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Computing Certain Fano 3-Folds

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We use the computer algebra system Magma to study graded rings of Fano 3-folds of index ≥ 3 in terms of their Hilbert series.

Key words: Fano 3-fold, Fano index, graded ring

1. Introduction

Fano 3-folds are, typically, the complex (projective) solution spaces of homogeneous polynomial equations of low degree in 5 variables. A quartic hypersurface is a classical example, for instance

$$X_4 = (x_0^4 + \cdots + x_4^4 = 0) \subset \mathbb{P}^4.$$

In this example, the canonical class K_{X_4} is represented simply by a hyperplane section $A = (x_0 = 0) \subset X_4$, and so X_4 has index (as defined below) equal to 1. The cubic hypersurface $X_3 = (x_0^3 + \cdots + x_4^3 = 0) \subset \mathbb{P}^4$ is also a Fano 3-fold, with $K_{X_3} = 2A$ and so index 2. Of course, there are more complicated examples involving more variables, including weighted variables; see [4] or [1] for an introduction to weighted projective space in this context.

By Suzuki [10], the Fano index is bounded $f \leq 19$ (and it does not take the values $12, 14, 15, 16, 18$). We study Fano 3-folds of index ≥ 3, especially the case $f = 3$ generalising the conic hypersurface $X_2 = (x_0^2 + \cdots + x_4^2 = 0) \subset \mathbb{P}^4$; see, for example, the lists of Iskovskikh and Prokhorov, [5], Table 12.2.

Furthermore, in notation explained in the following section, we list the number of possible numerical types (more precisely, of possible Hilbert series) of Fano 3-folds of each index $f = 3, \ldots, 19$. (The case of index ≥ 9 is already proved in [10].) We work over the complex number field $\mathbb{C}$ throughout.

Theorem 1. *For each $f = 3, \ldots, 19$, the number of power series that could be the Hilbert series of some X, A with X a Fano 3-fold of Fano index f and A a primitive ample divisor is:*

f	3	4	5	6	7	8	9	10	11	13	17	19
number of series	231	124	63	11	23	10	2	1	3	2	1	1
of which unstable	50	42	29	5	11	6	0	1	0	0	0	0

The second line of this table indicates the number of these series that cannot be realised by the Hilbert series of some Bogomolov–Kawamata stable Fano 3-fold X, A. (See Section 3*, Step* 1 (c)$^+$*, for a discussion of stability. There are no Fano 3-folds of indices $f = 12, 14, 15, 16, 18$.)*

Analogous methods for Fano 3-folds of index ≤ 2 work slightly differently: in those cases there is another discrete invariant, the genus, which does not play a role when $f \geq 3$. This is why we stop here at $f = 3$. The following theorem is a result of our classification; the proof is Step 2^+ of Section 3.

Theorem 2. *$H^0(X, \mathcal{O}(-K_X)) \neq 0$ for any Fano 3-fold X, A of index $f \geq 3$.*

A first analysis of the possible realisations of these Hilbert series in low codimension is in section 4 below. As with all the results in this paper, we used computer algebra—in our case, the Magma system [7]—in an essential way. But this analysis, and the list in codimension 4 especially, should be regarded only as a list of possible examples and not a proved classification. Tabulating these examples by codimension gives the following (in which a blank entry is a zero); all of these are stable.

f	3	4	5	6	7	8	9	10	11	13	17	19	total
codim 0	0	1	1	0	1	0	0	0	1	1	1	1	7
codim 1	7	2	5	1	4	3	2	0	2	1			27
codim 2	6	7	1	0	0								14
codim 3	0	0	2	0	0								2
codim 4	3	2	1	1	1								8

Text files with the Magma code to make the classification of Theorem 1 and with all the proposed models is at the webpage [3].

2. Definitions and tools

Fano 3-folds

A *Fano 3-fold* is a normal projective 3-fold X such that (a) $-K_X$ is ample, (b) $\rho(X) := \operatorname{rank} \operatorname{Pic}(X) = 1$, and (c) X has $\mathbb{Q}$-factorial terminal singularities. The *Fano index* $f = f(X)$ of a Fano 3-fold X is

$$f(X) = \max\{m \in \mathbb{Z}_{>0} \mid -K_X = mA \text{ for some Weil divisor } A\}$$

where equality of divisors denotes linear equivalence of some multiple. A Weil divisor A for which $-K_X = fA$ is called a *primitive ample divisor.*

Graded rings and Hilbert series

A Fano 3-fold X with primitive ample divisor A, which we denote by X, A from now on, has a graded ring

$$R(X,A) = \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nA)).$$

This graded ring is finitely generated, and $X \cong \operatorname{Proj} R(X,A)$. The *Hilbert series* $P_{X,A}(t)$ of X, A is defined to be that of the graded ring $R(X,A)$: thus $\dim H^0(X, \mathcal{O}_X(nA))$ is the coefficient of t^n in $P_{X,A}(t)$.

A choice of homogeneous generators for $R(X,A)$ determines a map

$$X \hookrightarrow \mathbb{P}^N = \mathbb{P}(a_0, \ldots, a_N)$$

into some weighted projective space (wps) $\mathbb{P}^N$, where $x_i \in H^0(X, \mathcal{O}_X(a_i A))$. With this embedding for a minimal set of generators in mind, we say that X, A has *codimension* $N - 3$.

Basket of singularities

We compute the Hilbert series of a Fano 3-fold X, A using the Riemann–Roch formula of Theorem 3 below. The singularities of X make a contribution to this formula in the following well-known sense (see Reid [8] (10.2)): for each singularity of X, there is a finite collection of quotient singularities whose invariants appear in the Riemann–Roch formula; the singularity itself does not appear explicitly in this formula. Thus the contribution of the singularities of X to its Hilbert series is computed using only a finite collection of quotient singularities. This collection is known as the *basket of singularities* of X.

We describe the quotient singularities that can arise in this context. Let the group $\mathbb{Z}/r$ of rth roots of unity act on $\mathbb{C}^3$ via the diagonal representation $\varepsilon \cdot (x,y,z) \mapsto (\varepsilon^a x, \varepsilon^b y, \varepsilon^c y)$. The (germ at the origin of the) quotient singularity $\mathbb{C}^3/(\mathbb{Z}/r)$ is denoted $\frac{1}{r}(a,b,c)$. By Suzuki [10] Lemma 1.2, when we work with Fano 3-folds of index f below, we may assume that $b = -a$, $c = f$ and that r is coprime to a, b, c. We abbreviate the notation $\frac{1}{r}(a, -a, f)$ to $[r, a]$; the index f is always clear from the context. Thus a *basket of singularities* is a collection (possibly with repeats) of singularity germs $[r, a]$.

There is an interesting question about baskets that we do not answer. If X admitted a deformation to a Fano 3-fold with (suitably polarised) quotient singularities, then these singularities would be its basket. We do not know such a deformation result in general, and so it is possible that there exists a Hilbert series in our final classification that is realised by some Fano 3-fold but not by a Fano 3-fold having only quotient singularities. Because of this possibility, it is important to emphasise again that, although our subsequent calculations use only quotient singularities, we do list the Hilbert series of every Fano 3-fold, not only of those having only quotient singularities.

The Riemann–Roch theorem

Suzuki proves the appropriate version of Riemann–Roch in this context, following Reid's plurigenus formula [8], to compute the dimensions of the graded pieces of $R(X, A)$.

For a singularity $p = \frac{1}{r}(a, -a, f)$, we define $i_p(n) = -n/f \bmod r$. By 'mod r', we always mean least residue modulo r, so that $0 \le i_p(n) < r$. Similarly, when r is clear from the context, the notation $\overline{c}$ denotes the least residue of c modulo r.

Theorem 3 ([10] Theorem 1.4). *Let X, A be a Fano 3-fold of index $f \ge 3$ and with basket $\mathcal{B}$. Then $p_n := \dim H^0(X, \mathcal{O}_X(nA))$ for any $n > -f$ is computed by*

$$p_n = 1 + \frac{n(n+f)(2n+f)}{12}A^3 + \frac{nAc_2(X)}{12} + \sum_{p=[r,a]\in\mathcal{B}} c_p(n), \tag{1}$$

where $c_p(k) = -i_p(k)\frac{r^2-1}{12r} + \sum_{j=1}^{i_p(k)-1} \frac{\overline{bj}(r-\overline{bj})}{2r}$ and $ab \equiv 1 \bmod r$.

Summing these as a Hilbert series gives

$$\begin{aligned} P_{X,A}(t) &= \frac{1}{1-t} + \frac{(f^2+3f+2)t + (8-2f^2)t^2 + (f^2-3f+2)t^3}{12(1-t)^4}A^3 \\ &\quad + \frac{t}{(1-t)^2}\frac{Ac_2(X)}{12} + \sum_{p\in\mathcal{B}} \frac{1}{1-t^r}\sum_{k=1}^{r-1} c_p(k)t^k. \end{aligned} \tag{2}$$

Kawamata computes $Ac_2(X) = (1/f)(-K_X c_2(X))$ in terms of $\mathcal{B}$:

Theorem 4 ([6]). *Let X, A be a Fano 3-fold with basket $\mathcal{B}$. Then*

$$-K_X c_2(X) = 24 - \sum_{[r,a]\in\mathcal{B}} \left(r - \frac{1}{r}\right).$$

3. The algorithm for $3 \le f \le 19$

We explain our algorithm for arbitrary $3 \le f \le 19$, and we give explicit results only in the case $f = 3$.

Step 1. Assembling possible baskets: A basket $\mathcal{B}$ comprising germs $[r, a]$ of a Fano 3-fold must satisfy several conditions.

Step 1 (a). Positive $Ac_2(X)$: Finiteness of the number is assured by Kawamata's condition ([6], Theorem 2):

$$-K_X c_2(X) > 0 \quad \text{or equivalently} \quad \sum_{[r,a]\in\mathcal{B}} \left(r - \frac{1}{r}\right) < 24.$$

Result. 2813 baskets satisfy Kawamata's condition. (Recall that results listed during this description refer to the calculation in the case $f = 3$.)

Step 1 (b). Positive degree: The degree A^3 of X, A can be computed from its basket $\mathcal{B}$ by setting $n = -1$ in equation (1) since $H^0(X, \mathcal{O}(-A)) = 0$. This degree must be strictly positive.

Result. 1295 of these baskets have $A^3 > 0$.

Step 1 (b)$^+$. Excess vanishing: This condition can be strengthened since furthermore $H^0(X, \mathcal{O}(nA)) = 0$ for each $n = -2, -3, \ldots, -f+1$. Enforcing this in equation (1) has a significant effect once $f \geq 5$.

Step 1 (c). Bogomolov–Kawamata bound: By Suzuki [10] Proposition 2.4 and a consideration of the stability of a tensor bundle in Kawamata [6] Proposition 1,

$$(4f^2 - 3f)A^3 \leq 4fAc_2(X).$$

Result. 231 of these baskets satisfy the Bogomolov–Kawamata bound.

Step 1 (c)$^+$. Imposing stability: This is an optional step, and we do not include it in our full classification. It imposes the stronger condition

$$f^2A^3 \leq 3Ac_2(X).$$

Fano 3-folds (or their baskets) that satisfy this stronger bound are called *Bogomolov–Kawamata stable*, being in the semistable part of Kawamata's analysis [6]. While it is expected that stable Fano 3-folds are the main case of the classification—in fact, we do not know any non-stable examples—this condition is not known to hold for all Fano 3-folds. All the examples we construct here are stable in this sense.

Result. 181 of these baskets are Bogomolov–Kawamata stable.

Step 2. Computing Hilbert series: For each basket in $\mathcal{B}$, compute a rational expression for $P(t)$ according to the formula (2). Since we also use a power series expression for $P(t)$ later, we convert this rational expression into a power series and record that (order 30 is sufficient for our calculations).

Step 2$^+$. Sections of $-K_X$: Theorem 2 follows at once from the list of Hilbert series. We simply confirm that in each case the coefficient of t^f is nonzero. Although we don't know that each of these Hilbert series is realised by a Fano 3-fold, certainly every Fano 3-fold (with $f \geq 3$) has Hilbert series among our list.

Step 3. Estimating the degrees of generators: Suppose $P(t) = 1 + p_1t + p_2t^2 + \cdots$ is the Hilbert series of some graded ring $R = \bigoplus_{d\geq 0} R_d$. The following is a standard method of guessing the degrees of some generators of a minimal generating set of R.

Certainly R must have p_1 generators of degree 1. (Of course, this number may be zero.) These generate at most a $q_2 = \frac{1}{2}p_1(p_1 - 1)$-dimensional subspace of R_2. If $p_2 - q_2 < 0$, then this routine stops. On the other hand, if $p_2 - q_2 \geq 0$, then R

must have at least $p_2 - q_2$ generators in degree 2. And so we continue into higher degree.

The calculation is made straightforward by the following observation. If $n_1, \dots, n_d$ are the numbers of generators so far in degrees 1 up to d, then the number of monomials in degree $d+1$ they determine (and so the maximum dimension space they could span in that degree) is the coefficient of t^{d+1} in the expansion

$$\frac{1}{(1-t)^{n_1}(1-t^2)^{n_2}\cdots(1-t^d)^{n_d}} = 1 + n_1 t + \left(\frac{1}{2}(n_1+1)n_1 + n_2\right)t^2 + \cdots.$$

Such type changing (from rational functions to power series) is included in most computer algebra systems, so this algorithm is easy to implement.

There are three important remarks. First, the graded ring $R(X, A)$ is certainly finitely generated, so this algorithm will stop. Usually it stops with some negative coefficient, but there is one special case. It can happen that after some point the expected number of generators $p_i - q_i$ in each degree i is zero. The power series expansion method fails here. But this only happens when the Hilbert series is that of a weighted projective space, and we can test this at the outset using the rational expression for $P(t)$.

Second, the assumption of generality (that the generators span a large space) can fail, and this will change the degrees occurring in a minimal generating set (although in small examples it will not reduce the number of generators). This is the main reason why our analysis is not a complete proof, although it is compelling.

Third, in most cases this algorithm will not determine a complete set of degrees for a minimal generating set. This is the main reason why we restrict our attention to low codimension when proposing models, which we do next.

Step 4. Confirming small cases: When the number of generators is small, we attempt to build a Fano 3-fold realising the given Hilbert series or prove that in fact more generators are needed. The methods we use are variations on a theme, and they can be automated after a few trial runs identify the behaviour that can arise. We describe the method in an example.

The basket $\mathcal{B} = \{[2,1],[3,1],[7,3]\}$ with index $f = 5$ determines the rational function

$$P = \frac{t^8 + t^5 + t^4 + t^3 + 1}{t^{13} - t^{12} - t^{11} + t^9 + t^8 - t^7 - t^6 + t^5 + t^4 - t^2 - t + 1}.$$

Expanded as a power series, this starts

$$1 + t + 2t^2 + 4t^3 + 6t^4 + 9t^5 + 13t^6 + 18t^7 + 24t^8 + \cdots.$$

The generator estimating routine above (called `FindFirstGenerators(P)` in Magma) predicts degrees $1, 2, 3, 3, 4, 5$. But P cannot be of the form

$$\frac{\text{polynomial in } t}{\prod_{d=1,2,3,3,4,5}(1-t^d)}$$

since the denominator of P still has a factor $t^6 + t^5 + t^4 + t^3 + t^2 + t + 1$ that does not occur in this denominator. The solution is clear: include 7 as the degree of a generator. From the Hilbert series point of view, this absorbs the excess factor in the denominator; from the basket point of view, this provides the cyclic group action to generate the contribution of the quotient singularity $[7,3] = \frac{1}{7}(3,8,5)$ in the basket.

The final form of the Hilbert series is thus

$$\frac{-t^{20} + t^{14} + t^{13} + t^{12} + t^{11} - t^9 - t^8 - t^7 - t^6 + 1}{\prod_{d=1,2,3,3,4,5,7}(1 - t^d)}$$

which suggests a variety defined by 5 equations of weights $6, 7, 8, 9, 10$:

$$X_{6,7,8,9,10} \subset \mathbb{P}^6(1,2,3,3,4,5,7).$$

In fact, these equations can be written as the five maximal Pfaffians of a skew 5×5 matrix, as in [1] Remark 1.8 or [9] Section 4, and it can be checked that this X is a Fano 3-fold with singularities equal to the basket.

4. Classification in low codimension

We distinguish between cases in codimension ≤ 3, where we can write down equations of Fano 3-folds and check their properties explicitly, and codimension 4, where calculations are more difficult. Tables of these results are given below, and the webpage [3] contains these and all other Hilbert series as Magma output, as well as the Magma code to generate them.

Examples in codimension at most 3

Only seven weighted projective spaces are themselves are Fano 3-folds. These are: $\mathbb{P}^3$ with $f = 4$; $\mathbb{P}(1,1,1,2)$ with $f = 5$; $\mathbb{P}(1,1,2,3)$ with $f = 7$; $\mathbb{P}(1,2,3,5)$ with $f = 11$; $\mathbb{P}(1,3,4,5)$ with $f = 13$; $\mathbb{P}(2,3,5,7)$ with $f = 17$; $\mathbb{P}(3,4,5,7)$ with $f = 19$.

For hypersurfaces or in codimension 2, listed in Tables 1 and 2, the equations are simply generic polynomials of the indicated degrees. Table 3 lists those in codimension 3; here one must build a 5×5 skew matrix of forms (as in [1] Remark 1.8), and then the equations are its five 4×4 Pfaffians. It is a mystery why there are so few families here for $f \geq 3$; by comparison, in the case $f = 1$ there are 70 families in codimension 3.

Examples in codimension 4 are more subtle

The Hilbert series routines and guesses of additional weights work in exactly the same way in codimension 4 as in lower codimension. But it is not easy to write down an example of a ring with given generator degrees in codimension 4. In other graded ring calculations, such as for K3 surfaces in [2], there is much use of projection and unprojection methods. But (Gorenstein) projection of a Fano of higher index does not result in another Fano. Nevertheless, the projection construction of a K3 surface section $S = (z = 0) \subset X$, where z is a variable in

Table 1. Fano 3-folds in codimension 1

f	Fano 3-fold $X \subset \mathbb{P}^4$	A^3	$Ac_2(X)$	Basket $\mathcal{B}$
3	$X_2 \subset \mathbb{P}^4$	2	8	no singularities
	$X_3 \subset \mathbb{P}(1,1,1,1,2)$	3/2	15/2	$[2,1]$
	$X_4 \subset \mathbb{P}(1,1,1,2,2)$	1	7/12	$2 \times [2,1]$
	$X_6 \subset \mathbb{P}(1,1,2,2,3)$	1/2	13/2	$3 \times [2,1]$
	$X_{12} \subset \mathbb{P}(1,2,3,4,5)$	1/10	49/10	$3 \times [2,1], [5,1]$
	$X_{15} \subset \mathbb{P}(1,2,3,5,7)$	1/14	73/14	$[2,1], [7,2]$
	$X_{21} \subset \mathbb{P}(1,3,5,7,8)$	1/40	151/40	$[5,2], [8,1]$
4	$X_4 \subset \mathbb{P}(1,1,1,2,3)$	2/3	16/3	$[3,1]$
	$X_6 \subset \mathbb{P}(1,1,2,3,3)$	1/3	14/3	$2 \times [3,1]$
5	$X_4 \subset \mathbb{P}(1,1,2,2,3)$	1/3	11/3	$2 \times [2,1], [3,1]$
	$X_6 \subset \mathbb{P}(1,1,2,3,4)$	1/4	15/4	$[2,1], [4,1]$
	$X_6 \subset \mathbb{P}(1,2,2,3,3)$	1/6	17/6	$[2,1]$
	$X_{10} \subset \mathbb{P}(1,2,3,4,5)$	1/12	35/12	$2 \times [2,1], [3,1], [4,1]$
	$X_{15} \subset \mathbb{P}(1,3,4,5,7)$	1/28	75/28	$[4,1], [7,3]$
6	$X_6 \subset \mathbb{P}(1,1,2,3,5)$	1/5	16/5	$[5,2]$
7	$X_6 \subset \mathbb{P}(1,2,2,3,5)$	1/10	21/10	$3 \times [2,1], [5,2]$
	$X_6 \subset \mathbb{P}(1,2,3,3,4)$	1/12	23/12	$[2,1], 2 \times [3,1], [4,1]$
	$X_8 \subset \mathbb{P}(1,2,3,4,5)$	1/15	29/15	$2 \times [2,1], [3,1], [5,1]$
	$X_{14} \subset \mathbb{P}(2,3,4,5,7)$	1/60	71/60	$3 \times [2,1], [3,1], [4,1], [5,2]$
8	$X_6 \subset \mathbb{P}(1,2,3,3,5)$	1/15	26/15	$2 \times [3,1], [5,2]$
	$X_{10} \subset \mathbb{P}(1,2,3,5,7)$	1/21	38/21	$[3,1], [7,2]$
	$X_{12} \subset \mathbb{P}(1,3,4,5,7)$	1/35	54/35	$[5,1], [7,3]$
9	$X_6 \subset \mathbb{P}(1,2,3,4,5)$	1/20	31/20	$[2,1], [4,1], [5,2]$
	$X_{12} \subset \mathbb{P}(2,3,4,5,7)$	1/70	61/70	$3 \times [2,1], [5,2], [7,3]$
11	$X_{12} \subset \mathbb{P}(1,4,5,6,7)$	1/70	69/70	$[2,1], [5,1], [7,1]$
	$X_{10} \subset \mathbb{P}(2,3,4,5,7)$	1/84	59/84	$2 \times [2,1], [3,1], [4,1], [7,2]$
13	$X_{12} \subset \mathbb{P}(3,4,5,6,7)$	1/210	89/210	$[2,1], 2 \times [3,1], [5,1], [7,3]$

degree f, can be a guide. We propose the list of examples in Table 4, although none has been constructed explicitly. As justification, we give an example to illustrate what goes wrong with the possible codimension 4 models that we have rejected—the proposals listed in Table 4 are exactly those candidates that do not suffer from this obstruction.

Let index $f = 4$ and basket $\mathcal{B} = \{[5,2]\}$; these (stable) data determine a Hilbert series $P(t)$. Suppose we can construct a Fano 3-fold X, A having Hilbert series P. Considerations as above suggest the degrees of a minimal set of eight generators for the ring $R(X, A)$ could be $1,1,1,2,2,3,4,5$ so that X is in codimension 4. And indeed there is a family of codimension 4 K3 surfaces in $\mathbb{P}^6(1,1,1,2,2,3,5)$ that could be the K3 sections $(z = 0) \subset X$, where z is the variable on X of degree 4. Now a typical such K3 surface S admits a projection to a K3 surface of codimension 3 in $\mathbb{P}^5(1,1,1,2,2,3)$—this is simply the elimination of the degree 5 variable w from the

Table 2. Fano 3-folds in codimension 2

f	Fano 3-fold $X \subset \mathbb{P}^5$	A^3	$Ac_2(X)$	Basket $\mathcal{B}$
3	$X_{6,6} \subset \mathbb{P}(1,1,2,3,3,5)$	2/5	32/5	$[5,2]$
	$X_{6,6} \subset \mathbb{P}(1,2,2,3,3,4)$	1/4	19/4	$4 \times [2,1], [4,1]$
	$X_{6,9} \subset \mathbb{P}(1,2,3,3,4,5)$	3/20	93/20	$[2,1], [4,1], [5,2]$
	$X_{12,15} \subset \mathbb{P}(1,3,4,5,6,11)$	1/22	85/22	$[2,1], [11,5]$
	$X_{9,12} \subset \mathbb{P}(2,3,3,4,5,7)$	3/70	183/70	$3 \times [2,1], [5,2], [7,3]$
	$X_{12,15} \subset \mathbb{P}(3,3,4,5,7,8)$	1/56	103/56	$[4,1], [7,3], [8,3]$
4	$X_{6,8} \subset \mathbb{P}(1,2,3,3,4,5)$	2/15	52/15	$2 \times [3,1], [5,2]$
	$X_{8,10} \subset \mathbb{P}(1,2,3,4,5,7)$	2/21	76/21	$[3,1], [7,2]$
	$X_{8,12} \subset \mathbb{P}(1,3,4,4,5,7)$	2/35	108/35	$[5,1], [7,3]$
	$X_{10,12} \subset \mathbb{P}(1,3,4,5,6,7)$	1/21	62/21	$2 \times [3,1], [7,1]$
	$X_{10,12} \subset \mathbb{P}(2,3,4,5,5,7)$	1/35	66/35	$2 \times [5,2], [7,2]$
	$X_{12,14} \subset \mathbb{P}(2,3,4,5,7,9)$	1/45	86/45	$[3,1], [5,2], [9,2]$
	$X_{18,20} \subset \mathbb{P}(4,5,6,7,9,11)$	1/231	206/231	$[3,1], [7,2], [11,5]$
5	$X_{10,15} \subset \mathbb{P}(2,3,5,5,7,8)$	1/56	87/56	$[2,1], [7,2], [8,3]$

Table 3. Fano 3-folds in codimension 3

f	Fano 3-fold $X \subset \mathbb{P}^6$	A^3	$Ac_2(X)$	Basket $\mathcal{B}$
5	$X_{6,7,8,9,10} \subset \mathbb{P}(1,2,3,3,4,5,7)$	5/42	109/42	$[2,1], [3,1], [7,3]$
	$X_{12,13,14,15,16} \subset \mathbb{P}(1,4,5,6,7,8,9)$	1/36	71/36	$[2,1], [4,1], [9,1]$

Table 4. Proposals for Fano 3-folds in codimension 4

f	Fano 3-fold $X \subset \mathbb{P}^7$	A^3	$Ac_2(X)$	Basket $\mathcal{B}$
3	$X \subset \mathbb{P}(1,1,1,1,1,3,3,4)$	9/4	27/4	$[4,1]$
	$X \subset \mathbb{P}(1,1,2,2,3,3,4,5)$	3/5	27/5	$[2,1], [2,1], [5,1]$
	$X \subset \mathbb{P}(1,1,2,2,3,3,5,7)$	4/7	40/7	$[7,2]$
4	$X \subset \mathbb{P}(1,1,2,2,3,3,4,5)$	7/15	62/15	$[3,1], [5,2]$
	$X \subset \mathbb{P}(1,1,2,2,3,4,5,7)$	3/7	30/7	$[7,2]$
5	$X \subset \mathbb{P}(1,1,2,3,3,4,5,7)$	2/7	24/7	$[7,3]$
6	$X \subset \mathbb{P}(1,2,3,4,5,5,6,7)$	3/35	72/35	$[5,2], [7,2]$

ideal defining S (using the Gröbner basis with respect to a standard lexicographic monomial order with w big, for instance). The image is in codimension 3, and its equations are the five Pfaffians of a skew 5×5 matrix of forms. Crucially, one calculates that the forms appearing in this matrix each have degree ≤ 3. That is not a problem at the level of the K3 surface, but the equations of the Fano 3-fold must involve the degree 4 variable: the analogous projection of X would have equations that do not involve the degree 4 variable, and this would force a non-terminal singularity onto X itself.

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