\|_{2,1,t} \leq C\lambda$. On the other hand, on this region, $\sigma_\chi \leq C$; in sum, we have $\|\sigma_\chi \tilde{\Pi}^\perp_{w'_\chi}(\delta_\lambda \mathcal{V}(w_\chi(s))\|_{2,1,t} \leq C_3\lambda$.
Case 2: In this region, the fact that y is in a standard d-b neighborhood plus the decay estimates in Proposition I.5.1.3 imply that for small enough λ,

$$\begin{aligned}
\|\Pi^\perp_{(u_0)_\gamma}\Big(\delta_\lambda \mathcal{V}(u_0(\gamma_{u_0})\Big)\|_{2,t} &\leq \lambda C_0(\|\Pi^\perp_{(u_0)_\gamma} T_{y,u_0(\gamma_{u_0})}\mathbf{e}_y\|_{2,t} + \|\mu(\gamma_{u_0})\|_{2,t}) \\
&\leq \lambda((1-(1+C_1^2\|b(\gamma_{u_0})\|_{2,t}^2)^{-1})^{1/2} + C''\gamma_{u_0}^{-1} \\
&\leq \lambda(C_1\|b(\gamma_{u_0})\|_{2,t} + C''\gamma_{u_0}^{-1}) \\
&\leq C_4\lambda\gamma_{u_0}^{-1},
\end{aligned}$$

where μ, b are defined by $\exp(y, \mu(\gamma)) = u_0(\gamma)$, $b(\gamma) = \Pi_{\mathbf{e}_y}\mu(\gamma)$, as in Section I.5. Meanwhile,

$$\begin{aligned}
&\left\|\partial_t[\Pi^\perp_{(u_0)_\gamma}\delta_\lambda \mathcal{V}(u_0(\gamma_{u_0}))]\right\|_{2,t} \\
&\quad\leq \|\partial_t[\delta_\lambda \mathcal{V}(u_0(\gamma_{u_0}))]\|_{2,t} + \left\|\partial_t(\Pi_{u_{\gamma_{u_0}}})(\delta_\lambda \mathcal{V}(u_0(\gamma_{u_0})))\right\|_{2,t} \\
&\quad\leq \lambda C_5\Big(\|\partial_t(u_0(\gamma_{u_0}))\|_{2,t} + \|\sigma_u \partial_t(u_0)_\gamma(\gamma_{u_0})\|_{2,t}\Big) \\
&\quad\leq C_6\lambda\gamma_{u_0}^{-1}.
\end{aligned}$$

On the other hand, we have from direct computation:

$$\begin{aligned}
\sigma_\chi^{-1}(s) &= \|w'_\chi(s)\|_{2,t} \\
&= \Pi_{(u_0)_\gamma(\gamma_{u_0}(s))}\Big(\delta_\lambda \mathcal{V}(u_0(\gamma_{u_0}(s)))\Big) + \|(u_0)_\gamma(\gamma_{u_0}(s))\|_{2,t}
\end{aligned} \tag{19}$$

when $\gamma_{u_0}^{-1}(0) \leq s \leq 0$. In particular,

$$C'(\lambda + \gamma_{u_0}(s)^{-2}) \geq \|w'_\chi(s)\|_{2,t} \geq C_7(\lambda + \gamma_{u_0}(s)^{-2}) \tag{20}$$

when $0 \geq s \geq \gamma_{u_0}^{-1}(\gamma_0)$.

In sum, in Case 2

$$\sigma_\chi(s)\Big\|\Pi^\perp_{w'_\chi}\delta_\lambda\mathcal{V}(w_\chi(s))\Big\|_{2,1,t} \leq \frac{C_8\lambda\gamma_{u_0}(s)^{-1}}{\lambda+\gamma_{u_0}(s)^{-2}} \leq C_9\lambda^{1/2}.$$

The last step above is obtained by a simple estimate of the critical value of the rational function. Combining the two cases, we have proved the lemma. □

2.5. Bounding linear and nonlinear terms. In the previous subsection, we obtained the estimate for the 0-th-order term of the expansion (1). We estimate the linear and nonlinear terms in this subsection. In our context, this means bounding $\hat{E}_{(\lambda,w_\chi)}$ and $\hat{n}_{(\lambda,w_\chi)}$. These are done, respectively, in Lemmas 2.5.1 and 2.5.2 below.

2.5.1. Lemma. *With respect to the norms $\hat{W}_\chi, L_\chi$ of Section 2.3, the deformation operator $\hat{E}_{(\lambda,w_\chi)}$ is bounded uniformly in λ.*

Proof. The uniform boundedness of E_{w_χ} follows from simple adaptation of I.5.2.3. We therefore just have to estimate the L_χ norm of

$$\hat{E}_{(\lambda,w_\chi)}(1,0) = Y_{(\lambda,w_\chi)},$$

where $Y_{(\lambda,w_\chi)}$ is as in I.(63). By the properties of $Y_{(\lambda,w_\chi)}$ listed following I.(63), $Y_{(\lambda,w_\chi)}$ is supported on $(-R_- - 1, R_+ + 1)\times S^1$, over which it has a λ-independent C^∞_ϵ bound. Also, from (19), we have $\sigma_\chi \leq C\lambda^{-1}$. These, together with Lemma 2.2.3, imply

$$\|\hat{E}_{(\lambda,w_\chi)}(1,0)\|_{L_\chi} \leq C'\lambda^{-1-1/(2p)} = C'\|(1,0)\|_{\hat{W}_\chi}.$$

for a λ-independent positive constant C'. □

Given $(\alpha,\xi)\in T_{(\lambda,w_\chi)}\mathcal{B}^\Lambda_P(\mathbf{x},\mathbf{z})$, let $\hat{n}_{(\lambda,w_\chi)}(\alpha,\xi)$ be

$$T^{-1}_{w_\chi,e(\lambda,w_\chi;\alpha,\xi)}\bar{\partial}_{J^\Lambda,X^\Lambda}(\lambda+\alpha,e(\lambda,w_\chi;\alpha,\xi)) - \bar{\partial}_{J_\lambda X_\lambda}w_\chi - \hat{E}_{(\lambda,w_\chi)}(\alpha,\xi).$$

2.5.2. Lemma. *There is a λ-independent constant C_n such that for any $\hat{\xi}=(\alpha,\xi), \hat{\eta}=(\alpha',\eta)\in\hat{W}_\chi$,*

$$\|\hat{n}_{(\lambda,w_\chi)}(\hat{\xi}) - \hat{n}_{(\lambda,w_\chi)}(\hat{\eta})\|_{L_\chi} \leq C_n(\|\hat{\xi}\|_{\hat{W}_\chi} + \|\hat{\eta}\|_{\hat{W}_\chi})\|\hat{\xi}-\hat{\eta}\|_{\hat{W}_\chi}.$$

Proof. These follow from direct computations, via the C^∞_ϵ-bounds of J, X. First, observe the pointwise estimate

$$\begin{aligned}&|\hat{n}_{(\lambda,w_\chi)}(\hat{\xi}) - \hat{n}_{(\lambda,w_\chi)}(\hat{\eta})|\\ &\quad\leq C_1(|\xi|+|\eta|)(|\xi-\eta|+|\nabla(\xi-\eta)|) + C_2(|\nabla\xi|+|\nabla\eta|)|\xi-\eta|\\ &\qquad+(|\alpha|+|\alpha'|)(|\alpha-\alpha'|)|Z_{\lambda\lambda}| + \Big((|\alpha|+|\alpha'|)|\xi-\eta| + |\alpha-\alpha'|(|\xi|+|\eta|)\Big)|Z_{\lambda w}|,\end{aligned}$$

where $Z_{\lambda\lambda}, Z_{\lambda w}$ are both supported on $(-R_- - 1, R_+ + 1)\times S^1$, over which they are $\partial_\lambda^2\check\theta_{X_\lambda}(w_\chi)$, $\partial_\lambda\nabla\check\theta_{X_\lambda}(w_\chi)/2$, respectively, up to ignorable terms. Estimating similarly to the proof of Lemma 2.5.1, we have

$$\|\sigma_\chi Z_{\lambda\lambda}\|_{p,1} + \|\sigma_\chi Z_{\lambda w}\|_{p,1} \leq C'\lambda^{-1-1/(2p)}.$$

Thus

$$\begin{aligned}
&\|\hat n_{(\lambda,w_\chi)}(\hat\xi) - \hat n_{(\lambda,w_\chi)}(\hat\eta)\|_{L_\chi} \leq \Big\|\sigma_\chi\Big(\hat n_{(\lambda,w_\chi)}(\hat\xi) - \hat n_{(\lambda,w_\chi)}(\hat\eta)\Big)\Big\|_{p,1}\\
&\quad\leq C_1(\|\sigma_\chi^{1/2}\xi\|_\infty + \|\sigma_\chi^{1/2}\eta\|_\infty)\|\xi-\eta\|_{W_\chi}\\
&\qquad + C_2(\|\xi\|_{W_\chi} + \|\eta\|_{W_\chi})\|\sigma_\chi^{1/2}(\xi-\eta)\|_\infty\\
&\qquad + C'\lambda^{-1-1/(2p)}(|\alpha|+|\alpha'|)(|\alpha-\alpha'|)\\
&\qquad + C'\lambda^{-1-1/(2p)}\Big((|\alpha|+|\alpha'|)\|\xi-\eta\|_\infty + |\alpha-\alpha'|(\|\xi\|_\infty+\|\eta\|_\infty)\Big)\\
&\quad\leq C_n(\|\hat\xi\|_{\hat W_\chi} + \|\hat\eta\|_{\hat W_\chi})\|\hat\xi-\hat\eta\|_{\hat W_\chi},
\end{aligned}$$

using a Sobolev inequality to bound the L^∞-norm by L^p_1-norm. □

3. Gluing at Deaths II: the Kuranishi structure

The purpose of this section is to introduce a K-model for the operator E_{w_χ}. By stabilization, this also yields a K-model for $\hat E_{(\lambda,w_\chi)}$. The main result is summarized in Proposition 3.1.3.

3.1. The generalized kernel and generalized cokernel. Given $y\in\mathcal{P}$, we denote by $\bar y$ the constant flow $\bar y(s) = y\ \forall s$.

We first partition the domain $\Theta = \mathbb{R}\times S^1$ into several regions, over which w_χ approximates either one of u_i or $\bar y$.

For a subdomain $\Theta'\subset\Theta$ and some norm L, we denote by $\|\xi\|_{L(\Theta')} := \|\xi\big|_{\Theta'}\|_L$.

3.1.1. Definition (Partioning Θ). Fix a small positive number $\epsilon > \lambda$. For $i = 0,\dots,k+1$, let

$$\mathfrak{r}_i := (2C_{u_i}\epsilon)^{1/2}(\lambda C'_y)^{-1/2},$$

where C_{u_i} is the constant in the bound $\|u_i'(s)\|_2 \geq C_{u_i}/s^2$ (cf. I.5.1.3). For $j = 1,\dots,k+1$, define

$$\begin{aligned}
\Theta_{yj} :=&[\mathfrak{s}_{j-},\mathfrak{s}_{j+}]\times S^1, \quad \textit{where}\\
&\mathfrak{s}_{j-} := \gamma^{-1}_{u_{j-1}}(\mathfrak{r}_{j-1}), \quad \mathfrak{s}_{j+} = \gamma^{-1}_{u_j}(-\mathfrak{r}_j).
\end{aligned}$$

Let Θ_{u_i} denote the $(i+1)$th component of $\Theta\backslash\bigcup_j\Theta_{yj}$, and let $\Theta'_{yj} = (\gamma^{-1}_{u_{j-1}}(\mathfrak{r}_{j-1}-1), \gamma^{-1}_{u_j}(-\mathfrak{r}_j+1))\times S^1 \supset \Theta_{yj}$.

Notice that the "length" of the region Θ_{yj}, $\mathfrak{s}_{j+} - \mathfrak{s}_{j-}$, is bounded as

$$C_1(\epsilon\lambda)^{-1/2} \leq \mathfrak{s}_{j+} - \mathfrak{s}_{j-} \leq C_2(\epsilon\lambda)^{-1/2}. \tag{21}$$

These inequalities follow from the arguments leading to (16), using, respectively, inequalities of the type of the left and the right inequalities in (14). The length of Θ'_{yj} satisfies similar bounds, with the constants C_1, C_2 above replaced by different constants C'_1, C'_2.

3.1.2. Definition (Bases for generalized kernel/cokernel). For $i = 1, \ldots, k+1$, let

$$\mathfrak{e}_{u_i} := \gamma_{u_i}^* u'_i \in W_\chi.$$

For $j = 1, \ldots, k+1$, define the following elements in L_χ:

$$\mathfrak{f}_j := C_j|\lambda|^{1+1/(2p)}\vartheta_{\Theta_{yj}} w'_\chi \|w_\chi\|_{2,t}^{-1},$$

where $\vartheta_{\Theta_{yj}}$ is a characteristic function supported on Θ_{yj} and C_j are constants chosen such that $\|\mathfrak{f}_j\|_{L_\chi} = 1$.

Let

$K_\chi := \mathrm{Span}\{\mathfrak{e}_{u_i}\}_{i\in\{0,\ldots,k+1\}} \subset W_\chi$;

$\hat{K}_\chi := \mathrm{Span}\{(1,0), (0, \mathfrak{e}_{u_i})\}_{i\in\{0,\ldots,k+1\}} \subset \hat{W}_\chi$;

$C_\chi := \mathrm{Span}\{\mathfrak{f}_j\}$, and

$$W'_\chi := \Big\{\xi \mid \langle(\gamma_{u_i}^{-1})^*\xi(0), \eta(0)\rangle_{2,t} = 0\, \forall \eta \in \ker E_{u_i}\, \forall i \in \{0, 1, \ldots, k+1\}\Big\} \subset W_\chi.$$

(Note that linearly independent elements in $\ker E_{u_i}$ restrict linearly independently to the circle $s = 0$, since they satisfy a homogeneous first-order differential equation.)

A quick computation shows that the W_χ-norm on K_χ and the L_χ-norm on C_χ are commensurate with the standard norm on Euclidean spaces with respect to the bases given above.

These are, respectively, fibers of Banach spaces bundles over the space of gluing parameters $\Xi(\mathbb{S})$, $K^\Xi, \hat{K}^\Xi, C^\Xi, W'^\Xi$, and $\hat{W}'^\Xi$.

Obviously, $W_\chi = K_\chi \oplus W'_\chi$ and $\hat{W}_\chi = \hat{K}_\chi \oplus W'_\chi$. Let $\tilde{W}_\chi := \mathbb{R}^{k+1} \oplus W'_\chi$, with the standard metric on $\mathbb{R}^{k+1}$. (As usual, we denote the norm on it by the same notation.) Let

$$\tilde{E}_\chi\colon \tilde{W}_\chi \to L_\chi, \quad \tilde{E}_\chi(\iota_1, \ldots, \iota_{k+1}, \xi) := E_{w_\chi}\xi + \sum_{j=1}^{k+1} \iota_j \mathfrak{f}_j.$$

A quick computation using (21) shows that this is a bounded operator. The rest of this section is devoted to proving the following.

3.1.3. Proposition. *For sufficiently small* λ_0*, the triples* $K^{\Xi}, C^{\Xi}, W'^{\Xi}$*, and* $\hat{K}^{\Xi}, C^{\Xi}$*,* W'^{Ξ} *are, respectively, K-models for the families of operators* $\{E_{w_\chi}\}_{\chi\in\Xi(\mathbb{S})}$ *and* $\{\hat{E}_{(\lambda,w_\chi)}\}_{\chi\in\Xi(\mathbb{S})}$*.*

In particular, there is an inverse $\tilde{G}_\chi\colon L_\chi \to \tilde{W}_\chi$ *of* $\tilde{E}_\chi$*, which is bounded uniformly in* λ*.*

We shall concentrate on proving the existence of a uniformly bounded $\tilde{G}_\chi$, since the rest of the assertions follow in a straightforward manner. The proof follows the "proof by contradiction" framework outlined in Section 1.2.3; since $\tilde{E}_\chi$ is Fredholm with ind $\tilde{E}_\chi = 0$, it suffices to show that there exists a λ-independent constant C, such that $\|\tilde{\xi}\|_{L_\chi} \leq C\|\tilde{E}_\chi\tilde{\xi}\|_{\tilde{W}_\chi}$ $\forall\tilde{\xi}\in\tilde{W}_\chi$.

Suppose the contrary that there exists a sequence

$$\{\tilde{\xi}_\lambda = (\iota_{1,\lambda},\dots,\iota_{k+1,\lambda},\xi_\lambda)\in\tilde{W}_\chi\}, \quad \text{with } \|\tilde{\xi}_\lambda\|_{\hat{W}_\chi} = 1 \quad \text{and}$$

$$\|\hat{E}_{(\lambda,w_\chi)}(\tilde{\xi}_\lambda)\|_{L_\chi} =: \varepsilon_E(\lambda) \to 0 \quad \text{where } \lambda\to 0. \tag{22}$$

We shall estimate $\tilde{\xi}_\lambda$ in terms of $\varepsilon_E(\lambda)$ over the various domains introduced in Definition 3.1.1 to obtain a contradiction.

3.2. Estimates over Θ_{u_i}. Given a diffeomorphism $\gamma\colon I\to\mathbb{R}$, let

$$\mathcal{T}^{\gamma}_{w,\underline{w}}\colon = (\gamma^{-1})^* T_{w,\underline{w}}\colon \Gamma(\Theta, w^*K) \to \Gamma(\gamma(\Theta), (\gamma^{-1})^*\underline{w}^*K).$$

By construction, for $\xi\in\Gamma(w_\chi^*K)$, $\mathcal{T}^{\gamma_{u_i}}_{w_\chi,\underline{w}_\chi}\xi \in \Gamma(u_i^*K)$.

Since the discussion in this subsection holds for all i, we shall often drop the index i. For instance, $u = u_i$ for some i.

In this subsection, the "transversal" or "longitudinal components" shall refer to the respective components of elements in $\Gamma(u^*K)$.

3.2.1. Comparing W_χ, L_χ-norms and W_u, L_u-norms. According to computation in the proof of Lemma 2.2.1 and the definition of Θ_u, in this region $|h_u(\gamma_u) - 1| \leq \epsilon$. Thus, σ_u and σ_χ are close in this region, and by direct computation we have the following lemma.

Lemma. *Suppose* $\xi\in\Gamma(w_\chi^*K)$ *is supported on* Θ_u*, and let* ϵ,λ *be as in Definition* 3.1.1*. Then*

$$(1-2\epsilon)\|\xi\|_{W_\chi} \leq \|\mathcal{T}^{\gamma_u}_{w_\chi,\underline{w}_\chi}\xi\|_{W_u} \leq (1+2\epsilon)\|\xi\|_{W_\chi};$$
$$(1-2\epsilon)\|\xi\|_{L_\chi} \leq \|\mathcal{T}^{\gamma_u}_{w_\chi,\underline{w}_\chi}\xi\|_{L_u} \leq (1+2\epsilon)\|\xi\|_{L_\chi},$$

and for some constant C*,*

$$\left\|\hat{E}_{(\lambda,w_\chi)}(\alpha,\xi)\right\|_{L_\chi} \geq (1+2\epsilon)^{-1}\left\|E_u(\mathcal{T}^{\gamma_u}_{w_\chi,\underline{w}_\chi}\xi) + \alpha\mathcal{T}^{\gamma_u}_{w_\chi,\underline{w}_\chi}Y_{(\lambda,w_\chi)}\right\|_{L_u} - (C\lambda+\epsilon)\,\|\xi\|_{W_\chi}.$$

Remark. The fact that y is in a standard d-b neighborhood (more precisely, the condition Definition I.5.3.1(2c)) is used here. In general, the last term on the right-hand side of the above inequality would be larger.

3.2.2. From $\xi_\lambda \in W'_\chi$ to $\bar\xi_{u_i,\lambda} \in W'_{u_i}$. For $i = 1, \dots, k$, and $\xi_\lambda \in W'_\chi$, let

$$\bar\xi_{u_i,\lambda} := \begin{cases} (\gamma_{u_i}^{-1})^*\xi_\lambda - \beta_i((\gamma_{u_i} g^{-1})^*\xi_\lambda)_T \\ \qquad -\theta_{i+}((\gamma_{u_i}^{-1})^*\xi_\lambda)_L - c_{i+}u'_i) - \theta_{i-}((\gamma_{u_i}^{-1})^*\xi_\lambda)_L - c_{i-}u'_i) \\ \qquad\qquad \text{on } (-\mathfrak{r}_i, \mathfrak{r}_i) \times S^1 \\ 0 \qquad\qquad \text{outside.} \end{cases} \tag{23}$$

where

- β_i is a smooth cutoff function in s supported away from $(-\mathfrak{r}_i+1, \mathfrak{r}_i-1)$, being 1 outside $(-\mathfrak{r}_i, \mathfrak{r}_i)$.
- $\theta_{i\pm}$ are characteristic functions of $(-\infty - \mathfrak{s}_i)$ and $(\mathfrak{s}_i, \infty)$, respectively.
- $c_{i\pm}$ are constants defined by

$$((\gamma_{u_i}^{-1})^*\xi_\lambda)_L(\pm\mathfrak{r}_i) = c_{i\pm}u'(\pm\mathfrak{r}_i). \tag{24}$$

For $i = 0$ or $k+1$ and similarly defined constants c_0, c_{k+1}, let

$$\bar\xi_{u_0,\lambda} := \begin{cases} \mathcal{T}^{\gamma_{u_0}}_{w_\chi,\underline{w}_\chi}\xi_\lambda - \beta(s - \mathfrak{r}_0 + 1)(\mathcal{T}^{\gamma_{u_0}}_{w_\chi,\underline{w}_\chi}\xi_\lambda)_T \\ \qquad - \theta(s - \mathfrak{r}_0)((\mathcal{T}^{\gamma_{u_0}}_{w_\chi,\underline{w}_\chi}\xi_\lambda)_L - c_0 u'_0) & \text{on } (-\infty, \mathfrak{r}_0) \times S^1 \\ 0 & \text{outside;} \end{cases}$$

$$\bar\xi_{u_{k+1},\lambda} := \begin{cases} \mathcal{T}^{\gamma_{u_{k+1}}}_{w_\chi,\underline{w}_\chi}\xi_\lambda - \beta(\mathfrak{r}_{k+1} - 1 - s)(\mathcal{T}^{\gamma_{u_{k+1}}}_{w_\chi,\underline{w}_\chi}\xi_\lambda)_T \\ \qquad -\theta(\mathfrak{r}_{k+1} - s)((\mathcal{T}^{\gamma_{u_{k+1}}}_{w_\chi,\underline{w}_\chi}\xi_\lambda)_L - c_{k+1}u'_{k+1}) \\ \qquad\qquad \text{on } (\mathfrak{r}_{k+1}, \infty) \times S^1 \\ 0 \qquad\qquad \text{outside,} \end{cases}$$

where β is the smooth cutoff function as in Part I and Section 1.2.2, θ is the characteristic function of $\mathbb{R}^+$.

Remark. The point of the above definition is to introduce cutoff on ξ_λ, while keeping the extra terms (arising from the cutoff) in $E_u(\bar\xi_\lambda)$ ignorable. The usual smooth cutoff works for the transversal direction, but not for the longitudinal component, over which the weight function is greater. Instead, we replace the longitudinal component over the cutoff region by a suitable multiple of u' determined by the matching condition (24) and make use of the fact that $E_u(u') = 0$.

3.2.3. Estimating $E_u\bar{\xi}_{u,\lambda}$. The estimate for all i is similar. Taking $i=0$, for example, a straightforward computation yields:

$$\begin{aligned} E_{u_0}(\bar{\xi}_{u_0,\lambda}) =&(1-\beta(s-\mathfrak{r}_0+1))E_{u_0}(\mathcal{T}^{\gamma_{u_0}}_{w_\chi,\underline{w}_\chi}\xi_{\lambda T}) \\ &+(1-\theta(s-\mathfrak{r}_0))E_{u_0}(\mathcal{T}^{\gamma_{u_0}}_{w_\chi,\underline{w}_\chi}\xi_{\lambda L}(\mathfrak{r}_0)) \\ &-\beta'(s-\mathfrak{r}_0+1)\mathcal{T}^{\gamma_{u_0}}_{w_\chi,\underline{w}_\chi}\xi_{\lambda T} \\ &-\delta(s-\mathfrak{r}_0)\Big(\mathcal{T}^{\gamma_{u_0}}_{w_\chi,\underline{w}_\chi}\xi_{\lambda L}-c_0u_0'\Big)\Big). \end{aligned}$$

The last term above has vanishing L_u-norm because of the condition (24); by Lemma 3.3.1, the L_u-norm of the penultimate term can be bounded by $C\varepsilon_0(\lambda)$, which goes to 0 as $\lambda\to 0$. Thus, by the previous lemma, we have for small λ that

$$\begin{aligned} &\|E_u\bar{\xi}_{u,\lambda}\|_{L_u(\gamma_u(\Theta_u))} \\ &\le \|E_u(\mathcal{T}^{\gamma_u}_{w_\chi,\underline{w}_\chi}\xi_\lambda)\|_{L_u(\gamma_u(\Theta_u))}+C\varepsilon_0(\lambda) \\ &\le (1+2\epsilon)\|E_{w_\chi}\xi_\lambda\|_{L_\chi(\Theta_u)}+(C'\lambda+\epsilon)\|\xi_\lambda\|_{W_\chi(\Theta_u)}+C\varepsilon_0(\lambda). \end{aligned} \tag{25}$$

In the last expression, the first term goes to zero because of (22) and the fact that over Θ_u, $\tilde{E}_\chi\tilde{\xi}_\lambda=E_{w_\chi}\xi_\lambda$. The second term is small since $\|\xi_\lambda\|_{W_\chi}\le 1$.

3.2.4. Estimating $\bar{\xi}_{u_i,\lambda}$. Since $\bar{\xi}_{u,\lambda}\in W_u'$, where

$$W_u' := \Big\{\xi|\,\xi\in W_u,\ \langle u'(0),\xi(0)\rangle_{2,t}=0\Big\},$$

by the right-invertibility of E_u, $\|\bar{\xi}_{u,\lambda}\|_{W_u}\le C\|E_u\bar{\xi}_{u,\lambda}\|_{L_u}\le\varepsilon$. In particular,

$$\|\mathcal{T}^{\gamma_u}_{w_\chi,\underline{w}_\chi}\xi_\lambda\|_{W_u(\gamma_u(\Theta_u))}\le\varepsilon_u \quad \text{when } \lambda\le\lambda_0 \text{ is sufficiently small,} \tag{26}$$

where ε_u is of the form

$$\varepsilon_u = C\varepsilon_0(\lambda)+C_2(\lambda+\epsilon)+2\varepsilon_E(\lambda),$$

which can be made arbitrarily small by choosing the small constants ϵ,λ appropriately.

3.3. Estimates over Θ_{yj}. The estimates over Θ_{yj} for different j are similar, so we shall drop the subscript j in the discussion below.

First, note that from the computation of (20) there exist constants C_M, C_m independent of λ such that

$$C_m\lambda^{-1}\le\sigma_\chi(s)\le C_M\lambda^{-1} \quad \text{on } \Theta_y. \tag{27}$$

We may therefore replace (modulo multiplication by a constant) the weights in the W_χ and L_χ norms by λ^{-1}.

3.3.1. Estimating the transversal component. In the transversal direction, the estimates are again similar to the standard case: By looking at the limit of $(\alpha_\lambda, \xi_\lambda)$, one has:

Lemma (Floer). *Let $(\alpha_\lambda, \xi_\lambda)$ be as in (22). Then for all sufficiently small λ,*

$$\|\xi_\lambda\|_{L^\infty(\Theta'_y)} \le \varepsilon_0(\lambda)\lambda^{1/2}$$

where $\varepsilon_0(\lambda)$ is a small positive number, $\lim_{\lambda\to 0}\varepsilon_0(\lambda) = 0$.

Proof. Let (s_λ, t_λ) be a maximum of $|\xi_\lambda|$ in Θ'_y. Consider a slight enlargement of Θ'_y, $\Theta''_y \supset \Theta'_y$, and let $C > 0$ be such that

$$[-C^{-1}(\epsilon\lambda)^{-1/2}, C^{-1}(\epsilon\lambda)^{-1/2}] \times S^1 \subset \Theta''_y.$$

Define

$$\varsigma_\lambda(s,t) := \lambda^{-1/2}\tilde{\beta}(C(\epsilon\lambda)^{1/2}s)\, T_{w_\chi \bar{y}}\xi_\lambda(s + s_\lambda, t) \quad \text{on } \Theta''_y,$$

where $\tilde{\beta}$ is a smooth cutoff function supported on $(-1,1)$ which equals 1 on $(-1/2, 1/2)$. By (22), $\|\varsigma_\lambda\|_{p,1}$ is uniformly bounded and thus by Sobolev embedding ς_λ converges in C_0 (taking a subsequence if necessary) to a ς_0, which satisfies $E_{\bar{y}}\varsigma_0 = 0$. (Note that the term involving ι dropped out because of the assumption $|\iota| \le \lambda^{1+1/(2p)}$.) Such a ς_0 must be identically zero cf. pp. 542–543 of [**6**]; so

$$\|\varsigma_\lambda\|_{L^\infty([-1,1]\times S^1)} \to 0 \quad \text{as } \lambda \to 0,$$

and thus $\|\xi_\lambda\|_{L^\infty(\Theta'_y)} < \varepsilon_0(\lambda)\lambda^{1/2}$. □

Remark. In fact, one may be more precise about the longitudinal component: by part (a) of Lemma 3.3.3, $\|\xi_{\lambda L}\|_{L^\infty(\Theta'_y)} \le C\lambda^{1/2+1/(2p)}\epsilon^{1/(2p)-1/2}$.

Lemma 3.3.1 tells us that $\|E_{w_\chi}(\beta_y\xi_\lambda)_T\|_{L_\chi} \to 0$, where β_y is a smooth cutoff function supported on Θ''_y with value 1 on Θ'_y, since contribution from the extra term due to the cutoff function goes to zero. Thus since E_{w_χ} is right-invertible (being close to a conjugation of $E_{\bar{y}}$) on the transversal subspace,

$$\|(\xi_\lambda)_T\|_{W_\chi(\Theta'_y)} \le C\varepsilon_E(\lambda) + C'\varepsilon_0(\lambda) \to 0 \quad \text{as } \lambda \to 0. \tag{28}$$

3.3.2. A useful normalizing function. The estimates for the longitudinal components hinge on the observation that, after certain normalization, E_{w_χ} behaves like the simple operator d/ds over the longitudinal components.

Definition. Let $\ell(s)$ be the positive real function such that

$$\langle w'_\chi, E_{w_\chi}(\ell \mathbf{e}_w)\rangle_{2,t} = 0 \quad \textit{and } \ell(\gamma_{u_0}^{-1}(0)) = 1, \textit{ where} \tag{29}$$

$$\mathbf{e}_w(s) := \|w'_\chi\|^{-1}_{2,t}(s)w'_\chi(s).$$

Lemma. *The function ℓ is always positive, and there are positive constants C_1, C_2 independent of λ such that*

$$0 < C_1 \leq \ell(s)\|w'_\chi\|_{2,t}(s)^{-1} \leq C_2 \quad \forall s \in [\gamma_{u_0}^{-1}(0), \gamma_{u_{k+1}}^{-1}(0)]. \tag{30}$$

Furthermore, over $[\mathfrak{s}_{j-}, \mathfrak{s}_{j+}]$ $\forall j$,

$$0 < C_1\lambda \leq \ell \leq C_2\lambda; \quad |\ell'| \leq C|\lambda|^{1/2}|\ell|. \tag{31}$$

Proof. ℓ satisfies a first-order linear differential equation, so its existence and uniqueness is obvious. It also follows that ℓ has no zeros, because otherwise it would be identically zero. The condition that $\ell(\gamma_{u_0}^{-1}(0)) = 1$ therefore implies that ℓ is always positive.

Equation (30) follows from the next claim by observing that $\ell\|w'_\chi\|_{2,t}^{-1}(\gamma_{u_0}^{-1}(0))$ is λ-independent.

Claim. Let $\mathfrak{s}_{0+} := \gamma_{u_0}^{-1}(0)$; $\mathfrak{s}_{k+2\,-} := \gamma_{u_{k+1}}^{-1}(0)$. Then

$$\left| \ln(\ell\|w'_\chi\|_{2,t}(r_1)) - \ln(\ell\|w'_\chi\|_{2,t}(r_2)) \right| \leq C$$

for a constant C independent of r_1, r_2, and λ when r_1, r_2 are both in

(a) $[\mathfrak{s}_{i+}, \mathfrak{s}_{i+1\,-}]$ for some $i \in \{0, 1, \ldots, k+1\}$ or
(b) $[\mathfrak{s}_{j-}, \mathfrak{s}_{j+}]$ for some $j \in \{1, 2, \ldots, k+1\}$.

Proof of the Claim. In case (a), set $u = u_i$ and drop the index i. In this case, $|\gamma_u| \leq C_i\lambda^{-1/2}$, γ'_u is close to 1, and $\|w'_\chi\|_{2,t}$ can be approximated by $\|u_\gamma(\gamma_u)\|_{2,t}$. We will therefore estimate $\ell\|u_\gamma\|_{2,t}^{-1}$ instead. In this region, rewrite (29) as

$$\frac{d}{ds}(\ln(\ell\|u_\gamma\|_{2,t}^{-1})) = (\gamma'_u - 1)\frac{d}{d\gamma}(\ln\|u_\gamma\|_{2,t}^{-1})$$

and integrate over s. Using the estimates in Section 2, it is easy to see that in this region, the L^∞-norm of the right-hand side of the above equation can be bounded by $C\lambda|\gamma| \leq C'\lambda^{1/2}$. On the other hand, the distance between r_1 and r_2 can be bounded by a multiple of $\lambda^{-1/2}$, so the claim is verified in this case.

In case (b), set $u = u_{j-1}$ or u_j depending on whether s is smaller or larger than $\mathfrak{l}_j$, and again drop the index $j-1$ or j. In this case, $|\gamma_u| \geq C_i\lambda^{-1/2}$, and $\lambda\|w'_\chi\|_{2,t}^{-1}$ can be bounded above and below independently of λ (cf. (27)) in this region, so it suffices to estimate the variation in ℓ. We write (29) in the form:

$$\frac{d}{ds}(\ln\,\ell) = -\|u_\gamma\|_{2,t}^{-1}(u_{\gamma\gamma})_L$$

in this case and again integrate over s. The claim then follows from the bound

$$\left|\frac{d}{ds}(\ln \ell)\right| \leq C_3|\gamma_u|^{-1} \leq C_3'\lambda^{1/2} \tag{32}$$

and bound of the distance between r_1 and r_2 can be bounded by $C_4\lambda^{-1/2}$. □

Continuing the proof of the lemma, the first inequality in (31) is the consequence of (30) and (27). The second inequality of (31) follows directly from (32) and the first inequality. □

3.3.3. Estimating the longitudinal direction. It is convenient to introduce the following.

Definition. For $j = 1, \ldots, k+1$, let the $\mathbb{R}$-valued function $f_j(s)$ be the unique solution of

$$\begin{aligned} E_{w_\chi}(f_j\ell\mathbf{e}_w) = f_j'\ell\mathbf{e}_w = \mathfrak{f}_j, \\ \ell f_j(\mathfrak{s}_{j-}) = 0. \end{aligned} \tag{33}$$

Also, let $\phi_\lambda(s)$, $\psi_{\lambda,i}(\gamma)$ be the $\mathbb{R}$-valued functions defined, respectively, by

$$\ell\phi_\lambda(s) = \langle \xi_\lambda(s), \mathbf{e}_w(s)\rangle_{2,t},$$
$$\psi_{\lambda,i}(\gamma) = \langle \mathcal{T}^{\gamma_{u_i}}_{w_\chi,\underline{w}_\chi}\xi_\lambda(\gamma), \mathbf{e}_{u_i}(\gamma)\rangle_{2,t}, \quad \textit{where } \mathbf{e}_{u_i} = \|u_i'\|_{2,t}^{-1}u_i'.$$

The estimates for the longitudinal components will be based on the following elementary lemma.

Lemma. *If* $q \in L^p_1([0,l])$, *then*

(a) $\|q\|_\infty \leq C_1 l^{1-1/p}\|q'\|_p + C_2 l^{-1/p}\|q\|_p$.

If furthermore $q(0) = 0$, *then in addition*:

(b) $\|q\|_\infty \leq C l^{1-1/p}\|q'\|_p$;

(c) $\|q\|_p \leq C' l\|q'\|_p$.

The positive constants C, C', C_1, C_2 *are independent of* q *and* l.

Let $\bar{\phi}_{\lambda,j} := \phi_\lambda + \iota_{\lambda,j}f_j$. Then by (31), (27), (21), and part (c) of the above lemma,

$$\begin{aligned} &\|(\xi_\lambda)_L + \iota_{\lambda,j}\ell f_j\mathbf{e}_w\|_{W_\chi(\Theta_{yj})} \\ &\leq C_1\left(\lambda^{1/2}\|\bar{\phi}_{\lambda,j}\|_{L^p(\Theta_{yj})} + \|\bar{\phi}'_{\lambda,j}\|_{L^p(\Theta_{y,j})}\right) \\ &\leq C_2\left(\epsilon^{-1/2}\|\bar{\phi}'_{\lambda,j}\|_{L^p(\Theta_{y,j})} + \lambda^{-1/2-1/(2p)}\epsilon^{-1/(2p)}|\psi_{\lambda,j-1}|(\mathfrak{r}_{j-1})\right) \\ &\leq C_3\left(\epsilon^{-1/2}\|(\tilde{E}_\chi\tilde{\xi}_{\lambda L})_L\|_{L_\chi(\Theta_{yj})} + \lambda^{-1/2-1/(2p)}\epsilon^{-1/(2p)}|\psi_{\lambda,j-1}|(\mathfrak{r}_{j-1})\right) \\ &\leq C_4\Big(\epsilon^{-1/2}\varepsilon_E + \epsilon^{-1/2-1/(2p)}\lambda^{1/2-1/(2p)}\varepsilon_0) \\ &\qquad + \lambda^{-1/2-1/(2p)}\epsilon^{-1/(2p)}|\psi_{\lambda,j-1}|(\mathfrak{r}_{j-1})\Big). \end{aligned} \tag{34}$$

In the last line above, ε_0 comes from Lemma 3.3.1 and an estimate for $(E_{w_\chi}\xi_{\lambda T})_L$ via a computation similar to I.(47). To estimate the last term above, note that from Lemma 3.2.1, we have

$$\begin{aligned}&\|E_{u_i}(\psi_{\lambda,i}\mathbf{e}_{u_i})\|_{L_{u_i}([0,\mathfrak{r}_i]\times S^1)}\\ &\quad\le (1+2\epsilon)\|\tilde{E}_\chi(\tilde{\xi}_\lambda)\|_{L_\chi}+(C\lambda+\epsilon)\|\xi_{\lambda L}\|_{W_\chi}+C'\|\mathcal{T}^{\gamma_{u_i}}_{w_\chi,\underline{w}_\chi}\xi_{\lambda T}\|_{W_{u_i}([0,\mathfrak{r}_i]\times S^1)}\\ &\quad\le C\varepsilon'_{u_i}\quad\to 0\quad\text{as } \lambda\to 0.\end{aligned}$$

In the above, we used (26) to estimate $\|\mathcal{T}^{\gamma_{u_i}}_{w_\chi,\underline{w}_\chi}\xi_T\|_{W_{u_i}([0,\mathfrak{r}_i]\times S^1)}$. On the other hand,

$$\|E_{u_i}(\psi_{\lambda,i}\mathbf{e}_{u_i})\|_{L_{u_i}([0,\mathfrak{r}_i]\times S^1)}\ge C\|(\sigma_{u_i}(\psi_{\lambda,i})_\gamma\|_{L^p([0,\mathfrak{r}_i])}.$$

Using Lemma 3.3.3 (b) and the fact that $\psi_{\lambda,i}(0)=0$ (because $\xi_\lambda\in W'_\chi$), the previous two inequalities imply:

$$\begin{aligned}|\psi_{\lambda,i}(\mathfrak{r}_i)|&\le C\lambda\mathfrak{r}_i^{1-1/p}\varepsilon'_{u_i}\\ &\le C'\lambda^{1/2+1/(2p)}\epsilon^{1/2-1/(2p)}\varepsilon'_{u_i}.\end{aligned}\tag{35}$$

Combining this with (34), we have

$$\begin{aligned}&\|(\xi_\lambda)_L+\iota_{\lambda,j}\ell f_j\mathbf{e}_w\|_{W_\chi(\Theta_{yj})}\\ &\qquad\le C_5\left(\epsilon^{-1/2}\varepsilon_E+\epsilon^{-1/2-1/(2p)}\lambda^{1/2-1/(2p)}\varepsilon_0+\epsilon^{1/2-1/p}\varepsilon'_{u_{j-1}}\right).\end{aligned}\tag{36}$$

3.4. Estimating ι_j and f_j. This subsection fills in the last ingredients for the proof of Proposition 3.1.3: estimates for ι_j and f_j (Lemmas 3.4.1 and 3.4.2 respectively). Combining these estimates with the estimates obtained in previous subsections, we finish the proof of Proposition 3.1.3 in Section 3.4.3.

3.4.1. Lemma. *Let λ, ϵ be small positive numbers as before. Then*

$$|\iota_{\lambda,j}|\le\varepsilon_{\iota,j}(\lambda,\epsilon),$$

where $\varepsilon_{\iota,j}>0$ can be made arbitrarily small as $\lambda\to 0$ by choosing ϵ appropriately.

Proof. This lemma follows from a lower bound on $\ell f_j(\mathfrak{s}_{j+})$, and a upper bound on $\iota_{\lambda,j}\ell f_j(\mathfrak{s}_{j+})$, given, respectively, in (37), (38) below.

Note from the defining equation for f_j and (31) that $|f'_j|\ge C\lambda^{1/(2p)}$ for some λ-independent constant C. Therefore by (21), (31), and the initial condition of f_j (33),

$$\begin{aligned}\ell f_j(\mathfrak{s}_{j+})&=\ell(\mathfrak{s}_{j+})\Big(f_j(\mathfrak{s}_{j+})-f_j(\mathfrak{s}_{j-})\Big)\\ &\ge C_1\lambda^{1+1/(2p)}(\mathfrak{s}_{j+}-\mathfrak{s}_{j-})\\ &\ge C_j\epsilon^{-1/2}\lambda^{1/2+1/(2p)}.\end{aligned}$$

A similar calculation establishes an analogous upper bound, and we have

$$C_j'\epsilon^{-1/2}\lambda^{1/2+1/(2p)} \geq \ell f_j(\mathfrak{s}_{j+}) \geq C_j\epsilon^{-1/2}\lambda^{1/2+1/(2p)}. \tag{37}$$

On the other hand,

$$|\iota_{\lambda,j}|\ell f_j(\mathfrak{s}_{j+}) \leq |\psi_{\lambda,j-1}(\mathfrak{r}_{j-1})| + |-\psi_{\lambda,j}(\mathfrak{r}_j)| + \Big|\ell\bar{\phi}_\lambda(\mathfrak{s}_{j+}) - \ell\bar{\phi}_\lambda(\mathfrak{s}_{j-})\Big|.$$

The first two terms on the RHS are already estimated in (35); the third term can be bounded by

$$\Big|\ell(\mathfrak{s}_{j+})\Big(\bar{\phi}_\lambda(\mathfrak{s}_{j+}) - \bar{\phi}_\lambda(\mathfrak{s}_{j-})\Big)\Big| + \Big|\Big(\ell(\mathfrak{s}_{j+}) - \ell(\mathfrak{s}_{j-})\Big)\bar{\phi}_\lambda(\mathfrak{s}_{j-})\Big|,$$

in which the first term may be bounded via (31), Lemma 3.3.3 (b) by

$$\begin{aligned}&C\lambda(\epsilon\lambda)^{-1/2+1/(2p)}\|\bar{\phi}_\lambda'\|_{L^p([\mathfrak{s}_{j-},\mathfrak{s}_{j+}])}\\ &\qquad\leq C'\lambda^{1/2+1/(2p)}(\epsilon^{-1/2+1/(2p)}\varepsilon_E + \epsilon^{-1/2}\lambda^{1/2-1/(2p)}\varepsilon_0),\end{aligned}$$

according to the computation in (34), lines 3–5.

The second term, via (35), the initial condition (33), and (31), may be bounded by

$$C''\lambda^{1/2+1/(2p)}\epsilon^{1/2-1/(2p)}\varepsilon'_{u_{j-1}}.$$

Summing up, we have

$$\begin{aligned}|\iota_{\lambda,j}|\ell f_j(\mathfrak{s}_{j+}) &\leq C_0\epsilon^{-1/2}\lambda^{1/2+1/(2p)}\cdot\\ &\qquad(\epsilon^{1/(2p)}\varepsilon_E + (\lambda+\epsilon)\epsilon^{1-1/(2p)} + \varepsilon_0(\lambda^{1/2-1/(2p)} + \epsilon^{1-1/(2p)})).\end{aligned} \tag{38}$$

Comparing with (37), we have an estimate for $\iota_{\lambda,j}$ as asserted in the lemma, with

$$\varepsilon(\epsilon,\lambda) = C_1(\epsilon^{1/(2p)}\varepsilon_E + (\lambda+\epsilon)\epsilon^{1-1/(2p)} + \varepsilon_0(\lambda^{1/2-1/(2p)} + \epsilon^{1-1/(2p)})),$$

which can be made arbitrarily small by choosing λ, ϵ appropriately. □

3.4.2. Lemma. *For $j = 1, \dots, k+1$,*

$$\lambda^{-1/2}\|\ell f_j\mathbf{e}_w\|_{L^p(\Theta_{yj})} + \lambda^{-1}\|(\ell f_j)'\mathbf{e}_w\|_{L^p(\Theta_{yj})} \leq C_j''\epsilon^{-1/2-1/(2p)}.$$

Proof. Note that since $f_j' = \ell^{-1}\langle\mathbf{e}_w, \mathfrak{f}_j\rangle_{2,t} > 0$, f_j is increasing, and thus the estimates leading to (37) imply that

$$\|\ell f_j\|_{L^\infty(\Theta_{yj})} \leq C_1\epsilon^{-1/2}\lambda^{1/2+1/(2p)}.$$

On the other hand, this and (31) yield

$$\begin{aligned}\|(\ell f_j)'\|_{L^\infty(\Theta_{yj})} &\leq \|\ell f_j'\|_{L^\infty(\Theta_{yj})} + \|\ell' f_{yj}\|_{L^\infty(\Theta_{yj})}\\ &\leq C_2(\lambda^{1+1/(2p)} + \lambda^{1/2}\|\ell f_j\|_{L^\infty(\Theta_{yj})})\\ &\leq C_3\epsilon^{-1/2}\lambda^{1+1/(2p)}.\end{aligned}$$

These two L^∞-bounds together with the length estimate for Θ_{yj} imply the lemma. □

3.4.3. Concluding the proof of Proposition 3.1.3. Now we have all the ingredients to finish the proof of the proposition.

By Lemmas 3.4.1 and 3.4.2 the ι-terms in (36) are ignorable as $\lambda \to 0$. They are bounded by expressions of the form

$$C_1(\varepsilon_E \epsilon^{-1/2} + (\lambda+\epsilon)\epsilon^{1/2-1/p} + \varepsilon_0(\lambda^{1/2-1/(2p)}\epsilon^{-1/2-1/(2p)} + \epsilon^{1/2-1/p})),$$

which can be made arbitrarily small by requiring, e.g.,

$$\begin{aligned} &\lambda = \lambda(\epsilon) \text{ is small enough such that} \\ &\lambda < \epsilon^5, \varepsilon_E(\lambda) + \varepsilon_0(\lambda) < \epsilon^3, \quad \text{and} \\ &\epsilon \to 0. \end{aligned} \tag{39}$$

Applying the comparison Lemma 3.2.1 to (26) for all i, and adding them to (28) and (36) for all j, we see that

$$\|\xi_\lambda\|_{W_\chi} \leq C_4(\epsilon^{-1/2}\varepsilon_E + \lambda + \epsilon + \varepsilon_0(1 + \epsilon^{-1/2-1/(2p)}\lambda^{1/2-1/(2p)})) \ll 1$$

by the same choice (39). Combining this with Lemma 3.4.1, we arrive at a contradiction to (22). □

4. Gluing at Deaths III: the gluing map

Sections 4.1 and 4.2 finish the proof of Proposition 2.1 (a): by further analyzing the Kuranishi map associated to the K-model of last section, we obtain a smooth gluing map, which is a local diffeomorphism. We show that this gluing map surjects to a neighborhood of the stratum $\mathbb{S}$ in Section 4.2.

In Section 4.3, we discuss the minor modification needed to obtain part (b) of Proposition 2.1, which glues broken orbits at $\lambda = 0$ to create new closed orbits for small $\lambda > 0$.

4.1. Understanding the Kuranishi map. In the previous section, we constructed the K-model for the family of deformation operators, $[K^\Xi \to C^\Xi]$. According to the discussion in Section 1.2.6, this yields a local description of the moduli space as the zero locus of an analytic map. In this subsection, we analyze this analytic map in more detail; this analysis enables us to show that the moduli space is in fact (Zariski) *smooth*.

Recall that Proposition 3.1.3 shows that we have decompositions:

$$W_\chi = \bigoplus_i \mathbb{R}\mathfrak{e}_{u_i} \oplus W'_\chi, \quad L_\chi = \bigoplus_j \mathbb{R}\mathfrak{f}_j \oplus E_{w_\chi}(W'_\chi), \tag{40}$$

and the (nonorthogonal) projection of L_χ to the $\mathfrak{f}_j$ direction is given by

$$P_{\mathfrak{f}_j} = \Pi_j \tilde{G}_\chi,$$

where Π_j is the projection to the jth $\mathbb{R}$-component of $\tilde{W}_\chi$. By Proposition 3.1.3, P_{f_j} has uniformly bounded operator norm. However, we need a finer estimate for P_{f_j} to understand the Kuranishi map. The next lemma is a useful tool for this purpose.

4.1.1. Lemma (Projection via integration). *Let $\eta \in L_\chi$, and as usual denotes*

$$\underline{\eta}_L(s) := \|w'_\chi(s)\|_{2,t}^{-1}\langle w'_\chi(s), \eta(s)\rangle_{2,t} \quad \textit{for } s \in [\gamma_{u_0}^{-1}(0), \gamma_{u_{k+1}}^{-1}(0)].$$

Then the projection $P_{f_j}\eta$ is bounded above and below by expressions of the form

$$C_{1\pm}\lambda^{1/2-1/(2p)}\int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}\underline{\eta}_L\, ds - C_{2\pm}\lambda^{1/2-1/(2p)}\|\eta\|_{L_\chi} \tag{41}$$

for λ-independent constants $C_{1\pm}, C_{2\pm}$.

In our later applications of this lemma, the second term in the above expression is typically dominated by the first term, and hence ignorable.

Proof. In accordance with decomposition (40), write

$$\eta = E_{w_\chi}\xi + \sum_{j=1}^{k+1}\iota_j\mathfrak{f}_j \tag{42}$$

for $(\iota_1, \ldots, \iota_{k+1}, \xi) \in \tilde{W}_\chi$. Thus

$$\begin{aligned}
&\int_{\mathrm{s}_{j-}}^{\mathrm{s}_{j+}} \ell^{-1} ds\, \iota_j \\
&= C_j^{-1}\lambda^{-1-1/(2p)}\left(\int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}\underline{\eta}_L ds - \int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}\underline{(E_{w_\chi}\xi)}_L\, ds\right) \qquad (43)\\
&= C_j^{-1}\lambda^{-1-1/(2p)}\left(\int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}\underline{\eta}_L ds - \int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}\underline{(E_{w_\chi}\xi_T)}_L\, ds\right).
\end{aligned}$$

The second identity above is due to the fact that $\xi_L(\gamma_{u_{j-1}}^{-1}(0)) = 0 = \xi_L(\gamma_{u_j}^{-1}(0))$: writing $\xi_L(s) = \ell\phi w'_\chi\|w'_\chi\|_{2,t}^{-1}(s)$, we see from the definition of ℓ that $\ell^{-1}\underline{(E_{w_\chi}\xi_L)}_L = \phi'$. Now integrate by parts, using the fact that since $\ell(s) \neq 0\,\forall s$, $\phi(\gamma_{u_{j-1}}^{-1}(0)) = 0 = \phi(\gamma_{u_j}^{-1}(0))$.

By (31) and (21), we have

$$C_h\lambda^{-3/2} \leq \int_{\mathrm{s}_{j-}}^{\mathrm{s}_{j+}} \ell^{-1}\, ds \leq C'_h\lambda^{-3/2}; \tag{44}$$

on the other hand, a computation similar to that leading to I.(47) yields

$$\begin{aligned}
\left|\ell^{-1}\underline{(E_{w_\chi}\xi_T)}_L\right|(s) &\leq C_1\Big(\sigma_\chi\|u_\gamma\|_{2,t}^{-1}(\gamma'_u+1)\|(u_{\gamma\gamma})_T\|_{2,t} + \lambda\Big)\|\xi_T\|_{2,t}(s)\\
&\leq C_2\|\xi_T\|_{2,t}(s),
\end{aligned}$$

where $u = u_{j-1}$ or u_j depending on whether $s < \mathfrak{l}_j$ or $> \mathfrak{l}_j$. So by the estimates for $\|w'_\chi\|_{2,t}$ and $\gamma_{u_i}^{-1}(0) - \gamma_{u_{i-1}}^{-1}(0)$, and the uniform boundedness of $\tilde{G}_\chi$,

$$\left|\int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}\underline{(E_{w_\chi}\xi_T)}_L\, ds\right| \leq C_3\|\sigma_\chi^{-1/2}\|_{L^q((\gamma_{u_{j-1}}^{-1}(0),\gamma_{u_j}^{-1}(0))\times S^1)}\|\xi_T\|_{W_\chi} \leq C_4\|\eta\|_{L_\chi}, \tag{45}$$

where $q^{-1} := 1-p^{-1}$. Putting (43), (44), (45) together, we arrive at (41). $\square$

Next, applying the recipe of Section 1.2.5 to the K-models given by Proposition 3.1.3, we look for solutions $(\alpha, \varphi_0, \dots, \varphi_{k+1}, \xi) \in W'_\chi \oplus \mathbb{R} \oplus \mathbb{R}^{k+2}$ of

$$P^c\Bigg(\bar{\partial}_{JX_\lambda} w_\chi + \hat{E}_{(\lambda,w_\chi)}(\alpha,\xi) + \sum_{i=0}^{k+1}\varphi_i E_{w_\chi}\mathfrak{e}_{u_i} + \hat{n}_{(\lambda,w_\chi)}\Big(\alpha, \xi + \sum_{i=0}^{k+1}\varphi_i\mathfrak{e}_{u_i}\Big)\Bigg) = 0; \tag{46}$$

$$P_{\mathrm{f}_j}\Bigg(\bar{\partial}_{JX_\lambda} w_\chi + \hat{E}_{(\lambda,w_\chi)}\Big(\alpha, \sum_{i=0}^{k+1}\varphi_i\mathfrak{e}_{u_i}\Big) + \hat{n}_{(\lambda,w_\chi)}\Big(\alpha, \xi + \sum_{i=0}^{k+1}\varphi_i\mathfrak{e}_{u_i}\Big)\Bigg) = 0, \tag{47}$$

where $P^c := 1 - \sum_j P_{\mathrm{f}_j}$.

4.1.2. Lemma (Solving the infinite-dimensional equation). *Given*

$$\hat{\varphi} := (\alpha, \varphi_0, \dots, \varphi_{k+1}) \in \mathbb{R}\oplus\mathbb{R}^{k+2}, \quad \textit{with } |\hat{\varphi}|^2 := |\lambda^{-3/2}\alpha|^2 + \sum_i |\varphi_i|^2 \ll 1,$$

(46) has a unique solution $\xi(\hat{\varphi})$, with

$$\|\xi(\hat{\varphi})\|_{W_\chi} \leq C_1\lambda^{1/2-1/(2p)} + C_2\lambda^{1/2-1/(2p)}|\hat{\varphi}| + C_3\lambda^{1-1/(2p)}|\hat{\varphi}|^2. \tag{48}$$

Furthermore, the solution $\xi(\hat{\varphi}')$ corresponding to another $\hat{\varphi}' = (\alpha', \varphi'_0, \dots, \varphi'_{k+1})$ satisfies

$$\|\xi(\hat{\varphi}) - \xi(\hat{\varphi}')\|_{W_\chi} \leq C_4\lambda^{1/2-1/(2p)}|\hat{\varphi} - \hat{\varphi}'|. \tag{49}$$

The significance of the factor $|\lambda|^{2/3}$ associated to α in the definition of $|\hat{\varphi}|$ will become clear in (70).

Proof. To apply the usual contraction mapping argument (Lemma 1.2.1) to (46), we need to estimate the "error" $\mathcal{F}$ and the nonlinear term N, and to show that the linearization has a uniformly bounded right inverse.

In this context, the error $\mathcal{F}$ consists of

$$P^c\bar{\partial}_{JX_\lambda}w_\chi + P^c\sum_{i=0}^{k+1}\varphi_i E_{w_\chi}\mathfrak{e}_{u_i} + \alpha P^c Y_{(\lambda,w_\chi)} + P^c\hat{n}_{(\lambda,w_\chi)}\left(\alpha, \sum_{i=0}^{k+1}\varphi_i\mathfrak{e}_{u_i}\right),$$

which we estimate term by term below. We shall drop all P^c from the terms, since by Proposition 3.1.3, it has uniformly bounded operator norm, and thus only affect the estimate by a λ-independent factor.

For the first term, note that $\|\bar{\partial}_{JX_\lambda}w_\chi\|_{L_\chi}$ is readily estimated by Proposition 2.4.1.

For the second term, we claim:

$$\|E_{w_\chi}\gamma_{u_i}^*(u_i)_\gamma\|_{L_\chi} \le C\lambda^{1/2-1/(2p)}. \tag{50}$$

We shall again suppress the subscript i below. Note that

$$E_{w_\chi}\gamma_u^* u_\gamma = (\gamma_u' - 1)u_{\gamma\gamma}(\gamma_u) + Z(u)u_\gamma(\gamma_u), \tag{51}$$

where Z arises from the difference between X_λ and X_0, and hence we have $\|Z(u)\|_\infty \le C\lambda$. Thus, by I.5.3.1 (2c) and routine estimates, the L_χ-norm of the second term above is also bounded by $C'\lambda$.

For the first term, note that the length of $[\gamma_u^{-1}(-\gamma_0), \gamma_u^{-1}(\gamma_0)]$ is bounded independently of λ; therefore, the L_χ-norm of it in this region is bounded by $C\lambda$. On the other hand, the length of the intervals where $|\gamma_u| \ge \gamma_0$ is bounded by $C'\lambda^{-1/2}$. When s is in these intervals, by the computations in case (2) in proof of Lemma 2.4.2, $\|\sigma_\chi(\gamma_u' - 1)u_{\gamma\gamma}(\gamma_u)\|_\infty \le C_3\lambda^{1/2}$. Together with the length estimate above, we see that the L_χ-norm in this region is bounded by $C''\lambda^{1/2-1/(2p)}$. Equation (50) is verified.

For the third term, recall the following estimate obtained in the proof of Lemma 2.5.1:

$$\|\alpha Y_{(\lambda,w_\chi)}\|_{L_\chi} \le C\|(\alpha,0)\|_{\hat{W}_\chi} \le C\lambda^{1/2-1/(2p)}|\hat{\varphi}|. \tag{52}$$

For the last term in the error, we have

$$\begin{aligned}
&\|\hat{n}_{(\lambda,w_\chi)}\left(\alpha, \sum_{i=0}^{k+1}\varphi_i\mathfrak{e}_{u_i}\right)\|_{L_\chi};\\
&\le C_2\sum_{i=0}^{k+1}\Big(\|n_{w_\chi}(\varphi_i\mathfrak{e}_{u_i})\|_{L_\chi} + \|\alpha\varphi_i\nabla_{\mathrm{e}_{u_i}}Y_{(\lambda,w_\chi)}\|_{L_\chi}\Big)\\
&\qquad + C_2'\alpha^2\|\partial_\lambda Y_{(\lambda,w_\chi)}\|_{L_\chi} + \text{higher order terms}\\
&\le C_3\sum_{i=0}^{k+1}\Big(\lambda|\varphi_i|^2 + |\alpha|\lambda^{-1/(2p)}|\varphi_i|\Big) + C_3'|\alpha|^2\lambda^{-1-1/(2p)}\\
&\le C_4\lambda|\hat{\varphi}|^2.
\end{aligned}$$

For the first inequality above, we used the fact that for different i, j, $\mathfrak{e}_{u_i}, \mathfrak{e}_{u_j}$ have disjoint supports. For the second inequality, we used the invariance of the flow equation under translation, which implies

$$n_{u_i}^{JX_0}(\varphi_i(u_i)_\gamma) = 0 \quad \forall \varphi_i \in \mathbb{R}. \tag{53}$$

Next, the linear term in (46) is of the form $E'_\chi \xi$, where E'_χ is E_{w_χ} perturbed by a term coming from $\hat{n}_{(\lambda, w_\chi)}(\alpha, \sum_{i=0}^{k+1} \varphi_i \mathfrak{e}_{u_i} + \xi)$, which has operator norm bounded by

$$C\left(\sum_{i=0}^{k+1} |\varphi_i| + \lambda^{-1/2}|\alpha|\right) \leq C|\hat{\varphi}|.$$

Thus, with the assumption that $|\hat{\varphi}| \ll 1$, E'_χ is uniformly right-invertible as E_{w_χ} is. By contraction mapping theorem and the error estimates above, we have an $\xi(\hat{\varphi})$ satisfying (48).

The estimate for the nonlinear term is not very different from that in Lemma 2.5.2, which we shall omit.

Finally, to estimate $\xi - \xi'$, where $\xi := \xi(\hat{\varphi}); \xi' := \xi(\hat{\varphi}')$, notice that it satisfies

$$\begin{aligned} E_{w_\chi}(\xi - \xi') = -P^c \Bigg(& \sum_{i=0}^{k+1} (\varphi_i - \varphi'_i) E_{w_\chi} \mathfrak{e}_{u_i}) + (\alpha - \alpha') Y_{(\lambda, w_\chi)} \\ & + \hat{n}_{(\lambda, w_\chi)}\left(\alpha, \xi + \sum_{i=0}^{k+1} \varphi_i \mathfrak{e}_{u_i}\right) - \hat{n}_{(\lambda, w_\chi)}\left(\alpha', \xi' + \sum_{i=0}^{k+1} \varphi'_i \mathfrak{e}_{u_i}\right)\Bigg). \end{aligned}$$

Thus, by Proposition 3.1.3,

$$\begin{aligned} \|\xi - \xi'\|_{W_\chi} \leq C' \Bigg(& \sum_{i=0}^{k+1} |\varphi_i - \varphi'_i| \|E_{w_\chi} \mathfrak{e}_{u_i}\|_{L_\chi} + |\alpha - \alpha'| \|Y_{(\lambda, w_\chi)}\|_{L_\chi} \\ & + \left\| \hat{n}_{(\lambda, w_\chi)}\left(\alpha, \xi + \sum_{i=0}^{k+1} \varphi_i \mathfrak{e}_{u_i}\right) - \hat{n}_{(\lambda, w_\chi)}\left(\alpha, \xi' + \sum_{i=0}^{k+1} \varphi'_i \mathfrak{e}_{u_i}\right) \right\|_{L_\chi} \Bigg) \end{aligned}$$

for a λ-independent constant C'. The first two terms inside the parenthesis may be bounded by $C_1 \lambda^{1/2 - 1/(2p)} |\hat{\varphi} - \hat{\varphi}'|$ according to (50) and (52).

The third term, by direct computation and (53) again, may be bounded by

$$\begin{aligned}
&\begin{pmatrix} \|\xi\|_{W_\chi}+\|\xi'\|_{W_\chi}+(|\alpha|+|\alpha'|)\lambda^{-1/2} \\ +\sum_{i=0}^{k+1}(|\varphi_i|+|\varphi_i'|)\|\mathfrak{e}_{u_i}\|_{W_\chi}\end{pmatrix}\cdot C_2\|\xi-\xi'\|_{W_\chi} \\
&\quad+\sum_{i=0}^{k+1}\begin{pmatrix} \|\xi\|_{W_\chi}+\|\xi'\|_{W_\chi}+(|\alpha|+|\alpha'|)\lambda^{-1/2} \\ +\lambda(|\varphi_i|+|\varphi_i'|)\|\mathfrak{e}_{u_i}\|_{W_\chi}\end{pmatrix}\cdot C_3\|\mathfrak{e}_{u_i}\|_{W_\chi}|\varphi_i-\varphi_i'| \\
&\quad+\begin{pmatrix} \lambda^{-1/2}(\|\xi\|_{W_\chi}+\|\xi'\|_{W_\chi})+\lambda^{-1-1/(2p)}(|\alpha|+|\alpha'|) \\ +\sum_{i=0}^{k+1}(|\varphi_i|+|\varphi_i'|)\lambda^{-1/2}\|\mathfrak{e}_{u_i}\|_{W_\chi}\end{pmatrix}\cdot C_4|\alpha-\alpha'| \\
&\le C_2'\varepsilon\|\xi-\xi'\|_{W_\chi}+C_3'\lambda^{1/2-1/(2p)}|\hat\varphi-\hat\varphi'|,
\end{aligned} \tag{54}$$

where $0<\varepsilon\ll 1$, and we have used (48) and the fact that $|\hat\varphi|\ll 1$ above. Now, the first term in the last expression above can be got rid of by a rearrangement argument, and we arrive at (49). □

Next, substitute $\xi(\hat\varphi)$ back in (47) to solve for $\hat\varphi$. To understand the behavior of the solutions, we estimate each term in the Kuranishi map in turn.

4.1.3. Lemma (Terms in the Kuranishi map). *Let* $q^{-1}:=1-p^{-1}$. *Then*:

(a) $|P_{f_j}\bar\partial_{JX_\lambda}w_\chi|\le C'\lambda^{1/q}$ *for any* $j\in\{1,\dots,k+1\}$;

(b) *For any* $i\in\{0,\dots,k+1\}$, $j\in\{1,\dots,k+1\}$,

$$|P_{f_j}E_{w_\chi}\mathfrak{e}_{u_i}|\le C\lambda^{1/q}\quad \text{if } j\neq i \text{ or } i+1;$$
$$-C_-'\lambda^{1/(2q)}\ge P_{f_i}(E_{w_\chi}\mathfrak{e}_{u_i})\ge -C_-\lambda^{1/(2q)};$$
$$C_+'\lambda^{1/(2q)}\ge P_{f_{i+1}}(E_{w_\chi}\mathfrak{e}_{u_i})\ge C_+\lambda^{1/(2q)}.$$

(c) *Let* $\hat\varphi,\hat\varphi'$, $\xi:=\xi(\hat\varphi),\xi':=\xi(\hat\varphi')$ *be as in the previous lemma. Then* $\forall j$,

$$\left|P_{f_j}\left(\hat n_{(\lambda,w_\chi)}\left(\alpha,\xi+\sum_{i=0}^{k+1}\varphi_i\mathfrak{e}_{u_i}\right)-\hat n_{(\lambda,w_\chi)}\left(\alpha',\xi'+\sum_{i=0}^{k+1}\varphi_i'\mathfrak{e}_{u_i}\right)\right)\right|\le C_n\lambda^{1/q}|\hat\varphi-\hat\varphi'|.$$

Proof. (a) Apply (41) with $\eta=\bar\partial_{JX_\lambda}w_\chi$. The integrals in the first terms vanish because by our definition of pregluing, $\bar\partial_{JX_\lambda}w_\chi$ has no longitudinal component in this region. On the other hand, $\|\eta\|_{L_\chi} \leq C\lambda^{1/(2q)}$ by Proposition 2.4.1.

(b) Let $\eta = E_{w_\chi}\mathfrak{e}_{u_i}$ in (41). By (50), the second term of (41) (multiple of $\lambda^{1/(2q)}\|\eta\|_{L_\chi}$) contributes a multiple of $\lambda^{1/q}$. If $j \neq i, i+1$, the first term of (41) (multiples of integrals) vanishes, because η is supported away from the interval of integration. These together imply the first line of (b).

For the other cases ($u = u_j$ or u_{j-1}), we shall show that

$$\pm C' \leq \int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}\underline{E_{w_\chi}\mathfrak{e}_u}_L \leq \pm C \quad (- \text{ when } u = u_j, + \text{ when } u = u_{j-1}).$$

This would imply that the first term of (41) is bounded below and above by positive multiples of $\pm\lambda^{1/(2q)}$, dominating the second term. The other two cases of (b) would then follow.

To see this, recall the computation of $E_{w_\chi}\mathfrak{e}_u$ from (51), and note that the longitudinal component of the second term vanishes because of I.5.3.1 (2c). Thus,

$$\int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}\underline{E_{w_\chi}\mathfrak{e}_u}_L = \int_{\gamma_{u_{j-1}}^{-1}(0)}^{\gamma_{u_j}^{-1}(0)} \ell^{-1}(\gamma_u' - 1)\underline{u_{\gamma\gamma}(\gamma_u)}_L.$$

To estimate the integral on the RHS, note that on the interval of integration, $\gamma_{u_j}, \gamma_{u_{j-1}}$ are negative/positive, respectively. Also,

$$\begin{cases} |\ell^{-1}(\gamma_u' - 1)\underline{u_{\gamma\gamma}}_L| \leq C_1\lambda & \text{when } |\gamma_u| \leq \gamma_0; \\ C_2'\lambda|\gamma_u| \leq |\ell^{-1}(\gamma_u' - 1)\underline{u_{\gamma\gamma}}_L| \leq C_2\lambda|\gamma_u| & \text{when } \gamma_0 \leq |\gamma_u| \leq \epsilon^{1/2}\lambda^{-1/2}; \\ C_3'|\gamma_u|^{-1} \leq |\ell^{-1}(\gamma_u' - 1)\underline{u_{\gamma\gamma}}_L| \leq C_3|\gamma_u|^{-1} & \text{when } |\gamma_u| \geq \epsilon^{1/2}\lambda^{-1/2}; \end{cases}$$

Furthermore, when $|\gamma_u| \geq \gamma_0$, the sign of $\ell^{-1}(\gamma_u' - 1)\underline{u_{\gamma\gamma}}_L$ is the sign of γ_u. Thus, the contribution to the first integral from the two regions where $|\gamma_u| \geq \gamma_0$ is bounded above and below by expressions of the form

$$\operatorname{sign}(\gamma_u)\, C_2''\Big(\int_{\gamma_0}^{\epsilon^{1/2}\lambda^{-1/2}} \lambda\gamma(\gamma')^{-1}\, d\gamma + \int_{\epsilon^{1/2}\lambda^{-1/2}}^{\infty} \gamma^{-1}(\gamma')^{-1}\, d\gamma\Big).$$

By our estimate for γ' in Section 2, this is in turn bounded above and below by $\operatorname{sign}(\gamma_u)\, C$, where $C > 0$ is a λ-independent constant, and the sign is $-$ when $u = u_j$; $+$ when $u = u_{j-1}$. Meanwhile, the contribution from the region where $|\gamma_u| \leq \gamma_0$ is bounded by $C'\lambda$; therefore ignorable. In sum, we have the claimed estimate, and hence the assertion (b).

(c) Let $\eta = \hat{n}_{(\lambda,w_\chi)}(\alpha, \xi + \sum_{i=0}^{k+1}\varphi_i\mathfrak{e}_{u_i}) - \hat{n}_{(\lambda,w_\chi)}(\alpha', \xi' + \sum_{i=0}^{k+1}\varphi_i'\mathfrak{e}_{u_i})$ in (41). The second term in it, by (54) and (49), is bounded by $C\lambda^{1/(2q)}\lambda^{1/(2q)}|\hat{\varphi} - \hat{\varphi}'|$. On the other hand, by Hölder inequality and the same direct computation that appeared in the end of the proof of last lemma, the first term of (41) can be bounded in absolute value by writing $I_j = \big(\gamma_{u_{j-1}}^{-1}(0), \gamma_{u_j}^{-1}(0)\big)$

$$\begin{aligned}
&C\lambda^{1/(2q)}\Bigg(\sum_i |\varphi_i - \varphi_i'|\Big(\lambda(|\varphi_i| + |\varphi_i'|)\int_{I_j} \ell^{-1}\|(u_i)_\gamma\|_{2,t}^2 ds \\
&\qquad\qquad + (|\alpha| + |\alpha'|)\int_{I_j} \ell^{-1}\underline{(u_i)_\gamma}_L ds\Big) \\
&\quad + (\|\xi\|_{W_\chi} + \|\xi'\|_{W_\chi})\Big(\|\xi - \xi'\|_{W_\chi}\lambda^{-1/2+1/p} \\
&\qquad + \sum_i |\varphi_i - \varphi_i'|\|\ell^{-1/2}(u_i)_\gamma\|_{L^q(I_j\times S^1)} + |\alpha - \alpha'|\,\|\ell^{-1/2}\|_{L^q(I_j)}\Big) \\
&\quad + |\alpha - \alpha'|\Big((|\alpha| + |\alpha'|)\int_{I_j} \ell^{-1}\,ds + \sum_i (|\varphi_i| + |\varphi_i'|)\int_{I_j} \ell^{-1}\underline{(u_i)_\gamma}_L ds\Big) \\
&\quad + \|\xi - \xi'\|_{W_\chi}\Big(\sum_i (|\varphi_i| + |\varphi_i'|)\|\ell^{-1/2}(u_i)_\gamma\|_{L^q(I_j\times S^1)} \\
&\qquad\qquad + (|\alpha| + |\alpha'|)\|\ell^{-1/2}\|_{L^q(I_j)}\Big)\Bigg) \\
&\le C'\lambda^{1/(2q)}\Big(\lambda^{1/(2q)}|\hat\varphi - \hat\varphi'| + (\varepsilon_1 + C''\lambda^{1/(2p)})\|\xi - \xi'\|_{W_\chi}\Big) \\
&\le C_3\lambda^{1/q}|\hat\varphi - \hat\varphi'|.
\end{aligned}$$

In the above we again used (49), the estimate for ℓ in Lemma 3.3.2, and the estimates for γ_{u_i} and σ_χ in section 2. Summing up, this gives us assertion (c). □

4.1.4. Constructing the gluing map. It follows immediately from the previous lemma that the linearization of the Kuranishi map is surjective, and hence the moduli space is (Zariski) smooth. More precisely, choose

$$Q_\chi := \mathrm{Span}\{\mathfrak{e}_{u_1}, \dots, \mathfrak{e}_{u_{k+1}}\} \subset K_\chi \quad Q^\Xi := \bigcup_\chi Q_\chi.$$

The reductions of the K-models $[K_\chi \to C_\chi]_{W_\chi'}$, $[\hat K_\chi \to C_\chi]_{W_\chi'}$ by Q_χ give respectively the standard K-models for E_{w_χ} and $\hat E_{(\lambda, w_\chi)}$:

$$[\ker E_{w_\chi} \to *]_{Q_\chi \oplus W_\chi'}, \quad [\ker \hat E_{(\lambda, w_\chi)} \to *]_{Q_\chi \oplus W_\chi'}.$$

Indeed, from Lemma 4.1.3 (b), we see that the $(k+1)\times(k+1)$-matrix $E = (E_{ji})$,

$$E_{ji} := \lambda^{-1/(2q)} P_{\mathfrak{f}_j}(E_{w_\chi}\mathfrak{e}_{u_i}), \quad i, j \in \{1, \dots, k+1\}$$

is, up to ignorable terms, of the form

$$\begin{pmatrix} - & 0 & \cdots & \cdots & 0 \\ + & - & 0 & \cdots & 0 \\ 0 & + & \ddots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & + & - \end{pmatrix} \quad (+/- \text{ denote positive/negative numbers of } O(1)).$$

Thus, it has a uniformly bounded inverse, denoted (G_{ij}). Restricted to Q_χ (i.e., setting $\alpha = \varphi_0 = 0$), (47) can be rewritten in the form

$$\vec{\varphi} = \Psi(\vec{\varphi}), \quad \text{where } \vec{\varphi} := (\varphi_1, \dots, \varphi_{k+1}), \tag{55}$$

and $\Psi\colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is the map given by

$$(\Psi(\vec{\varphi}))_i = -\sum_{j=1}^{k+1} G_{ij} \lambda^{-1/(2q)} P_{\mathfrak{f}_j} \left(\bar{\partial}_{JX_\lambda} w_\chi + n_{w_\chi} \left(\xi(\vec{\varphi}) + \sum_{l=1}^{k+1} \varphi_l \mathfrak{e}_{u_l} \right) \right).$$

Note from the uniform boundedness of (G_{ij}) and Lemmas 4.1.2, 4.1.3 (a) that

$$\begin{aligned} |\Psi(\vec{0})| &\le \sum_j C\lambda^{-1/(2q)} \Big(|P_{\mathfrak{f}_j} \bar{\partial}_{JX_\lambda} w_\chi| + \|n_{w_\chi}(\xi(\vec{0}))\|_{L_\chi} \Big) \\ &\le C_2 \lambda^{-1/(2q)} (\lambda^{1/q} + \|\xi(\vec{0})\|^2_{W_\chi}) \\ &\le C_2' \lambda^{1/(2q)} \ll 1. \end{aligned} \tag{56}$$

On the other hand, Lemmas 4.1.2 and 4.1.3 (c) and the uniform boundedness of (G_{ij}) again imply:

$$|\Psi(\vec{\varphi}) - \Psi(\vec{\varphi}')| \le K|\vec{\varphi} - \vec{\varphi}'| \quad \text{for a positive constant } K \le C\lambda^{1/(2q)} \ll 1.$$

Thus by the contraction mapping theorem, we have a unique solution of (55) among all small enough $\vec{\varphi}$.

To summarize, for sufficiently small $\lambda_0 > 0$, there is a universal positive constant C_w, such that for all $\chi \in \Xi(\mathbb{S})$, there is a unique

$$(\vec{\varphi}_\chi, \xi_\chi) \in Q_\chi \oplus W'_\chi \quad \text{with } \|\xi_\chi\|^2_{W_\chi} + |\vec{\varphi}_\chi|^2 \le C_w,$$

which solves (46), (47). In fact, the solution satisfies

$$\|\xi_\chi + \sum_i \varphi_{\chi,i} \mathfrak{e}_{u_i}\|^2_{W_\chi} \le \|\xi_\chi\|^2_{W_\chi} + C'|\vec{\varphi}_\chi|^2 \le C\lambda^{1/q},$$

because of (56) and (48).

We define the gluing map to be the map from $\Xi(\mathbb{S})$ to $\hat{\mathcal{M}}^\Lambda_P$ sending

$$\chi \mapsto \exp\Big(w_\chi, \xi_\chi + \sum_{i=1}^{k+1} \varphi_{\chi,i} \mathfrak{e}_{u_i} \Big).$$

4.2. Surjectivity of the gluing map. With the gluing map constructed above, the standard arguments outlined in Section 1.2.7 shows that it is a local diffeomorphism onto $\hat{\mathcal{M}}_P^{\Lambda}(\mathbf{x},\mathbf{z})\cap U_\varepsilon$, the intersection of the moduli space with a tubular neighborhood U_ε of the image of the pregluing map in $\mathcal{B}$-*topology*.

As our goal is instead to show that the gluing map is a local diffeomorphism onto (the interior of) a neighborhood of $\mathbb{S}\subset\hat{\mathcal{M}}_P^{\Lambda,+}(\mathbf{x},\mathbf{z})$ in the coarser *chain topology*, we need to show that the latter neighborhood in fact lies in U_ε. This is done via the following variant of decay estimates for flows near y.

4.2.1. w in terms of $\tilde{w}$ and $\tilde{\xi}$. Let $(\lambda,\hat{w})\in\hat{\mathcal{M}}_P^{(-\lambda_0,\lambda_0),1}(\mathbf{x},\mathbf{z};\mathrm{wt}_{-\langle\mathcal{Y}\rangle,e_{\mathcal{P}}}\leq\Re)$ be in a chain-topology neighborhood of $\mathbb{S}\subset\hat{\mathcal{M}}_P^{\Lambda,+}(\mathbf{x},\mathbf{z})$. Namely, $|\lambda|\ll 1$, and there is a broken trajectory $\hat{\chi}:=\{u_0,u_1,\dots,u_{k+1}\}\in\mathbb{S}$, which is close to $\hat{w}$ in chain topology. Let $\chi:=(\hat{\chi},\lambda)\in\Xi(\mathbb{S})$. We may find a representative w of $\hat{w}$, such that

$$w(s)=\exp(\tilde{w}(s),\tilde{\xi}(s)),$$

where w, $\tilde{w}$ are chosen such that

- $\tilde{w}(s)=y$ at $s=\tilde{\mathfrak{l}}_1,\dots,\tilde{\mathfrak{l}}_{k+1}$; $\tilde{\mathfrak{l}}_1<\tilde{\mathfrak{l}}_2<\cdots<\tilde{\mathfrak{l}}_{k+1}$ subdivide $\mathbb{R}$ into $k+2$ open intervals I_i, $i=0,1,\dots,k+1$;
- $\tilde{w}(s)=u_i(\tilde{\gamma}_{u_i}(s))$ over I_i, where $\tilde{\gamma}_{u_i}:I_i\to\mathbb{R}$ are homeomorphisms determined by:

$$\begin{cases}\Pi_{\mathbf{e}_y}\tilde{\zeta}(s)=\Pi_{\mathbf{e}_y}\zeta(s) & \text{for } s\in\bigcup_i[\tilde{\gamma}_{u_{i-1}}^{-1}(\gamma_0),\tilde{\gamma}_{u_i}^{-1}(-\gamma_0)],\\ \tilde{\gamma}'_{u_i}(s)=1 & \text{for } s\in\mathbb{R}\backslash\bigcup_i[\tilde{\gamma}_{u_{i-1}}^{-1}(\gamma_0),\tilde{\gamma}_{u_i}^{-1}(-\gamma_0)],\end{cases}\tag{57}$$

  with ζ, $\tilde{\zeta}$ given by $\tilde{w}(s)=\exp(y,\tilde{\zeta}(s))$, $w(s)=\exp(y,\zeta(s))$, and γ_0 the large positive constant in Section 2.4.
- $$\gamma_{u_0}^{-1}(0)=\tilde{\gamma}_{u_0}^{-1}(0);\quad\langle(u_0)_\gamma(0),\tilde{\gamma}_{u_0}^{*}\tilde{\xi}(0)\rangle_{2,t}=0.\tag{58}$$

Because of elliptic regularity and the fact that (λ,w) is close to $\mathbb{S}$ in chain topology, we may assume without loss of generality that

$$\|\tilde{\xi}(s)\|_{\infty,1,t}+\|\tilde{\xi}'(s)\|_{\infty,1,t}<\varepsilon\ \forall s\quad\text{for } |\lambda|^{1/2}<\varepsilon\ll 1;$$
$$\tilde{\gamma}_{u_j}^{-1}(0)-\tilde{\gamma}_{u_{j-1}}^{-1}(0)>\varepsilon^{-1}\quad\forall j\in\{1,\dots,k+1\}.$$

4.2.2. Estimating $\tilde{\xi}$. Because of the large weights near y, we need the following more refined pointwise estimate for $\tilde{\xi}$ near y.

Lemma. *Let $(\lambda,w)\in\mathcal{M}_P^{\Lambda}$ be close to $\mathbb{S}$ in chain topology, and let y be a death as before. Then $\lambda>0$. Furthermore, in the notation of Section* 4.2.1, *there is a small positive constant $\varepsilon_0=\varepsilon_0(\gamma_0^{-1},\varepsilon)$ independent of w, such that*

$$\|\tilde{\xi}(s)\|_{2,2,t}+\|\tilde{\xi}'(s)\|_{2,1,t}\leq\varepsilon_0(\|\Pi_{\mathbf{e}_y}\tilde{\zeta}(s)\|_{2,1,t}^4+|\lambda|)\tag{59}$$

for all $s\in\bigcup_j[\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0),\tilde{\gamma}_{u_j}^{-1}(-\gamma_0)]$.

Proof. Let $s \in \bigcup_j [\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0), \tilde{\gamma}_{u_j}^{-1}(-\gamma_0)]$ throughout this proof. In fact, it suffices to consider only one j.

To estimate ξ, it is equivalent to estimate $\zeta - \tilde{\zeta}$, which we denote by c. The assumption (57) implies that $c \in \ker A_y^{\perp}$.

Write $b := \Pi_{\ker A_y}\tilde{\zeta}$, and let Z: $\ker A_y \to \ker A_y^{\perp}$ be such that $\tilde{\zeta} = (1+Z)b$. Similar to I.(38), I.(39), the flow equation can be rewritten as:

$$-\frac{d\tilde{\zeta}}{ds} = (1+\nabla_b Z)(\lambda C_y'\mathbf{e}_y + \Pi_{\ker A_y}\hat{n}_{(0,y)}(\lambda, \tilde{\zeta}+c)); \tag{60}$$

$$\begin{aligned} -\frac{dc}{ds} = {} & A_y c - \nabla_b Z(\lambda C_y'\mathbf{e}_y) \\ & + (1 - \Pi_{\ker A_y} - \nabla_b Z \Pi_{\ker A_y})(\hat{n}_{(0,y)}(\lambda, \tilde{\zeta}+c) - n_y(\tilde{\zeta})). \end{aligned} \tag{61}$$

Taking the L_t^2-inner product of (61) with c and rearranging like the proof of Sublemma I.5.1.7, we get (adopting the notation of I.5.1)

$$\frac{d\|c_+\|_{2,t}}{ds} \geq \mu_+\|c_+\|_{2,t} - \epsilon_+\|c\|_{2,t} - C_+|\lambda|\|\tilde{\zeta}\|_{2,1,t},$$
$$\frac{d\|c_-\|_{2,t}}{ds} \leq -\mu_-\|c_-\|_{2,t} + \epsilon_-\|c\|_{2,t} + C_-|\lambda|\|\tilde{\zeta}\|_{2,1,t}.$$

Subtracting a suitable multiple of the first inequality from the second, one obtains:

$$(\|c_-\|_{2,t} - \epsilon_-'\|c_+\|_{2,t})' \leq -\mu_-'\|c_-\|_{2,t} + C_-'\gamma_0^{-1}|\lambda|.$$

Taking convolution product with the integral kernel of $d/ds + \mu_-'$ on both sides, one gets

$$\|c_-\|_{2,t} - \epsilon_-'\|c_+\|_{2,t} \leq C_-\varepsilon \mathrm{e}^{-\mu'(s-\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0))} + C_-''\gamma_0^{-1}|\lambda|,$$

and similarly,

$$\|c_+\|_{2,t} - \epsilon_+'\|c_-\|_{2,t} \leq C_+\varepsilon \mathrm{e}^{-\mu'(\tilde{\gamma}_{u_j}^{-1}(-\gamma_0)-s)} + C_+''\gamma_0^{-1}|\lambda|.$$

Adding the above two inequalities, we get

$$\|c\|_{2,t} \leq C\varepsilon\Big(\mathrm{e}^{-\mu'(s-\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0))} + \mathrm{e}^{-\mu'(\tilde{\gamma}_{u_j}^{-1}(-\gamma_0)-s)}\Big) + C''\gamma_0^{-1}|\lambda|.$$

This may be improved to give a similar estimate for $\|c\|_{2,1,t}$ using (61) by the same elliptic bootstrapping and Sobolev embedding argument as in I.5.1.7.

On the other hand, write $b(s) = \underline{b}(s)\mathbf{e}_y$ as usual, and notice that by taking $\Pi_{\ker A_y}$ of (60), $b(s)$ satisfies:

$$-b'(s) = \lambda C_y'\mathbf{e}_y + \Pi_{\ker A_y}\hat{n}_{(0,y)}(\lambda, \tilde{\zeta}+c). \tag{62}$$

Integrating this equation, it is easy to see that $\lambda < 0$ would contradict the fact that, due to the proximity of w and $\tilde{w}$,

$$\underline{b}(\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0)) > 0, \quad \underline{b}(\tilde{\gamma}_{u_j}^{-1}(-\gamma_0)) < 0, \quad \tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0) < \tilde{\gamma}_{u_j}^{-1}(-\gamma_0).$$

Thus, λ must be positive.

On the other hand, as $\lambda > 0$, (62) implies that $\underline{b}(s)$ decreases monotonically with s. We now claim that

$$\mathrm{e}^{-\mu'(s-\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0))} + \mathrm{e}^{-\mu'(\tilde{\gamma}_{u_j}^{-1}(-\gamma_0)-s)} \leq C_e(\underline{b}(s)^4 + |\lambda|).$$

Combined with the above estimates for c, this would then imply the second assertion of the lemma.

To prove the claim, note that by symmetry and the decay/growth behavior of the two terms on the LHS, it suffices to show that

$$\underline{b}^4(s) \geq C_1\mathrm{e}^{-\mu'(\tilde{\gamma}_{u_j}^{-1}(-\gamma_0)-s)} \quad \text{when } -1 \ll \underline{b}(s) \leq -|\lambda|^{1/2};$$
$$\underline{b}^4(s) \geq C_2\mathrm{e}^{-\mu'(s-\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0))} \quad \text{when } 1 \gg \underline{b}(s) \geq |\lambda|^{1/2}$$

for s-independent constants C_1, C_2. We shall only demonstrate the second inequality since the first is similar. When $\underline{b}(s) \geq |\lambda|^{1/2}$, (62) together with the above estimate for $\|c\|_{2,1,t}$ imply that

$$(\underline{b}^4)' \geq -\mu'\underline{b}^4 - C_b\varepsilon^4\mathrm{e}^{-4\mu'(s-\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0))}.$$

Taking convolution product with the integral kernel of $d/ds + \mu'$, we get in this region

$$\underline{b}^4(s) \geq C_6\mathrm{e}^{-\mu'(s-\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0))} - C_7\varepsilon^4\mathrm{e}^{-4\mu'(s-\tilde{\gamma}_{u_{j-1}}^{-1}(\gamma_0))},$$

and hence the claim. □

4.2.3. From $\tilde{w}$ to w_χ. Next, notice that $\tilde{w}$ differs from the pregluing w_χ by a reparameterization. We shall estimate the difference between $\tilde{w}$ and w_χ by estimating the difference between $\tilde{\gamma}_{u_i}^{-1}$ and $\gamma_{u_i}^{-1}$.

Similar to γ_{u_i} (see Section 2.2.1), $\tilde{\gamma}_{u_i}$ satisfies:

$$\Big\langle u_\gamma(\tilde{\gamma}_{u_i}), -(\tilde{\gamma}'_{u_i} - 1)u_\gamma(\tilde{\gamma}_{u_i}) + \lambda Y_{(0,\tilde{w})} + E_{\tilde{w}}\tilde{\xi} + \tilde{n}(\lambda, \tilde{\xi})\Big\rangle_{2,t} = 0,$$

where $\tilde{n}$ is some nonlinear term in $\lambda, \tilde{\xi}$. By (57), when $\tilde{\gamma}_{u_i}(s) \geq |\gamma_0|$,

$$\begin{aligned}&\|u_\gamma(\tilde{\gamma}_{u_i})\|_{2,t}^{-1}|\langle u_\gamma(\tilde{\gamma}_{u_i}), E_{\tilde{w}}\tilde{\xi} + \tilde{n}(\lambda, \tilde{\xi})\rangle_{2,t}|\\ &\qquad \leq C_8(\|\tilde{\xi}\|_{2,1,t} + \|\tilde{\xi}'\|_{2,t})/\tilde{\gamma}_{u_i} + C_8'(\|\tilde{\xi}\|_{2,1,t} + \lambda)^2.\end{aligned}$$

Write $\gamma_{\lambda,u_i} = \gamma_{u_i}$ in (11) to emphasize the parameter λ used in the definition, and let

$$\Delta_{s,i}(\gamma) := \gamma_{\lambda,u_i}^{-1}(\gamma) - \tilde{\gamma}_{u_i}^{-1}(\gamma).$$

Comparing the defining equations for γ_{λ,u_i} and $\tilde{\gamma}_{\lambda,u_i}$, and using (59) and the above estimate for $\|u_\gamma(\tilde{\gamma}_{u_i})\|_{2,t}^{-1}|\langle u_\gamma(\tilde{\gamma}_{u_i}), E_{\tilde{w}}\tilde{\xi} + \tilde{n}(\lambda, \tilde{\xi})\rangle_{2,t}|$, we find that for all $i \in \{0, \dots, k\}$,

$$|\Delta_{s,i}(\gamma_+) - \Delta_{s,i}(\gamma_-)| \leq$$

$$\begin{cases} \left| \int_0^{\gamma_0} (O(\varepsilon) + O(\lambda))\, d\gamma \right| \leq C_1\varepsilon & \text{if } 0 \leq \gamma_+, \gamma_- \leq \gamma_0; \\ \left| \int_{\gamma_0}^{\mathfrak{r}_i} C_7(\varepsilon\gamma(\lambda + \gamma^{-4}) + (\lambda + \gamma^{-4})^2\gamma^2)\, d\gamma \right| \leq C_6\varepsilon & \text{if } \gamma_0 < \gamma_+, \gamma_- \leq \mathfrak{r}_i; \\ \left| \int_{\mathfrak{r}_i}^{\infty} \frac{C_5(\varepsilon\gamma(\lambda+\gamma^{-4})+(\lambda+\gamma^{-4})^2\gamma^2)\, d\gamma}{(1+\lambda\gamma^2)^2} \right| \leq C_4\varepsilon & \text{if } \gamma_+, \gamma_- \geq \mathfrak{r}_i. \end{cases} \tag{63}$$

Similar estimates hold for negative γ_+, γ_- when $i \in \{1, \dots, k+1\}$. For $i = 0$ and any two negative γ_+, γ_-, or for $i = k+1$ and any two positive γ_+, γ_-, the estimate in the first case above holds.

Combining this with the initial conditions from (58):

$$\Delta_{s,0}(0) = 0; \quad \Delta_{s,i-1}(\infty) = \Delta_{s,i}(-\infty)\ \forall i \in \{1, \dots, k+1\},$$

we have

$$|\Delta_{s,i}(\gamma)| \leq C\varepsilon\ \forall \gamma, i \text{ for a } \lambda\text{-independent constant } C > 0.$$

Applying the mean value theorem and the estimate for w'_χ in (20), and recalling the assumption (58), we see that $\tilde{w} = \exp(w_\chi, \tilde{\xi}_\chi)$ for an $\tilde{\xi}_\chi$ with:

$$\begin{aligned} &\langle (u_0)_\gamma, (0), (\gamma_{u_0}^{-1})^* \tilde{\xi}_\chi(0) \rangle_{2,t} = 0; \\ \|\tilde{\xi}_\chi\|_{2,1,t} \leq & C'(\lambda + \varepsilon\gamma_{u_i}^{-2}) \quad \text{on } I_i \cap \bigcup_j [\gamma_{u_{j-1}}^{-1}(\gamma_0), \gamma_{u_j}^{-1}(-\gamma_0)]\ \forall i. \end{aligned} \tag{64}$$

4.2.4. From pointwise estimates to $\hat{W}_\chi$ estimates. Recall that our goal is to show that given $(\lambda, \hat{w}) \in \hat{\mathcal{M}}_P^\Lambda$ in a chain-topology neighborhood of $\mathbb{S}$, as prescribed in Section 4.2.1, we may write

$$(\lambda, w) = e(\lambda', w_{\chi'}; \alpha_{\chi'}, \xi_{\chi'}) \tag{65}$$

for some $\chi' = (\underline{\chi}, \lambda') \in \Xi(\mathbb{S})$, $\underline{\chi} := \{\hat{u}_0, \dots, \hat{u}_{k+1}\}$, with $(\alpha_{\chi'}, \xi_{\chi'})$ satisfying

$$(1)\ \|(\alpha_{\chi'}, \xi_{\chi'})\|_{\hat{W}_{\chi'}} \leq C\varepsilon; \quad (2)\ (\alpha_{\chi'}, \xi_{\chi'}) \in B_{\chi'}, \tag{66}$$

where C is a λ-independent constant, and $B_{\chi'}$ is the B-space chosen so that $[\ker \hat{E}_{(\lambda', w_{\chi'})} \to *]_{B_{\chi'}}$ forms a K-model for $\hat{E}_{(\lambda', w_{\chi'})}$.

Lemma. *Suppose the $(\alpha_{\chi'}, \xi_{\chi'})$ given in (65) satisfies*

$$\begin{aligned} |\alpha_{\chi'}| &\leq C'\lambda^{3/2}\varepsilon; \\ \|\xi_{\chi'}\|_{2,1,t} &\leq C'_\xi(\lambda + \varepsilon\gamma_{u_i}^{-2}) \quad \text{on } I_i \cap \bigcup_j [\gamma_{u_{j-1}}^{-1}(\gamma_0), \gamma_{u_j}^{-1}(-\gamma_0)]\ \forall i \end{aligned} \tag{67}$$

for λ-independent constants C', C'_ξ. Then (66.1) holds.

Proof. First, notice that the assumption on $\alpha_{\chi'}$ implies $\|(\alpha_{\chi'},0)\|_{\hat{W}_{\chi'}} \leq C_1\lambda^{1/2-1/(2p)}\varepsilon$. On the other hand, the assumption that $(\lambda,\hat{w})$ is close to χ' in chain topology implies that over $\Theta^c := \Theta\backslash\bigcup_j[\gamma_{u_{j-1}}^{-1}(\gamma_0),\gamma_{u_j}^{-1}(-\gamma_0)]\times S^1$,

$$\|\xi_{\chi'}\|_{W_{\chi'}(\Theta^c)} \leq C_2\|\xi_{\chi'}\|_{L_1^p(\Theta^c)} \leq C_3\varepsilon \quad \text{for } \lambda\text{-independent constants } C_2, C_3.$$

Thus, it remains to estimate $\|\xi_{\chi'}\|_{W_{\chi'}([\gamma_{u_{j-1}}^{-1}(\gamma_0),\gamma_{u_j}^{-1}(-\gamma_0)]\times S^1)}$. We shall focus on estimates on the region $[\gamma_{u_i}^{-1}(\gamma_0),\gamma_{u_i}^{-1}(\infty))\times S^1$ for an $i\in\{0,\dots,k\}$, since estimates on the rest are similar.

By the definition of γ_{u_i}, on this region the flow equation has the form:

$$(\bar{\partial}_{JX_{\lambda'}}w_{\chi'})_{T_w} + \hat{E}_{(\lambda',w_{\chi'})}(\alpha_{\chi'},\xi_{\chi'}) + \hat{n}_{(\lambda',w_{\chi'})}(\alpha_{\chi'},\xi_{\chi'}) = 0. \tag{68}$$

(T_w here means the transverse component with respect to $w'_{\chi'}$, in contrast to T_y below).

Subdivide the region again into $[\gamma_{u_i}^{-1}(\mathfrak{r}_i),\gamma_{u_i}^{-1}(\infty))\times S^1$ and the rest.

Over the first region, namely when $\gamma_{u_i}\geq\mathfrak{r}_i$, in place of $\xi_{\chi'}$ and its $W_{\chi'}$-norm, it is equivalent to estimate

$$\xi_0 := T_{w_{\chi'},\bar{y}}\xi_{\chi'}\in\Gamma(\bar{y}^*K) \quad \text{in the norm}$$
$$\|\xi_0\|_{W_0^{\lambda'}} := (\lambda')^{-1/2}\|\xi_0\|_{p,1} + (\lambda')^{-1}\|(\xi_0)'_{L_y}\|_p,$$

where $(\xi_0)_{L_y}(s) = \langle\mathbf{e}_y,\xi_0(s)\rangle_{2,t}\mathbf{e}_y$ is the "longitudinal direction with respect to y." Notice that $\mathbf{e}_y$ differs from the original longitudinal direction $T_{w_{\chi'}(s),y}w'_{\chi'}(s)\|w'_{\chi'}(s)\|_{2,t}^{-1}$ by an ignorable factor of $C(\lambda'/\epsilon)^{1/2}$. Let the transversal direction T_y and the $L_0^{\lambda'}$-norm be similarly defined.

Rewriting the flow equation in terms of the above transverse and longitudinal directions, we have:

$$\begin{aligned}
E_{\bar{y}}(\xi_{0T_y}) &= -\alpha_{\chi'}(T_{w_{\chi'},\bar{y}}Y_{(\lambda,w_{\chi'})})_{T_y} + Z_{T_y} + \Upsilon_{T_y}\\
&\qquad - (T_{w_{\chi'},\bar{y}}\hat{n}_{(\lambda',w_{\chi'})}(\alpha_{\chi'},\xi_{\chi'}))_{T_y},\\
E_{\bar{y}}(\xi_{0L_y}) &= -\alpha_{\chi'}(T_{w_{\chi},\bar{y}}Y_{(\lambda,w_{\chi'})})_{L_y} + Z_{L_y} + \Upsilon_{L_y}\\
&\qquad - (T_{w_{\chi'},\bar{y}}\hat{n}_{(\lambda',w_{\chi'})}(\alpha_{\chi'},\xi_{\chi'}))_{L_y},
\end{aligned}\tag{69}$$

where

- Z_{T_y}, Z_{L_y} come from the difference between $E_{\bar{y}}$ and $E_{w_{\chi'}}$. Thus, their $L_{1,t}^2$-norms are bounded by $C\|\xi_{\chi'}\|_{\infty,1,t}\gamma_{u_i}^{-1}$;
- Υ_T, Υ_L are terms coming from $(\bar{\partial}_{JX_{\lambda'}}w_{\chi'})_{T_w}$. The computation in the proof of Lemma 2.4.2 shows that $\|\Upsilon_{T_y}\|_{2,1,t}$ is bounded by $C_1\lambda'\gamma_{u_i}^{-1}$, while $\|\Upsilon_{L_y}\|_{2,1,t}$ is bounded by $C_2\lambda'\gamma_{u_i}^{-2}$.

Now, the length estimate for $[\gamma_{u_i}^{-1}(\mathfrak{r}_i), \gamma_{u_i}^{-1}(\infty)) \times S^1$ (cf. (21)) and the assumption (67) yield

$$(\lambda')^{-1/2}\Big(\|\xi_0\|_{L^p([\gamma_{u_i}^{-1}(\mathfrak{r}_i),\gamma_{u_i}^{-1}(\infty))\times S^1)} + \|\dot{\xi}_0\|_{L^p([\gamma_{u_i}^{-1}(\mathfrak{r}_i),\gamma_{u_i}^{-1}(\infty))\times S^1)}\Big) \\ \leq C'(\lambda')^{1/2-1/(2p)} \ll \varepsilon.$$

In addition, the second line of (69) and the above estimates for terms therein, combined with (67) and the length estimate for this region yield

$$(\lambda')^{-1}\|\xi'_{0L_y}\|_{L^p([\gamma_{u_i}^{-1}(\mathfrak{r}_i),\gamma_{u_i}^{-1}(\infty))\times S^1)} \leq C'_L(\lambda')^{1/2-1/(2p)} \ll \varepsilon.$$

In sum, we have

$$\|\xi_{\chi'}\|_{W_{\chi'}([\gamma_{u_i}^{-1}(\mathfrak{r}_i),\gamma_{u_i}^{-1}(\infty))\times S^1)} \leq C_1\|\xi_0\|_{W_0^{\lambda'}([\gamma_{u_i}^{-1}(\mathfrak{r}_i),\gamma_{u_i}^{-1}(\infty))\times S^1)} \ll \varepsilon.$$

To estimate on the second region, namely on $[\gamma_{u_i}^{-1}(\gamma_0), \gamma_{u_i}^{-1}(\mathfrak{r}_i)] \times S^1$, let $\beta_i^+(s)$ be a smooth cutoff function with:

- support on $[\gamma_{u_i}^{-1}(\gamma_0) - 1, (1+\epsilon)\gamma_{u_i}^{-1}(\mathfrak{r}_i)] =: \Theta_{\beta_i^+}$;
- value 1 over $[\gamma_{u_i}^{-1}(\gamma_0), \gamma_{u_i}^{-1}(\mathfrak{r}_i)]$, and
- $|(\beta_i^+)'| < C'\lambda^{1/2}$ on $[\gamma_{u_1}^{-1}(\mathfrak{r}_i), (1+\epsilon)\gamma_{u_i}^{-1}(\mathfrak{r}_i)]$.

Notice that $(\gamma_{u_i}^{-1})^*(\beta_i^+\xi_{\chi'}) \in W'_{u_i}$, and since $E_{u_i}|_{W'_{u_i}}$ an isomorphism, we have from Lemma 3.2.1 and (68) that

$$\begin{aligned}\|\beta_i^+\xi_{\chi'}\|_{W_{\chi'}} &\leq \|E_{w_\chi}(\beta_i^+\xi_{\chi'})\|_{L_{\chi'}} \\ &\leq |\alpha_{\chi'}|\|\beta_i^+ Y_{(\lambda',w_{\chi'})}\|_{L_{\chi'}} + \|\beta_i^+(\bar{\partial}_{JX_{\lambda'}}w_{\chi'})_{T_w}\|_{L_{\chi'}} \\ &\qquad + \|\beta_i^+\hat{n}_{(\lambda',w_{\chi'})}(\alpha_{\chi'},\xi_{\chi'})\|_{L_{\chi'}} + \|(\beta_i^+)'\xi_{\chi'}\|_{L_{\chi'}}.\end{aligned}$$

By assumption (67) and the length estimate $|\gamma_{u_i}^{-1}(\mathfrak{r}_i) - \gamma_{u_i}^{-1}(\gamma_0)| \leq C'(\lambda')^{-1/2}$, we may bound each terms on the RHS as follows:

- The first term may be bounded by $C_1|\lambda'|^{1/2-1/(2p)}$.
- The second term is already estimated to be small in Proposition 2.4.1.
- The computation in the proof of Lemma 2.5.2 shows that

$$\begin{aligned}&\|\beta_i^+\hat{n}_{(\lambda',w_{\chi'})}(\alpha_{\chi'},\xi_{\chi'})\|_{L_{\chi'}} \\ &\leq C_n\begin{pmatrix}\|(\alpha_{\chi'},0)\|^2_{\hat{W}_{\chi'}} + \|(\alpha_{\chi'},0)\|_{\hat{W}_{\chi'}}\cdot\|\beta_i^+\xi_{\chi'}\|_{W_{\chi'}} \\ + \|\sigma_{\chi'}^{1/2}\xi_{\chi'}\|_{L^\infty(\Theta_{\beta_i^+})}\cdot\|\beta_i^+\xi_{\chi'}\|_{W_{\chi'}}\end{pmatrix} \\ &\leq C'_n\Big(C_1(|\lambda'|^{1/2-1/(2p)})^2 + C_2(|\lambda'|^{1/2-1/(2p)} + \varepsilon\gamma_0^{-1} + \lambda^{1/2})\|\beta_i^+\xi_{\chi'}\|_{W_{\chi'}}\Big).\end{aligned}$$

- By the defining properties of β_i^+,

$$\|(\beta_i^+)'\xi_{\chi'}\|_{L_{\chi'}} \leq C''|\lambda'|^{1/2-1/(2p)}.$$

Collecting all the above and rearranging, we obtain

$$\|\xi_{\chi'}\|_{W_{\chi'}([\gamma_{u_i}^{-1}(\gamma_0),\gamma_{u_i}^{-1}(\mathfrak{r}_i)]\times S^1)} \leq C_i\varepsilon.$$

Now that we have the estimates for the $W_{\chi'}$-norm over all the various regions, we conclude $\|\xi_{\chi'}\|_{W_{\chi'}} \leq C\varepsilon$, and hence the claim of the lemma. □

4.2.5. Concluding the proof of Proposition 2.1 (a). Recall from Section 4.1.1 the K-model for $\hat{E}_{(\lambda',w_{\chi'})}$: $[\ker \hat{E}_{(\lambda',w_{\chi'})} \to *]_{B_{\chi'}}$, where $B_{\chi'}$ was chosen to be the following subspace of $\hat{W}_{\chi'}$:

$$B_{\chi'} = \Big\{(0,\xi_{\chi'}) \,\Big|\, \langle (u_0)_\gamma(0), (\gamma_{u_0}^{-1})^*\xi_{\chi'}(0)\rangle_{2,t} = 0\Big\}.$$

Thus, setting $\lambda' = \lambda$ and $\chi' = \chi$, $(\alpha_{\chi'},\xi_{\chi'}) = (0,\xi_\chi)$, and ξ_χ can be expressed in terms of $\tilde{\xi}_\chi$ (cf. Section 4.2.3) and $\tilde{\xi}$ (cf. Section 4.2.2). In particular, by (58) and the first line of (64), (66.2) holds. On the other hand, combining Lemma 4.2.2 and the second line of (64), we see that the assumption (67) holds, and therefore Lemma 4.2.4 implies the validity of (66.1). The arguments in Section 1.2.7 then complete the last step of the proof of Proposition 2.1 (a). □

4.3. Gluing broken orbits. We now discuss the modification needed for the proof of Proposition 2.1 (b).

Given a broken orbit $\{\hat{u}_1,\hat{u}_2,\ldots,\hat{u}_k\}$ connected at y, and an $\lambda \in (0,\lambda_0)$, the pregluing w_χ associated to $\chi = (\{\hat{u}_1,\hat{u}_2,\ldots,\hat{u}_k\},\lambda) \in \Xi(\mathbb{S})$ is given by

$$w_\chi = \underline{w}_\chi,$$

where $\underline{w}_\chi$ is given by the same formula (11), except that now $i \in \{1,\ldots,k\}$ only, and instead of taking values in $\mathbb{R}$, s now takes value in $\mathbb{R}/T_\chi\mathbb{Z}$, where

$$T_\chi := \mathfrak{l}_{k+1}.$$

With this explained, the material in Sections 2 and 3 transfers directly to the case of broken orbits, but the discussion in Sections 4.1 and 4.2 above requires the following modification.

4.3.1. Constructing the gluing map. At a closed orbit $(\lambda,(T,w)) \in \mathcal{B}_O^\Lambda = \Lambda\times\mathcal{B}_O$, the deformation operator is $\hat{D}_{(\lambda,(T,w))}$: $\mathbb{R}_\alpha \oplus \mathbb{R}_\varrho \oplus L_1^p(w^*K) \to L^p(w^*K)$,

$$\hat{D}_{(\lambda,(T,w))}(\alpha,\varrho,\xi) = \alpha\partial_\lambda\check{\theta}_{X_\lambda} + \tilde{D}_{(T,w)}(\varrho,\xi),$$

namely, it is a rank-2 stabilization of D_w (cf. Section 3.3.1). ($\mathbb{R}_\alpha$, $\mathbb{R}_\varrho$ above, respectively, parameterize variation in λ and in the period T). In our context, this operator is a map between the weighted spaces

$$\hat{W}_\chi := \mathbb{R}_\alpha \oplus \mathbb{R}_\varrho \oplus W_\chi \quad \text{and} \quad L_\chi.$$

Let $K_\chi = \mathrm{Span}\{\mathfrak{e}_{u_i}\}_{i=1}^k$, $C_\chi = \mathrm{Span}\{\mathfrak{f}_j\}_{j=1}^k$. The analog of Proposition 3.1.3 shows that $[K_\chi \to C_\chi]$ forms a K-model for D_{w_χ}, which induces a K-model for $\hat{D}_{(\lambda,(T_\chi,w_\chi))}$ by stabilization.

However, since s is now periodic instead of real, the matrix $\lambda^{-1/2+1/(2p)}(P_{\mathfrak{f}_j} D_{w_\chi}\mathfrak{e}_{u_i})$ is no longer (approximately) triangular, and hence not clearly uniformly invertible. Consequently, it is no longer clear that, with the choice of Q_χ in Section 4.1, reduction by Q_χ gives another K-model for the deformation operator. Instead, use the following subspace $Q_{O,\chi} \subset \hat{W}_\chi$:

$$Q_{O,\chi} = \mathbb{R}_\alpha \oplus * \oplus \mathrm{Span}\{\mathfrak{e}_{u_i}\}_{i=1}^{k-1}.$$

Note that from (52) and the uniform boundedness of $P_{\mathfrak{f}_j}$

$$C_{\alpha-}\lambda^{1/2-1/(2p)} \leq \lambda^{3/2} P_{\mathfrak{f}_j}\partial_\lambda \check{\theta}_{X_\lambda}(w_\chi) \leq C_{\alpha+}\lambda^{1/2-1/(2p)}$$

for constants $C_{\alpha\pm} > 0$ independent of λ. Supplementing Lemma 4.1.3 (b) with this additional estimate, we see that the matrix representation of the operator

$$\lambda^{-1/2+1/(2p)}\Pi_{C_\chi}\hat{D}_{(\lambda,(T_\chi,w_\chi))}|_{Q_{O,\chi}}$$

with respect to the bases

$$\Big\{(\lambda^{3/2},0,0),(0,0,\mathfrak{e}_1),\dots,(0,0,\mathfrak{e}_{k-1})\Big\}, \quad \{\mathfrak{f}_1,\dots,\mathfrak{f}_k\}$$

is, modulo ignorable terms, of the form

$$\begin{pmatrix} + & - & 0 & \cdots & 0 \\ + & + & - & 0 & 0 \\ + & 0 & + & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & - \\ + & 0 & \cdots & 0 & + \end{pmatrix} \quad (+/- \text{ denote positive/negative numbers of } O(1)),$$

which is easily seen to have a uniformly bounded right inverse. Thus, the rest of Section 4.1 may be repeated with Q_χ replaced by $Q_{O,\chi}$ to define a gluing map in this case, which is also a local diffeomorphism.

4.3.2. Surjectivity of the gluing map. As the choice of $Q_{\chi'}$ is changed, the definition of the B-space $B_{\chi'}$ changes accordingly. In this situation,

$$B_{\chi'} = \Big\{(\alpha',0,\xi_{\chi'}) \,\Big|\, \langle (u_k)_\gamma(0), (\gamma_{u_k}^{-1})^*\xi_{\chi'}(0)\rangle_{2,t} = 0\Big\} \subset \hat{W}_{\chi'}.$$

(Note that in the case of broken orbits, $i \in \mathbb{Z}/k\mathbb{Z}$, thus $u_0 = u_k$.) The work in Section 4.2 needs corresponding modification.

Given a $(\lambda,(T,w)) \in \hat{\mathcal{M}}_O^\Lambda$ close to the broken orbit $\{\hat{u}_1,\dots,\hat{u}_k\}$, one may define $\tilde{w}$ and $\tilde{\gamma}_{u_i}$ in essentially the same way as Section 4.2.1, and the estimates in Section 4.2.2 still hold. However, for $\chi = (\{\hat{u}_1,\dots,\hat{u}_k\},\lambda)$, the period T_χ of the pregluing w_χ differs from those of $\tilde{w}$ or w. Thus, instead of comparing with w_χ, we compare w or $\tilde{w}$ with $w_{\chi'}$, where $\chi' = (\{\hat{u}_1, \ldots, \hat{u}_k\}, \lambda')$, and $\lambda' = \lambda + \alpha'$ is chosen so that the period of $w_{\chi'}$ agrees with the period of w (which is also the period of $\tilde{w}$). With this choice of χ', assumption (58), together with the definition of $B_{\chi'}$ above, implies (66.2).

Moreover, the length estimates in Section 2.2.1 show that

$$C_-\lambda^{-1/2} \leq T_\chi \leq C_+\lambda^{-1/2};$$

combining with the estimate for the difference in periods of w and w_χ given by (63), we have:

$$|\alpha'| \leq C\lambda^{3/2}\varepsilon. \tag{70}$$

For such $\lambda' = \lambda + \alpha'$, the difference $\gamma^{-1}_{\lambda', u_i} - \tilde{\gamma}^{-1}_{u_i}$ satisfies estimates similar to (63). Thus, (67) holds for this choice of χ', which in turn implies (66.1), via Lemma 4.2.4.

5. Gluing at births

The purpose of this section is to prove Proposition 5.1 below. The proof is in many ways similar to the proof of Proposition 2.1, but simpler in Step 2, since here we glue only a single flow line, and the generalized cokernel in this case is trivial.

5.1. Statement of the gluing theorem. The next proposition verifies part of (RHFS2c, 3c) for admissible (J, X)-homotopies.

Proposition. *Let (J^Λ, X^Λ) be an admissible (J, X)-homotopy connecting two regular pairs, and $\mathbf{x}, \mathbf{z}$ be two path components of $\mathcal{P}^\Lambda \backslash \mathcal{P}^{\Lambda,\deg}$. Then a chain-topology neighborhood of $\mathbb{J}_P(\Lambda, \mathbf{x}, \mathbf{z}; \Re)$ in $\hat{\mathcal{M}}^{\Lambda,1,+}_P(\mathbf{x}, \mathbf{z}; \mathrm{wt}_{-\langle \mathcal{Y}\rangle, e_{\mathcal{P}}} \leq \Re)$ is l.m.b. along $\mathbb{J}_P(\Lambda, \mathbf{x}, \mathbf{z}; \Re)$.*

Furthermore, Π_Λ maps these neighborhoods to birth-neighborhoods.

We shall restrict our attention to

$$\hat{u} \in \hat{\mathcal{M}}^0_{P,\lambda}((\underline{\mathbf{x}, [\mathbf{w}]}), (\underline{\mathbf{z}, [\mathbf{v}]})) \cap \mathbb{J}_P(\Lambda, \mathbf{x}, \mathbf{z}; \Re),$$

where one of x_λ and z_λ is a death–birth, and $\mathrm{gr}_+((x_\lambda, [w_\lambda]), (z_\lambda, [v_\lambda])) = \mathrm{gr}((\mathbf{x}, [\mathbf{w}]), (\mathbf{z}, [\mathbf{v}]))$. The other cases follow either from standard gluing theory or from structure theory of parameterized moduli spaces, since the flow lines decay exponentially to the critical points in these cases.

Without loss of generality, assume as in Sections 2–4 that $\lambda = 0$, that $z_\lambda = y$ is in a standard death–birth neighborhood, and that the (J, X)-homotopy is oriented such that $C'_y > 0$. Under these assumptions, a birth neighborhood is $(-\lambda_0, 0) \subset \Lambda$, for a small $\lambda_0 > 0$.

Our goal is thus to construct a gluing map from $\Xi(\mathbb{S})$ to $\hat{\mathcal{M}}^{\Lambda,1}_P(\mathbf{x}, \mathbf{z}; \mathrm{wt}_{-\langle \mathcal{Y}\rangle, e_{\mathcal{P}}} \leq \Re)$, where

$$\mathbb{S} = \hat{\mathcal{M}}^0_{P,0}(\underline{\mathbf{x}}, \underline{\mathbf{y}}; \mathrm{wt}_{-\langle \mathcal{Y}\rangle, e_{\mathcal{Y}}} \leq \Re); \quad \Xi(\mathbb{S}) = \mathbb{S} \times (-\lambda_0, 0) \quad \text{for a small } \lambda_0 > 0.$$

We shall again focus on a single $\hat{u} \in \mathbb{S}$, since in this case $\mathbb{S}$ also consists of finitely many isolated points. Notice that when $x_0 = y$, $\hat{u}$ can be the constant flow at y, $\bar{y}$. The argument required for this case is somewhat different from the other cases. We discuss this case in Section 5.3, and the other cases in Section 5.2.

5.2. When $u \neq \bar{y}$. Assume without loss of generality that $x_0 \neq y$ is nondegenerate, so that we may concentrate on the region where $s > 0$.

5.2.1. Pregluing. Let $\chi := (\lambda, \hat{u}) \in \Xi(\mathbb{S})$ as above, and let u be a centered representative. Write

$$u(s) = \exp(y, \mu(s)) \quad \text{for large } s,$$

and as in I.5.3.2, let

$$y_{\lambda-} = \exp(y, \eta_{\lambda-}) \in \mathcal{P}_\lambda$$

be the critical point near y of index $\mathrm{ind}_-(y)$. Note that $\langle \mathbf{e}_y, \mu\rangle_{2,t}(s) > 0$ is a decreasing function for large s, sending (s_0, ∞) to $(C, 0)$ for some positive numbers s_0, C. Since $\langle \mathbf{e}_y, \eta_{\lambda-}\rangle_{2,t}$ is a small positive number, it equals $\langle \mathbf{e}_y, \mu(s)\rangle_{2,t}$ for certain large $s = \breve{\gamma}_\lambda$. From the estimates in Lemma I.5.3.2 and Proposition I.5.1.3, we have

$$C|\lambda|^{-1/2} \leq \breve{\gamma}_\chi \leq C'|\lambda|^{-1/2}.$$

Let $R < \breve{\gamma}_\chi - 1$ be a λ-independent large positive number such that $u(s)$ is close to y for $s \geq R$, and set $R_\pm = \pm C_0|\lambda|^{-1/2}$ for some λ-independent constant $C_0 > 0$. Define $u_\lambda \in \Gamma((-\infty, \breve{\gamma}_\chi) \times S^1, p_2^* T_f)$ by

$$u_\lambda(s) := \begin{cases} e_{R_-,R_+}(0, u; \lambda, 0) & \text{when } s \leq R/2, \\ \exp\Big(y, \mu(s) + \beta(s-R)\Pi^\perp_{\ker A_y}\eta_{\lambda-}\Big) & \text{when } s \geq R/2, \end{cases} \tag{71}$$

Lemma. *There is a function $\gamma_\chi(s)$ defining a homeomorphism from $\mathbb{R}$ to $(-\infty, \breve{\gamma}_\chi)$, such that*

$$\begin{cases} \langle w'_\chi(s), \bar{\partial}_{J,X_\lambda} w_\chi(s)\rangle_{2,t} = 0; & \textit{when } s \in [\gamma_\chi^{-1}(0), \infty); \\ \gamma'_\chi = 1 & \textit{otherwise.} \end{cases} \tag{72}$$

Proof. Write $\frac{d\gamma_\chi}{ds} = h(\gamma_\chi)$, where

$$h(\gamma) := \begin{cases} 1 - \langle (u_\lambda)_\gamma, \bar{\partial}_{JX_\lambda}(u_\lambda)\rangle_{2,t} \|(u_\lambda)_\gamma\|^{-2}_{2,t} & \text{when } s \in [\gamma_\chi^{-1}(0), \infty); \\ 1 & \text{otherwise.} \end{cases}$$

We now examine the behavior of $\bar{\partial}_{JX_\lambda} u_\lambda(\gamma)$ near $\gamma = \breve{\gamma}_\chi$. Here since u_λ is close to $y_{\lambda-}$, expanding $\bar{\partial}_{JX_\lambda}$ about $y_{\lambda-}$ and writing $\exp(y_{\lambda-}, \mu_\lambda(\gamma)) = u_\lambda(\gamma)$, we have

$$T_{u_\lambda, y_{\lambda-}} \bar{\partial}_{JX_\lambda} u_\lambda(\gamma) = (\mu_\lambda)_\gamma + A_{y_{\lambda-}}\mu_\lambda + n_{y_{\lambda-}}(\mu_\lambda).$$

By definition, $\mu_\lambda(\breve{\gamma}_\chi) = 0$; hence $h(\breve{\gamma}_\chi) = 0$. Thus,

$$h(\gamma) = \Big\langle (u_\lambda)_\gamma(\gamma), T_{y_{\lambda-},u_\lambda} A_{y_{\lambda-}} T^{-1}_{y_\lambda,u_\lambda}((\breve{\gamma}_\chi - \gamma)((u_\lambda)_\gamma(\gamma))\Big\rangle_{2,t} \|(u_\lambda)_\gamma(\gamma)\|^{-2}_{2,t} + O\Big(|\breve{\gamma}_\chi - \gamma|^2 \|(u_\lambda)_\gamma(\gamma)\|_{2,1,t}\Big).$$

By the estimate for minimal eigenvalue of $A_{y_{\lambda-}}$ in I.5.3.2, this is bounded above and below by multiples of $|\lambda|^{1/2}(\breve{\gamma}_\chi - \gamma)$. Integrating like (15), we see that for large s

$$C_5' \mathrm{e}^{-c_6'|\lambda|^{1/2}s} \le \breve{\gamma}_\chi - \gamma_\chi \le C_5 \mathrm{e}^{-c_6|\lambda|^{1/2}s}, \tag{73}$$

while on the other end $\gamma_\chi(s) = s + c_\lambda$ for some constant c_λ. We define γ_χ such that $\gamma_\chi(s) = s$ for $s < 0$. □

Definition. The *pregluing* w_χ corresponding to gluing data $\chi = (\lambda, u)$ is

$$w_\chi(s) := u_\lambda(\gamma_\chi(s)).$$

5.2.2. The weighted norms. The norms W_χ, L_χ here are defined by the same formulae in Definition 2.3.1, with the *weight function* σ_χ replaced by

$$\sigma_\chi(s) := \begin{cases} \|w'_\chi(\gamma_\chi^{-1}(0))\|^{-1}_{2,t} & \text{when } s \le \gamma_\chi^{-1}(0), \\ \|w'_\chi(s)\|^{-1}_{2,t} & \text{when } \gamma_\chi^{-1}(0) \le s \le \gamma_\chi^{-1}(\mathfrak{r}_\chi), \\ \|w'_\chi(\gamma_\chi^{-1}(\mathfrak{r}_\chi))\|^{-1}_{2,t} & \text{when } s \ge \gamma_\chi^{-1}(\mathfrak{r}_\chi), \end{cases}$$

where $\mathfrak{r}_\chi = \mathfrak{r}_\chi(\lambda, \epsilon) = C_\tau(\lambda/\epsilon)^{-1/2} < \breve{\gamma}_\chi$ is chosen such that $1 - h(s) \le \epsilon$ where $s \le \mathfrak{r}_\chi$ for a small positive number ϵ.

We shall frequently call on the following useful facts.

(a) In this case, $\gamma'_\chi \le 1$.
(b) $\|\Pi^\perp_{\ker A_y} \eta_{\lambda-}\|_{2,2,t} \le C|\lambda|$ by I.(55), Lemma I.5.3.2, and the decay estimates in Proposition I.5.1.3.
(c) $\sigma_\chi \le C|\lambda|^{-1}$.

In particular, Fact (b) often implies that in addition to estimates analogous to those in the proof of Proposition 2.1, the extra terms introduced by the cutoff function β in the definition of u_λ is usually ignorable.

5.2.3. Error estimate. Proceeding to Step 1 of the gluing theory, we have Lemma 5.2.3.

Lemma. $\|\bar{\partial}_{JX_\lambda} w_\chi\|_{L_\chi} \le C\lambda^{1/2-1/(2p)}$.

Proof. Consider the two regions (a) $\gamma_\chi^{-1}(-R_-) \le s \le \gamma_\chi^{-1}(R)$, (b) $\gamma_\chi^{-1}(R) \le s \le \infty$ separately. The point is to expand $\bar{\partial}_{JX_\lambda} w_\chi(s) = \tilde{\Pi}^\perp_{u'_\lambda}(\bar{\partial}_{JH_\lambda} u_\lambda(\gamma_\chi(s)))$ differently in the two regions: expand u_λ about u in region (a) and about y_λ in region (b).

In region (a), modulo terms coming from $\beta(\gamma_\chi - R)\Pi^\perp_{\ker A_y}\eta_{\lambda-}$, the estimate of the norm is entirely parallel to that in Proposition 2.4.1. The time w_χ spends in this region is

$$\gamma_\chi^{-1}(R) - \gamma_\chi^{-1}(R_-) \leq C|\lambda|^{-1/2};$$

the L^∞-norm of $\sigma_\chi\bar\partial_{JX_\lambda}w_\chi$ can be estimated as in Case 1 of Lemma 2.4.2, with γ_0 replaced by R.

On the other hand, since σ_χ has a λ-independent uniform bound in this region, and the L^p_t-norm of the contribution to $\bar\partial_{JX_\lambda}w_\chi$ from the extra terms introduced by $\beta\Pi^\perp_{\ker A_y}\eta_{\lambda-}$ can be bounded by $C\|\Pi^\perp_{\ker A_y}\eta_{\lambda-}\|_{2,1,t} \leq C'|\lambda|$, the contribution from these terms to $\|\sigma_\chi\bar\partial_{JX_\lambda}w_\chi\|_p$ is thus bounded by $C|\lambda|$.

For region (b), w_χ spends infinite amount of time here; however

$$\begin{aligned}\Pi^\perp_{(u_\lambda)_\gamma}\bar\partial_{JX_\lambda}u_\lambda(\gamma) = \Pi^\perp_{(u_\lambda)_\gamma}\Big((\delta\gamma)T_{y_{\lambda-},u_\lambda}A_{y_\lambda}(T_{y_{\lambda-},u_\lambda})^{-1}(u_\lambda)_\gamma(\bar\gamma))\\ + T_{y_{\lambda-},u_\lambda}n_{y_\lambda}((\delta\gamma)(T_{y_{\lambda-},u_\lambda})^{-1}(u_\lambda)_\gamma(\bar\gamma))\Big),\end{aligned}$$

where $\delta\gamma := \breve\gamma_\chi - \gamma$; $\gamma \leq \bar\gamma \leq \breve\gamma_\chi$. On the other hand, in this region, $\sigma_\chi(s) \leq C|\lambda|^{-1}$. Thus by Lemma I.5.3.2, on this region $\|\sigma_\chi\bar\partial_{JX_\lambda}w_\chi\|_p$ is bounded by $C\|(\delta\gamma)\|_p|\lambda|^{1/2} \leq C'|\lambda|^{1/2-1/(2p)}$, since $\delta\gamma \leq C_5 e^{-C_6|\lambda|^{1/2}s}$ by (73). □

5.2.4. Existence and uniform boundedness of the right inverse G_χ: $L_\chi \to W_\chi$ of E_{w_χ}. We now proceed to Step 2 of the proof. In this case, $W'_\chi \subset W_\chi$ is

$$W'_\chi := \Big\{\xi \in W_\chi \,|\, \langle\nu(0), \xi(\gamma_\chi^{-1}(0))\rangle_{2,t} = 0 \text{ for all } \nu \in \ker E_u\Big\},$$

and we aim to show that there is a uniformly bounded isomorphism G_χ: $L_\chi \to W'_\chi$ which is a right inverse of E_{w_χ}. Assume the opposite that there is a sequence $\{\xi_\lambda \in W'_\chi\}_\lambda$ satisfying

$$\begin{aligned}&\|\xi_\lambda\|_{W_\chi} = 1;\\ &\|E_{w_\chi}\xi_\lambda\|_{L_\chi} =: \varepsilon_E(\lambda) \to 0 \quad \text{when } \lambda \to 0.\end{aligned} \tag{74}$$

Divide $\Theta = \mathbb{R} \times S^1$ into two parts Θ_u, Θ_{y-}, separated by the line $s = \gamma_\chi^{-1}(\mathfrak{r}_\chi)$. Let $\Theta'_u := (-\infty, \gamma_\chi^{-1}(\mathfrak{r}_\chi) + 1) \times S^1 \supset \Theta_u$; $\Theta'_{y-} := (\gamma_\chi^{-1}(\mathfrak{r}_\chi - 1), \infty) \times S^1 \supset \Theta_{y-}$.

On $\underline{\Theta'_u}$, we define $\xi_{\lambda,u} \in \Gamma(u^*K)$ by

$$T_{u,u_\lambda}\xi_{\lambda,u}(\gamma_\chi(s)) = \xi_\lambda(s).$$

Let $(\xi_{\lambda,u})_L$ be the projection of $\xi_{\lambda,u}$ to the direction of u' and let $(\xi_{\lambda,u})_T = \xi_{\lambda,u} - (\xi_{\lambda,u})_L$. Let β_u be a smooth cutoff function supported on $\gamma_\chi(\Theta'_u)$ with value 1 on $\gamma_\chi(\Theta_u)$. Arguing as in the proof of Proposition 3.1.3, one obtains:

$$\begin{aligned}\|\xi_\lambda\|_{W_\chi(\Theta_u)} &\le C\|\xi_{\lambda,u}\|_{W_u(\gamma_\chi(\Theta_u))}\\ &\le C'\|\beta_u E_u \xi_{\lambda,u}\|_{L_u(\gamma_\chi(\Theta'_u))} + C\|\beta'_u(\xi_{\lambda,u})_T\|_{L_u(\gamma_\chi(\Theta'_u))}\\ &\le C'(1+2\epsilon)\|E_{w_\chi}(\xi_\lambda)\|_{L_\chi(\Theta')} + C''(\epsilon+|\lambda|^{1/2})\|\xi_\lambda\|_{W_\chi}\\ &\qquad + C\|\beta'_u(\xi_{\lambda,u})_T\|_{L_u(\gamma_\chi(\Theta'_u))}\\ &\le 2C'\varepsilon_E + C''(\epsilon+|\lambda|^{1/2}) + C\varepsilon_0.\end{aligned}\tag{75}$$

(Note in comparison with (25), the second term in the third line above has a worse factor of $|\lambda|^{1/2}$ instead of $|\lambda|$; this arises from the difference between u and u_λ.)

On the other hand, on $\underline{\Theta'_{y-}}$ we consider $\xi_{\lambda,y-} \in \Gamma(\bar{y}^*_{\lambda-}K)$ defined by

$$T_{\bar{y}_{\lambda-},w_\chi}\xi_{\lambda,y-} = \xi_\lambda.$$

Let $\mathbf{e}_{y_{\lambda-}}$ be the unit eigenvector associated with the minimal eigenvalue of $A_{y_{\lambda-}}$, which goes to $\mathbf{e}_y$ as $\lambda \to 0$. By I.(57), $\mathbf{e}_{y_{\lambda-}}$ differs from $T_{w_\chi,y_{\lambda-}}w'_\chi/\|w'_\chi\|_{2,t}(s)$ by $O(|\lambda|^{1/2})$ for $s \in \Theta'_{y-}$. Let

$$(\xi_{\lambda,y-})_L := \Pi_{\mathbf{e}_{y_{\lambda-}}}\xi_{\lambda,y-} = \underline{(\xi_{\lambda,y-})}_L\mathbf{e}_{y_{\lambda-}},$$

and $(\xi_{\lambda,y-})_T = \xi_{\lambda,y-} - (\xi_{\lambda,y-})_L$. The above observation about $\mathbf{e}_{y_{\lambda-}}$, together with the fact that in this region σ_χ is bounded above and below by multiples of $|\lambda|^{-1}$, implies that to estimate $\|\xi\|_{W_\chi(\Theta_{y-})}$ or $\|\xi\|_{L_\chi(\Theta_{y-})}$, it is equivalent to estimate $\|\xi_{y-}\|_{W_{y-}(\Theta_{y-})}$ or $\|\xi_{y-}\|_{L_{y-}(\Theta_{y-})}$, where

$$\begin{aligned}\|\xi_{y-}\|_{W_{y-}} &:= |\lambda|^{-1/2}\|\xi_{y-}\|_{p,1} + |\lambda|^{-1}\|(\xi_{y-})'_L\|_p,\\ \|\xi_{y-}\|_{L_{y-}} &:= |\lambda|^{-1/2}\|\xi_{y-}\|_{p,1} + |\lambda|^{-1}\|(\xi_{y-})_L\|_p.\end{aligned}$$

We have a refined version of Lemma 3.3.1 in this case.

Lemma (**Refining Floer's lemma**). *Let ξ_λ be as in* (74). *Then for all sufficiently small λ,*

$$|\lambda|^{1/(2p)}\|\xi_{\lambda,y-}\|_{L^\infty(\Theta'_{y-})} + \|(\xi_{\lambda,y-})_L\|_{L^\infty(\Theta'_{y-})} \le \varepsilon_0(\lambda)|\lambda|^{1/2+1/(2p)},$$

where $\varepsilon_0(\lambda)$ is a small positive number, with $\lim_{\lambda\to 0}\varepsilon_0(\lambda) = 0$.

Proof. The estimate for $\|\xi_{\lambda,y-}\|_{L^\infty(\Theta'_{y-}})$ follows easily from the argument for Lemma 3.3.1. The longitudinal component has a more refined bound because it has a better bound on the Sobolev norm. Let

$$\tilde{\varsigma}_\lambda(\tau) := \lambda^{-1/2-1/(2p)}\underline{(\xi_{\lambda,y-})}_L(\lambda^{-1/2}(\tau+s_\lambda)) \quad \text{over } [1,\infty),$$

where s_λ are constants chosen such that $\lambda^{-1/2}(1+s_\lambda) = \gamma_\chi^{-1}(\mathfrak{r}_\chi - 1)$. Then by (74), $\|\tilde{\varsigma}_\lambda\|_{L^p_1([1,\infty))}$ is bounded (note the rescaling contributes a factor of

$(\lambda)^{-1/(2p)}$ to the L^p_1-norm). Thus (again after possibly taking a subsequence) $\tilde\varsigma_\lambda$ converges in C_0 to $\tilde\varsigma_0$, and $s_\lambda \to s_0$. $\tilde\varsigma_0$ satisfies an equation of the form

$$\frac{d\tilde\varsigma_0}{d\tau} + \chi\tilde\varsigma_0 = 0, \quad \text{where } \chi \sim C_1 + C_2 \mathrm{e}^{-\nu'\tau}. \tag{76}$$

(The assumption of being in a standard d-b neighborhood is used to simplify the differential equation above. Notice also that $(\xi_\lambda)_T$ does not appear in this equation, because by the L^∞_t estimate for ξ_λ, its contribution vanishes as $\lambda \to 0$.) Thus, $\|\tilde\varsigma_0\|_\infty \leq |\tilde\varsigma_0(1)|$. Meanwhile, $\tilde\varsigma_0(1) = 0$ since by the argument for (35), $\|(\xi_\lambda)_L(\gamma_\chi^{-1}(\mathfrak{r}_\chi - 1))\|_{\infty,t} \leq C\lambda^{1/2+1/(2p)}\varepsilon_u$. □

Let β_{y-} be a cutoff function on $\mathbb{R}$ which vanishes in $(-\infty, \mathfrak{r}_\chi - 1]$ and is 1 on $[\mathfrak{r}_\chi, \infty)$. We may estimate the longitudinal component as:

$$\begin{aligned}
&|\lambda|^{-1}\|(\xi_{\lambda,y-})'_L\|_{L^p(\Theta_{y-})} + |\lambda|^{-1/2}\|(\xi_{\lambda,y-})_L\|_{L^p(\Theta_{y-})}\\
&\leq C|\lambda|^{-1}\|E_{y_\lambda}(\beta_{y-}(\gamma_\chi)(\xi_{\lambda,y-})_L)\|_{L^p(\Theta'_{y-})}\\
&\leq C|\lambda|^{-1}\|\beta_{y-}(\gamma_\chi)(E_{y_{\lambda-}}(\xi_{\lambda,y-})_L)_L\|_{L^p(\Theta'_{y-})}\\
&\qquad + C'|\lambda|^{-1}\|(\beta_{y-})_\gamma(\gamma_\chi)\gamma'_\chi(\xi_{\lambda,y-})_L\|_{L^p(\Theta'_{y-})}\\
&\leq C_1\|\beta_{y-}\circ\gamma_\chi E_{w_\chi}\xi_\lambda\|_{L_\chi(\Theta'_{y-})} \qquad (77)\\
&\qquad + C_2|\lambda|^{-1/2}\|\beta_{y-}(\gamma_\chi)\mathrm{e}^{-C_6|\lambda|^{-1/2}(s-\gamma_\chi^{-1}(\mathfrak{r}_\chi))}(\xi_{\lambda,y-})_L\|_{L^p(\Theta'_{y-})}\\
&\qquad + C'|\lambda|^{-1}\|(\beta_{y-})_\gamma(\gamma_\chi)\gamma'_\chi(\xi_{\lambda,y-})_L\|_{L^p(\Theta'_{y-})}\\
&\qquad + C_3\|\beta_{y-}(\gamma_\chi)\mathrm{e}^{-C_6|\lambda|^{-1/2}(s-\gamma_\chi^{-1}(\mathfrak{r}_\chi))}(\xi_{\lambda,y-})_T\|_{L^p(\Theta'_{y-})}\\
&\leq C'_1\varepsilon_E + C'_2\varepsilon_0.
\end{aligned}$$

The first inequality above follows from the eigenvalue estimate for $A_{y_{\lambda-}}$ in I.5.3.2. The second term in the penultimate expression above comes from the difference between E_{w_χ} and (a conjugate of) $E_{y_{\lambda-}}$, while the last term arises from $(T^{-1}_{\bar y_{\lambda-},w_\chi}E_{w_\chi}T_{\bar y_{\lambda-},w_\chi}(\xi_{\lambda,y-})_T)_L$. (Note that this term would have an extra factor of $|\lambda|^{-1/2}$ if I.5.3.1 (1c) is not assumed.) We have also used Lemma 5.2.4 and the estimates that in this region,

$$|(\beta_{y-})_\gamma\gamma'_\chi| \leq C|\lambda|^{1/2}\exp(-C_6|\lambda|^{1/2}(s-\gamma_\chi^{-1}(\mathfrak{r}_\chi))) \quad \text{and that}$$
$$\|\mu_\lambda(\gamma_\chi(s))\|_{2,2,t} \leq C'|\lambda|^{1/2}\exp(-C_6|\lambda|^{1/2}(s-\gamma_\chi^{-1}(\mathfrak{r}_\chi)),$$

which in turn follows from the computation in the proof of Lemma 5.2.1.

Similarly, the transversal direction can be estimated by:

$$\begin{aligned}
&|\lambda|^{-1/2}(\|(\xi_{\lambda,y-})'_T\|_{L^p(\Theta_{y-})} + \|(\xi_{\lambda,y-})_T\|_{L^p(\Theta_{y-})})\\
&\leq C\|\beta_{y-}(\gamma_\chi)E_{w_\chi}\xi_\lambda\|_{L_\chi(\Theta'_{y-})} + C'|\lambda|^{1/2-1/(2p)}\varepsilon_0(\lambda) \qquad (78)\\
&\leq C''\varepsilon_E + C'|\lambda|^{1/2-1/(2p)}\varepsilon_0.
\end{aligned}$$

Combining (77), (78), and (75), we obtain $\|\xi_\lambda\|_{W_\chi} \ll 1$ for all large enough λ, and hence the desired contradiction.

5.2.5. Surjectivity of the gluing map. Estimates for the nonlinear terms required for Step 3 in this case are not very different from those discussed in Section 2.5, and hence will be omitted. The argument in Section 1.2.1 then defines a gluing map, which is a local diffeomorphism onto a $\mathcal{B}$-topology neighborhood of the image of pregluing map. Again, we need to show that the latter neighborhood contains a chain-topology neighborhood of $\mathbb{S}$.

To adapt the proof in Section 4.2, given $(\lambda, \hat{w}) \in \hat{\mathcal{M}}^{1,\Lambda}(\mathbf{x}, \mathbf{y}_{\lambda-})$ close to $\hat{u} \in \mathbb{S}$ in the chain-topology neighborhood, we may again choose a representative w and $\tilde{w}$ as in Section 4.2.1, satisfying conditions similar to (57) and (58).

- $\tilde{w}(s) := u_\lambda(\tilde{\gamma}_\chi(s))$, where $\tilde{\gamma}_\chi\colon \mathbb{R} \to (-\infty, \breve{\gamma}_\chi)$ is a homeomorphism determined by

$$\Pi_{\mathbf{e}_y}\tilde{\zeta}_\lambda(s) = \Pi_{\mathbf{e}_y}\zeta(s) \quad \forall s \in [\tilde{\gamma}_\chi^{-1}(R+1), \infty), \tag{79}$$

  and ζ, $\tilde{\zeta}_\lambda$ are defined by $w(s) = \exp(y, \zeta(s))$, $\tilde{w}(s) = \exp(y, \tilde{\zeta}_\lambda(s))$ as in Section 4.2.1.
- $\gamma_\chi^{-1}(0) = \tilde{\gamma}_\chi^{-1}(0)$; $\quad \langle u_\gamma(0), \tilde{\gamma}_\chi^*\tilde{\xi}(0)\rangle_{2,t} = 0$.

Equation (59) is in this case replaced by the following lemma.

Lemma. $\forall s \in [\tilde{\gamma}_\chi^{-1}(R+1), \infty)$,

$$\begin{aligned} &\|\tilde{\xi}(s)\|_{2,2,t} + \|\tilde{\xi}'(s)\|_{2,1,t} \\ &\quad \le C(|\lambda| + \|\Pi_{\mathbf{e}_y}(\tilde{\zeta}_\lambda(s) - \tilde{\zeta}_\lambda(\infty))\|_{2,t}^3)\|\Pi_{\mathbf{e}_y}(\tilde{\zeta}_\lambda(s) - \tilde{\zeta}_\lambda(\infty))\|_{2,t}. \end{aligned} \tag{80}$$

Proof. Write $u(\tilde{\gamma}_\chi(s)) = \exp(y, \tilde{\zeta}(s))$, and let $b(s) := \Pi_{\mathbf{e}_y}\tilde{\zeta}(s)$, $c(s) := \zeta(s) - \tilde{\zeta}(s)$. Note that $\Pi_{\mathbf{e}_y}\tilde{\zeta} = \Pi_{\mathbf{e}_y}\tilde{\zeta}_\lambda$, and on this region $\tilde{\zeta}_\lambda - \tilde{\zeta} = \eta_{\lambda-}$ $\forall s$. The functions $b(s), c(s)$ still satisfy (62), (61). However, we want to estimate instead

$$c_d(s) := \zeta(s) - \tilde{\zeta}_\lambda(s) = c(s) - \Pi^\perp_{\ker A_y}\eta_{\lambda-}.$$

From the definitions, estimates for c_d would imply similar estimates for $\tilde{\xi}$.

Let $\underline{b}_d(s) := \langle \mathbf{e}_y, \tilde{\zeta}_\lambda(s) - \tilde{\zeta}_\lambda(\infty)\rangle_{2,t}$; $b_d(s) = \underline{b}_d(s)\mathbf{e}_y$. Noting that

$$\begin{aligned} &-A_y\Pi^\perp_{\ker A_y}\eta_{\lambda-} \\ &\quad = (1 - \Pi_{\ker A_y} - \nabla_{b(\infty)}Z\Pi_{\ker A_y})\Big(\hat{n}_{(0,y)}(\lambda, \tilde{\zeta}(\infty) + c(\infty)) - n_y(\tilde{\zeta}(\infty))\Big) \\ &\qquad - \nabla_{b(\infty)}Z(\lambda C_y'\mathbf{e}_y), \end{aligned}$$

we see that (61) may be rewritten in terms of c_d as:

$$\begin{aligned} -c_d' &= A_y c_d - \nabla_{b_d}Z\Big(\lambda C_y'\mathbf{e}_y + \Pi_{\ker A_y}(\hat{n}_{(0,y)}(\lambda, \tilde{\zeta}_\lambda(\infty)) - n_y(\tilde{\zeta}(\infty)))\Big) \\ &\quad + (1 - \Pi_{\ker A_y} - \nabla_b Z\Pi_{\ker A_y})\begin{pmatrix} \hat{n}_{(0,y)}(\lambda, \tilde{\zeta}_\lambda + c_d) - n_y(\tilde{\zeta}) \\ - \hat{n}_{(0,y)}(\lambda, \tilde{\zeta}_\lambda(\infty)) + n_y(\tilde{\zeta}(\infty)) \end{pmatrix} \end{aligned} \tag{81}$$

By the nature of u, u_λ, and $\hat{n}_{(0,y)}$, this leads to the familiar estimates:

$$\|c_{d+}\|'_{2,t} \geq \nu_+\|c_{d+}\|_{2,t} - \epsilon_+\|c_d\|_{2,t} - C_+|\lambda \underline{b}_d|;$$
$$\|c_{d-}\|'_{2,t} \leq -\nu_-\|c_{d-}\|_{2,t} + \epsilon_-\|c_d\|_{2,t} + C_-|\lambda \underline{b}_d|. \tag{82}$$

Subtracting the two inequalities, we get

$$(\|c_{d+}\|_{2,t} - \|c_{d-}\|_{2,t})' \geq \nu'(\|c_{d+}\|_{2,t} - \|c_{d-}\|_{2,t}) - C'|\lambda \underline{b}_d|.$$

Taking convolution product with the integral kernel of $d/ds - \nu'$, we find that for $s \geq s_0$

$$\begin{aligned}&\|c_{d+}\|_{2,t}(s) - \|c_{d-}\|_{2,t}(s)\\ &\quad\geq (\|c_{d+}\|_{2,t}(s_0) - \|c_{d-}\|_{2,t}(s_0))\mathrm{e}^{\nu'(s-s_0)} - C'\int_{s_0}^{s} |\lambda \underline{b}_d(\underline{s})|\mathrm{e}^{\nu'(s-\underline{s})}\, d\underline{s},\end{aligned}$$

and since $\underline{b}_d(s) > 0$ decreases with s, this implies that for all large enough s,

$$\|c_{d+}\|_{2,t}(s) \leq \|c_{d-}\|_{2,t}(s) + C''|\lambda \underline{b}_d(s)|, \tag{83}$$

otherwise $\|c_{d+}\|_{2,t}(s) - \|c_{d-}\|_{2,t}(s)$ would be growing exponentially as $s \to \infty$, contradicting the fact that by construction, $\lim_{s\to\infty} \|c_d(s)\|_{2,t} = 0$.

Plugging in this back to (82), we get

$$\|c_{d-}\|'_{2,t} \leq -\nu'_-\|c_{d-}\|_{2,t} + C'_-|\lambda \underline{b}_d|, \tag{84}$$

where ν'_- is a positive numbers close to ν_-. Taking convolution product with the integral kernel of $d/ds + \nu'$,

$$\|c_{d-}(s)\|_{2,t}(s) \leq C_0\mathrm{e}^{-\nu'_- s} + \int_{s_0}^{s} |\lambda|\underline{b}_d(\underline{s})\mathrm{e}^{\nu'_-(\underline{s}-s)}\, d\underline{s}. \tag{85}$$

We claim that there is a positive constant ν''_- slightly smaller than ν'_- such that

$$\underline{b}_d(\underline{s}) \leq 2\underline{b}_d(s)\mathrm{e}^{\nu''_-(s-\underline{s})/4} \quad \text{for } s_0 \leq \underline{s} \leq s. \tag{86}$$

Using this in the integrand in (85), we arrive at

$$\begin{aligned}\|c_{d-}(s)\|_{2,t}(s) &\leq C_0\mathrm{e}^{-\nu'_- s} + C'_0|\lambda|\underline{b}_d(s)\\ &\leq C_d\underline{b}_d(s)(\underline{b}_d^3(s) + |\lambda|).\end{aligned}$$

(In the second step above, we used (86) again to bound

$$\mathrm{e}^{-\nu'_- s} = (\mathrm{e}^{-\nu''_- s/4})^4 \leq C_8\underline{b}_d(s)^4 \quad \text{for large } s.)$$

Combining with (83), we obtain a similar estimate for $\|c_d(s)\|_{2,t}$:

$$\begin{aligned}\|c_d(s)\|_{2,t}(s) &\leq C_1\mathrm{e}^{-\nu'_- s} + C'_1|\lambda|\underline{b}_d(s)\\ &\leq C'_d\underline{b}_d(s)(\underline{b}_d^3(s) + |\lambda|).\end{aligned} \tag{87}$$

We now return to verify the claim (86). To see this, note that projecting the flow equation to ker A_y, we have

$$-b_d' = \Pi_{\ker A_y}\Big(\hat{n}_{(0,y)}(\lambda, \tilde{\zeta}_\lambda + c_d) - \hat{n}_{(0,y)}(\lambda, \tilde{\zeta}_\lambda(\infty))\Big).$$

Then the properties of $\hat{n}_{(0,y)}$, $\tilde{\zeta}_\lambda(\infty)$ and u again give the estimate:

$$\underline{b}_d' \geq -\varepsilon''(\underline{b}_d + \|c_d\|_{2,t}) \tag{88}$$

for a small positive constant ε''. Subtracting a small multiple of this from (84) and using (83), we have

$$(\|c_{d-}\|_{2,t} - \varepsilon_1 \underline{b}_d)' \leq -\nu''(\|c_{d-}\|_{2,t} - \varepsilon_1 \underline{b}_d).$$

Taking convolution product with the integral kernel of $d/ds + \nu''$, we have

$$\|c_{d-}\|_{2,t} \leq \varepsilon_1 \underline{b}_d + C_1 e^{-\nu'' s}.$$

Plugging this back in (83) and (88), we get

$$\underline{b}_d' \geq -\frac{\nu''}{4}\underline{b}_d - \varepsilon_2 \mathrm{e}^{-\nu'' s}.$$

Now taking convolution product with the integral kernel of $d/ds + \nu''/4$, we have for $\underline{s} < s$:

$$\begin{aligned}\underline{b}_d(s) &\geq \underline{b}_d(\underline{s})\mathrm{e}^{\nu''(\underline{s}-s)/4} - \frac{4\varepsilon_2}{3\nu''}(\mathrm{e}^{-3\nu''\underline{s}/4} - \mathrm{e}^{-3\nu'' s/4})\mathrm{e}^{-\nu'' s/4} \\ &\geq \frac{1}{2}\underline{b}_d(\underline{s})\mathrm{e}^{\nu''(\underline{s}-s)/4}.\end{aligned}$$

To obtain the second inequality above, first use the first inequality and the fact that $s > \underline{s} \geq s_0 \gg 1$ to obtain $\underline{b}_d(s) \geq C\mathrm{e}^{-\nu'' s}$; then use this (with s replaced by $\underline{s}$) to estimate

$$\frac{4\varepsilon_2}{3\nu''}(\mathrm{e}^{-3\nu''\underline{s}/4} - \mathrm{e}^{-3\nu'' s/4}) \leq \frac{\underline{b}_d(\underline{s})}{2}\mathrm{e}^{\nu''\underline{s}/4}.$$

Claim verified.

Next, to get estimates for higher derivatives of c_d from (87), we need to elliptic bootstrap using (81) and apply Sobolev embedding as in the proof of Lemma I.5.1.7. To obtain the estimates claimed in the lemma, we need to bound the average of $\underline{b}_d$ in an interval about s in terms of $\underline{b}_d(s)$. This is obtained using (86) and the fact that $\underline{b}_d$ is decreasing. □

Next we compare $\tilde{w}(s)$ with the pregluing $w_\chi(s)$ to get a pointwise estimate of $\xi(s)$, as in Section 4.2.3. In this case, the first two formulas of (63) are still valid (with $\mathfrak{r}_i$ there replaced by $\mathfrak{r}_\chi$) by arguments similar to those in Section 4.2, but the third needs to be modified. In this region (where $\gamma \geq \mathfrak{r}_\chi$), we need to expand about y_λ instead of u_λ as in the proof of Lemma 5.2.1, keeping in mind that μ_λ is of order $\lambda^{1/2}$ while $(\mu_\lambda)_\gamma$ is of order λ. Recall that γ_χ satisfies the equation $\gamma'_\chi = h(\gamma_\chi)$, with h given by (73). The function $\tilde{\gamma}_\chi$ satisfies a similar equation:

$$\tilde{\gamma}'_\chi = h(\tilde{\gamma}_\chi) + \|(\mu_\lambda)_\gamma(\tilde{\gamma}_\chi)\|_{2,t}^{-2}\Big\langle(\mu_\lambda)_\gamma(\tilde{\gamma}_\chi), E_{y_\lambda}T_{w_\chi,y_\lambda}\tilde{\xi}(s) + o(\|\tilde{\xi}(s)\|_{2,1,t})\Big\rangle_{2,t}.$$

By (80) and (79), the absolute value of this can be bounded by

$$\begin{aligned} &C_1|\lambda|^{-1/2}(\|\mu_\lambda(\tilde{\gamma}_\chi)\|_{2,1,t}^3 + |\lambda|)\|\mu_\lambda(\tilde{\gamma}_\chi)\|_{2,1,t} \\ &\quad \le C_2|\lambda|^{3/2}(\breve{\gamma}_\chi - \tilde{\gamma}_\chi)(1+\lambda^2(\breve{\gamma}_\chi - \tilde{\gamma}_\chi)^3). \end{aligned}$$

Recall also the estimate for h from Section 5.2.1; we then obtain

$$\begin{aligned} |\Delta_s(\gamma) - \Delta_s(\mathfrak{r}_\chi)| &\le C_3\int_{\mathfrak{r}_\chi}^{\gamma}\Big|\frac{|\lambda|^{3/2}(\breve{\gamma}_\chi - \gamma)(1+\lambda^2(\breve{\gamma}_\chi-\gamma)^3)}{|\lambda|(\breve{\gamma}_\chi-\gamma)^2}\,d\gamma\Big| \\ &\le C_4|\lambda|^{1/2}\Big(\lambda^2(\breve{\gamma}_\chi-\gamma)^2 + |\ln(\breve{\gamma}_\chi-\gamma)|\Big)\Big|_{\mathfrak{r}_\chi}^{\gamma}. \end{aligned}$$

Using this and the facts that in this region

$$\begin{aligned} \|w'_\chi\|_{2,2,t} &\le C_5|\lambda|\mathrm{e}^{-C_6|\lambda|^{1/2}(s-\gamma_\chi^{-1}(\mathfrak{r}_\chi))} \quad \text{and} \\ \breve{\gamma}_\chi - \gamma_\chi(s) &\le C'_5|\lambda|^{-1/2}\mathrm{e}^{-C_6|\lambda|^{1/2}(s-\gamma_\chi^{-1}(\mathfrak{r}_\chi))}, \end{aligned} \tag{89}$$

we can bound

$$\|\tilde{\xi}_\chi\|_{2,2,t} \le \varepsilon_7|\lambda|\mathrm{e}^{-C'_6|\lambda|^{1/2}(s-\gamma_\chi^{-1}(\mathfrak{r}_\chi))},$$

where $\tilde{\xi}_\chi$ is defined by $\tilde{w}(s) = \exp(w_\chi(s), \tilde{\xi}_\chi(s))$ as in Section 4.2. Recall also that $w(s) = \exp(w_\chi(s), \xi_\chi(s))$. Combining the above estimate with (80) and the other two lines of (63), we have

$$\begin{aligned} &\|\xi_\chi(s)\|_{2,2,t} \\ &\le \begin{cases} \varepsilon_7|\lambda|\mathrm{e}^{-C'_6|\lambda|^{1/2}(s-\gamma_\chi^{-1}(\mathfrak{r}_\chi))} & \text{if } s \in [\gamma_\chi^{-1}(\mathfrak{r}_\chi), \infty), \\ \varepsilon_8|\lambda| & \text{if } s \text{ is between } \gamma_\chi^{-1}(\mathfrak{r}_\chi) \text{ and } \gamma_\chi^{-1}(R), \\ \varepsilon(\lambda) & \text{otherwise.} \end{cases} \end{aligned} \tag{90}$$

So over $(-\infty, \gamma_\chi^{-1}(\mathfrak{r}_\chi)] \times S^1 \subset \Theta$, we can estimate $\|\xi_\chi\|_{W_\chi}$ by the same argument as in Section 4.2. The estimate over $\Theta_{y-} := [\gamma_\chi^{-1}(\mathfrak{r}_\chi), \infty) \times S^1$ is replaced by the following. By (90),

$$|\lambda|^{-1/2}\|\xi_\chi\|_{L^p(\Theta_{y-})} + |\lambda|^{-1/2}\|\dot{\xi}_\chi\|_{L^p(\Theta_{y-})} \le \varepsilon_9|\lambda|^{1/2-1/(2p)} \ll 1. \tag{91}$$

Next, to estimate ξ'_χ, it is equivalent to estimate $\xi'_{\chi,y-}$, which is obtained by expanding the flow equation about $y_{\lambda-}$. Here, we have an equation similar to (69), with $\bar{y}$ replaced by $\bar{y}_{\lambda-}$, and $\alpha = 0$. Using the error estimate in this region in the proof of Lemma 5.2.3, (89), Lemma I.5.3.2, we find

$$|\lambda|^{-1/2}\|(\xi_{\chi,y-})'_T\|_{L^p(\Theta_{y-})}|\lambda|^{-1}\|(\xi_{\chi,y-})'_L\|_{L^p(\Theta_{y-})} \le C_9|\lambda|^{1/2-1/(2p)} \ll 1.$$

This together with (91) shows that $\|\xi_\chi\|_{W_\chi(\Theta_{y-})} \ll 1$. Now one may follow the argument in Section 1.2.7 to complete Step 4 of the proof of the gluing theorem.

5.3. When $u = \bar{y}$. We now assume that $x_0 = z_0 = y$, and $u = \bar{y}$, the constant flow at y.

5.3.1. The pregluing. Let $\underline{b}^0_\lambda(s)$ be the solution of

$$-(\underline{b}^0_\lambda)' = C'_y\lambda + C_y(\underline{b}^0_\lambda)^2, \quad \text{with } \underline{b}^0_\lambda(0) = 0, \tag{92}$$

where C_y, C'_y are as defined in I.5.3.1. In other words, there are positive λ-independent constants C_0, C', such that

$$\underline{b}^0_\lambda(s) = C_0|\lambda|^{1/2}\tanh(C'|\lambda|^{1/2}s). \tag{93}$$

Let $b^0_\lambda(s) := \underline{b}^0_\lambda(s)\mathbf{e}_y$. Denote by $\underline{b}^{0\pm}_\lambda = \lim_{s\to\pm\infty}\underline{b}^0_\lambda(s) = \pm C_0|\lambda|^{1/2}$. Let

$$\tilde{\zeta}_\lambda := b^0_\lambda + \beta_+(\eta_{\lambda+} - b^{0+}_\lambda) + \beta_-(\eta_{\lambda-} - b^{0-}_\lambda),$$

where $\eta_{\lambda\pm}$ are defined by $\exp(y, \eta_{\lambda\pm}) = y_{\lambda\pm}$, and $\beta_-(s) := \beta(|\lambda|^{-1} + s)$; $\beta_+(s) := \beta(|\lambda|^{-1} - s)$.

We define the *pregluing* w_λ in this case by

$$w_\lambda := \exp(y, \tilde{\zeta}_\lambda).$$

5.3.2. The weighted norms. Recall the definition of W_{y-}, L_{y-} from Section 5.2.4. Let W_{y+}, L_{y+} and W_y, L_y be similarly defined for elements in $\Gamma(\bar{y}^*_{\lambda+}K)$ and $\Gamma(\bar{y}^*K)$, respectively, with longitudinal directions given by $\mathbf{e}_{y_{\lambda+}}$ and $\mathbf{e}_y$.

Via the map $T_{y,w_\lambda}\colon \Gamma(\bar{y}^*K) \to \Gamma((w_\lambda)^*K)$, the norms W_y, L_y on $\Gamma(\bar{y}^*K)$ induce norms on $\Gamma((w_\lambda)^*K)$, which we denote by W_λ, L_λ. The associated spaces shall be the domain and range for E_{w_λ}.

By the estimates for $\eta_{\lambda\pm}$, it is easy to see that the induced norms on $\Gamma((w_\lambda)^*K)$ via $T_{y_{\lambda\pm},w_\lambda}$ from $W_{y\pm}$, $L_{y\pm}$ are commensurate with W_λ, L_λ.

5.3.3. Error estimates. Divide Θ into three regions: Θ_a, Θ_b, and Θ_c corresponding to $s < -|\lambda|$, $|s| < |\lambda|$, and $s > |\lambda|$, respectively. We will expand $\bar\partial_{JX_\lambda}w_\lambda$ around $y_{\lambda+}, y, y_{\lambda-}$, respectively, in the three regions.

Over Θ_b, using (92) and the fact of y being in a standard d-b neighborhood, we have

$$\begin{aligned}
&(T_{y,w_\lambda})^{-1}\bar\partial_{JX_\lambda}w_\lambda\\
&= E_y\tilde{\zeta}_\lambda + n_y(\tilde{\zeta}_\lambda) + (T_{y,w_\lambda})^{-1}\delta_\lambda\mathcal{V}(w_\lambda)\\
&= -\beta'_+(\eta_{\lambda+} - b^{0+}_\lambda) + \beta'_-(\eta_{\lambda-} - b^{0-}_\lambda) + \beta_+A_y(\eta_{\lambda+} - b^{0+}_\lambda)\\
&\quad + \beta_-A_y(\eta_{\lambda-} - b^{0-}_\lambda) + C_y\langle\mathbf{e}_y, 2b^0_\lambda + \delta\rangle_{2,t}\delta + \Pi^\perp_{\mathbf{e}_y}O((\|\tilde{\zeta}_\lambda\|_{2,1,t} + |\lambda|)^2),
\end{aligned} \tag{94}$$

where $\delta := \beta_+(\eta_{\lambda+} - b^{0+}_\lambda) + \beta_-(\eta_{\lambda-} - b^{0-}_\lambda)$.

From the estimates for $\eta_{\lambda\pm}$ in the proof of Lemma I.5.3.2, one sees that

$$|\lambda|^{-1}\|\delta_L\|_{2,1,t}+|\lambda|^{-1/2}\|\delta_T\|_{2,1,t}\leq C|\lambda|^{1/2}\quad\forall s, \tag{95}$$

and therefore from (94)

$$\|\bar\partial_{JX_\lambda}(w_\lambda)\|_{L_\lambda(\Theta_b)}\leq C|\lambda|^{1/2-1/p}.$$

The estimates on Θ_a and Θ_c are similar, so we shall focus on Θ_c. In this region, writing $w_\lambda(s)=\exp(y_{\lambda+},\mu_{\lambda+}(s))$, we have

$$(T_{y,w_\lambda})^{-1}\bar\partial_{JX_\lambda}w_\lambda=E_{y_{\lambda+}}\mu_{\lambda+}+n_{y_{\lambda+}}(\mu_{\lambda+}). \tag{96}$$

From the definition of $\mu_{\lambda+}$, $\|\mu_{\lambda+}(s)\|_{p,1,t}\leq C\|b_\lambda^{0+}-b_\lambda^0(s)\|_{p,1,t}$. So by (93), (96),

$$\|\bar\partial_{JX_\lambda}(w_\lambda)\|_{L_\lambda(\Theta_c)}\leq C|\lambda|^{-1/2-1/p}\mathrm{e}^{-C'|\lambda|^{-1}}\to 0\quad\text{as }\lambda\to 0.$$

5.3.4. Existence and uniform boundedness of right inverse to $E_{w_\lambda}\colon W_\lambda\to L_\lambda$. In this case, let

$$W'_\lambda:=\{\xi\in W_\lambda\,|\,\xi_L(0)=0\}.$$

Again assume the existence of a sequence $\{\xi_\lambda\in W'_\lambda\}_\lambda$ satisfying (74), with the obvious modification.

Divide Θ into three regions Θ_y, $\Theta_{y\pm}$ corresponding, respectively, to the three possibilities: $|s|\leq|\lambda|^{-1/2}$, $\pm s\geq|\lambda|^{-1/2}$. Let $\Theta'_y\supset\Theta_y$ be the region in which $|s|<(1+\varepsilon)|\lambda|^{-1/2}$; let $\Theta'_{y\pm}\supset\Theta_{y\pm}$ be the region in which $\pm s>(1-\varepsilon)|\lambda|^{-1/2}$, where $1\gg\varepsilon>0$. Instead of estimating $\|\xi_\lambda\|_{W_\lambda}$, we shall estimate

$$\xi_{\lambda,y}:=T_{w_\lambda,y}\xi_\lambda,\quad \xi_{\lambda,y\pm}:=T_{w_\lambda,y\pm}\xi_\lambda$$

in W_y or $W_{y\pm}$-norm over Θ_y and $\Theta_{y\pm}$, respectively. First, observe the following analog of Lemma 5.2.4 over Θ_y and $\Theta_{y\pm}$.

Lemma (Analog of Floer's lemma). *Let ξ_λ be as in (74) with W_χ, W'_χ, L_χ replaced by $W_\lambda, W'_\lambda, L_\lambda$ respectively. Then for all sufficiently small λ, there is a small positive number, $\varepsilon_0(\lambda)$, $\lim_{\lambda\to 0}\varepsilon_0(\lambda)=0$, such that*

(a) $|\lambda|^{1/(2p)}\|\xi_{\lambda,y}\|_{L^\infty(\Theta'_y)}+\|(\xi_{\lambda,y})_L\|_{L^\infty(\Theta'_y)}\leq\varepsilon_0(\lambda)|\lambda|^{1/2+1/(2p)}$;

(b) $|\lambda|^{1/(2p)}\|\xi_{\lambda,y\pm}\|_{L^\infty(\Theta'_{y\pm})}+\|(\xi_{\lambda,y\pm})_L\|_{L^\infty(\Theta'_{y\pm})}\leq\varepsilon_0(\lambda)|\lambda|^{1/2+1/(2p)}$.

Proof. The L^∞_t-estimate for ξ_λ (and hence $\xi_{\lambda,y}$ and $\xi_{\lambda,y\pm}$) is now routine. The estimates for the longitudinal components follow the rescaling argument in the proof of Lemma 5.2.4, with the following modifications.

On Θ'_y, one may similarly define a sequence $\tilde\varsigma_\lambda$ of L^p_1-bounded functions on $[-1,1]$, which converges to $\tilde\varsigma_0$ which satisfy also an equation of the form (76), but now $\chi\sim C\,\tanh(C'\tau)$. Because $(\xi_\lambda)_L(0)=0$, $\tilde\varsigma_0(0)=0$, and thus $\tilde\varsigma_0=0$. This proves part (a) above.

On $\Theta'_{y\pm}$, we have another version of $\tilde{\varsigma}_\lambda$ and $\tilde{\varsigma}_0$ (which are now functions on $[1,\infty)$, $(-\infty,-1]$, respectively), and the argument in the proof of Lemma 5.2.4 again gives a bound on $\|\tilde{\varsigma}_0\|_\infty$ by $|\tilde{\varsigma}_0|(\pm 1)$, which vanishes by the estimate for $\|(\xi_\lambda)_L\|_{L^\infty(\Theta'_y)}$ obtained in part (a). This proves part (b). □

We now return to estimate $\|\xi_{\lambda,y}\|_{W_y}(\Theta_y)$ and $\|\xi_{\lambda,y\pm}\|_{W_{y\pm}}(\Theta_{y\pm})$.

On Θ'_y: let χ_y be a smooth cutoff function which vanishes outside $(-1,1)$ and let $\beta_y(s) := \chi_y(s/|\lambda|^{-1/2})$. Since $(\xi_{\lambda,y})_L(0) = 0$, applying Lemma 3.3.3 (c) to the longitudinal component, one may estimate:

$$
\begin{aligned}
\|\beta_y\xi_{\lambda,y}\|_{W_y(\Theta_y)} &\le C\|E_{\bar{y}}(\beta_y\xi_{\lambda,y})\|_{L_\lambda(\Theta'_y)}\\
&\le C_1\|E_{w_\lambda}\xi_\lambda\|_{L_\lambda(\Theta'_y)} + C_2(|\lambda|^{1/2}\|\beta_y\xi_\lambda\|_{W_\lambda(\Theta'_y)}\\
&\qquad + \|\beta_y(\xi_{\lambda,y})_L\|_{W_y(\Theta'_y)}) + C_3\|\beta'_y\xi_\lambda\|_{L_\lambda(\Theta'_y)}\\
&\le C_1\varepsilon_E + C_4(|\lambda|^{1/2} + \varepsilon_0).
\end{aligned}
\tag{97}
$$

In the above, the second term in the penultimate line came from the difference between E_{w_χ} and the conjugate of E_y, using the fact that $\|\tilde{\zeta}_\lambda\|_{\infty,1,t} \le C|\lambda|^{1/2}$. The last line used Lemma 5.3.4 (a) and the equation for $(\xi_{\lambda,y})_L$.

On $\Theta'_{y\pm}$ one may estimate similarly. Let $\beta_{y\pm}$ be smooth cutoff functions supported on $\Theta'_{y\pm}$ with value 1 over $\Theta_{y\pm}$ and $|\beta'_{y\pm}| \le C|\lambda|^{1/2}$. By the eigenvalue estimate for $A_{y_{\lambda\pm}}$ in I.5.3.2,

$$\|\beta_{y\pm}\xi_{\lambda,y\pm}\|_{W_{y\pm}} \le C\|E_{y_{\lambda\pm}}(\beta_{y\pm}\xi_{\lambda,y\pm})\|_{L_\pm}.$$

The RHS can be estimated like (77), (78) using Lemma 5.3.4 (b) below.

Finally, from the estimates for $\xi_{\lambda,y}$ and $\xi_{\lambda,y\pm}$ above, we obtain the desired contradiction that $\|\xi_\lambda\|_{W_\lambda} \ll 1$.

5.3.5. Surjectivity of the gluing map. We have the routine estimate for the nonlinear term to define the gluing map. The main isssue is again to show that the gluing map surjects to a neighborhood of $\mathbb{S}$ in the parameterized moduli space of broken trajectories.

Let $(\lambda,\hat{w}) \in \hat{\mathcal{M}}_P^{\Lambda,1}(\mathbf{y}_+,\mathbf{y}_-)$ be in a chain-topology neighborhood of $\bar{y} \in \hat{\mathcal{M}}_P^{\Lambda,1,+}$. Choose a representative w of $\hat{w}$ such that writing

$$w(s) = \exp(y,\zeta(s)), \quad w_\lambda(s) = \exp(y,\tilde{\zeta}(s)),$$

the difference $\eta := \zeta - \tilde{\zeta}$ satisfies $\eta_L(0) = 0$. Writing $w(s) = \exp(w_\lambda(s),\xi(s))$, we want to show that $\|\xi\|_{W_\lambda} \ll 1$; equivalently, it suffices to estimate η.

Let $\epsilon' \gg |\lambda|^{1/2}$ be a small positive number. Consider the three regions $\Theta_a^{\epsilon'} = (-\infty,-\epsilon'|\lambda|^{-1}]\times S^1$, $\Theta_b^{\epsilon'} = [-\epsilon'|\lambda|^{-1},\epsilon'|\lambda|^{-1}]\times S^1$, $\Theta_c^{\epsilon'} = [\epsilon'|\lambda|^{-1},\infty)\times S^1$ separately.

From the flow equation and the definition of w_λ, we find that η_L, η_T satisfy, respectively:

$$\underline{\eta}_L' + C_y(\underline{\tilde{\zeta}}_L + \underline{\zeta}_L)\underline{\eta}_L = O\Big((|\lambda| + \|\eta_T(s)\|_{2,1,t})^2 + \|\zeta_L(s)\|_{2,1,t}^4\Big); \tag{98}$$

$$\eta_T'(s) + A_y\eta_T(s) = O((|\lambda| + \|\zeta\|_{2,1,t})^2). \tag{99}$$

(In the usual notation, $\eta_L =: \underline{\eta}_L\mathbf{e}_y$; $\zeta_L =: \underline{\zeta}_L\mathbf{e}_y$.)

Equation (99) and the fact that $\eta(\infty) = \eta(-\infty) = 0$ imply:

$$\|\eta_T\|_{2,2,t} \leq C_1(|\lambda| + \|\zeta_L\|_\infty)^2 \quad \forall s. \tag{100}$$

The argument to get this estimate should be by-now familiar to the reader (cf., e.g., the proof of Lemma 5.2.5): Take L^2_t-inner product of (99) with η_{T-}, η_{T+}, respectively, and integrate over s, one obtains

$$\|\eta_T\|_2 \leq C_2(|\lambda| + \sup_s \|\zeta_L\|_{2,t}(s))^2.$$

Then apply the usual elliptic bootstrapping and Sobolev embedding to get estimates on higher derivatives. Finally, observe that on the 1-dimensional subspace of longitudal direction, the various norms are all commensurate.

On the other hand, ζ_L satisfies

$$\underline{\zeta}_L' + C_y\underline{\zeta}_L^2 + C_y'\lambda = O\Big((|\lambda| + \|\eta_T\|_{2,1,t} + \|\delta_T\|_{2,1,t} + \|\zeta\|_{2,1,t}^2)^2\Big),$$

with $\zeta_L'(\infty) = 0 = \zeta_L'(-\infty)$ (so when $|\underline{\zeta}_L|(s)$ reaches maximum, $\underline{\zeta}_L' = 0$). Combining this with (95) and (100), we have

$$|\zeta_L|_\infty \leq C_L|\lambda|^{1/2}.$$

Plugging this back in (100), we get

$$\|\eta_T\|_{2,2,t} \leq C_1|\lambda|. \tag{101}$$

Using these L^∞ estimates for η_T, ζ_L and multiplying (98) with $\underline{\eta}_L$, we obtain

$$-C'|\lambda|^2 \leq |\underline{\eta}_L|' + C_y(\underline{\tilde{\zeta}}_L + \underline{\zeta}_L)|\underline{\eta}_L| \leq C|\lambda|^2.$$

Now since $\underline{\tilde{\zeta}}_L$, $\underline{\zeta}_L$ are both ≥ 0 when $s \geq 0$, and are both ≤ 0 when $s \leq 0$, we see that

$$|\underline{\eta}_L|' \leq C|\lambda|^2 \quad \text{when } s \geq 0;$$
$$-|\underline{\eta}_L|' \leq C|\lambda|^2 \quad \text{when } s \leq 0.$$

Integrating using the initial condition that $\eta_L(0) = 0$, we see that

$$\|\eta_L\|_{L^\infty(\Theta_b^{\epsilon'})} \leq C_1\epsilon'|\lambda|. \tag{102}$$

Combining this with (101), we get

$$\|\xi\|_{W_\lambda(\Theta_b^{\epsilon'})} \leq C_2{\epsilon'}^{1+1/(2p)}|\lambda|^{1/2-1/p} \to 0 \quad \text{as } \lambda \to 0.$$

We now turn to estimating ξ over $\Theta_a^{\epsilon'}$ and $\Theta_c^{\epsilon'}$. We shall only consider $\Theta_c^{\epsilon'}$ since the other works by analogy. On this region, writing $w_\lambda(s) = \exp(y_{\lambda-}, \tilde{\zeta}_{\lambda-}(s))$, we have from the definition of w_λ that

$$\|\tilde{\zeta}_{\lambda-}\|_{2,2,t}(s) \leq C_3|\lambda|^{3/2}\mathrm{e}^{-C_6|\lambda|^{1/2}(s-\epsilon'|\lambda|^{-1})} + \text{terms involving } \delta.$$

We may ignore the terms involving δ since by (95), its contribution to the W-norm is at most of order $|\lambda|^{1/2}$. On the other hand, by standard estimates $w(s)$ has the same exponential decay behavior in this region, and so combining the estimate for $\zeta_{\lambda-}(\epsilon'|\lambda|^{-1})$ above and (101), (102), we have on this region

$$\|\zeta_{\lambda-}\|_{2,2,t}(s) \leq C_4\epsilon'|\lambda|\mathrm{e}^{-C_6'|\lambda|^{1/2}(s-\epsilon'|\lambda|^{-1})},$$

where $\zeta_{\lambda-}$ is defined by $\exp(y_{\lambda-}, \zeta_{\lambda-}) = w(s)$. Together with the previous expression, we obtain a pointwise estimate for $\xi(s)$ on this region, which, when combined with (98), (99), yields

$$\|\xi\|_{W_\lambda(\Theta_c^{\epsilon'})} \leq C_5\epsilon'|\lambda|^{1/2-1/(2p)} \ll 1.$$

6. The handleslide bifurcation

The purpose of this section is to verify the bifurcation behavior at handleslides predicted in I.4.3, namely, Propositions 6.1.1, 6.1.2 below.

6.1. Summary of results. Combined with the previous gluing theorems: Propositions 2.1, 5.1, the following proposition completes the verification of (RHFS2c), (RHFS3c) for admissible (J, X)-homotopies.

6.1.1. Proposition. *Let* (J^Λ, X^Λ) *be an admissible* (J, X)*-homotopy connecting two regular pairs, and* $\mathbf{x}, \mathbf{z}$ *be two path components of* $\mathcal{P}^\Lambda \backslash \mathcal{P}^{\Lambda,\deg}$. *Then*:

- (a) *a chain-topology neighborhood of* $\mathbb{T}_{P,\text{hs-s}}(\mathbf{x}, \mathbf{z}; \Re)$ *in* $\hat{\mathcal{M}}_P^{\Lambda,1,+}(\mathbf{x}, \mathbf{z}; \mathrm{wt}_{-\mathcal{Y},e_{\mathcal{P}}} \leq \Re)$ *is l.m.b. along* $\mathbb{T}_{P,\text{hs-s}}(\mathbf{x}, \mathbf{z}; \Re)$;
- (b) *a chain-topology neighborhood of* $\mathbb{T}_{O,\text{hs-s}}(\Re)$ *in* $\hat{\mathcal{M}}_O^{\Lambda,1,+}(\mathrm{wt}_{-\mathcal{Y},e_{\mathcal{P}}} \leq \Re)$ *is l.m.b. along* $\mathbb{T}_{O,\text{hs-s}}(\Re)$.

The proof follows the standard gluing construction outlined in Section 1.2, and shall be omitted. A description of the relevant K-models will be given in Section 7.3.2 and 7.3.3. A result analogous to part (a) above is also given by Proposition 4.2 of [**6**].

The rest of this section will be devoted to the proof of Proposition 6.1.2.

6.1.2. Proposition. *Let* (J^Λ, X^Λ) *be an admissible* (J, X)*-homotopy, and* $u \in \hat{\mathcal{M}}_P^{\Lambda,0}(\mathbf{x}, \mathbf{x})$. *Then (NEP) holds for* u.

Without loss of generality, we restrict our attention to a $J|X$-homotopy without death–birth bifurcations throughout this section.

6.2. Nonequivariant perturbations on finite-cyclic covers. This subsection contains the main body of the proof of Proposition 6.1.2. We first discuss a simpler situation in which the nonequivariant perturbation may be obtained from a vector field on a finite-cyclic covering of M. In general, we need to resort to nonlocal perturbations.

6.2.1. A special case: local perturbations from finite-cyclic covers of M. If a finite-cyclic cover $\mathcal{C}^{\nu,m} \to \mathcal{C}$ is (a path component of) the pull-back bundle of a finite-cyclic cover $\hat{M} \to M$ via $e_f\colon \mathcal{C} \to M$ (cf. I.3.1.1), then a nonequivariant function or vector field on $\hat{M}$ may induce a nonequivariant function or vector field on $\mathcal{C}^{\nu e,m}$.

Example. Assume the conditions of Corollary I.2.2.5 (namely, M is monotone, f is symplectic isotopic to id, and γ_0 is the trace of a point under the symplectic isotopy). We claim that in this case, for any $m \in \mathbb{Z}^+$ not dividing $\mathrm{div}([u])$, there exists a u-breaking m-cyclic cover of $\mathcal{C}$ via the above pull-back construction. Thus, in this case, Proposition 6.1.2 may be proven by simply repeating the argument for Proposition I.6.2.2 for nonequivariant Hamiltonian perturbations over finite-cyclic covers of M. (In fact, only Lemma I.6.2.5 needs to be redone.)

To see the claim, recall that in this case,

$$\mathfrak{H} = H_1(\mathcal{C};\mathbb{Z}) = \pi_2(M) \oplus H_1(M;\mathbb{Z}), \quad \textit{and} \quad e_{f*} = 0 \oplus \mathrm{id}$$

with respect to this decomposition. Notice that $e_{f*}([u])$ is a nontorsion element in $H_1(M;\mathbb{Z})$. Otherwise, by the commutative diagram from I.(12),

$$k[u] = b \in \ker\, c_1\Big|_{\pi_2(M)} \quad \textit{for some } k \in \mathbb{Z}^+.$$

But then by monotonicity of M,

$$[\mathcal{Y}_X](k[u]) = \omega(b) - e_f^*\theta_X(k[u]) = 0,$$

contradicting the fact that u has positive energy.

Thus, for any $m \in \mathbb{Z}^+$ not dividing $\mathrm{div}([u])$, one may simply set $\nu_M \in H^1(M;\mathbb{Z})$ to be a primitive class with $\nu_M(e_{f*}([u])) = \mathrm{div}([u])$ and take $\mathcal{C}^{\nu,m} = e_f^* M^{\nu_M,m}$. Furthermore, such a finite-cyclic cover is always $\mathfrak{H}$-adapted, and u-breaking if m does not divide $\mathrm{div}([u])$.

However, this simple construction does not give all the u-breaking finite covers we need.

6.2.2. The general case: nonlocal perturbations. Let $\mathcal{C}^{\nu,m}$ be a u-breaking, $\mathfrak{H}$-adapted m-cyclic cover of $\mathcal{C}$ introduced in I.4.4.5.

We shall often make use of the following convenient description of $\mathcal{C}^{\nu,m}$:

$$\mathcal{C}^{\nu,m} = \Big\{(z,[\mu]) \,|\, z \in \mathcal{C}, \mu\colon [0,1] \to \mathcal{C}; \mu(0) = \gamma_0, \mu(1) = z\Big\}/\sim,$$

where $(z_1,[\mu_1]) \sim (z_2,[\mu_2])$ iff $z_1 = z_2$ and $\nu([\mu_1-\mu_2]) = 0 \mod m$. Such an equivalence class shall be denoted by a pair $(z,[\mu]_m)$.

Recall that $\nu \in \mathrm{Hom}(\mathfrak{H},\mathbb{Z})$. The fact that $\mathcal{C}^{\nu,m}$ is u-breaking implies that ν is nontorsion. Thus, one may find a class $\nu_2 \in H^2(T_f;\mathbb{R})$ extending ν by linearity, that is, satisfying $\nu_2((i_{\mathfrak{H}} \ker\nu)\otimes\mathbb{R}) = 0$, and $\nu_2(i_{\mathfrak{H}}[u]) = \mathrm{div}(u)$ where $i_{\mathfrak{H}}\colon \mathfrak{H} \hookrightarrow H_2(T_f;\mathbb{Z})$ is the inclusion. Let ω_ν be a smooth closed 2-form on T_f in the cohomology class ν_2.

The 2-form ω_ν defines an $\mathbb{R}$-valued function Ω_ν on $\tilde{\mathcal{C}}$, by setting

$$\Omega_\nu(z,[\mu]) := \int_{[0,1]\times S^1_1} \mu^*\omega_\nu.$$

This induces an $\mathbb{R}/m\mathbb{Z}$-valued function on $\mathcal{C}^{\nu,m}$, which we shall denote by the same notation.

Definition (**A class of nonlocal perturbations**). Let $\chi\colon \mathbb{R}/m\mathbb{Z} \to \mathbb{R}$ be a smooth function, and let $P \in \mathcal{H}$. We define the formal vector field $\wp_{\chi P}$ on $\mathcal{C}^{\nu,m}$ by

$$\wp_{\chi P}(z,[\mu]_m) := \chi(\Omega_\nu(z,[\mu]_m))\nabla P(z). \tag{103}$$

For a path $(u(s),[\mu(s)]_m)$ in $\mathcal{C}^{\nu,m}$, let

$$\bar\partial^{\chi P}_{JX}(u,[\mu]_m) := \bar\partial_{JX}u + \wp_{\chi P}(u,[\mu]_m).$$

Similarly, for a smooth function $\chi^\Lambda\colon \Lambda\times\mathbb{R}/m\mathbb{Z} \to \mathbb{R}$ and $P^\Lambda \in \mathcal{H}^\Lambda$, one may define a path of formal vector fields $\{\wp_{\chi_\lambda P_\lambda}\}_{\lambda\in\Lambda}$ and the section $\bar\partial^{\chi^\Lambda P^\Lambda}_{J^\Lambda X^\Lambda}$ on $\mathcal{B}^\Lambda_P$ or $\mathcal{B}^\Lambda_O$.

For the rest of this section, a "*χP-perturbed flow*" or simply a "*perturbed flow*" will refer to a solution of $\bar\partial^{\chi P}_{JX}(u,[\mu]_m) = 0$. One may define the moduli spaces of such flows, $\mathcal{M}_{P;\nu,m}(J,X;\chi,P), \mathcal{M}_{O;\nu,m}(J,X;\chi,P)$, etc., and their parameterized versions, in the usual manner (cf. I.2.1.2, I.4.3.1). Notice that if one chooses $P \in V^k_\delta(J,X)$ and $P^\Lambda \in V^{\Lambda;k,\kappa}_\delta(J^\Lambda,X^\Lambda)$, then

$$\mathcal{P}(X;\chi,P) = \mathcal{P}(X); \quad \mathcal{P}^\Lambda(X^\Lambda;\chi^\Lambda,P^\Lambda) = \mathcal{P}^\Lambda(X^\Lambda),$$

and in both equalities, the former is nondegenerate iff the latter is. We shall show in the next subsection that in this case, when χ, χ^Λ are sufficiently small, and if (J,X) is regular and (J^Λ,X^Λ) admissible, then the moduli spaces of χP-perturbed flows and their parameterized versions satisfy all the usually expected regularity and compactness properties, as described by (FS2), (FS3) and (RHFS2*), (RHFS3*).

Proof of Proposition 6.1.2. Let $\Re \in \mathbb{R}^+$ and $\mathcal{C}^{\nu,m}$ be fixed as in the statement of (NEP). Without loss of generality, assume $\Pi_\Lambda u = 0$.

The admissible (J,X)-homotopy (J^Λ,X^Λ) induces a homotopy of formal flows on $\mathcal{C}^{\nu,m}$, which satisfies all the properties listed in I.6.2.3 for admissibility, except for Property (8) (injectivity of $\Pi_\Lambda|_{\hat{\mathfrak{M}}^{\Lambda,0}_P}$): at $\lambda = 0$, there are

m distinct elements in $\hat{\mathcal{M}}^0_P(J_\lambda, X_\lambda)$, which are precisely the m different lifts of u.

We write this induced homotopy of vector fields as $\{\mathcal{V}^{\nu,m}(J_\lambda, X_\lambda)\}_{\lambda\in\Lambda}$.

To achieve Property (8), we shall consider homotopy of vector fields on $\mathcal{C}^{\nu,m}$ of the form

$$\{\mathcal{V}^{\nu,m}(J_\lambda, X_\lambda; \chi_\lambda, P_\lambda)\}_{\lambda\in\Lambda} := \{\mathcal{V}^{\nu,m}(J_\lambda, X_\lambda) + \wp_{\chi_\lambda P_\lambda}\}_{\lambda\in\Lambda},$$

where

$$P^\Lambda \in V_\delta^{\Lambda;k,\kappa}(J^\Lambda, X^\Lambda). \tag{104}$$

In fact, since $\hat{\mathcal{M}}^{\Lambda,0}_P(J^\Lambda, X^\Lambda; \mathrm{wt}_{-\langle\mathcal{Y}\rangle,e_{\mathcal{P}}} \leq \Re)$ consists of finitely many points, each projecting under Π_Λ to distinct values, we may assume that

$$P_\lambda = 0 \quad \text{for } \lambda \in \Lambda\backslash S, \tag{105}$$

where S is a small interval about $\Pi_\Lambda(u) = 0$, so that

$$S \cap (\Lambda_{\mathrm{db}} \cup \Pi_\Lambda(\hat{\mathcal{M}}^{\Lambda,0}_P(J^\Lambda, X^\Lambda; \mathrm{wt}_{-\langle\mathcal{Y}\rangle,e_{\mathcal{P}}} \leq \Re)\backslash\{u\}) = \emptyset.$$

Such perturbed homotopy of formal flows might no longer be co-directional; however, Properties (1)–(6) of I.6.2.3 are preserved. Moreover, we shall see in the next subsection that as long as χ is sufficiently small in C_ϵ-norm, the parameterized moduli spaces remain $\Re$-regular (i.e., $\Re$-truncated version of I.6.2.3 (7) holds).

We now describe an explicit choice of χ^Λ, P^Λ among all those satisfying both (104), (105), so that I.6.2.3 (8) may be achieved. For this purpose, the argument in the proof of Lemma I.6.2.5 is revised as follows.

Replace u_n there by u, let B be a small neighborhood in $Q_1 \cap Q_2 \subset \mathbb{R}\times S^1$. Let $P_0 \in V_\delta^k(J_0, X_0)$ be supported in a small neighborhood $\mathfrak{B} \subset T_f$, such that $u^{-1}(\mathfrak{B}) \subset B$, similar to the definition of $\underline{H}_{\lambda_n}$ in I.6.2.5. Let P^Λ be an extension of P_0 satisfying (104) and (105), which is in turn the analog of $\underline{H}^\Lambda$ in I.6.2.5.

Let $\tilde{u}_1, \dots, \tilde{u}_m$ be the m distinct lifts of u in $\mathcal{C}^{\nu,m}$. With the above choice of P_0, the perturbation $\wp_{\chi_0 P_0}(\tilde{u}_i(s))$ is nontrivial only when s is in the small interval

$$I_B := \mathrm{pr}_1(B),$$

where $\mathrm{pr}_1 : \mathbb{R}\times S^1 \to \mathbb{R}$ denotes the projection. By construction, the values of Ω_ν at different lifts of a point in $\mathcal{C}$ differ by multiples of m. Thus if I_B is sufficiently small, the image $\Omega_\nu(\tilde{u}_i(I_B))$ forms disjoint intervals in $\mathbb{R}$ for different $\tilde{u}_i$. We denote the interval corresponding to $\tilde{u}_i$ by $\mathcal{I}_i$ and choose χ_0 such that

$$\chi_0(\phi) = C_i \quad \text{when } \phi \in \mathcal{I}_i,\ i = 1, 2, \dots, m,$$

where C_i are distinct constants and χ_0 is very small in C_ϵ-norm. With this choice,

$$\wp_{\chi_0 P_0}(\tilde{u}_i(s)) = C_i \nabla P_0(\tilde{u}_i(s)),$$

and the analog of I.(71) now reads

$$E_{\tilde{u}_i}\xi + \alpha_i Y_{\tilde{u}_i} + C_i \nabla P_0(\tilde{u}_i) = 0,$$

this, together with the contraction mapping theorem, shows that $\tilde{u}_i$ perturbs into a $\tilde{v}_i$, so that $\Pi_\Lambda \tilde{v}_i - \Pi_\Lambda u$ are, up to higher order correction terms, proportional to C_i. Hence the perturbed flows have distinct values under Π_Λ.

As remarked before, the regularity of $\Re$-truncated moduli spaces is unaffected by this perturbation, and thus $\{\mathcal{V}^{\nu,m}(J_\lambda, X_\lambda; \chi_\lambda, P_\lambda)\}_{\lambda\in\Lambda}$ satisfies all the $\Re$-truncated versions of I.6.2.3 (1)–(8). In particular, it has all the $\Re$-truncated versions of the properties (RHFS*), except for (RHFS2c), (RHFS3c), and (RHFS4).

To see that the remaining properties also hold, we need to verify that the gluing theorems proven in previous sections still hold. (The arguments for (RHFS4) to appear in Section 7 below depend on the perturbation only through the existence of Fredholm theory, and the linear gluing theorem in Section 1.2.4.)

By construction, $S \cap \Lambda_{\mathrm{db}} = \emptyset$. Thus, no gluing for births or deaths (as in Sections 2–5) is necessary.

The proofs of standard gluing theorems such as Proposition 6.1.1, Lemma 1.2.4 do require updates. However, because of our choice of P^Λ, $\wp_{\chi_\lambda P_\lambda}$ vanishes near the critical point x_λ. Thus, we have the usual exponential decay of flows to critical points and the same error estimates. Only two facts need to be verified for the standard arguments sketched in Section 1.2 to go through:

(1) Fredholm theory and surjectivity of the perturbed version of deformation operators $E_{\tilde{v}}^{J_\lambda, X_\lambda; \chi_\lambda, P_\lambda}$, $\hat{E}_{\tilde{u}}^{J_\lambda, X_\lambda; \chi_\lambda, P_\lambda}$, where $\tilde{u}$, $\tilde{v}$ are the perturbed flows to be glued;
(2) the usual quadratic bound on the nonlinear term N_{w_χ}, namely, (3).

We shall verify these in the next subsection.

□

6.3. $\Re$-truncated moduli spaces of perturbed flows. The structure theory of the moduli spaces of such perturbed flows is not covered in the literature or in the discussion of Part I. We need to start from scratch and check the foundation of this more general theory. The major components of the expected structure theory are examined in turn below.

We have already mentioned the following basic fact:

6.3.1. Fact (Exponential decay). A perturbed flow decays exponentially to a nondegenerate critical point.

This is due to our choice that $P^\Lambda \in V_\delta^{\Lambda;k,\kappa}(J^\Lambda, X^\Lambda)$, in particular, P_λ vanishes up to kth order at the critical points. In fact, this also shows that a perturbed flow decays polynomially to a good minimally degenerate critical point as described in Section I.5. However, we do not need this fact.

6.3.2. Fredholm theory. Consider the linearization of $\bar{\partial}^{\chi P}_{JX}(u,[\mu]_m)$. We denote it by $E^{J,X;\chi,P}_{(u,[\mu]_m)}$ or $D^{J,X;\chi,P}_{(u,[\mu]_m)}$, depending on whether $(u,[\mu]_m)$ is an connecting flow line or an orbit. In addition to the well-understood $E^{J,X}_{(u,[\mu]_m)}(\xi)$ or $D^{J,X}_{(u,[\mu]_m)}(\xi)$, it has the following extra terms due to $\wp_{\chi P}$:

$$\begin{aligned}(106)\qquad &\chi(\Omega_\nu(u,[\mu]_m))\nabla_\xi\nabla P(u)\\ &\quad+\chi'(\Omega_\nu(u,[\mu]_m))\int_{S^1}\omega_\nu(\xi,\partial_t u)\,dt\,\nabla P(u).\end{aligned}$$

Observing that the first term is a 0-th-order multiplicative operator, and the second term is a mixture which is infinitely smoothing in t and 0-th-order in s, this implies that $D^{J,X;\chi,P}_{(u,[\mu]_m)}$ is still Fredholm. To see that $E^{J,X;\chi,P}_{(u,[\mu]_m)}$ is Fredholm, we use in addition Fact 6.3.1 above, and the usual excision argument for Fredholmness in this situation.

The deformation operators for parameterized moduli spaces are finite-rank stabilizations of the above operators, and their Fredholmness is thus evident from the above discussion.

6.3.3. Estimating the nonlinear term. The contribution of the perturbation to the nonlinear term $N^{J,X;\chi,P}_{(u,[\mu]_m)}(\xi)$ is

$$\begin{aligned}&\chi(\Omega_\nu(u,[\mu]_m))\nabla_\xi\nabla_\xi\nabla P(u)\\ &\quad+2\chi'(\Omega_\nu(u,[\mu]_m))\int_{S^1}\omega_\nu(\xi,\partial_t u)\,dt\,\nabla_\xi\nabla P(u)\\ &\quad+\chi''(\Omega_\nu(u,[\mu]_m))\Big(\int_{S^1}\omega_\nu(\xi,\partial_t u)\,dt\Big)^2\nabla P(u)\\ &\quad+\chi'(\Omega_\nu(u,[\mu]_m))\int_{S^1}\omega_\nu(\xi,\partial_t \xi)\,dt\,\nabla P(u).\end{aligned}$$

It is straightforward to check that each term above may be bound by

$$C\|\xi\|_{C^0}\|\xi\|_{L^p_1}\le C'\|\xi\|^2_{L^p_1}.$$

We omit the straightforward estimate for the parameterized version.

6.3.4. Compactness. Let us go over the main ingredients in the usual proof one by one.

- *Elliptic regularity.* By the above estimate on the nonlinear term, and the form of (106), the elliptic bootstrapping argument still hold, provided a C^0 bound can be established. The latter relies on the Gromov compactness.

- *Gromov compactness.* Going through the rescaling argument, we note that the extra term $\wp_{\chi P}$ disappears in the limit, and therefore again (local) compactness is lost only through bubbling off honest holomorphic spheres. This possibility is eliminated via transversality as in Section 3 of Part I.
- *Energy bound.* With this definition of nonlocal perturbations, there might not be a good action functional for the perturbed flows.[1] However, we still have the requisite energy bound for perturbed flows with weight $\leq \Re$. Let $(u,[\mu]_m)$ be such a χP-perturbed flow. Then

(107)
$$\begin{aligned}\mathcal{E}(u,[\mu]_m) &= \|\partial_s u\|_2^2\\ &= \alpha\,\mathrm{wt}_{-\langle\mathcal{Y}\rangle,e_{\mathcal{P}}}(u,[\mu]_m) + \int \Big\langle \partial_s u, \chi(\Omega_\nu(u,[\mu]))\nabla P(u)\Big\rangle_{2,t}(s)\,ds.\end{aligned}$$

On the other hand,

Lemma. *Let $(u,[\mu]_m)$ be a χP-perturbed flow (either a connecting flow line or an orbit). Then there is a constant C independent of s or $(u,[\mu]_m)$, such that*
$$\|\nabla P(u)\|_{2,t}(s) \leq C\|\partial_s u\|_{2,t}(s)\ \forall s.$$

Proof. This follows from the L_1^∞-boundedness of P, the fact that P vanishes to up order $k>2$ at the critical points, and the following.

Palais–Smale condition. *There exists an $\varepsilon'>0$ such that for any $(z,[\mu]_m)\in\mathcal{C}^{\nu,m}$ with $\|J(z)\partial_t z+\check{\theta}_X(z)+\wp_{\chi P}(z,[\mu]_m)\|_{2,t}\leq\varepsilon'$, there is a critical point z_0 such that $z=\exp(z_0,\xi)$ for a small ξ, and*
$$\|J(z)\partial_t z+\check{\theta}_X(z)+\wp_{\chi P}(z,[\mu]_m)\|_{2,t}\geq \begin{cases} C_1\|\xi\|_{2,t} & \text{when } z_0 \text{ is nondegenerate}\\ C_2\|\xi\|_{2,t}^2 & \text{when } z_0 \text{ is minimally degenerate}\\ & \quad\text{in a standard d-b neighborhood.}\end{cases}$$

This in turn follows from the Ascoli–Arzela argument as in the unperturbed case, since by our condition on P, $\wp_{\chi P}$ can be ignored near critical points. □

Thus, if $\|\chi\|_{C_\epsilon}\leq\varepsilon$, the absolute value of the last term in (107) can be bounded by $C\varepsilon\|\partial_s u\|_2^2$, and by rearranging,
$$\mathcal{E}(u,[\mu]_m)\leq(1-C\varepsilon)^{-1}\alpha\,\mathrm{wt}_{-\langle\mathcal{Y}\rangle,e_{\mathcal{P}}}(u,[\mu]_m)\leq(1-C\varepsilon)^{-1}\alpha\Re.$$

- *Global compactness* (for $\hat{\mathcal{M}}_P,\hat{\mathcal{M}}_P^\Lambda$). As in the unperturbed case, to go from local compactness to global compactness, we just need in addition Fact 6.3.1.

[1] We may easily modify the definition of $\wp_{\chi P}$ so that there is; however we would run into difficulty with Gromov compactness.

6.3.5. Transversality. The transversality arguments in Part I uses a unique continuation theorem extensively. However, Aronszajn's theorem or the Carleman similarity principle used in [**7**] is no longer applicable as the nonlocal term is introduced. While it might be possible to prove a unique continuation result in this situation, we choose not to develop a general theory here. Instead, for the purpose of proving Proposition 6.1.2, we only need the following.

Claim. *Let* (J, X) *be regular,* χ *be sufficiently small* (*in* C_ϵ *norm*), $P \in V_\delta^k(J, X)$ *and* $i \leq 2$. *Then for* $\mathcal{M} = \mathcal{M}_P$ *or* $\mathcal{M}_O$, $\mathcal{M}^i_{\nu,m}(J, X; \chi, P; \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re)$ *is* (*Zariski*) *smooth. Similarly for the parameterized versions.*

Take $\mathcal{M}_P$, for example; the arguments for $\mathcal{M}_O$ or the parameterized versions are similar. Due to Lemma 1.2.4, for regular (J, X) and u in a neighborhood of a lower-dimensional strata of $\hat{\mathcal{M}}_P^{i-1,+}(J, X; \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re)$, the deformation operator E_u has a uniformly bounded right inverse. Combining this with the compactness of $\hat{\mathcal{M}}_P^{i-1,+}(J, X; \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re)$, there is a small number $\delta > 0$ such that any element in $\{\mathfrak{D} \,|\, \mathfrak{D} - E_v^{JX}\| < \delta, v \in \hat{\mathcal{M}}_P^{i-1}(J, X; \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re)\}$ is surjective. In particular, there is a $\delta' = \delta'(\delta)$, such that for any element w in

$$\Big\{ \exp(v, \xi) \,|\, \|\xi\|_{\infty,1} < \delta',\ v \in \hat{\mathcal{M}}_P^{i-1}(J, X; \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re) \Big\},$$

$E^{JX;\chi P}_{(w,[\mu_w]_m)}$ is surjective for any lift $(w, [\mu_w]_m)$ of w in $\mathcal{C}^{\nu,m}$, $\forall \chi$ with $\|\chi\|_{C_\epsilon} < \delta'$. Thus, the claim follows from the following.

Lemma. *Fix* P, i, *and* $\Re$ *as above. Then there is an* $\varepsilon' > 0$ *such that for all* χ *with* $\|\chi\|_{C_\epsilon} < \varepsilon'$, *any element* $(u, [\mu]) \in \hat{\mathcal{M}}^{i-1}_{P;\nu,m}(J, X; \chi, P, \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re)$ *is close to* $\hat{\mathcal{M}}_P^{i-1}(J, X, \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re)$ *in the sense that*

$$(*) \quad u = \exp(v, \xi),\ \|\xi\|_{\infty,1} < \delta' \quad \textit{for some } v \in \hat{\mathcal{M}}_P^{i-1}(J, X, \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re).$$

Proof. Suppose the contrary. Then there exists a sequence $\{\chi_n\}$ such that $\lim_{n\to\infty} \|\chi_n\|_{C_\epsilon} = 0$, and a sequence

$$\big\{(u_n, [\mu_n]_m)\big\}_n \subset \hat{\mathcal{M}}^k_{P,m}(J, X; \chi, P; \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re)$$

such that none of them satisfies (*). By Gromov compactness theorem, such a sequence $\big\{(u_n, [\mu_n]_m)\big\}$ must weakly converge to an element v in $\hat{\mathcal{M}}^k_P(J, X; \mathrm{wt}_{-\langle\mathcal{P}\rangle,e_\mathcal{P}} \leq \Re)$ together with some bubbles. Since by the regularity of (J, X), there is no such bubble, $(u_n, [\mu_n]_m)$ are close to v, contradicting our assumption. □

This also shows that when χ is sufficiently small and (J, X) regular, these χP-perturbed flows avoid pseudo-holomorphic spheres, as in the case before perturbation.

7. Orientation and signs

In this section, we tie up the last loose end of this article by addressing all orientation issues so far ignored: we verify (FS4) for the Floer theory described in Section I.3 and show that an admissible (J, X)-homotopy satisfies (RHFS4).

In Section 7.2, we show that the various moduli spaces $\mathcal{M}_P(x, y)$, $\mathcal{M}_O^1$ and their parameterized variants are orientable; furthermore, we introduce the notions of coherent orientations for $\mathcal{M}_P, \mathcal{M}_P^\Lambda$ and grading-compatible orientations for $\mathcal{M}_O^1, \mathcal{M}_O^{\Lambda,2}$ and show that these moduli spaces may be endowed with such orientations. This completes the verification that the formal flow associated to a regular pair (J, X) forms a Floer system. The coherence of orientation is determined by linearized versions of the gluing theorems proven in the previous sections; this is, in fact, why we have postponed this discussion. Compared with the full gluing theory, major simplifications for these linearized versions arise from the fact that we may substitute the complicated polynomially weighted Sobolev spaces used in Sections 2–5 by larger, exponentially weighted versions, due to the removal of constraints from nonlinear aspects of general gluing theory. Furthermore, deformation operators between these exponentially weighted spaces are conjugate to deformation operators between the usual L^p_k spaces with perturbation by asympotically constant 0-th-order terms, making it possible to work only with the ordinary Sobolev norms throughout this section.

In Section 7.3, we verify the signs in the expressions for $\mathbb{T}_{P,\mathrm{db}}, \ldots, \mathbb{T}_{O,\text{hs-s}}$ given in (RHFS4) (cf. Section I.4.3.7). This is obtained by examining the orientations of the K-models used in the proofs of the gluing theorems in previous sections. With this done, the verification that admissible (J, X)-homotopies satisfy the assumptions of Proposition I.4.6.3 is complete, which in turn implies the general invariance theorem, Theorem I.4.1.1.

7.1. Basic notions and conventions. We first review some basic materials to fix terminology and conventions.

7.1.1. Orientation for direct sums and exact sequences. Given a direct sum of an ordered k-tuple of oriented vector spaces, $E = E_1 \oplus \cdots \oplus E_k$, we orient it by $e_1 \wedge \cdots \wedge e_k \in \det E$, where $e_i \in \det E_i$ orients E_i.

An exact sequence of finite-dimensional vector spaces

$$0 \to E_1 \xrightarrow{i_1} F_1 \xrightarrow{j_1} E_2 \xrightarrow{i_2} F_2 \cdots \xrightarrow{j_{n-1}} E_n \xrightarrow{j_n} F_n \to 0$$

determines an isomorphism $\bigotimes_k \det E_k \simeq \bigotimes_k \det F_k$, by writing

$$E_k = B_k^E \oplus j_{k-1} B_{k-1}^F, \quad F_k = i_k B_k^E \oplus B_k^F$$

for appropriate oriented subspaces B_k^E, B_k^F, over which i_k, j_k restrict to isomorphisms.

7.1.2. Orientation for determinant lines and K-models. Given a Fredholm operator $\mathfrak{D}\colon E \to F$, the determinant line

$$\det \mathfrak{D} := \det \ker \mathfrak{D} \otimes \det(\operatorname{coker} \mathfrak{D})^*.$$

It is well known that for a continuous family (in operator norm) of Fredholm operators, the determinant lines above form a real line bundle over the parameter space, namely the determinant line bundle. We use $\mathfrak{or}(\mathfrak{D})$ to denote the space of possible orientations for $\det \mathfrak{D}$ when it is orientable, and similarly, $\mathfrak{or}(\mathfrak{D}^\Lambda)$ denotes the space of possible trivializations of the determinant line bundle for the family $\mathfrak{D}^\Lambda$ when it is orientable. These are affine spaces under $\mathbb{Z}/2\mathbb{Z}$. If $\mathfrak{D}_{\lambda_1}, \mathfrak{D}_{\lambda_2}$ are elements of the family of operators $\mathfrak{D}^\Lambda$, we say that $\mathfrak{o}_1 \in \mathfrak{or}(\mathfrak{D}_{\lambda_1})$ and $\mathfrak{o}_2 \in \mathfrak{or}(\mathfrak{D}_{\lambda_2})$ are *correlated* via $\mathfrak{D}^\Lambda$ if they are restrictions of the same trivialization $\mathfrak{O} \in \mathfrak{or}(\mathfrak{D}^\Lambda)$. They are said to be of *relative sign* $\rho \in \{\pm 1\}$ (with respect to $\mathfrak{D}^\Lambda$), denoted $[\mathfrak{o}_1/\mathfrak{o}_2]$, if $\mathfrak{o}_1$ and $\rho\mathfrak{o}_2$ are correlated.

It is convenient to describe the orientation of $\det \mathfrak{D}$ in terms of K-models. Recall the definition of oriented K-models and the exact sequence (6) from Section 1.2.3. This exact sequence induces the isomorphism:

$$\det \mathfrak{D} \simeq \det K \otimes \det C^*.$$

Thus, an orientation of a K-model for $\mathfrak{D}$ decides an orientation for $\det \mathfrak{D}$. Given an orientation of $\det \mathfrak{D}$, an oriented K-model $[K; C]$ of $\mathfrak{D}$ is said to be *compatible* with this orientation, if the orientation of $[K; C]$ induces the orientation of $\det \mathfrak{D}$.

Two K-models of $\mathfrak{D}$ are said to be *co-oriented* if they give rise to the same orientation of $\det \mathfrak{D}$. Let $[\mathfrak{D}_{\lambda_1}\colon K_{\lambda_1} \to C_{\lambda_1}]_{B_{\lambda_1}}$, $[\mathfrak{D}_{\lambda_2}\colon K_{\lambda_2} \to C_{\lambda_2}]_{B_{\lambda_2}}$ be fibers of a family K-model for $\mathfrak{D}^\Lambda$. They are said to be *mutually co-oriented* via the family $\mathfrak{D}^\Lambda$ if they are, respectively, compatible with orientations of $\det \mathfrak{D}_{\lambda_1}$ and $\det \mathfrak{D}_{\lambda_2}$ correlated by $\mathfrak{D}^\Lambda$.

7.1.3. Induced orientation of a stabilization. Let $\hat{\mathfrak{D}}_\Psi\colon \mathbb{R}^k \oplus E \to F$ be a stabilization of $\mathfrak{D}\colon E \to F$; recall the definition of stabilized K-models from Section 1.2.3.

Given an orientation $\mathfrak{o} \in \mathfrak{or}(\mathfrak{D})$, we define the *induced orientation* $\hat{\mathfrak{o}} \in \mathfrak{or}(\hat{\mathfrak{D}}_\Psi)$ from $\mathfrak{o}$ as follows. Given an oriented K-model $[\mathfrak{D}\colon K \to C]_B$ compatible with $\mathfrak{o}$, let $\hat{\mathfrak{o}}$ be the orientation given by the stabilization $[\hat{\mathfrak{D}}_\Psi\colon \hat{K} \to C]_{\hat{B}}$, where $\hat{K}$ is oriented as

$$\hat{K} = (-1)^{k \operatorname{ind} \mathfrak{D}}\, \mathbb{R}^k \oplus K.$$

7.1.4. Reduction of oriented K-models. Let the K-model $[K' \to C']$ be a reduction of another K-model, $[K \to C]$, by Q (cf. Section 1.2.3). Then the orientation of one K-model induces an orientation of the other via writing

$$K = K' \oplus Q; \quad C = C' \oplus \Pi_C \mathfrak{D}(Q)$$

as oriented spaces. Note that changing the orientation of Q results in a co-oriented K-model.

7.1.5. Orientation for glued K-models. Recall the definitions and notations in Section 1.2.4. Given an ordered k-tuple of finite-dimensional subspaces $K_1, \dots, K_k$ in E or F, and sufficiently large $R_1, \dots, R_k$, we orient the glued space $K_1 \#_{R_1} \cdots \#_{R_{k-1}} K_k$ or $K_1 \#_{R_1} \cdots \#_{R_{k-1}} K_k \#_{R_k}$ by its natural isomorphism with $K_1 \oplus K_2 \oplus \cdots \oplus K_k$.

Let $\mathfrak{D}_1$, $\mathfrak{D}_2$ be an ordered pair of glueable Floer-type operators, and let $\mathfrak{D}$ be a cyclically glueable Floer-type operator. Given $\mathfrak{o}_1 \in \mathfrak{or}(\mathfrak{D}_1)$, $\mathfrak{o}_2 \in \mathfrak{or}(\mathfrak{D}_2)$, $\mathfrak{o} \in \mathfrak{or}(\mathfrak{D})$, we define $\mathfrak{o}_1 \#_R \mathfrak{o}_2 \in \mathfrak{or}(\mathfrak{D}_1 \#_R \mathfrak{D}_2)$, $\mathfrak{o}_{\#_R} \in \mathfrak{or}(\mathfrak{D}\#_R)$ as follows. Let $[K_1 \to C_1]$, $[K_2 \to C_2]$, $[K \to C]$ be oriented K-models compatible with $\mathfrak{o}_1$, $\mathfrak{o}_2$, and $\mathfrak{o}$, respectively. Then the *induced orientation*, $\mathfrak{o}_1 \#_R \mathfrak{o}_2$ and $\mathfrak{o}_{\#R}$, are, respectively, the orientation given by the oriented K-models

$$\Big[(-1)^{\dim(C_1)\cdot \operatorname{ind} \mathfrak{D}_2} K_1 \#_R K_2 \to C_1 \#_R C_2\Big], \quad [K_{\#R} \to C_{\#R}]. \tag{108}$$

The orientations for the generalized kernels and generalized cokernels of stabilized, reduced, or glued K-models given above are chosen such that co-oriented K-models give rise to co-oriented stabilized, reduced, or glued K-models. Thus, we have well-defined homomorphisms of affine spaces under $\mathbb{Z}/2\mathbb{Z}$:

$$\begin{aligned} s_\Psi\colon\ \mathfrak{or}(\mathfrak{D}) &\to \mathfrak{or}(\hat{\mathfrak{D}}_\Psi), \\ \mathfrak{g}_R\colon\ \mathfrak{or}(\mathfrak{D}_1) \times \mathfrak{or}(\mathfrak{D}_2) &\to \mathfrak{or}(\mathfrak{D}_1 \#_R \mathfrak{D}_2), \\ \mathfrak{sg}_R\colon\ \mathfrak{or}(\mathfrak{D}) &\to \mathfrak{or}(\mathfrak{D}_{\#R}), \end{aligned}$$

sending $\mathfrak{o}$ to $\hat{\mathfrak{o}}$, $\mathfrak{o}_1 \times \mathfrak{o}_2$ to $\mathfrak{o}_1 \#_R \mathfrak{o}_2$, and $\mathfrak{o}$ to $\mathfrak{o}_{\#R}$, respectively. We call s_Ψ the *stabilization isomorphism*, and $\mathfrak{g}_R$, $\mathfrak{sg}_R$ the *gluing homomorphisms.* As a consequence of the independence of K-models, the above constructions also work in the family setting to define induced orientations for the determinant line bundles of stabilized or glued operators. In addition, the gluing homomorphisms above may be extended to be defined for arbitrary k-tuple of glueable or cyclically glueable Floer-type operators, by observing that any glued operator or cyclically glued operator can be obtained by a combination of translation and the two gluing operations discussed above. Morever, with the above definition, it is straightforward to check that the oriented K-model for the same glued operator obtained from different combinations are actually the same.

Remarks. (1) Alternatively, one may define induced orientation for stabilization by the oriented K-model $[\mathbb{R}^k \oplus K \to C]$ instead. We have so chosen our definition because in our context, det $\mathfrak{D}_\Psi$ gives the orientation of a fiber bundle, where $\mathbb{R}^k$ corresponds to the tangent space of the base. We prefer

the "fiber-first" convention for orienting a fiber bundle. With our definitions, the gluing homomorphism commutes with the (rank k) stabilization isomorphism on $\mathfrak{D}_2$, but commutes with stabilization on $\mathfrak{D}_1$ modulo the sign $(-1)^{k \operatorname{ind} \mathfrak{D}_2}$.
(2) The definitions of the orientation for a stabilization and glued operators in [**4**] differ from ours. Their definitions have the following disadvantage. Given an orientation of a determinant line bundle for a family $\{\mathfrak{D}_\lambda\}_{\lambda\in\Lambda}$, the stabilization isomorphism of [**4**] gives a *possibly discontinuous*, nowhere vanishing section of the determinant line bundle of the stabilized family $\{\hat{\mathfrak{D}}_{\lambda,\Psi}\}_{\lambda\in\Lambda}$. Furthermore, the gluing morphisms in [**4**] commute with stabilization *only up to a sign* depending on the dimension of $\mathbb{R}^k$.

7.2. Orienting moduli spaces. This subsection addresses the orientability issues required by (FS4) and (RHFS4).

By an *orientation* of a moduli space $\mathcal{M} = \mathcal{M}_P$ or $\mathcal{M}_O$, we mean the following. Notice that the configuration spaces $\mathcal{B}^k_P(x, y)$, $\mathcal{B}^k_O$ parameterize families of deformation operators, $\{E_u \,|\, u \in \mathcal{B}^k_P(x, y)\}$, $\{\tilde{D}_{(T,u)} \,|\, (T, u) \in \mathcal{B}^k_O\}$. Thus, they carry determinant line bundles, which we denote by $L\mathcal{B}^k_P(x, y)$, $L\mathcal{B}_O$. The moduli space $\mathcal{M} \subset \mathcal{B} = \mathcal{B}^k_P(x, y)$ or $\mathcal{B}^k_O$ parameterizes a subfamily of deformation operators, and thus carries a determinant line bundle $L\mathcal{M}$, which is the pull-back of $L\mathcal{B}$. An orientation of $\mathcal{M}$ will mean a trivialization of $L\mathcal{M}$. In this article, this will always be the pull-back of a trivialization of $L\mathcal{B}$, and we shall therefore focus on orienting $L\mathcal{B}$ for various configuration spaces $\mathcal{B}$. Similarly, parameterized moduli spaces $\mathcal{M}^\Lambda$ will be oriented by orienting $L\mathcal{B}^\Lambda$. Since the deformation operators for $\mathcal{M}^\Lambda$ are stabilizations of those for $\mathcal{M}$, this also orients the fiber moduli spaces $\mathcal{M}_\lambda$ for $\lambda \in \Lambda$.

Notice that the above definition does not require nondegeneracy of the moduli spaces $\mathcal{M}$, and hence we make no such assumptions in this subsection. Nevertheless, when $\mathcal{M}$ is nondegenerate, the determinant line for the relevant deformation operator $\det \mathfrak{D}_u = \det T_u\mathcal{M}$ at any $u \in \mathcal{M}$. In this case, this definition agrees with the usual definition of the orientation of a manifold.

We do, however, assume nondegeneracy of the spaces of critical points. Namely, we assume (FS1) for a Floer theory $(\mathcal{C}, \mathfrak{H}, \operatorname{ind}; \mathcal{Y}_\chi, \mathcal{V}_\chi)$, and assume (RHFS1*) for a CHFS throughout this subsection.

We now begin with some general discussion on abstract Floer theories in Sections 7.2.1–7.2.4.

7.2.1. General strategy for orientability. Below we roughly outline a scheme to establish orientability of $L\mathcal{B}$, which is particularly useful for symplectic Floer theories, when the configuration spaces have complicated topology. To begin, construct a map

$$m\colon \mathcal{B} \to \Sigma/G,$$

where Σ is a contractible space parameterizing certain operators and G is a suitable automorphism group. The map m is typically defined by identifying the deformation operator at $u \in \mathcal{B}$ to an operator in Σ, after certain trivialization is chosen. G is usually the group of automorphisms relating different possible trivializations.

The space Σ parameterizes a trivial determinant line bundle $L\Sigma$, over which the action by G extends. Moreover, $(L\Sigma)/G = L(\Sigma/G)$ and $L\mathcal{B} = m^*L(\Sigma/G)$. One next shows that G induces trivial actions on the determinant lines. Thus $L(\Sigma/G)$, and hence also $L\mathcal{B}$, are trivial.

In family settings, $\mathcal{B}$ and Σ above are both replaced by bundles $\mathcal{B}^\Lambda$, Σ^Λ over the parameter space Λ, and m above will be a bundle map.

In the case $\mathcal{B} = \mathcal{B}_P(x,y)$, in order for the deformation operator to be Fredholm, x, $y \in \mathcal{P}$ have to be nondegenerate. More generally, one may consider exponentially weighted versions of deformation operators $E_u^{(\sigma_1,\sigma_2)}$ (cf. Section I.3.2.3) instead of E_u. When x, y are, respectively, σ_1-weighted nondegenerate and σ_2-weighted nondegenerate, this defines another determinant line bundle over $\mathcal{B}_P(x,y)$, which we denote by $L^{(\sigma_1,\sigma_2)}\mathcal{B}_P(x,y)$. Under this weighted-nondegeneracy condition on x, y, the determinant line bundle $L^{(\sigma_1,\sigma_2)}\mathcal{B}_P(x,y)$ is independent of small perturbations to the weights σ_1, σ_2.

These weighted versions are useful for dealing with the case when one of x, y is minimally degenerate: in this case, the deformation operator E_u is defined as a map between complicated polynomially weighted Sobolev spaces (cf. Section I.5). However, we showed in Section I.5.2.5 that E_u has identical kernel and cokernel as $E_u^{(-\sigma,\sigma)}$ for any small positive σ. As we are only concerned with the *linear* aspect (Kuranishi structure) of the Floer theory, there is thus no harm in replacing E_u by the simpler $E_u^{(-\sigma,\sigma)}$: the orientation of $\mathcal{M}_P(x,y)$ in this case will be given by an orientation of $L^{(-\sigma,\sigma)}\mathcal{B}_P(x,y)$.

Turning now to the family situation of a CHFS $\{\mathcal{V}_\lambda\}_{\lambda\in\Lambda}$ satisfying properties (RHFS1*), given $\mathbf{x},\mathbf{y} \in \aleph_\Lambda$ and an interval $S \subset \Lambda$, let $\mathcal{B}_P^S(\underline{\mathbf{x}},\underline{\mathbf{y}}) = \bigcup_{\lambda\in S\cap\bar{\Lambda}_{\mathbf{x}}\cap\bar{\Lambda}_{\mathbf{y}}} \mathcal{B}_P(x_\lambda,y_\lambda)$. Under the assumption (RHFS1*), one may choose a set of intervals $\{S_i\}$ covering Λ, such that each S_i contains at most one death–birth, and all overlaps $S_i \cap S_j$ for different i,j contains no death–birth. Over each S_i, one may choose appropriate weights $\sigma_{x,i}$, $\sigma_{y,i}$ with small absolute value, such that x_λ is $\sigma_{x,i}$-weighted nondegenerate and y_λ is $\sigma_{y,i}$-weighted nondegenerate for all $\lambda \in S_i \cap \bar{\Lambda}_{\mathbf{x}} \cap \bar{\Lambda}_{\mathbf{y}}$. An orientation of $L^{(\sigma_{x,i},\sigma_{y_i})}\mathcal{B}_P^{S_i}(\underline{\mathbf{x}},\underline{\mathbf{y}})$ determines an orientation of $\mathcal{M}_P^{S_i}(\underline{\mathbf{x}},\underline{\mathbf{y}})$ as well as ones for its fibers $\mathcal{M}_{P,\lambda}(\underline{\mathbf{x}},\underline{\mathbf{y}})$, which agree with our previous discussion on orienting $\mathcal{M}_P(x,y)$ for nondegenerate or minimally degenerate x, y. Note again that the precise values of the weights $\sigma_{x,i}$, $\sigma_{y,i}$ are immaterial; in particular, when S_i contains no death–birth, they can be chosen to be 0. Otherwise, only the signs of these weights matter.

Lastly, we may patch up the determinant line bundles $L^{(\sigma_{x,i},\sigma_{y_i})}\mathcal{B}_P^{S_i}(\underline{\mathbf{x}},\underline{\mathbf{y}})$ for all intervals S_i to define a determinant line bundle $L\mathcal{B}_P^{\Lambda}(\underline{\mathbf{x}},\underline{\mathbf{y}})$ over $\mathcal{B}_P^{\Lambda}(\underline{\mathbf{x}},\underline{\mathbf{y}})$, by observing that, since for all $\lambda \in \bigcup_{i,j}\overline{S_i \cap S_j}$, x_λ, y_λ are nondegenerate, determinant line bundles with different weights over $\mathcal{B}_P^{\overline{S_i\cap S_j}}(\underline{\mathbf{x}},\underline{\mathbf{y}})$ can be identified.

More concretely, in Section 7.2.5 we shall apply the above general scheme to the specific Floer theory described in Section I.3. See also [**8**] for its application in other versions of symplectic Floer theories. In gauge-theoretic settings, the configuration space $\mathcal{B}$ itself has the structure of $\mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is an affine space and $\mathcal{G}$ is the gauge group, which is often connected under the assumption of simple-connectivity of the underlying manifold. Thus, much of the above scheme also carries over to this context.

7.2.2. Coherent orientations for $L\mathcal{B}_P$. Assuming the orientability of $L\mathcal{B}_P^k(x,y)$ and $L\mathcal{B}_P^{\Lambda,k+1}(\mathbf{x},\mathbf{y})$ for any pair of $x,y\in\mathcal{P}$ or $\mathbf{x},\mathbf{y}\in\aleph_\Lambda$ and any $k\in\mathbb{Z}$, we explain in Sections 7.2.2–7.2.3 how the orientations of all these should be related, so as to endow the moduli spaces of broken trajectories with a correct oriented manifold-with-corners structure.

Notation. Given a determinant line bundle LQ over a parameter space Q, $LQ\backslash$zero section contracts to a $\mathbb{Z}/2\mathbb{Z}$-bundle over Q, which we denote by $\mathfrak{Or}(LQ)$. LQ is orientable if $\mathfrak{Or}(LQ)$ is trivial; in this case, the space of sections of $\mathfrak{Or}(LQ)$ is denoted $\mathfrak{or}(LQ)$: this $\mathbb{Z}/2\mathbb{Z}$-torsor is the space of possible orientations for LQ.

Recall the continuity of the gluing homomorphism $\mathfrak{g}_R(\mathfrak{D}_1,\mathfrak{D}_2)$ in $\mathfrak{D}_1$, $\mathfrak{D}_2$, and R. Thus, it defines a gluing homomorphism

$$\mathfrak{g}\colon\ \mathfrak{or}(L\mathcal{B}_P^{k_1}(z_1,z_2))\times\mathfrak{or}(L\mathcal{B}_P^{k_2}(z_2,z_3))\to\mathfrak{or}(L\mathcal{B}_P^{k_1+k_2}(z_1,z_3))$$

for any $z_1,z_2,z_3\in\mathcal{P}$ and $k_1,k_2\in\mathbb{Z}$. We write $\mathfrak{g}(o_1,o_2)=o_1\#o_2$.

Definition. Let $(\mathcal{C},\mathfrak{H},\mathrm{ind};\mathcal{Y}_\chi,\mathcal{V}_\chi)$ be a Floer theory satisfying (FS1); in particular, $\mathcal{P}$ consists of finitely many nondegenerate elements. A *coherent orientation* of

$$L\mathcal{B}_P=\coprod_{k\in\mathbb{Z}}\coprod_{x,y\in\mathcal{P}}L\mathcal{B}_P^k(x,y)$$

is a section, $\mathfrak{s}$, of $\mathfrak{Or}(L\mathcal{B}_P)$, so that for all $k_1,k_2\in\mathbb{Z}$, $z_1,z_2,z_3\in\mathcal{P}$,

$$\mathfrak{s}|_{\mathcal{B}_P^{k_1}(z_1,z_2)}\#\mathfrak{s}|_{\mathcal{B}_P^{k_2}(z_2,z_3)}=\mathfrak{s}|_{\mathcal{B}_P^{k_1+k_2}(z_1,z_3)}. \tag{109}$$

A moment's thought (or cf. [**4**]) reveals that coherent orientations always exist. Fixing an $x\in\mathcal{P}$, a coherent orientation for $L\mathcal{B}_P$ is determined by choosing an element in $\mathfrak{or}(L\mathcal{B}_P^{k_y}(x,y))$ for each $y\neq x$ and an integer $k=\mathrm{gr}(x,y)\mod 2\mathbb{N}_\psi$, and in the case when $\mathbb{N}_\psi\neq 0$, an additional element of $\mathfrak{or}(L\mathcal{B}_P^{2\mathbb{N}_\psi}(x,x))$. The cases of $\mathrm{card}(\mathcal{P})=0$, or $\mathrm{card}(\mathcal{P})=1$ and $\mathbb{N}_\psi=0$ are excluded: in the first case, $\mathcal{B}_P$ is empty, while in the second case, there is no nonconstant connecting flow line. Thus, there is nothing to orient in these cases.

The following fact is immediate from the definition of coherent orientation, but shall be important later.

Lemma. *Let $\mathfrak{s}$ be an arbitrary coherent orientation of $L\mathcal{B}_P$. Then for any $x \in \mathcal{P}$, $\mathfrak{s}|_{\mathcal{B}^0_P(x,x)}$ is the canonical orientation of $L\mathcal{B}^0_P(x,x)$.*

In the above, the *canonical orientation* of $L\mathcal{B}^0_P(x,x)$ is that determined by the canonical orientation of det $E_{\bar{x}}$, the latter being due to the identification of the kernel and cokernel of $E_{\bar{x}} = d/ds + A_x$ via the facts that ker $E_{\bar{x}} =$ ker A_x, coker $E_{\bar{x}} =$ coker A_x, and that A_x is self-adjoint.

Proof. The coherence condition (109) requires the gluing maps

$$\begin{aligned}\mathfrak{g}(\mathfrak{s}|_{\mathcal{B}^0_P(x,x)}, -)&\colon \ \mathfrak{or}(L\mathcal{B}^k_P(x,y)) \to \mathfrak{or}(L\mathcal{B}^k_P(x,y));\\ \mathfrak{g}(-, \mathfrak{s}|_{\mathcal{B}^0_P(x,x)})&\colon \ \mathfrak{or}(L\mathcal{B}^{k'}_P(z,x)) \to \mathfrak{or}(L\mathcal{B}^{k'}_P(z,x))\end{aligned}$$

to be the identity map. □

7.2.3. Coherent orientations for $L\mathcal{B}^\Lambda_P$. Given a CHFS $\{(\mathcal{C}, \mathfrak{H}, \mathrm{ind}; \mathcal{Y}_\lambda, \mathcal{V}_\lambda)\}_{\lambda\in\Lambda}$ satisfying (RHFS1*), we aim to orient $L\mathcal{B}^\Lambda_P$, where

$$\mathcal{B}^\Lambda_P = \coprod_{k\in\mathbb{Z}} \coprod_{\mathbf{x},\mathbf{y}\in\aleph_\Lambda} \mathcal{B}^{\Lambda,k}_P(\underline{\mathbf{x}}, \underline{\mathbf{y}}).$$

There is a natural projection map $\Pi_\Lambda\colon \mathcal{B}^\Lambda_P \to \Lambda$, whose fiber over $\lambda \in \Lambda$ is:

- $\mathcal{B}_{P,\lambda} = \coprod_{x_\lambda,y_\lambda\in\mathcal{P}_\lambda} \mathcal{B}_P(x_\lambda, y_\lambda)$, when λ is not a death–birth,
- $\mathcal{B}_{P,\lambda} = \coprod_{x_\lambda,y_\lambda\in\mathcal{P}_\lambda\setminus\{z_\lambda\}\cup\{z_{\lambda+},z_{\lambda-}\}} \mathcal{B}_P(x_\lambda, y_\lambda)$, when $\mathcal{P}_\lambda$ contains a unique minimally degenerate critical point z_λ.

The elements $z_{\lambda+}, z_{\lambda-}$ should be regarded as the end points of the two path components $\mathbf{z}_+, \mathbf{z}_-$ of $\mathcal{P}^\Lambda\setminus\mathcal{P}^{\Lambda,\deg}$ connected at z_λ, with $\mathrm{ind}(\mathbf{z}_+) = \mathrm{ind}_+(z)$, $\mathrm{ind}(\mathbf{z}_-) = \mathrm{ind}_-(z)$ respectively. We write

$$L\mathcal{B}_P(x_\lambda, z_{\lambda\pm}) = \rho^*_\lambda L\mathcal{B}^\Lambda_P(\underline{\mathbf{x}}, \underline{\mathbf{z}}_\pm) = L^{(\sigma_x,\mp\sigma)}\mathcal{B}_P(x_\lambda, z_\lambda), \quad \text{for } 0 < \sigma \ll 1,$$

where $\rho_\lambda : \mathcal{B}_{P,\lambda} \hookrightarrow \mathcal{B}^\Lambda_P$ is the inclusion.

First, observe that it is also useful to identify det $E^{(\sigma_1,\sigma_2)}_u$ with det $E^{[\sigma_1,\sigma_2]}_u$, where

$$E^{[\sigma_1,\sigma_2]}_u := \varsigma^{\sigma_1,\sigma_2} E_u (\varsigma^{\sigma_1,\sigma_2})^{-1} = E_u + (\sigma_2 s\beta(s) + \sigma_1 s\beta(-s))'$$

is a map between ordinary Sobolev spaces. With this identification, one may extend the gluing homomorphism to the weighted case:

$$\mathfrak{g}\colon \mathfrak{or}(L^{(\sigma_1,\sigma_2)}\mathcal{B}_P^{k_1}(z_1,z_2))\times\mathfrak{or}(L^{(\sigma_2,\sigma_3)}\mathcal{B}_P^{k_2}(z_2,z_3))\to\mathfrak{or}(L^{(\sigma_1,\sigma_3)}\mathcal{B}_P^{k_1+k_2}(z_1,z_3)).$$

For any triple $\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3\in\aleph_\Lambda$ with $\bar\Lambda_{\mathbf{z}_1}\cap\bar\Lambda_{\mathbf{z}_2}\cap\bar\Lambda_{\mathbf{z}_3}\neq\emptyset$, and any pair of integers k_1,k_2, one has also the parameterized version of gluing homomorphism

$$\mathfrak{g}^\Lambda\colon \mathfrak{or}(L\mathcal{B}_P^{\Lambda,k_1}(\underline{\mathbf{z}}_1,\underline{\mathbf{z}}_2))\times\mathfrak{or}(L\mathcal{B}_P^{\Lambda,k_2}(\underline{\mathbf{z}}_2,\underline{\mathbf{z}}_3))\to\mathfrak{or}(L\mathcal{B}_P^{\Lambda,k_1+k_2}(\underline{\mathbf{z}}_1,\underline{\mathbf{z}}_3))$$

extending the gluing homomorphism $\mathfrak{g}$ over the fibers $L\mathcal{B}_{P,\lambda}$.

Definition. Given a CHFS satisfying (RHFS1*) as above, a *coherent orientation* of $L\mathcal{B}_P^\Lambda$ is a section, $\mathfrak{s}$, of $\mathfrak{Or}(L\mathcal{B}_P^\Lambda)$, so that:

(1) For any triple $\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3\in\aleph_\Lambda$ with $\bar\Lambda_{\mathbf{z}_1}\cap\bar\Lambda_{\mathbf{z}_2}\cap\bar\Lambda_{\mathbf{z}_3}\neq\emptyset$, and any pair of integers k_1,k_2,

$$\mathfrak{s}|_{\mathcal{B}_P^{\Lambda,k_1}(\underline{\mathbf{z}}_1,\underline{\mathbf{z}}_2)}\#\mathfrak{s}|_{\mathcal{B}_P^{\Lambda,k_2}(\underline{\mathbf{z}}_2,\underline{\mathbf{z}}_3)}=\mathfrak{s}|_{\mathcal{B}_P^{\Lambda,k_1+k_2}(\underline{\mathbf{z}}_1,\underline{\mathbf{z}}_3)};$$

(2) For all $y_\lambda\in\mathcal{P}^{\Lambda,\deg}$, $\mathfrak{s}|_{\mathcal{B}_P^1(y+,y-))}$ is the *standard orientation* of $L^{(-\sigma,\sigma)}\mathcal{B}_P^1(y_\lambda,y_\lambda)$, namely, the orientation given by the oriented K-model $[E_{\bar y_\lambda}^{(-\sigma,\sigma)}\colon \mathbb{R}\mathbf{e}_{y_\lambda}\to *]_B$.

Again, it is easy to see that such coherent orientation always exists. When $\mathbb{N}_\psi\neq 0$, there are $\mathrm{card}(\aleph_\Lambda)$ possible coherent orientations. When $\mathbb{N}_\psi=0$, there are $\mathrm{card}(\aleph_\Lambda)-1$ of them. Condition 2 in the above definition is imposed so that the short flow line between the two new critical points $y_{\lambda\pm}$ described in Section 5.3 has positive sign.

7.2.4. Grading-compatible orientation for $L\mathcal{B}_O^1$. The definition of our invariant I_F involves both $\mathcal{M}_O^1$ and $\mathcal{M}_P^1$, which are related by gluing elements in $\mathcal{M}_P^0(x,x)$ during a CHFS. It is thus crucial to orient $\mathcal{M}_P^0(x,x)$ and $\mathcal{M}_O^1$ consistently. The notion of "grading compatible orientation" describes such a suitable compatibility relation. More generally, one may consider compatibility conditions relating the orientations of higher-dimensional moduli spaces $\mathcal{M}_O^{2k\mathbb{N}_\psi+1}$, and $\mathcal{M}_P(x,x)^{2k\mathbb{N}_\psi}$, but this does not concern us, since our invariant involves only low-dimensional moduli spaces.

Let $\mathcal{B}_O^k\subset\mathcal{B}_O$ be the subset consisting of elements (T,u) with $\mathrm{gr}(u)=k-1$, and $L\mathcal{B}_O^k$ be the determinant line bundle of the family of deformation operators $\tilde D_{(T,u)}$. Assume that $L\mathcal{B}_O^1$ is orientable.

Recall that the relative grading gr in a Floer theory is typically defined via spectral flow by identifying deformation operators A_x with elements in a space of self-adjoint operators Σ_C (cf. Section I.3.1.4 for the version relevant to this article). On the other hand, the orientation of $L\mathcal{B}_O^1$ is defined by a map $m_O\colon \mathcal{B}_O^1\to\Sigma_O^1/G_O$, where Σ_O^1 is a space of Fredholm operators of index 1, which includes rank-1 stablizations of the operator

$$D_{\mathbb{A};T}:=\partial_s+\mathbb{A},\quad s\in S_T^1$$

for any surjective $\mathbb{A}\in\Sigma_C$ and $T\in\mathbb{R}^+$.

Definition. For a Floer theory $(\mathcal{C}, \mathfrak{H}, \mathrm{ind}; \mathcal{Y}_\chi, \mathcal{V}_\chi)$, the *grading-compatible* orientation of $L\mathcal{B}^1_O$, or more generally $L\Sigma^1_O/G_O$ (also called the orientation *compatible* with the absolute $\mathbb{Z}/2\mathbb{Z}$-grading ind), is the orientation given by the canonical orientation of det $\tilde{D}_{\mathbb{A},T}$, where $\tilde{D}_{\mathbb{A},T} \in \Sigma_O$ is the rank-1 stabilization of $D_{\mathbb{A},T}$ by the zero map, and $\mathbb{A}$ is a surjective operator in Σ_C of even index.

In the above, the canonical orientation of $\tilde{D}_{\mathbb{A},T}$ is the stabilization of the canonical orientation of $D_{\mathbb{A},T}$, which in turn is defined in the same way as the canonical orientation of $E_{\bar{x}}$ (cf. Section 7.2.2). Note that the choice of $\mathbb{A}$ and T do not matter in the above definition: as on varies T, $D_{\mathbb{A},T}$ remains surjective; on the other hand, the independence of the choice of $\mathbb{A}$ is a consequence of the following basic Lemma.

Lemma. *For any two surjective operators* $\mathbb{A}, \mathbb{A}' \in \Sigma_C$*, the canonical orientations of* det $D_{\mathbb{A},T}$ *and* det $D_{\mathbb{A}',T}$ *are of relative sign* $(-1)^{\mathrm{gr}(\mathbb{A},\mathbb{A}')}$ *with respective to the family* $\{D_{\mathbb{A},T} | \mathbb{A} \in \Sigma_C\}$*, where* $\mathrm{gr}(\mathbb{A}, \mathbb{A}')$ *denotes the relative index between* $\mathbb{A}$ *and* $\mathbb{A}'$*.*

An immediate corollary is as follows.

Corollary. *Suppose that* $L\Sigma^1_O/G_O$ *is orientable. Then for any surjective* $\mathbb{A} \in \Sigma_C$ *and* $T \in \mathbb{R}^+$*, the relative sign between the grading-compatible orientation of* $L\Sigma^1_O/G_O$ *and the canonical orientation of* det $D_{\mathbb{A},T}$ *is* $(-1)^{\mathrm{ind}\, A}$*.*

7.2.5. Orientability in symplectic Floer theory. We now apply the general strategy described in Section 7.2.1 to establish the orientability of moduli spaces for the specific version of Floer theory considered in this article.

(1) *Orienting* $L\mathcal{B}_P$. This follows from [**4**], which we now review in our terminology. Let $J \in \mathcal{J}^{\mathrm{reg}}_K$, X be J-nondegenerate (cf. Section I.3.2.1), and Σ_C be as in Section I.3.1.4. Given two self-adjoint, surjective operators $A_-, A_+ \in \Sigma_C$, let $\Sigma_P(A_-, A_+)$ be the space of operators of the form:[2]

$$\partial_s + J(s,t)\partial_t + \nu(s,t)\colon L^p_1(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n}),$$

where J is a smooth complex structure on the trivial $\mathbb{R}^{2n}$-bundle over $\mathbb{R} \times S^1$, compatible with the standard symplecic structure on $\mathbb{R}^{2n}$. ν is a smooth matrix-valued function, and both J and ν extend smoothly over the cylinder $[-\infty, +\infty] \times S^1$ that compactifies $\mathbb{R} \times S^1$. Furthermore, over the two circles at infinity,

$$J(-\infty,t)\partial_t + \nu(-\infty,t) = A_-; \quad J(\infty,t)\partial_t + \nu(\infty,t) = A_+.$$

[2] $\Sigma_P(A_-, A_+)$ is basically the space Θ in Proposition 7 of [**4**].

The contractibility of $\Sigma_P(A_-, A_+)$ follows from well-known contractibility of the space of complex structures, and the fact that ν lies in a vector space.

Next, denote by $\mathfrak{T}_x$ the space of unitary trivializations of x^*K for $x \in \mathcal{P}$ and let $\mathfrak{T} = \coprod_{x\in\mathcal{P}} \mathfrak{T}_x$. This is a $C^\infty(S^1, U(n))$-bundle over $\mathcal{P}$. Fix a $g \in C^\infty(S^1, U(n))$ and a section $\Phi: \mathcal{P} \to \mathfrak{T}$, such that the inclusion $\mathfrak{S}: = \{g^k\Phi(x) \mid k \in \mathbb{Z}, x \in \mathcal{P}\} \hookrightarrow \mathfrak{T}$ induces an isomorphism $i_\pi: \mathfrak{S} = \pi_0\mathfrak{S} \to \pi_0\mathfrak{T}$.

Recall from Section I.3.1.4 that we have a bundle map (over $\mathcal{P}$) from $\tilde{\mathcal{P}}$ to $\pi_0\mathfrak{T}$: from a fixed unitary trivialization of γ_0^*K and a path $w \subset \mathcal{C}$ from γ_0 to $x \in \mathcal{P}$, we extend the trivialization over w^*K to obtain a homotopy class of trivializations of x^*K. If w' is another path in the same equivalence class, i.e., $\mathrm{im}[w - w'] = 0$, then $(w - w')^*K$ is trivial, since $c_1^f(\mathrm{im}[w - w']) = 0$. Hence w, w' induce the same homotopy class of trivializations of x^*K. Composing this map with i_π^{-1}, we have a map assigning to each $(x, [w]) \in \tilde{\mathcal{P}}$ a trivialization $\Phi_{x,[w]} \in \mathfrak{S}_x$. Let

$$\mathbb{A}_{(x,[w])} := \Phi_{x,[w]} A_x \Phi_{x,[w]}^{-1}.$$

We have a map $m_P: \mathcal{B}_P^k((x, [w]), (y, [v])) \to \Sigma_P(\mathbb{A}_{(x,[w])}, \mathbb{A}_{(y,[v])})/G_P$, defined as follows.

A $u \in \mathcal{B}_P((x, [w]), (y, [v]))$, together with a trivialization Φ_u of the symplectic vector bundle u^*K that restricts, respectively, to $\Phi_{(x,[w])}$ and $\Phi_{(y,[v])}$ at the circles at $-\infty$ and ∞, assigns an element in Σ_P. Namely, $\mathbb{E}_u := \Phi_{u*} E_u \Phi_{u*}^{-1}$.

The space of such trivializations Φ_u is an affine space under

$$G_P = \Big\{\Psi \,\Big|\, \Psi \in C^\infty([-\infty, \infty] \times S^1, \mathrm{Sp}_n), \Psi|_{\{\pm\infty\}\times S^1} = 1\Big\}.$$

Let $m_P(u)$ be the G_P-orbit of $\mathbb{E}_u$. It is shown in Lemma 13 of [**4**] that any orbit of G_P in $\mathfrak{Or}(L\Sigma_P(\mathbb{A}_{(x,[w])}, \mathbb{A}_{(y,[v])}))$ is contained in a single-path component of $\mathfrak{Or}(L\Sigma_P(\mathbb{A}_{(x,[w])}, \mathbb{A}_{(y,[v])}))$; thus, $L(\Sigma_P(\mathbb{A}_{(x,[w])}, \mathbb{A}_{(y,[v])})/G_P)$ is trivial; hence so is $L\mathcal{B}_P((x, [w]), (y, [v]))$.

Notice that by definition, for any $(z_1, [v_1]), (z_2, [v_2]), (z_2, [v_3]) \in \tilde{\mathcal{P}}$, $(\mathbb{E}_1, \mathbb{E}_2) \in \Sigma_P(\mathbb{A}_{(z_1,[v_1])}, \mathbb{A}_{(z_2,[v_2])})) \times \Sigma_P(\mathbb{A}_{(z_2,[v_2])}, \mathbb{A}_{(z_3,[v_3])}))$ is glueable. This gives rise to a gluing homomorphism

$$\begin{aligned}\mathfrak{or}(L\mathcal{B}_P((z_1, [v_1]), (z_2, [v_2])) \times \mathfrak{or}(L\mathcal{B}_P((z_2, [v_2]), (z_3, [v_3])) \\ \to \mathfrak{or}(L\mathcal{B}_P((z_1, [v_1]), (z_3, [v_3])).\end{aligned}$$

Lastly, observe that if $\mathrm{gr}((x, [w), (y, [v])) = \mathrm{gr}((x, [w'], (y, [v'])) = k$, the spaces $\Sigma_P(\mathbb{A}_{(x,[w])}, \mathbb{A}_{(y,[v])}))$ and $\Sigma_P(\mathbb{A}_{(x,[w'])}, \mathbb{A}_{(y,[v'])}))$ may be identified via conjugation by $\bar{g}^i$ for some $i \in \mathbb{Z}$ and $\bar{g} \in C^\infty([-\infty, \infty] \times S^1, \mathrm{Sp}_n)$, $\bar{g}(s, t) := g(t)$. Thus, the above discussion in fact verifies the orientability of $L\mathcal{B}_P^k(x, y)$ for any $x, y \in \mathcal{P}$, $k \in \mathbb{Z}$, and the gluing homomorphism above gives the gluing homomorphism $\mathfrak{g}$ described in Section 7.2.3.

(2) *Orienting* $L\mathcal{B}_P^\Lambda$. Suppose (J^Λ, X^Λ) generates a CHFS satisfying the properties (RHFS1*). Let $(\mathbf{x},[\mathbf{w}]),(\mathbf{y},[\mathbf{v}])$ be two path components of the space $\tilde{\mathcal{P}}^\Lambda/\tilde{\mathcal{P}}^{\Lambda,\deg}$.

The deformation operator of $\mathcal{M}_P^\Lambda$ at u_λ, $\hat{E}_{u_\lambda}$, may be regarded as a stabilization of E_{u_λ}. Because of the stabilization isomorphism for families, to orient the determinant line bundle $\det\{\hat{E}_{u_\lambda}\}_{u_\lambda\in\mathcal{B}_P^\Lambda((\mathbf{x},[\mathbf{w}]),(\mathbf{y},[\mathbf{v}]))}$, it is equivalent to orient the determinant line bundle

$$L\mathcal{B}_P^\Lambda((\mathbf{x},[\mathbf{w}]),(\mathbf{y},[\mathbf{v}])) := \det\{E_{u_\lambda}\}_{u_\lambda\in\mathcal{B}_P^\Lambda((\mathbf{x},[\mathbf{w}]),(\mathbf{y},[\mathbf{v}]))}.$$

This can be oriented by repeating part (1) above, replacing Σ_P by the parameterized version:

$$\Sigma_P(\mathbf{A}_-,\mathbf{A}_+) := \bigcup_{\lambda\in\Lambda_{\mathbf{x}}\cap\Lambda_{\mathbf{y}}} \Sigma_P(\mathbb{A}_{(x_\lambda,[w_\lambda])},\mathbb{A}_{(y_\lambda,[v_\lambda])}),$$

which is a Σ_P-bundle over $\Lambda_{\mathbf{x}}\cap\Lambda_{\mathbf{y}}$. In the above, $\mathbb{A}_{(x_\lambda,[w_\lambda])}$ $\forall(x_\lambda,[w_\lambda])\in\tilde{\mathcal{P}}$ is defined via a smooth section $\Phi^\Lambda\colon \mathcal{P}^\Lambda\to\mathfrak{T}^\Lambda$, $\mathfrak{T}^\Lambda := \bigcup_{x_\lambda\in\mathcal{P}^\Lambda}\mathfrak{T}_{x_\lambda}$. This is again a contractible space, since it is a bundle with contractible fibers and base.

As in (1), this in turn demonstrates the orientability of $L\mathcal{B}^{\Lambda,k+1}(\mathbf{x},\mathbf{y})$ and defines the parameterized version of gluing homomorphism $\mathfrak{g}^\Lambda$ described in Section 7.2.3. Now one may follow the arguments in Section 7.2.3 to define a coherent orientation of $L\mathcal{B}_P^\Lambda$.

(3) *Orienting* $L\mathcal{B}_O^1$ *and* $L\mathcal{B}_O^{\Lambda,2}$. Since $\tilde{D}_{(T,u)}$ is a rank-1 stabilization of $D_{(T,u)}$, it is equivalent to orient $\underline{L}\mathcal{B}_O^1 := \det\{D_{(T,u)}\}_{(T,u)\in\mathcal{B}_O^1}$.

Similarly to parts (1), (2) above, we introduce a map $m_O\colon \mathcal{B}_O^1\to\Sigma_O^1/G_O$, where Σ_O^1 is the space of rank-1 stabilizations of operators of the form:

$$\bar{\partial}_{J,\nu;T} := \partial_s + J(s,t)\partial_t + \nu(s,t)\colon L_1^p(S_T^1\times S^1;\mathbb{R}^{2n})\to L^p(S_T^1\times S^1;\mathbb{R}^{2n})$$

for some $T\in\mathbb{R}^+$, with J,ν defined similarly to part (1). The determinant line bundle $L\Sigma_O^1$ is canonically oriented as follows. Note that Σ_O^1 contracts to the subspace consisting of complex linear $\bar{\partial}_{J,\nu;T}$, which we denote by Σ_O'. However, $L\Sigma_O'$ is canonically oriented by the complex linearity of kernels and cokernels. Next, note that u^*K is trivial for any $(T,u)\in\mathcal{B}_O^1$. Given a $(u,T)\in\mathcal{B}_O^1$ and a trivialization Φ_u of u^*K, one has

$$\tilde{\mathbb{D}}_{(T,u)} := \Phi_{u*}\tilde{D}_{(T,u)}\tilde{\Phi}_{u*}^{-1}\in\Sigma_O,$$

where $\tilde{\Phi}_{u*}^{-1} := 1\oplus\Phi_{u*}^{-1}$ as an endomorphism of $\mathbb{R}\oplus L_1^p(S_T^1\times S^1;\mathbb{R}^{2n})$. This defines m_O.

It is not hard to see that G_O acts trivially on the $\mathbb{Z}/2\mathbb{Z}$-bundle $\mathfrak{Or}(L\Sigma_O^1)$ by conjugation: by continuation (cf. the commutative diagram in p. 28 of [**4**]), it suffices to check this for $\mathfrak{Or}(L\Sigma_O')$. However, for $\mathfrak{Or}(L\Sigma_O')$, this is obvious, again by the complex linearity of elements in Σ_O'.

The orientability of $L\mathcal{B}_O^{\Lambda,2}$ follows immediately from that of $L\mathcal{B}_O^1$, since $\mathcal{B}_O^{\Lambda,2} = \mathcal{B}_O^1 \times \Lambda$ by definition.

Finally, note that for this version of symplectic Floer theory, the canonical orientation of $L\Sigma_O^1/G_O$ is compatible with the mod 2 Conley–Zehnder index ind: the former is given by the canonical orientation of det $\tilde{D}_{\mathbb{A}_0,T}$, where $\mathbb{A}_0$ is such that $D_{\mathbb{A}_0,T}$ is complex linear. By the definition of Conley–Zehnder index (cf. Section I.3.1.4), $\mathrm{CZ}(\mathbb{A}_0)$ is even.

7.3. The signs. It was shown in [**4**] that with a coherent orientation for $\mathcal{M}_P$, the Floer complex indeed satisfies $\tilde{\partial}_F^2 = 0$. In this subsection, we generalize this result to the setting of CHFSs and verify the second statement of (RHFS4). Namely, we show that with $\mathcal{M}_P^\Lambda$ endowed with coherent orientations and $\mathcal{M}_O^{\Lambda,2}$ endowed with the grading-compatible orientation, the various 0-dimensional strata $\mathbb{J}_P$, $\mathbb{T}_{P,\text{hs-s}}$, $\mathbb{T}_{P,\text{db}}$ in $\hat{\mathcal{M}}_P^{\Lambda,1,+}$ and their analogs for broken orbits are expressed in terms of products of 0-dimensional moduli spaces, with relative signs given by the formulae (I.28–I.33).

As the signs for $\mathbb{J}_P$, $\mathbb{J}_O$ given in (I.28, 31) follow immediately from the definition of coherent orientation, we shall concentrate on the signs for $\mathbb{T}_{P,\text{hs-s}}$, $\mathbb{T}_{O,\text{hs-s}}$, $\mathbb{T}_{P,\text{db}}$, and $\mathbb{T}_{O,\text{db}}$: the formulae (I.30, 33, 29, 32) are, respectively, rephrased in terms of the gluing theorems Propositions 2.1, 6.1.1 in Lemmas 7.3.2–7.3.5 below.

We assume throughout this subsection that $L\Sigma_P/G_P, L\Sigma_O/G_O$ and their parameterized versions are endowed with coherent orientations/grading-compatible orientations, and all the oriented K-models are compatible with these orientations, unless otherwise specified.

The results and arguments in this subsection apply to general Floer theory, in which the relevant moduli spaces are oriented according to the scheme in Sections 7.2.1–7.2.4 above.

7.3.1. Preparations. (1) *Signs of flowlines.* The sign of a flow in a 0-dimensional reduced moduli space, $\hat{u} \in \mathcal{M}^1/\mathbb{R}$, in general, means the relative sign $[u']/\ker \mathfrak{D}_u$ for any representative $u \in \mathcal{M}^1$ in the unreduced moduli space, where $\mathfrak{D}_u$ is the deformation operator of $\mathcal{M}^1$. It will be denotedy by $\mathrm{sign}(u)$.

(2) *Trivializations of deformation operators.* Instead of working with the deformation operators $E_u, \tilde{D}_{(T,u)}$ and their parameterized versions, it is often more convenient to work with their corresponding operators in Σ_P or Σ_O via lifts of the maps m_P, m_O. These will be denoted by boldface letters such as $\mathbb{E}_u, \tilde{\mathbb{D}}_{(T,u)}$. When $L\Sigma_P/G_P$, $L\Sigma_O/G_O$ are orientable, the choice of liftings does not matter. We shall also omit specifying the class $[v]$ in $\mathbb{A}_{(y,[v])} =: \mathbb{A}_y$, when the precise choice is immaterial.

For the symplectic Floer theory discussed in this article, this means replacing the deformation operators by their conjugates by trivializations of u^*K, namely Φ_u (cf. the definition of $\mathbb{E}_u$, $\mathbb{D}_u$ in Section 7.1.5). We write $(f)_\Phi := \Phi_{u*}f$, e.g., $f = u'$ for $u \in \mathcal{B}_P$ or $\mathcal{B}_O$.

The families of operators considered in the rest of this subsection will always be subfamilies of various versions of Σ_P, Σ_O. Thus, we shall refer to the correlation and relative signs of orientations of determinant lines, or mutual co-orientation of K-models without specifying the family.

The following consistency conditions on the choice of liftings will be assumed in the following discussion:

(a) for a subfamily $U \subset \mathcal{B}$, the lifting $\tilde{m}$: $U \to \Sigma$ is continuous;
(b) the liftings are "coherent" in the sense that they are consistent with gluing.

7.3.2. Signs for $\mathbb{T}_{P,\text{hs-s}}$. To verify the sign in (I.30), we need to examine oriented K-models for the gluing theorem, Proposition 6.1.1 (a). Let (J^Λ, X^Λ) be an admissible (J, X)-homotopy, for any $\mathbf{x}_1, \mathbf{x}_2 \in \aleph_\Lambda$ and $R_0 \gg 1$, the (omitted) proof of Proposition 6.1.1 (a) defines a gluing map

$$\mathrm{Gl}_{P,\text{hs}}(\mathbf{x}_1, \mathbf{x}_2; \Re)\colon \mathbb{T}_{P,\text{hs-s}}(\mathbf{x_1}, \mathbf{x_2}; \Re) \times (R_0, \infty) \to \mathcal{M}_P^{\Lambda,2}(\mathbf{x}_1, \mathbf{x}_2; \mathrm{wt}_{-\mathcal{Y},e_{\mathcal{P}}} \leq \Re).$$

Let $(\lambda_0, \hat{u}) \in \hat{\mathcal{M}}_P^{\Lambda,0}(\mathbf{x}, \mathbf{y}; J^\Lambda, X^\Lambda)$ be a handleslide. Without loss of generality, assume $\lambda_0 = 0$. Let $\mathbf{q}, \mathbf{z} \in \aleph_\Lambda$ be of indices $\mathrm{ind}\,\mathbf{y} + 1$ and $\mathrm{ind}\,\mathbf{y} - 1$, respectively. Let

$$\hat{v}_- \in \hat{\mathcal{M}}_P^0(q_0, x_0; J_0, X_0), \quad \hat{v}_+ \in \hat{\mathcal{M}}_P^0(y_0, z_0; J_0, X_0)$$

and v_-, v_+, u be centered representatives of $\hat{v}_-, \hat{v}_+, \hat{u}$, respectively. Let

$$w_{\#-}(R) = v_- \#_R u, \quad w_{\#+}(R) = u \#_R v_+$$

be the pregluings defined in Section 1.2.2, and let

$$\begin{aligned}(\lambda_-(R), w_-(R)) &:= \mathrm{Gl}_{P,\text{hs}}(\mathbf{q}, \mathbf{y}; \Re)(\{\hat{v}_-, \hat{u}\}, R),\\ (\lambda_+(R), w_+(R)) &:= \mathrm{Gl}_{P,\text{hs}}(\mathbf{x}, \mathbf{z}; \Re)(\{\hat{u}, \hat{v}_+\}, R).\end{aligned}$$

be the images of the gluing map obtained by further perturbing $w_{\#-}(R)$ and $w_{\#+}(R)$, respectively. To simplify notation, we shall omit R when there is no danger of confusion.

Lemma. *Let $u, v_\pm, (\lambda_\pm, w_\pm)$ be as above. Then*

$$\begin{aligned}&(1)\ -\mathrm{sign}(\lambda_-)\,\mathrm{sign}(w_-) = \mathrm{sign}(v_-)\,\mathrm{sign}(u)\\ &(2)\ \mathrm{sign}(\lambda_+)\,\mathrm{sign}(w_+) = \mathrm{sign}(u)\,\mathrm{sign}(v_+).\end{aligned} \tag{110}$$

Proof. We shall focus on case (1) below, since case (2) is entirely parallel. According to Sections 7.1.3, 7.1.5, and the choice of coherent orientation, we have the oriented K-models:

(i±) $[\hat{\mathbb{E}}_{(0,w_{\#\pm})}: \hat{K}_{\#\pm} \to C_{\#\pm}]$, where

$$\hat{K}_{\#-} = -\mathbb{R} \oplus (\ker \mathbb{E}_{v_-} \#_R \ker \mathbb{E}_u), \quad C_{\#-} = *\#_R \operatorname{coker} \mathbb{E}_u;$$
$$\hat{K}_{\#+} = \mathbb{R} \oplus (\ker \mathbb{E}_u \#_R \ker \mathbb{E}_{v_+}), \quad C_{\#+} = \operatorname{coker} \mathbb{E}_u \#_R * .$$

(ii) $[\hat{\mathbb{E}}_{(0,u)}\colon \mathbb{R} \oplus (\ker \mathbb{E}_u) \to \operatorname{coker} \mathbb{E}_u]$.

Since u is by assumption a nondegenerate element of $\mathcal{M}_P^{\Lambda,1}$, the standard oriented K-model for $\hat{\mathbb{E}}_{(0,u)}$ may be viewed as a reduction of the oriented K-model (ii) by $-\mathbb{R}$, taking

$$\operatorname{coker} \mathbb{E}_u = -\mathbb{R}(Y_{(0,u)})_\Phi \quad \text{and}$$
$$\ker \mathbb{E}_u = \ker \hat{\mathbb{E}}_{(0,u)} = \operatorname{sign}(u)\mathbb{R}(u')_\Phi$$

as oriented spaces. (Recall that $Y_{(0,u)}$ is a cutoff version of $\partial_\lambda \mathcal{V}_\lambda$ appearing in the definition of $\hat{\mathbb{E}}_{(0,u)}$, cf. I.6.1.5.)

Next, decompose $\hat{K}_{\#-}$ into the direct sum of the ordered triple of oriented subspaces

$$* \oplus (\ker \mathbb{E}_{v_-} \#_R *), \quad \mathbb{R} \oplus *, \quad \text{and } * \oplus (* \#_R \ker \mathbb{E}_u).$$

By Lemma 1.2.4 (2), for large R, the restriction of $\Pi_{C_{\#-}} \hat{\mathbb{E}}_{(0,w_{\#-})}$ to the first and last subspaces are small, while its restriction to the second subspace approximates the multiplication by $\Pi_{\operatorname{coker} \mathbb{E}_u} \tilde{Y}$ (under the natural identification of the domain and range spaces), where $\tilde{Y}$ is another cutoff version of $\partial_\lambda \mathcal{V}_\lambda$ which agrees with $(Y_{(0,u)})_\Phi$ except in the region where $s \ll -1$. Let $Y_\nu := \nu(Y_{(0,u)})_\Phi + (1-\nu)\tilde{Y}$ for $\nu \in [0,1]$, and $\hat{\mathbb{E}}_\nu$ be the rank-1 stabilization of $\mathbb{E}_u$ by multiplication by Y_ν. By the surjectivity of $\partial_s + \mathbb{A}_y$ and $\hat{\mathbb{E}}_0 = \mathbb{E}_{(0,u)}$, and an excision argument (as outlined in Section 1.2.5), $\hat{\mathbb{E}}_\nu$ has uniformly bounded right inveres. Thus, we may conclude that $\Pi_{\operatorname{coker} \mathbb{E}_u} \tilde{Y}$, and hence also $\Pi_{C_{\#-}} \hat{\mathbb{E}}_{(0,w_{\#-})}|_{\mathbb{R}\oplus *}$, are positive of $O(1)$. This implies that the reduction of the oriented K-model (i−) by $-\mathbb{R} \oplus *$ is equivalent to the standard oriented K-model of $\hat{\mathbb{E}}_{(0,w_{\#-})}$, which is in turn equivalent to the standard oriented K-model of $\hat{\mathbb{E}}_{(\lambda_-,w_-)}$, due to the the proximity between w_- and $w_{\#-}$. In other words, the projection

$$\Pi_{K_{\#-}}\colon \ker \hat{\mathbb{E}}_{(\lambda_-,w_-)} \to \ker \mathbb{E}_{v_-} \#_R \ker \hat{\mathbb{E}}_{(0,u)} =: K_{\#-}.$$

is an orientation-preserving isomorphism. We have the following ordered bases compatible with the former and latter oriented spaces above:

$$\Big\{ \operatorname{sign}(w_-)(0, w_-'),\, (\partial_R \lambda_-)^{-1}(\partial_R \lambda_-, \partial_R w_-) \Big\},$$
$$\Big\{ \operatorname{sign}(v_-)(0, v_-'),\, \operatorname{sign}(u)(0, u') \Big\}.$$

Observing that $\operatorname{sign}(\partial_R\lambda_-) = -\operatorname{sign}(\lambda_-)$, and recalling the description of $\Pi_{\iota^j_\# K_J}$ in Section 1.2.4, one finds that with respect to these bases, $\Pi_{K_{\#-}}$ is a matrix of the form

$$\begin{pmatrix} C_1 \operatorname{sign}(w_-)\operatorname{sign}(v_-) & C_1' \operatorname{sign}(\lambda_-)\operatorname{sign}(v_-) \\ C_2 \operatorname{sign}(w_-)\operatorname{sign}(u) & -C_2' \operatorname{sign}(\lambda_-)\operatorname{sign}(u) \end{pmatrix} \quad \text{for } C_1, C_1', C_2, C_2' \in \mathbb{R}^+.$$

Equation (110.1) follows from the requirement that this matrix has positive determinant. □

7.3.3. Signs for $\mathbb{T}_{O,\text{hs-s}}$. To verify the sign in (I.33), we examine the oriented K-model for the gluing theorem Proposition 6.1.1 (b). Let y, u be as in Section 7.3.2, but now assume that the handleslide u is of Type II, namely, $x = y$. Let $w_\#(R) = u_{\#R}$ be the glued orbit introduced in Section 1.2.2, and let $(\lambda(R), (T(R), w(R))) := \mathrm{Gl}_{O,\text{hs}}(\hat{u}, R)$ be the image of the gluing map obtained by perturbing $w_\#(R)$, where $\mathrm{Gl}_{O,\text{hs}} : \mathbb{T}_{O,\text{hs-s}}(\Re) \times (R_0, \infty) \to \hat{\mathcal{M}}^{\Lambda,1}_O(\mathrm{wt}_{-\mathcal{Y},e_{\mathcal{P}}} \leq \Re)$ is the gluing map in the (omitted) proof of Proposition 6.1.1 (b).

Lemma. *In the above notation*, $\operatorname{sign}(w) = (-1)^{\operatorname{ind} y} \operatorname{sign}(\lambda)\operatorname{sign}(u)$.

Proof. According to Corollary 7.2.4, $\operatorname{sign}(w) = (-1)^{\operatorname{ind} y}[(w')_\Phi]/\ker^{\mathfrak{o}_y} \tilde{\mathbb{D}}_{(T,w)}$, where $\ker^{\mathfrak{o}_y} \tilde{\mathbb{D}}_{(T,w)}$ denotes $\ker \tilde{\mathbb{D}}_{(T,w)}$ endowed with the orientation correlated to the canonical orientation of $\tilde{\mathbb{D}}_{\mathbb{A}_y,T}$. We compute $[(w')_\Phi]/\ker^{\mathfrak{o}_y} \tilde{\mathbb{D}}_{(T,w)}$ in two steps.

Step 1. The relative sign $[(w')_\Phi]/\ker^{\mathfrak{o}_y} \acute{\mathbb{D}}_{(\lambda,w)}$. Let $\acute{D}_{(\lambda,w)}$ be the rank-1 stabilization of D_w defined by

$$\acute{D}_{(\lambda,w)}(\alpha,\xi) = \alpha\partial_\lambda \mathcal{V}_\lambda(w) + D_w\xi.$$

Performing cyclic gluing to the oriented K-model $[\hat{\mathbb{E}}_{(\lambda,u)} : \mathbb{R} \oplus \ker \mathbb{E}_u \to \operatorname{coker} \mathbb{E}_u]$, we obtain an oriented K-model for $\acute{\mathbb{D}}_{w_\#}$, a rank-1 stabilization of $\mathbb{D}_{w_\#}$ by multiplication with a cutoff version of $\partial_\lambda \mathcal{V}_\lambda(w_\#)$. The argument in Section 7.3.2 shows that a reduction of this oriented K-model by $\mathbb{R}$ is equivalent to a standard K-model for $\acute{\mathbb{D}}_{w_\#}$, which is in turn equivalent to a standard K-model for $\acute{\mathbb{D}}_{(\lambda,w)}$. Moreover, according to the continuity of gluing homomorphisms and Lemma 7.2.2, the orientation of this standard K-model is correlated to the canonical orientation of $\tilde{\mathbb{D}}_{\mathbb{A}_y,T}$. In other words,

$$[(w')_\Phi]/\ker^{\mathfrak{o}_y} \acute{\mathbb{D}}_{(\lambda,w)} = \operatorname{sign}(u).$$

Step 2. The relative sign $\ker^{\mathfrak{o}_y} \acute{D}_{(\lambda,w)}/\ker^{\mathfrak{o}_y} \tilde{D}_{(T,w)}$. Notice that the operators $\acute{D}_{(\lambda,w)}$, $\tilde{D}_{(T,w)}$ have a common stabilization, namely $\hat{D}_{(\lambda,(T,w))} = (\partial_\lambda \mathcal{V}_\lambda, (-w'/T, D_w))$. The 2-dimensional space $\ker \hat{D}_{(\lambda,(T,w))}$ is spanned by

$\{\partial_R(\lambda,(T,w)),\ (0,(0,w'))\}$. This means $\Pi_{\mathrm{coker}\ D_w}\Big(\partial_R\lambda\partial_\lambda\mathcal{V}_\lambda+\partial_R T(-w'/T)\Big)$ $=0$, and hence the relative sign is computed by

$$\ker^{\mathfrak{o}_y}\acute{D}_{(\lambda,w)}/\ker^{\mathfrak{o}_y}\tilde{D}_{(T,w)}=-\operatorname{sign}(\partial_R\lambda)/\operatorname{sign}(\partial_R T)=\operatorname{sign}(\lambda).$$

Finally, the Lemma is obtained taking the product of the relative signs obtained in Steps 1 and 2 above with $(-1)^{\mathrm{ind}\ y}$. □

7.3.4. Signs for $\mathbb{T}_{P,\mathrm{db}}$. To verify the signs in (I.29), we need to analyze the orientation of the K-model for the gluing theorem Proposition 2.1 (a). Let $\lambda,u_0,\dots,u_{k+1}$ be as in Section 2.2, and let (λ,w) be the image of $(\{u_0,u_1,\dots,u_{k+1}\},\lambda)$ under the gluing map defined in Section 4.1.4.

Lemma. *Under the assumptions in Section* 2.1 *and in the above notation*,

$$\operatorname{sign}(w)=(-1)^{k+1}\prod_{i=0}^{k+1}\operatorname{sign}(u_i).$$

Proof. As explained earlier in this section, since we work with the ordinary Sobolev norms instead of the complicated polynomially weighted ones, it is convenient to replace the delicate pregluing w_χ defined in Section 2.2 by ordinary glued trajectories or orbits. Choose $R_i,\ R_i',\ i\in\{1,\dots,k+1\}$ and L appropriately so that:

$$w_\#:=\tau_L\Big(u_0\#_{R_1}\bar{y}\#_{R_1'}u_1\#_{R_2}\cdots\#_{R_{k+1}'}u_{k+1}\Big)$$

is pointwise close to w and w_χ: more precisely, $w_\#(s),\ w(s),\ w_\chi(s)$ are C_ϵ-close to each other $\forall s$, and

$$w_\chi(\gamma_{u_i}^{-1}(0))=w_\#(\gamma_{u_i}^{-1}(0)).$$

As explained in Sections 7.2.1 and 7.2.3, the deformation operator in Σ_P corresponding to $w_\#$ is:

$$\mathbb{E}^\sigma_{w_\#}:=\mathbb{E}^{[0,\sigma]}_{u_0}\#_{R_1}\mathbb{E}^{[\sigma,-\sigma]}_{\bar{y}}\#_{R_1'}\mathbb{E}^{[-\sigma,\sigma]}_{u_1}\#_{R_2}\mathbb{E}^{[\sigma,-\sigma]}_{\bar{y}}\#_{R_2'}\cdots\#_{R_k'}\mathbb{E}^{[-\sigma,0]}_{u_{k+1}}$$

for a small $\sigma>0$. Let $\bar{\mathbf{e}}^\sigma_y(s):=\varsigma^{-\sigma,\sigma}(s)\mathbf{e}_y$, and recall that

$$\ker\ \mathbb{E}^{[-\sigma,\sigma]}_{\bar{y}}=\operatorname{coker}\ \mathbb{E}^{[\sigma,-\sigma]}_{\bar{y}}=\mathbb{R}\bar{\mathbf{e}}^\sigma_y;\quad \operatorname{coker}\ \mathbb{E}^{[-\sigma,\sigma]}_{\bar{y}}=\ker\ \mathbb{E}^{[\sigma,-\sigma]}_{\bar{y}}=*.$$

Then by Lemma 1.2.4, we have the following oriented K-model for $\mathbb{E}_{w_\#}$ compatible with the coherent orientation:

$$[\,\mathbb{E}^\sigma_{w_\#}\colon K_\#\to C_\#],\quad \text{where } C_\#=*\#_{R_1}\mathbb{R}\bar{\mathbf{e}}^\sigma_y\#_{R_1'}*\cdots\#_{R_k'}*,$$

$$K_\#=(-1)^{k+1}\ker\ \mathbb{E}^{[0,\sigma]}_{u_0}\#_{R_1}*\#_{R_1'}\ker\ \mathbb{E}^{[-\sigma,\sigma]}_{u_1}\#_{R_2}*\#_{R_2'}\cdots\#_{R_k'}\ker\mathbb{E}^{[-\sigma,0]}_{u_{k+1}}.$$

On the other hand, in Section 3, we constructed the following K-model:

$$[\mathbb{E}_{w_\chi}\colon K_\chi\to C_\chi]=\Big[\mathbb{R}\mathfrak{e}_{u_0}\oplus\cdots\oplus\mathbb{R}\mathfrak{e}_{u_{k+1}}\to\mathbb{R}\mathfrak{f}_1\oplus\cdots\oplus\mathbb{R}\mathfrak{f}_{k+1}\Big].$$

($\mathbb{E}_{w_\chi}$ is now considered as an operator between ordinary Sobolev spaces. As remarked before, the polynomially weighted spaces are commensurate with the ordinary Sobolev spaces, and we do not need uniform boundedness of right inverses in this section. We have also suppressed the subscript Φ and written $\mathfrak{e}_{u_i} = (\mathfrak{e}_{u_i})_\Phi, \mathfrak{f}_j = (\mathfrak{f}_j)_\Phi$ above for simplicity.)

Using the descriptions of Π_{K_χ} and Π_{C_χ} given in Section 1.2.4 and Lemma 4.1.1 and the proximity between $\mathbb{E}^\sigma_{w_\#}$ and $\mathbb{E}_{w_\chi}$, one may easily check that the oriented K-model $[K_\# \to C_\#]$ is equivalent to

$$\Big[(-1)^{k+1}\prod_{i=0}^{k+1}\operatorname{sign}(u_i)K_\chi \to C_\chi\Big],$$

implying that the latter is also compatible with the coherent orientation.

Next, observe that $(w')_\Phi$ projects positively to all $\mathfrak{e}_{u_i}$. This, together with the form of $\Pi_{C_\chi}\mathbb{E}_\chi|_{K_\chi}$ given in Lemma 4.1.3 (b), implies that the reduction of the above oriented K-model,

$$\Big[(-1)^{k+1}\prod_{i=0}^{k+1}\operatorname{sign}(u_i)\,\mathbb{R}(w')_\Phi \to \,*\Big],$$

is equivalent to the standard oriented K-model for $\mathbb{E}_{w_\chi}$, which is in turn equivalent to the standard oriented K-model for $\mathbb{E}_w$, due to the proximity between w and w_χ. These observations immediately imply the lemma. □

7.3.5. Signs for $\mathbb{T}_{O,\mathrm{db}}$. To verify the sign in (I.32), we examine the orientation of the K-model in the proof of Proposition 2.1 (b). Let $\{\hat{u}_1,\dots,\hat{u}_k\}$ be a broken orbit, u_i be the centered representative of $\hat{u}_i$, and $(\lambda,(T,w))$ be the image of $(\{\hat{u}_1,\dots,\hat{u}_k\},\lambda')$ under the gluing map defined in Section 4.3.1.

Lemma. *Under the assumptions in Section* 2.1 *and in the above notation*,

$$\operatorname{sign}(w) = (-1)^{\mathrm{ind}_-\, y+k}\prod_{i=1}^{k}\operatorname{sign}(u_i).$$

Proof. As argued in the proof of Lemma 7.3.3,

$$\operatorname{sign}(w) = (-1)^{\mathrm{ind}_-\, y}\operatorname{sign}(\lambda)(w')_\Phi/\ker^{\mathfrak{o}_{y-}}\acute{\mathbb{D}}_{(\lambda,w)}, \tag{111}$$

where the superscript $\mathfrak{o}_{y-}$ indicates the orientation correlated with the canonical orientation of $\tilde{\mathbb{D}}_{\mathbb{A}_y+\sigma,T}$. (Recall the definition $\mathrm{ind}_-\, y = \mathrm{ind}(\mathbb{A}_y + \sigma)$.) According to the assumption of Section 2.1, $\operatorname{sign}(\lambda) = 1$.

Instead of working with the standard K-model for $\tilde{\mathbb{D}}_{\mathbb{A}_y+\sigma,T}$, we find it easier to work with the following mutually co-oriented K-model: $[\acute{\mathbb{D}}_y\colon -(\mathbf{e}_y)_\Phi \to *]$, where $\acute{\mathbb{D}}_y$ is the stabilization of $\mathbb{D}_{\mathbb{A}_y,T}$ by multiplication with $(\mathbf{e}_y)_\Phi$. To see that they are indeed co-oriented, observe that the interpolation between them, $\acute{\mathbb{D}}_\nu = ((1-\nu)(\mathbf{e}_y)_\Phi, \mathbb{D}_{\mathbb{A}_y+\nu\sigma,T})$, is surjective $\forall\nu\in[0,1]$ and has the following continuous basis for the kernel: $\xi_\nu := (\nu\sigma, -(1-\nu)(\mathbf{e}_y)_\Phi)$.

We now consider mutually co-oriented K-models for two operators approximating $\mathbb{D}_{(\lambda,w)}$ and $\acute{\mathbb{D}}_y$, respectively. The proximity of the operators implies that these K-models also form mutually co-oriented K-models for to $\acute{\mathbb{D}}_{(\lambda,w)}$ and $\acute{\mathbb{D}}_y$, respectively. Choose an glued orbit

$$w_\# = \tau_L(\bar{y}\#_{R_1}u_1\#_{R'_1}\bar{y}\#_{R_2}\cdots\#_{R_k}u_k\#_{R'_k})$$

appproximating w and w_χ pointwise in the sense of Section 7.3.4, and let

$$\begin{aligned}\acute{\mathbb{D}}^\sigma_{w\#} &:= \Big(\partial_\lambda\mathcal{V}_\lambda(w_\#),\, \mathbb{E}^{[\sigma,-\sigma]}_{\bar{y}}\#_{R_1}\mathbb{E}^{[-\sigma,\sigma]}_{u_1}\#_{R'_1}\mathbb{E}^{[\sigma,-\sigma]}_{\bar{y}}\#_{R_2}\mathbb{E}^{[-\sigma,\sigma]}_{u_2}\cdots\mathbb{E}^{[-\sigma,\sigma]}_{u_k}\#_{R'_k}\Big)\\ \acute{\mathbb{D}}^\sigma_{y\#} &:= \Big(-(\mathbf{e}_y)_\Phi,\, \mathbb{E}^{[\sigma,-\sigma]}_{\bar{y}}\#_{R_1}\mathbb{E}^{[-\sigma,\sigma]}_{\bar{y}}\#_{R'_1}\mathbb{E}^{[\sigma,-\sigma]}_{\bar{y}}\#_{R_2}\mathbb{E}^{[-\sigma,\sigma]}_{\bar{y}}\cdots\mathbb{E}^{[-\sigma,\sigma]}_{\bar{y}}\#_{R'_k}\Big).\end{aligned}$$

Since $[\mathbb{E}^{[-\sigma,\sigma]}_{u_i}\colon \ker E^{[-\sigma,\sigma]}_{u_i}\to *]$ and $[\mathbb{E}^{[-\sigma,\sigma]}_{\bar{y}}\colon \mathbb{R}\bar{\mathbf{e}}^\sigma_y\to *]$ are mutually co-oriented K-models (by coherent orientation), the continuity of gluing homorphisms and stabilization imply that we have the mutually co-oriented K-models $[\acute{\mathbb{D}}^\sigma_{w\#}\colon \hat{K}_{w\#}\to C_\#]^{\mathfrak{gl}}$, $[\acute{\mathbb{D}}^\sigma_{y\#}\colon \hat{K}_{y\#}\to C_\#]^{\mathfrak{gl}}$, where

$$\begin{aligned}\hat{K}_{w\#} =& (-1)^{k+1}\mathbb{R}\oplus(*\#_{R_1}\ker\mathbb{E}^{[-\sigma,\sigma]}_{u_1}\#_{R'_1}*\#_{R_2}\ker\mathbb{E}^{[-\sigma,\sigma]}_{u_2}\cdots\ker\mathbb{E}^{[-\sigma,\sigma]}_{u_k}\#_{R'_k})\\ C_\# =& (\mathbb{R}\hat{\mathbf{e}}^\sigma_y)\#_{R_1}*\#_{R'_1}(\mathbb{R}\hat{\mathbf{e}}^\sigma_y)\cdots*\#_{R'_k},\\ \hat{K}_{y\#} =& (-1)^{k+1}\mathbb{R}\oplus(*\#_{R_1}(\mathbb{R}\hat{\mathbf{e}}^\sigma_y)\#_{R'_1}*\#_{R_2}(\mathbb{R}\hat{\mathbf{e}}^\sigma_y)\cdots(\mathbb{R}\hat{\mathbf{e}}^\sigma_y)\#_{R'_k}).\end{aligned}$$

Note that the orientation of these K-models is different from the grading-compatible orientation or the $\mathfrak{o}_{y-}$ orientation. We call it the "glued orientation," indicated by the superscript $\mathfrak{gl}$ above.

We now compute the sign of $(w')_\Phi$ relative to the $\mathfrak{gl}$-orientation above. As in Section 7.3.4, this is done by comparing the glued K-model above to $[\acute{\mathbb{D}}_{(\lambda,w_\chi)}\colon \hat{K}_\chi\to C_\chi] := [\mathbb{R}_\alpha\oplus\mathbb{R}\mathfrak{e}_{u_1}\oplus\cdots\oplus\mathbb{R}\mathfrak{e}_{u_k}\to\mathbb{R}\mathfrak{f}_1\oplus\cdots\oplus\mathbb{R}\mathfrak{f}_k]$, constructed previously in Section 4.3. In this case, we find $[\acute{\mathbb{D}}^\sigma_{w\#}\colon \hat{K}_{w\#}\to C_{w\#}]^{\mathfrak{gl}}$ equivalent to

$$[\acute{\mathbb{D}}_{(\lambda,w_\chi)}\colon (-1)^{k+1}\Pi_i(\mathrm{sign}(u_i))\,\hat{K}_\chi\to C_\chi].$$

On the other hand, in Lemma 4.1.3 (b), $\Pi_{C_\chi}\acute{\mathbb{D}}_{w_\chi}|_{\hat{K}_\chi}$ is computed in the bases $\{1,\mathfrak{e}_{u_1},\dots\mathfrak{e}_{u_k}\}$, $\{\mathfrak{f}_j\}$ to be of the form:

$$\begin{pmatrix} + & - & 0 & \cdots & \cdots & + \\ + & + & - & 0 & \cdots & 0 \\ + & 0 & + & \ddots & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ + & 0 & \cdots & \cdots & + & - \end{pmatrix} \qquad (+/- \text{ denote positive/negative numbers.})$$

modulo ignorable terms. Combining this with the fact that, in terms of the same basis,

$$\Pi_{\hat{K}_\chi}(w')_\Phi = (0, +, +, \cdots, +),$$

we see that

$$[(w')_\Phi]/\ker^{\mathfrak{gl}} \acute{\mathbb{D}}_{(\lambda,w)} = -\prod_{i=1}^{k} \operatorname{sign}(u_i). \tag{112}$$

Next, we need to find the relative sign between the $\mathfrak{gl}$ and $\mathfrak{o}_y$ orientations. For this purpose, we compute explicitly the form of the operator $\Pi_{C_\#}\acute{\mathbb{D}}_{y\#}|_{\hat{K}_{y\#}}$. In terms of the bases $\{1, e_1, \ldots, e_k,\}$ and $\{f_j\}$, where

$$e_i := *\#_{R_1} * \cdots * \#_{R_i}\bar{\mathbf{e}}_y^\sigma \#_{R'_i} * \cdots * \#_{R'_k};$$
$$f_j := *\#_{R_1} * \cdots * \#_{R'_{j-1}}\bar{\mathbf{e}}_y^\sigma \#_{R_j} * \cdots * \#_{R'_k},$$

it has the following form:

$$\begin{pmatrix} + & + & 0 & \cdots & \cdots & - \\ + & - & + & 0 & \cdots & 0 \\ + & 0 & - & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ + & 0 & \cdots & 0 & - & + \end{pmatrix}.$$

Combining this with the facts that, in the same basis,

$$\Pi_{\hat{K}_{y\#}}(-(\mathbf{e}_y)_\Phi) = (0, -, -, \cdots, -),$$

we have by the proximity between $\acute{\mathbb{D}}_{y\#}$ and $\acute{\mathbb{D}}_y$ that

$$\operatorname{sign}(\mathfrak{o}_{y-}/\mathfrak{gl}) = -[(\mathbf{e}_y)_\Phi]/\ker^{\mathfrak{gl}} \acute{\mathbb{D}}_y = (-1)^{k+1}.$$

The Lemma now follows by combining this with (112) and (111). □

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