All Harmonic Numbers Less than 10^{14}

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Received October 31, 2006
Revised March 22, 2007

A positive integer n is said to be *harmonic* if the harmonic mean $H(n)$ of its positive divisors is an integer. Ore proved that every perfect number is harmonic and conjectured that there exist no odd harmonic numbers greater than 1. In this article, we give the list of all harmonic numbers up to 10^{14}. In particular, we find that there exist no nontrivial odd harmonic numbers less than 10^{14}.

Key words: harmonic number, perfect number, Ore's conjecture

1. Introduction

A positive integer n is said to be *perfect* if $\sigma(n) = 2n$, where $\sigma(n)$ denotes the sum of the positive divisors of n. It is an open problem whether or not an odd perfect number exists. In 1948, Ore [7] introduced the concept of *harmonic numbers*. In general, the *harmonic mean* of positive numbers $a_1, \ldots, a_k$ is defined by

$$\left(\sum_{i=1}^{k} \frac{1}{a_i}\right)^{-1}.$$

A positive integer n is said to be harmonic if the harmonic mean of its positive divisors

$$H(n) = \frac{n\tau(n)}{\sigma(n)}$$

is an integer, where $\tau(n)$ denotes the number of the positive divisors of n. We call 1 the trivial harmonic number. Ore proved that every perfect number is harmonic. Many harmonic numbers are known; however, no nontrivial odd harmonic numbers have been discovered. Ore conjectured that every nontrivial harmonic number is even. If Ore's conjecture is true, then it follows that no odd perfect numbers exist.

Ore listed all harmonic numbers up to 10^4 and this list was extended by Garcia [4] to 10^7, by Cohen [2] to $2 \cdot 10^9$, and by Sorli [8] to 10^{12}. They used the algorithm described in [4]. In this article, we give the list of all harmonic numbers up to 10^{14} using an improved algorithm.

Theorem 1.1. *Let n be harmonic numbers less than 10^{14}. Then n is one of the 937 numbers in* §5.

In particular, we find that there exist no nontrivial odd harmonic numbers less than 10^{14}. Note that Sorli [8] showed that there exist no nontrivial odd harmonic numbers less than 10^{15} by another method. Cohen and Sorli [3] introduced the concept of harmonic seeds.

Definition. *Let d be a divisor of an integer n. We call d a unitary divisor of n and n a unitary multiple of d if $(d, n/d) = 1$. A harmonic number is called harmonic seed if it does not have a smaller proper unitary divisor which is harmonic (we call a unitary divisor d proper if $d > 1$).*

Cohen and Sorli [3] gave the list of all harmonic seeds up to 10^{12}, and this list was extended by Sorli [8] to 10^{15}. Note that our list is consistent with these lists. Sorli's list contains no odd harmonic seeds greater than 1. Since every harmonic number is a unitary multiple of a seed, it follows that there exist no nontrivial odd harmonic numbers less than 10^{15}.

Theorem 1.1 is a consequence of the following two lemmas. The lists mentioned in the statements are available on the webpage `http://www.ma.noda.tus.ac.jp/u/tg/html/harmonic-e.html`.

Lemma 1.2. *Let n be harmonic and $H(n)^{4.55} > n$. Then n is one of the 1643 numbers in the list which is available on the webpage.*

Lemma 1.3. *Let n be harmonic and $H(n) \leq 1200$. Then n is one of the 1376 numbers in the list which is available on the webpage.*

Proof of Theorem 1.1. Let

$$\begin{aligned}
\mathcal{H} &= \{n \in \mathbb{N} \mid H(n) \in \mathbb{N}\}, \\
\mathcal{H}_1 &= \{n \in \mathcal{H} \mid n < 10^{14}\}, \\
\mathcal{H}_2 &= \{n \in \mathcal{H} \mid H(n)^{4.55} > n\}, \\
\mathcal{H}_3 &= \{n \in \mathcal{H} \mid H(n) \leq 1200\}.
\end{aligned}$$

We can easily see that $\mathcal{H}_1 \subset \mathcal{H}_2 \cap \mathcal{H}_3$. Indeed, suppose that $n \in \mathcal{H}_1$. If $n \notin \mathcal{H}_3$, then $H(n)^{4.55} > 1200^{4.55} > 10^{14} > n$, and hence $n \in \mathcal{H}_2$. By Lemmas 1.2 and 1.3, the finite sets $\mathcal{H}_2$ and $\mathcal{H}_3$ are known. Therefore we can give the set $\mathcal{H}_1$. □

Note that Lemma 1.2 is an example of the following theorem.

Theorem 1.4. *For any real number α, there exist only finitely many positive integers n satisfying $H(n)^{\alpha} > n$.*

In §2, we give a proof of Theorem 1.4. The proof describes the algorithm to show Lemma 1.2.

Remark. It is well known that a harmonic mean is equal to or less than a geometric mean. Since the geometric mean of positive divisors of n is $\sqrt{n}$, there exist no harmonic numbers n satisfying $H(n)^2 > n$.

Shibata and the first author [5] gave the list of all harmonic numbers n with $H(n) \leq 300$. Lemma 1.3 is the extended result. In §3, we discuss the algorithm to show it.

2. Proof of Theorem 1.4 and Lemma 1.2

Proof of Theorem 1.4. From the remark after Theorem 1.4, we may assume that $\alpha > 2$. Now, let us fix a real number α and define $f(\alpha, n) = H(n)^\alpha/n$. Then $H(n)^\alpha > n$ if and only if $f(\alpha, n) > 1$. Note that f is multiplicative in the second variable, that is, $f(\alpha, nm) = f(\alpha, n)f(\alpha, m)$ when $(n, m) = 1$. For a prime p and a positive integer e,

$$f(\alpha, p^e) = \frac{p^{(\alpha-1)e}(e+1)^\alpha}{(p^e + p^{e-1} + \cdots + 1)^\alpha}.$$

Since $\alpha > 2$, it is clear that $f(\alpha, p^e)$ is monotone decreasing as a function of p and of e for sufficiently large p and e. Furthermore, we have $\lim_{p\to\infty} f(\alpha, p^e) = 0$ and $\lim_{e\to\infty} f(\alpha, p^e) = 0$. Hence there are only finitely many prime powers p^e satisfying $f(\alpha, p^e) > 1$. Let $\mathscr{L}$ be the set of integers whose all prime components satisfy this condition. Since $\mathscr{L}$ is finite, there exists the maximum value $\max_{n\in\mathscr{L}} f(\alpha, n)$. Let A be this maximum value. There are also only finitely many prime powers q^f satisfying $f(\alpha, q^f) > 1/A$. Let $\mathscr{L}'$ be the set of integers whose all prime components satisfy this condition, and $\mathscr{M}$ be the set of integers n satisfying the required condition $f(\alpha, n) > 1$. It is easy to show that $\mathscr{M} \subset \mathscr{L}'$. Since $\mathscr{L}'$ is finite, $\mathscr{M}$ is also finite, and hence the proof is complete. □

In the rest of this section, we demonstrate this procedure for the case of $\alpha = 4.55$. We first give some useful lemmas.

Lemma 2.1. *If $p > ((e+2)/(e+1))^\alpha$, then $f(\alpha, p^e) > f(\alpha, p^{e+1})$. In particular, if $p > (3/2)^\alpha$, then $f(\alpha, p^e) > f(\alpha, p^{e+1})$ for any positive integer e.*

Proof. Suppose that $p > ((e+2)/(e+1))^\alpha$. Then we have

$$\begin{aligned}\frac{f(\alpha, p^e)}{f(\alpha, p^{e+1})} &= \frac{p^{(\alpha-1)e}(p-1)^\alpha(e+1)^\alpha}{(p^{e+1}-1)^\alpha} \cdot \frac{(p^{e+2}-1)^\alpha}{p^{(\alpha-1)(e+1)}(p-1)^\alpha(e+2)^\alpha} \\ &= p\left(\frac{p^{e+2}-1}{p^{e+2}-p}\right)^\alpha \left(\frac{e+1}{e+2}\right)^\alpha > p\left(\frac{e+1}{e+2}\right)^\alpha > 1,\end{aligned}$$

and hence $f(\alpha, p^e) > f(\alpha, p^{e+1})$. □

Lemma 2.2. *Let p, q be primes and $\alpha - 1 \leq p < q$. Then $f(\alpha, p) > f(\alpha, q)$.*

Proof. By a direct calculation, we have

$$\frac{\partial f(\alpha, p)}{\partial p} = \frac{2^\alpha p^{\alpha-2}}{(p+1)^{\alpha+1}}(\alpha - p - 1).$$

Hence the lemma holds. □

Let $f(n) = f(4.55, n) = H(n)^{4.55}/n$. By a direct calculation, we have the following table.

Table 1. Values of $f(p^e)$

p^e	$f(p^e)$	p^e	$f(p^e)$	p^e	$f(p^e)$	p^e	$f(p^e)$
2	1.85114	2^9	2.97148	3^4	3.01135	7	1.82274
2^2	2.90414	2^{10}	2.28725	3^5	2.27237	7^2	1.52008
2^3	3.92759	2^{11}	1.69723	3^6	1.52109	7^3	0.79486
2^4	4.66921	2^{12}	1.22077	3^7	0.92960	11	1.43336
2^5	4.97589	2^{13}	0.85492	5	2.04381	11^2	0.79664
2^6	4.83980	3	2.10908	5^2	2.22788	13	1.28618
2^7	4.36412	3^2	3.09044	5^3	1.60208	17	1.06240
2^8	3.69607	3^3	3.39892	5^4	0.87927	19	0.97628

By Lemma 2.2 and Table 1, $f(p)$ is monotone decreasing for $p \geq 3$. By Lemma 2.1, we have

- $f(2^5) \geq f(2^e)$ for $e \geq 5$,
- $f(3^3) \geq f(3^e)$ for $e \geq 3$,
- $f(5^2) \geq f(5^e)$ for $e \geq 2$,
- $f(p) \geq f(p^e)$ for $e \geq 1$, $p \geq 7$.

Hence it follows that

$$\mathscr{L} = \left\{ 2^{\varepsilon_1}3^{\varepsilon_2}5^{\varepsilon_3}7^{\varepsilon_4}11^{\varepsilon_5}13^{\varepsilon_6}17^{\varepsilon_7} \;\middle|\; \begin{array}{l} 0 \leq \varepsilon_1 \leq 12,\ 0 \leq \varepsilon_2 \leq 6,\ 0 \leq \varepsilon_3 \leq 3, \\ 0 \leq \varepsilon_4 \leq 2,\ 0 \leq \varepsilon_5, \varepsilon_6, \varepsilon_7 \leq 1 \end{array} \right\},$$

and $A := \max_{n \in \mathscr{L}} f(n) = f(2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17) < 4.9759 \cdot 3.399 \cdot 2.2279 \cdot 1.8228 \cdot 1.4334 \cdot 1.2862 \cdot 1.0625 < 134.55$. Therefore, if $f(n) > 1$, then every prime component p^e of n satisfies $f(p^e) > 1/A > 1/134.55 > 0.007432$. A computer search showed that $\mathscr{L}'$ is the set of integers whose all prime components are in the set

$$\left\{ \begin{array}{l} 2^e\ (1 \leq e \leq 23),\ 3^e\ (1 \leq e \leq 13),\ 5^e\ (1 \leq e \leq 8), \\ 7^e\ (1 \leq e \leq 7),\ 11^e\ (1 \leq e \leq 5),\ 13^e, 17^e, 19^e\ (1 \leq e \leq 4), \\ 23^e, 29^e, 31^e, 37^e\ (1 \leq e \leq 3),\ p^2\ (41 \leq p \leq 137),\ p\ (41 \leq p \leq 3137) \end{array} \right\}.$$

We have to select harmonic numbers from $\mathscr{L}'$. Using the method of Garcia [4], we can do it and show Lemma 1.2. Using Mathematica® and a machine with a processor Pentium M 1.2 GHz, the authors needed about 3 hours to do this computation. In the case of $\alpha = 4$, we needed only ten seconds. These data show that the amount of computation increases very rapidly with α.

Table 2. More values of $f(p^e)$

p^e	$f(p^e)$	p^e	$f(p^e)$	p^e	$f(p^e)$	p^e	$f(p^e)$
2^{23}	0.00969	11^5	0.01397	23^3	0.03684	41^2	0.07880
2^{24}	0.00583	11^6	0.00256	23^4	0.00442	41^3	0.00711
3^{13}	0.01625	13^4	0.03684	29^3	0.01917	43^2	0.07202
3^{14}	0.00741	13^5	0.00649	29^4	0.00182	137^2	0.00763
5^8	0.02037	17^4	0.01376	31^3	0.01586	139^2	0.00742
5^9	0.00658	17^5	0.00185	31^4	0.00141	3137	0.00745
7^7	0.00774	19^4	0.00908	37^3	0.00956	3163	0.00739
7^8	0.00188	19^5	0.00109	37^4	0.00071		

3. Algorithm to show Lemma 1.3

We first summarize the algorithm described in [5]. The algorithm is to find all harmonic numbers n satisfying $H(n) = c$ for a given integer c. In this section, we say that an integer n has a *type of exponents* $(e_1, \ldots, e_r)$ when the factorization of n is $p_1^{e_1} \cdots p_r^{e_r}$ with $e_1 \geq \cdots \geq e_r$. If n has the type of exponents $(e_1, \ldots, e_r)$, then

$$c = H(n) \geq H(2^{e_1} 3^{e_2} \cdots q_r^{e_r}),$$

where q_r is the rth prime (see Lemmas 2.1 and 4.2 in [5]). If $H(n) = c$ and n has the type of exponents $(e_1, \ldots, e_r)$, then

$$S(n) := \frac{\sigma(n)}{n} = \frac{\tau(n)}{H(n)} = \frac{(e_1+1)\cdots(e_r+1)}{c}.$$

Hence we need to find all integers n which have the type $(e_1, \ldots, e_r)$ and satisfy $S(n) = d$ for a given rational number d. It is clear that the function S is multiplicative, that is, $S(mn) = S(m)S(n)$ when $(m, n) = 1$. Furthermore, if p, q are primes and $p < q$, then

$$1 < \frac{q+1}{q} \leq S(q^e) < S(p^e) < \frac{p}{p-1}.$$

Hence, if p is a smallest prime dividing n, then

$$d = S(n) < S(p^{e_1}) \cdots S(p^{e_r}) < \left(\frac{p}{p-1}\right)^r.$$

Therefore it is necessary that

$$p < \frac{\sqrt[r]{d}}{\sqrt[r]{d} - 1}.$$

The basic algorithm to find all integers n which have the type $(e_1, \ldots, e_r)$ and satisfy $S(n) = d$, is described as follows. Note that this procedure will end since the number of the second terms decreases for each step of the subroutine.

Basic algorithm

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go to subroutine *Sub* $(1, (e_1, \ldots, e_r), d)$
Sub $(N, (f_1, \ldots, f_t), D)$
for $p = 2$ to $\sqrt[t]{D}/(\sqrt[t]{D} - 1)$ **do**
  if p is prime **then**
    for $i = 1$ to t **do**
      if $t = 1$ and $S(p^{f_1}) = D$ **then print** Np^{f_i}
      elseif $S(p^{f_i}) < D$ **then**
        go to subroutine *Sub* $(Np^{f_i}, (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_t), D/S(p^{f_i}))$
      endif
    endfor
  endif
endfor

The problem with this algorithm is that the bound $\sqrt[t]{D}/(\sqrt[t]{D}-1)$ is large when D is close to 1.

Example 1. Let p_1, p_2, p_3 be odd primes with $p_1 < p_2 < p_3$. Consider the equation $H(2^3 p_1 p_2 p_3) = 17$. Since $H(2^3) = 2^5/(3 \cdot 5)$, we have $H(p_1 p_2 p_3) = 17 \cdot 3 \cdot 5/2^5$. Therefore $S(p_1 p_2 p_3) = 2^8/(3 \cdot 5 \cdot 17) = 1.0039 \cdots$, and the upper bound of p_1 is a little large. It is about 767.

Example 2. Let l, p_1, p_2 be odd primes with $p_1 < p_2$. Consider the equation $H(2^{l-1} p_1 p_2) = 2l$. Since $H(2^{l-1}) = 2^{l-1} l/(2^l - 1)$, it follows that $H(p_1 p_2) = 2l(2^l - 1)/(2^{l-1} l) = (2^l - 1)/2^{l-2}$. Therefore $S(p_1 p_2) = 2^l/(2^l - 1)$ and it is very close to 1, the bound of p_1 is large. For example, if $l = 47$, then the bound is greater than 10^{14}.

Using the following proposition, we can improve the algorithm.

Proposition 3.1. *Suppose that n has a type of exponents $(e_1, \ldots, e_r)$ and $H(n) = a/b$ $(a, b \in \mathbb{N}, (a, b) = 1)$. Let m be the greatest common divisor of a and $\tau(n)$ $(= (e_1 + 1) \cdots (e_r + 1))$. Then a/m divides n.*

Proof. From $H(n) = n\tau(n)/\sigma(n) = a/b$, we have $(a/m) \cdot \sigma(n) = nb\tau(n)/m$. Since a/m is coprime to $b\tau(n)/m$, it follows that a/m divides n. □

Let us go back to Example 1. From $H(2^3 p_1 p_2 p_3) = 17$, we have $H(p_1 p_2 p_3) = 17/H(2^3) = (3 \cdot 5 \cdot 17)/2^5$. By Proposition 3.1, it is necessary that $p_1 p_2 p_3 = 3 \cdot 5 \cdot 17$; however, $H(2^3 \cdot 3 \cdot 5 \cdot 17) \neq 17$. Hence no solutions exist. This method is often very effective, but sometimes it has the opposite effect.

Example 3. Let p_1, p_2 be odd primes with $p_1 < p_2$. Consider the equation $H(5^{100} p_1^2 p_2) = 303$. From the equation, we have $H(p_1^2 p_2) = 3\sigma(5^{100})/5^{100}$. Hence it is necessary that $\sigma(5^{100}) \mid p_1^2 p_2$ by Proposition 3.1. However $\sigma(5^{100})$ is a 70-digit integer, and it is slightly difficult to find a prime factor of the integer*. Hence we

*According to the book [1], the factorization of $\sigma(5^{100})$ is given by $5937018283241 \cdot 34344873113965898214738541 21 \cdot 48359315388774726502953690742 1$.

should use the upper bound of p_1. Let $D = S(p_1^2 p_2)$. Then it is necessary that $p_1 < \sqrt{D}/(\sqrt{D}-1) = 4.77\cdots$, and hence $p_1 = 2$ or 3. his is contradictory to the condition $\sigma(5^{100}) \mid p_1^2 p_2$ since $\sigma(5^{100})$ is not divisible by either 2 or 3. Therefore no solutions exist.

In this way, we have the following algorithm.

Improved algorithm

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go to subroutine $Sub\ (1, (e_1, \ldots, e_r), d)$
$Sub\ (N, (f_1, \ldots, f_t), D)$
$M \leftarrow$ the denominator of D
if M has a prime factor less than 2^{17}, $M < 10^{50}$ or the smallest prime factor of M is known **then**
  $p \leftarrow$ the smallest prime factor of M
  for $i = 1$ to t **do**
    if $t = 1$ and $S(p^{f_1}) = D$ **then print** Np^{f_i}
    elseif $S(p^{f_i}) < D$ **then**
      go to subroutine $Sub\ (Np^{f_i}, (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_t), D/S(p^{f_i}))$
    endif
  endfor
else then
  for $p = 2$ to $\sqrt[t]{D}/(\sqrt[t]{D}-1)$ **do**
    if p is prime **then**
      for $i = 1$ to t **do**
        if $t = 1$ and $S(p^{f_1}) = D$ **then print** Np^{f_i}
        elseif $S(p^{f_i}) < D$ **then**
          go to subroutine $Sub\ (Np^{f_i}, (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_t), D/S(p^{f_i}))$
        endif
      endfor
    endif
  endfor
endif

In this algorithm, we factor integers less than 10^{50}. It is possible to change this value. The authors used UBASIC to factor integers. We can find a prime factor less than 2^{17} using the command `prmdiv` (trial division), and the program `MPQX3` (multiple polynomial quadratic sieve, implemented by Y. Kida and A. Yamasaki) enable us to factor 50-digit integers within a few seconds.

Unfortunately, both a/m and $\sqrt[t]{D}/(\sqrt[t]{D}-1)$ are sometimes very large. Consider Example 2 again. From the equation $H(2^{l-1}p_1p_2) = 2l$, it follows that $H(p_1p_2) = (2^l-1)/2^{l-2}$. By Proposition 3.1, it is necessary that $2^l - 1 \mid p_1p_2$. If $l = 47$, then $2^{47}-1 \mid p_1p_2$ is impossible since $2^{47}-1 = 2351 \cdot 4513 \cdot 13264529$. When l is large, it is difficult to factor the Mersenne number $2^l - 1$. Hence the following proposition is useful.

Proposition 3.2 ([2], [5]). *Let p be prime. If $H(n) = p$, $2p$ or $3p$, then n is an even perfect number or $p \mid n$.*

From this proposition, it is necessary that $p_1 = l$ or $p_2 = l$. Recall that $2^l - 1 \mid p_1p_2$. Since $l \nmid 2^l - 1$ from Fermat's Little Theorem, it is necessary that $2^l - 1$ is prime and $n = 2^{l-1}l(2^l - 1)$. However $H(n)$ is not integral for this n, and hence no solutions exist.

The following example is one of the most unfortunate cases.

Example 4. Let p_1, p_2 be odd primes with $p_1 < p_2$. Consider the equation $H(2^{l-1}p_1p_2p_3) = 4l$. From this equation, we have $H(p_1p_2p_3) = (2^l - 1)/2^{l-3}$ and $S(p_1p_2p_3) = 2^l/(2^l - 1)$. Both the number $2^l - 1$ and the upper bound of p_1 are large. Furthermore we cannot apply Proposition 3.2 in this case. In such a case, we can refer to known prime factors of Mersenne numbers $2^l - 1$. As of March 2007, at least one prime factor of $2^l - 1$ for $l < 1061$ is known (cf. [1] or recent webpages†). No factors of $2^{1061} - 1$ are known, and hence it is unknown whether or not $H(2^{1060}p_1p_2p_3) = 4 \cdot 1060$ has a solution.

The authors used UBASIC and it took about 10 hours to show Lemma 1.3.

4. Numerical data

In this section, we give some interesting examples and numerical data.

Let x, y be harmonic numbers. In this section, we write $x \preceq y$ if $x \neq 1$ and x is a unitary divisor of y. If $x \preceq y$ and $x \neq y$, then we write $x \prec y$. Clearly, $x \prec y$, $y \prec z$ implies $x \prec z$. In other words, the notation $\prec$ is the partial order relationship in the set of harmonic numbers. A harmonic number n is a seed if it is minimal for this order, that is, there exist no harmonic numbers n' satisfying $n' \prec n$. Every harmonic number is a unitary multiple of a certain harmonic seed. Cohen and Sorli [3] raised following question.

Problem 1. Does every harmonic number have a unique harmonic seed?

From Lemma 1.3, we find that the answer is no. The harmonic number $n_0 = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31^2 \cdot 83 \cdot 331$ with $H(n_0) = 525$ has two harmonic seeds $n_1 = 2^4 \cdot 3 \cdot 7^2 \cdot 19 \cdot 31^2 \cdot 83 \cdot 331$ with $H(n_1) = 217$, and $n_2 = 2^4 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 31^2 \cdot 83 \cdot 331$ with $H(n_2) = 350$. In other words, $n_1, n_2 \prec n_0$ and there are no harmonic numbers n satisfying $n \prec n_1$ or $n \prec n_2$. If m is a harmonic number and $n_0 \prec m$, then m also has two harmonic seeds n_1 and n_2. There are many such harmonic numbers: $13n_0$, $29n_0$, $41n_0$, and so on.

A positive integer n is said to be *arithmetic* if the arithmetic mean of its positive divisors, $A(n) = \sigma(n)/\tau(n)$, is an integer. Ore conjectured that all harmonic numbers n are perfect or arithmetic; however, he soon found the counterexample 950976 ($H(950976) = 27$, $A(950976) = 105664/3$). Cohen [2] showed that n is arithmetic if $H(n)$ is a prime and n is not an even perfect number. Shibata and

†See, for example, P. Leyland's page `http://www.leyland.vispa.com/numth/factorization/cunningham/2-.txt`.

the first author [5] showed that n is arithmetic if $H(n)$ is the double of a prime. They raised following question (cf. [6, B2]).

Problem 2. Assume that $H(n)$ is the triple of a prime. Is n arithmetic?

From Lemma 1.3, we find that the answer is also no. The harmonic number $n = 2^8 \cdot 7 \cdot 19^2 \cdot 37 \cdot 73 \cdot 113 \cdot 127$ with $H(n) = 3 \cdot 113$ is not arithmetic ($A(n) = 221908282624/3$).

Let $N(x) = \#\{n \in \mathbb{N} \mid H(n) \in \mathbb{N},\ H(n) \le x\}$ and $N'(x) = \#\{n \in \mathbb{N} \mid n$ is a seed, $H(n) \le x\}$ for a real number x. From Lemma 1.3, we see that $N(1200) = 1376$, $N'(1200) = 188$.

Problem 3. How does the number $N(x)$ increase when x increases? What is the order of $N(x)$ as $x \to \infty$? How about $N'(x)$?

The number $N(x)$ (resp. $N'(x)$) seems to be close to $x^{1.015}$ (resp. $x^{0.74}$).

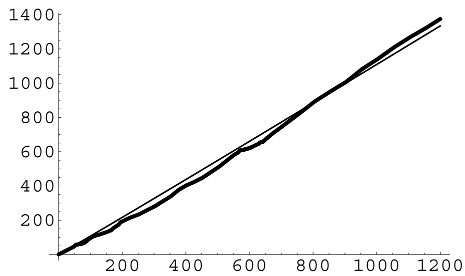

Fig. 1. The graphs of $N(x)$ and $x^{1.015}$

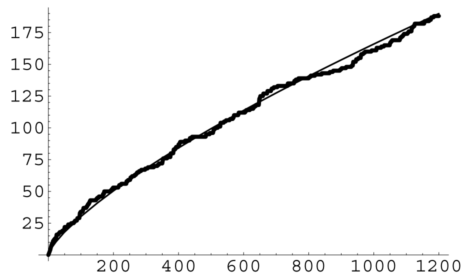

Fig. 2. The graphs of $N'(x)$ and $x^{0.74}$

The following problem is also still open.

Problem 4. Are there infinitely many harmonic numbers? How about seeds?

The above graphs are evidence that there seem to exist infinitely many harmonic numbers and seeds.

Let $M(c) = \#\{n \in \mathbb{N} \mid H(n) = c\}$ and $M'(c) = \#\{n \in \mathbb{N} \mid n \text{ is a seed}, H(n) = c\}$ for an integer c. We find that $M(1155) = 7$ and $M'(648) = 5$. In fact,

$$\begin{aligned} 1155 &= H(30233275380000) = H(30368564724960) = H(32752714995000) \\ &= H(256304066430720) = H(557957132902400) = H(5685642164944896) \\ &= H(16919757239726880), \\ 648 &= H(513480135168) = H(1058501001600) = H(1085239701000) \\ &= H(1599300612000) = H(10881843388416). \end{aligned}$$

From Lemma 1.3, if $c \leq 1200$, then $M(c) \leq 7$ and $M'(c) \leq 5$.

Problem 5. Can $M(c)$ be arbitrarily large? How about $M'(c)$?

5. Table of harmonic numbers

The table in this section is the list of all harmonic numbers up to 10^{14}. In this table, harmonic seeds are marked with an asterisk. The authors think that the list can be extended by improving the computer programs. For the most recent results, see the webpage `http://www.ma.noda.tus.ac.jp/tg/html/harmonic-e.html`.

n	H	n	H	n	H	n	H
1	1	753480	46	*33550336	13	301953024	27
*6	2	950976	27	37035180	102	318177800	73
*28	3	*1089270	42	44660070	82	318729600	168
140	5	1421280	47	*45532800	96	*326781000	168
*270	6	1539720	47	46683000	114	400851360	184
*496	5	2178540	54	50401728	53	407386980	187
*672	8	*2229500	35	*52141320	108	423184320	89
*1638	9	2290260	41	56511000	115	428972544	156
2970	11	*2457000	60	69266400	105	447828480	152
*6200	10	2845800	51	71253000	116	*459818240	96
*8128	7	4358600	37	75038600	91	*481572000	168
8190	15	*4713984	48	80832960	85	499974930	153
18600	15	4754880	45	*81695250	105	500860800	176
*18620	14	5772200	49	90409410	83	513513000	209
27846	17	*6051500	50	108421632	92	526480500	145
*30240	24	*8506400	49	110583200	91	540277920	186
*32760	24	8872200	53	*115048440	78	559903400	97
55860	21	11981970	77	115462620	106	623397600	189
105664	13	14303520	86	137891520	87	*644271264	117
117800	19	15495480	86	*142990848	120	675347400	189
167400	27	16166592	51	144963000	118	714954240	200
*173600	25	*17428320	96	163390500	135	758951424	161
237510	29	18154500	75	164989440	140	766284288	132
242060	26	*23088800	70	191711520	176	819131040	188
332640	44	23569920	80	221557248	94	825120800	97
360360	44	23963940	99	233103780	107	886402440	204
539400	29	27027000	110	*255428096	88	900463200	195
695520	46	*29410290	81	287425800	101	995248800	189
726180	39	32997888	84	300154400	130	1047254400	184

n	H
1162161000	215
1199250360	207
1265532840	143
*1307124000	240
1352913408	164
1379454720	144
*1381161600	240
1509765120	45
1558745370	159
*1630964808	99
1632825792	101
1727271000	222
1862023680	158
*1867650048	128
2008725600	203
2140041600	188
2144862720	260
2369162250	203
2481357060	201
2701389600	270
2705020500	149
2716826112	228
2738824704	166
2763489960	212
2777638500	255
2839922400	205
*2876211000	150
2945943000	218
3134799360	266
3209343200	139
3221356320	195
3288789504	230
3328809120	191
3349505250	205
3506025600	308
3594591000	308
3702033720	213
3740553180	202
3831421440	220
4143484800	312
4146734592	232
4720896180	197
4738324500	261
5058000640	176
5133201408	51
5275179000	226
5297292000	308
5510647296	167
5579121240	214
5943057120	341
6720569856	235
7279591410	163
7330780800	322
7515963000	322
8104168800	351
8154824040	165
8243595360	344
*8410907232	171
8436460032	236
*8589869056	17
*8628633000	195
8659696500	265
8696764800	191
*8698459616	121
9866368512	299
*10200236032	96
10575819520	184
10597041000	227
10597759200	357
10952611488	221
10983408128	172
11076156000	322
11296276992	237
11480905800	357
12941019000	229
13067913600	328
13073550336	224
13398021000	328
13581986600	181
13584130560	380
13660770240	169
*14182439040	384
14254365440	186
14378364000	440
14541754500	267
14980291200	329
15174001920	264
15192777600	440
15358707000	329
16003510272	53
16569653760	296
16919229600	357
17624538624	253
18999981000	407
*19017782784	336
*19209881600	256
19744452000	328
20015559200	181
20387256120	391
21537014400	344
21611457280	188
21943595520	392
22633884000	329
22933532160	278
23450730240	272
23855232960	173
24362612820	211
25559301600	369
25666007040	85
26113432800	377
26242070400	456
26454556800	332
27122823000	332
27689243400	369
27726401736	187
29715285600	495
30063852000	460
30600708096	144
31638321000	275
31727458560	276
31766716800	460
32950224384	258
32956953120	366
33040072800	371
34174812672	239
34482792960	396
*35032757760	392
35793412200	371
37906596000	464
39970476000	332
40053686400	464
40520844000	465
40752391680	494
40805200800	369
42054536160	285
42763096320	279
43783188480	87
*43861478400	264
43952044500	269
44184172032	309
45578332800	572
45923623200	510
50497467930	303
51001180160	160
52748186400	371
53227843200	334
53621568000	500
54572427000	334
54648009000	285
56481384960	395
*57575890944	192
57629644800	384
*57648181500	273
57897151488	248
59388963480	402
61434828000	470
62487000576	437
64834371840	282
64914595200	470
*66433720320	224
67622100480	302
*71271827200	270
*73924348400	125
77120316000	472
*77924700000	375
78340298400	522
80422524000	334
80533908000	375
80551516500	493
*81417705600	484
81488534400	472
83410119000	290
*84418425000	375
87825283840	191
89526646440	404
93419333280	377
95088913920	560
95300150400	598
97941285120	284
100383241728	262
100522566144	444
103262796000	474
108061356200	193
109111766400	474
*109585986048	324
*110886522600	155
112202596352	176
115987576320	518
*123014892000	484
*124406100000	375
126090783000	438
133410461184	311
134369095680	89
*137438691328	19
137770869600	663
142275893760	398
142985422944	323
143173648800	530
147112449120	367
150115204512	233
150759100800	602
151955343540	373
153003540480	240
154567413000	602
*156473635500	390
156798019840	341
159248314400	193
159381986400	531
164297299320	411
164751121920	430
169696449000	295
169956154368	416
*183694492800	672
194743785600	611
201532767744	263
*206166804480	384
213815481600	405
217494027520	344
220524885504	326
220920860160	515
*221908282624	171
227783556000	602
234605428736	184
236489897160	319
237191556096	254
240423674400	534
250230357000	377
*271309925250	405
280541488500	505
285266741760	728
287879454720	320
288662774400	836

n	H	n	H	n	H	n	H
289048687200	535	753132796416	458	1578475971072	664	2915401724928	446
292337717760	314	765181053000	443	1584792261000	551	2965353955200	904
307001350656	452	779729094144	656	*1599300612000	648	3076882754400	1005
307030348800	462	783990099200	495	1626268644000	614	3105356994432	616
311203567584	333	793104238080	759	1656012758400	872	3175969724928	668
312402636000	478	819730138500	519	1681994012160	439	3218345676000	652
321300067176	197	*830350521000	756	1683038945280	440	3238966130400	981
326196097920	736	861743282400	957	1708842189600	707	*3321402084000	1080
330097622400	478	863638364160	416	1721209990500	715	3356538237000	389
336607789056	264	869516291840	366	1773515487744	471	3377333836800	847
341519256000	325	888875820360	327	1784852619264	372	3398177502720	776
349002044160	506	888988066400	277	1801169758080	762	3448576989000	545
350280184800	389	893835790848	658	1862961762816	612	3500961340800	946
362526484320	671	906550977024	331	1886043571200	516	3519081431040	460
384342364800	367	*945884459520	756	1888271330400	699	3522876144480	675
403031236608	336	950432517216	339	1919938116096	463	3531726240768	488
405280060416	434	970956604800	888	1924339334400	729	3607776900000	725
410240742912	453	995024181060	401	1948245082112	191	*3622293071600	245
417624936960	436	*997978703400	279	1959868310400	1118	3634863187200	765
426778934400	618	1018809792000	950	1987794251520	524	3772440804608	323
*428440390560	546	*1058501001600	648	2015156183040	560	3777406841600	530
428555439000	298	1070373679200	707	2020639420800	1232	3881325763840	367
429520946400	689	1076349859200	614	2021976333000	555	*3946161492000	735
434508127200	697	*1085239701000	648	2033105289600	510	3962552630400	906
437409004032	644	1103539437000	614	2051203714560	755	3990762504960	526
439655610240	744	1109541413120	285	2059445329920	434	3991394534400	858
*443622427776	352	1135890756000	869	2061489484800	517	4029093232640	316
465036042240	392	*1144136294400	350	2066882988800	522	4205037804800	531
*469420906500	507	1159571485800	707	2070303429600	729	4224973334400	1288
470717137800	697	1161528261600	409	2096328767456	241	4240965560832	669
479411093504	188	1175104476000	899	*2112394079250	585	*4314435969536	385
482476262400	484	*1179832600464	217	2128528765440	776	4346661822720	548
483548738400	537	1200229430400	869	2130069916800	652	*4409499089268	147
494122282290	317	1209584724480	584	2172650274816	284	4437102673920	464
502612830720	740	1211621062400	510	2183877423000	652	4517245877760	786
505159855200	935	1219581548640	551	*2198278051200	1080	4537735429500	754
*513480135168	648	1233377308800	893	*2236152828000	529	4603679570880	337
518453342208	101	1253107608480	389	2259816300000	725	4612268729250	765
520212037632	272	1288623772800	622	2267834849280	704	4638285943200	1010
547929930240	540	1324245491712	368	2312019021312	851	4660073935104	378
583096381560	422	1325481830400	736	2335483332000	725	4694568278400	1133
586207480320	748	*1330464844800	660	2363575441500	533	4712844296160	1001
603567619200	874	1331785072800	986	2439654963200	508	4713692054400	643
616719527424	454	1369947647250	409	2448134325000	725	4741836503040	736
618269652000	860	1377031864320	432	2448278300160	781	4752162586080	565
626112396000	479	1386998613000	803	2468667064500	521	4824711643136	344
633926092800	704	1413817996500	509	2471771484000	915	4832764209000	643
652482082560	516	1438233280512	282	2520477679104	621	4903097162600	361
653289436800	860	1447428787200	600	2567400675840	1080	4959751305600	1296
661576406400	479	*1480003190400	529	2608548875520	549	5085231579136	37
666574634880	752	1482760097280	774	2627456832000	980	*5111051997870	366
*677701763200	340	1507838492160	962	2644660418400	979	5148385482240	758
693688413600	697	*1517389419000	529	2677752441000	735	5268640785408	806
703816286208	276	*1542738616320	352	2706066874368	376	5289640356000	946
704575228896	405	*1553357978368	252	2708593305600	752	5290460648928	629
713178090240	517	1556017837920	555	2708845856640	764	5431874152320	766
726673802400	538	1567241676000	872	2709493768800	1003	5469709639680	608
726972637440	527	1571198926080	536	*2827553208480	686	5681022328800	701

n	H	n	H	n	H	n	H
5745853670400	524	10410668674560	1026	17505483899904	824	27184083544800	1041
5808057260544	636	10434320851500	543	17550753948000	926	27188110404000	1224
5853911263200	985	*10461217539500	305	17566056012960	1066	27214447163904	1272
5914045683000	457	10670692032000	995	17592306732000	1188	27258821990400	1376
*5914410203520	936	10680522652800	1628	18218458487040	562	27261634143744	617
5956949980800	908	*10711009764000	1050	18297947606400	918	27501146956800	1140
6045468549120	728	10799170314240	616	18449074917000	1224	27628679988000	919
*6073712944992	693	*10881843388416	648	18536508900000	745	27717383688960	566
6175225017000	565	10996995170304	382	18544856803200	926	*28103080287744	496
6200648966400	783	11007262156800	764	18942468120576	658	29040286302720	1060
6312101796000	878	11332220524800	795	19029577862400	801	29193612739200	919
6343192620800	534	*11484718245000	1125	19075764394368	688	29382474401280	1442
6352588408320	554	11535568819200	526	19098061983000	1449	29495815011600	525
6355147895040	542	*11567890545120	1053	19172121516800	538	29646588972000	964
6669629366400	878	11610780300000	745	19621667049600	964	30209639896800	1055
6734495875072	49	11643511017600	1188	20193653718784	468	*30233275380000	1155
6764077878600	305	11725700507136	642	20432681637984	783	30368564724960	1155
6793110213120	788	11810043108864	1242	*20662005324800	506	30676980297600	1336
*6844445080704	684	11937636711000	1188	20663813681280	1457	31094717121000	569
6884622108000	916	11977778891232	765	20746479283200	946	31671732879360	828
7121968308000	643	11999552292000	745	21204827804160	1115	32133029292000	1365
7131668544000	1400	12087279697920	474	*21590959104000	800	32176700980480	551
7191166402560	470	12412499299200	1634	*21733758429600	434	32327865884160	1062
7274578147200	916	*12452007204000	936	21738589593600	665	32713768684800	1377
7318964889600	762	12493968334848	651	21755342568960	1449	*32752714995000	1155
*7322605472000	672	12578345325000	745	21967816416000	1008	33451592638464	664
7338147328512	876	12588244300800	902	22047495446340	245	34044371361000	1476
7512024199680	1106	12602388395520	1035	22051566231552	383	34222225403520	1140
7531474204800	1312	12757657068800	537	22072153958400	766	34854206521344	536
7574491607040	173	12876333500800	646	22332001910400	1702	35085648124800	1337
7626085510400	535	13202304998400	903	*22385029489560	198	35137010809600	986
7741979148288	506	13217359034880	1112	22717860433632	657	35195158303200	1377
7761092320800	1014	13230227556000	662	22735712876800	957	35560552416480	1079
7766789891840	420	13327831686400	935	22742476922880	632	35727233502464	483
7780605009408	639	13552871623200	1038	23300369675520	630	36457089596928	1278
7867987832250	783	*13661860101120	1056	23375124208800	1018	36501751345920	563
*8449576317000	936	13914857829600	1313	23409541693440	816	36567846174720	478
8467093071360	1022	14115958857000	1428	23814974355480	802	36690736642560	1460
8468207666688	514	14379426038250	795	23819044650240	557	36783914076000	1242
8633641161600	1316	14474134929408	516	*23885971200000	960	37342487131488	795
8729162297856	1224	14635113292800	1210	23906526134400	911	37643864076000	929
8756458300800	662	*14747907505800	434	23929031075040	569	37695962304000	1850
8867577438720	785	14814719631360	446	24133566352896	1269	38287967477760	1064
8924263096320	620	14873771827200	650	24146583347250	801	38583480499200	1276
8977654413000	662	15007087898880	989	24345523036800	1242	38629000502400	969
9027208888320	472	15147350507520	792	*24613169545216	285	38781262840320	1476
9068974548480	789	15246642902400	1328	24722083685376	854	38903025047040	1065
9231944494500	767	15337823806032	403	24960513123000	1242	39275901181440	1474
9269718441984	644	*15462510336000	960	25075635936512	339	39377859655680	764
9314808814080	1020	15820566085632	517	25206921653760	1462	40220975692800	917
9564679210240	671	15889967976960	1044	25278832051200	1254	40369640927616	1144
9689839810560	752	16080035811840	1428	25483518950400	913	40815295466400	1042
*9831938337200	350	16212258972000	942	26025228028800	1142	40963871894400	1338
10112079035520	1426	16524280700928	656	26183184168960	762	41051610243072	713
10132001510400	1050	*16924847940000	1125	26407085632256	476	41975434828800	1160
10256659997220	421	16965637957800	527	26757162432000	998	42848120544768	1173
*10297226649600	630	17086937762048	462	26772789288960	806	*43180427911400	403
10341947847528	373	17130547324800	942	26818992224640	1464	43588078934400	1661

n	H	n	H	n	H	n	H
43645811489280	2040	54942374462976	1284	68029152998400	1175	81178611180800	1005
43865704602720	1331	55062424216800	1853	68919093243000	1494	*84761657875440	651
43905339878400	1573	56087667603968	715	69775118828800	979	84762932436000	1272
*43947421401888	216	56100553084800	1272	70879832150400	1752	85372015528800	1859
44008577613000	1484	56221571976570	671	70902132973056	647	85398298444800	1180
44095620366336	524	56261841199200	1059	71485642681600	1003	85434688810240	770
44345330883840	1037	56463835428000	2040	71681373036000	1769	85454920812800	981
44531496801792	886	56483529822720	794	71969788628304	427	85626151750656	722
44621315481600	900	57209988326400	1290	72370674647040	860	86100192346112	367
45108496097280	776	57516364550400	807	72874680721920	1512	86887495094400	2408
45124517299680	1073	57517704153000	1272	72982369892250	807	87548446375936	91
45406018134400	670	58628502535680	1070	72982688808960	848	88527848521728	526
45634960425600	1739	58637313657000	1073	73288695889920	479	88847505747000	1498
*46013471418096	558	58954886991000	1491	74454619599360	2088	88954751508480	824
*46353444300800	760	58960053398400	986	75153571113600	1278	89163516169728	887
46796172040800	1019	*58991630023200	620	75167128764000	1269	89229599877600	1943
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47824897105920	682	59888894456160	1275	76392247932000	2070	*90134334505600	952
*47911115564928	1008	60580961156352	702	77052018771000	1278	90606219580800	1202
48765763791360	1068	60868244209500	778	77212128389760	1368	91501705658880	626
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50560395177600	2070	61986015974400	759	78497425843200	1298	96320660436000	2088
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51633280258560	1448	63687677113088	492	78958268284896	1287	97769262366720	852
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*53092467020880	651	64720497623040	846	79221256896768	714	98079457512960	1074
54409216942080	1080	65058512238720	1716	*79708161843200	660		
54557264361600	2376	67157796625920	1491	80508613785600	1397		
*54934276752360	252	67306216513536	668	80548660192000	1232		

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