Homology, Homotopy and Applications, vol. 15(1), 2013, pp.1–25

INTEGRAL EXCISION FOR K-THEORY

BJØRN IAN DUNDAS AND HARALD ØYEN KITTANG

(*communicated by Daniel Grayson*)

Abstract

If $\mathcal{A}$ is a homotopy cartesian square of ring spectra satisfying connectivity hypotheses, then the cube induced by Goodwillie's integral cyclotomic trace $K(\mathcal{A}) \to TC(\mathcal{A})$ is homotopy cartesian. In other words, the homotopy fiber of the cyclotomic trace satisfies excision.

The method of proof gives as a spin-off new proofs of some old results, as well as some new results, about periodic cyclic homology, and — more relevantly for our current application — the $\mathbf{T}$-Tate spectrum of topological Hochschild homology, where $\mathbf{T}$ is the circle group.

1. Introduction

Algebraic K-theory is an important invariant that can be approached from widely different angles. There are structural theorems cutting calculations into smaller, and hopefully more manageable pieces, and there are approximations by theories that are more open themselves to calculation. The aim of this paper is to explain how these two approaches can be combined in a certain situation.

Algebraic K-theory satisfies the Mayer-Vietoris property for Zariski open imbeddings of schemes [**18**]. For closed imbeddings this generally fails, which is bad, for instance, if you want to analyze a singularity where open covers are of little help.

On the other hand, it is sometimes possible to approximate algebraic K-theory through the cyclotomic trace $trc\colon K \to TC$ to topological cyclic homology. Topological cyclic homology lacks some of the structural properties of algebraic K-theory, but one can hope to calculate TC in a given situation.

This paper proves that the difference between K-theory and topological cyclic homology, that is, the homotopy fiber of the cyclotomic trace hofib_{trc}, has the Mayer-Vietoris property for closed imbeddings. The importance of this is that K-theory is wedged in a fiber sequence

$$\mathrm{hofib}_{trc} \to K \to TC,$$

where the fiber is structurally accessible and the base functor is accessible through

---

The first author was supported by the RCN grant "Topology". He also wishes to thank Stanford University for its hospitality during the final typing of this manuscript.
Received September 20, 2010, revised June 6, 2012; published on March 1, 2013.
2000 Mathematics Subject Classification: 19D55, 13D15, 14A20, 55P43.
Key words and phrases: excision in algebraic K-theory, derived algebraic geometry, ring spectrum, cyclotomic trace.
Article available at `http://intlpress.com/HHA/v15/n1/a1` and `doi:10.4310/HHA.2013.v15.n1.a1`

calculations in stable homotopy theory. More concretely, this means that if you have a closed cover, then algebraic K-theory can be recovered from topological cyclic homology and the hyper homology of algebraic K-theory with respect to the closed cover.

When trying to generalize algebraic geometry to ring spectra, certain obstacles are met. Most successful approaches have focused on connective (also called (-1)-connected, i.e., the negative homotopy groups vanish) ring spectra and have translated the crucial geometric invariants through the path component functor π_0. Also, the translation between rings and schemes requires some care. In particular, a pushout of affine schemes is, in general, not an affine scheme. When one of the maps involved is a closed embedding, things work out [**17**], and that is the context we are concerned with in this paper.

Theorem 1.1. *Let*

$$\mathcal{A} = \left\{ \begin{array}{ccc} A^0 & \longrightarrow & A^1 \\ \big\downarrow & & \big\downarrow{\scriptstyle f^1} \\ A^2 & \xrightarrow[f^2]{} & A^{12} \end{array} \right\}$$

be a homotopy cartesian square of connective ring spectra and 0*-connected maps. Then the resulting cube*

$$trc_{\mathcal{A}} \colon K(\mathcal{A}) \to TC(\mathcal{A})$$

is homotopy cartesian.

Remark 1.2.

1. The topological cyclic homology in question is Goodwillie's integral version. We will recall the necessary details when we need them in Section 3. If we take the profinite completion, then Theorem 1.1 is a special case of [**8**], which itself is an extension of the discrete case established by Geisser and Hesselholt [**10**]. If we replace topological cyclic homology with negative cyclic homology and work rationally, then it is closely related to Cortiñas' result [**4**]. The proof of Theorem 1.1 relies on these results.
2. Theorem 1.1 says that, under the given connectivity hypotheses, the homotopy fiber of the cyclotomic trace satisfies excision: it preserves homotopy cartesian squares. In the commutative case, the provision that the maps are 0-connected assures the connection to geometry: $Spec(\pi_0 f^j)$ are closed imbeddings, and so affine results are geometrically interesting. Note, however, that our ring spectra are not assumed to be commutative.
3. It would be desirable to have a statement where just one of the maps, say f^1, were 0-connected (as was the case in [**4, 8**] and [**10**]). With the present line of proof this is not obtainable, essentially because of a technicality (Ext-completion of infinite sums of torsion modules need not be torsion), which vanishes under certain finiteness conditions. We have refrained from pursuing this issue since it would significantly lengthen the exposition.

1.1. Notation

The category of finite sets and injections is denoted $\mathcal{I}$. If X is a spectrum, then $X^{\frown}$ is its profinite completion and $X_{(0)}$ its rationalization. If $\mathcal{X}$ is a cube of spectra, then $\operatorname{ifib}\mathcal{X}$ is the iterated homotopy fiber. If M is a simplicial abelian group, then HM is the associated Eilenberg-Mac Lane spectrum. The results in this paper are independent of choice of framework for symmetric monoidal smash products, but for concreteness, "spaces" are simplicial sets and the spectra are supposed to be simplicial functors (i.e., simplicially enriched functors from finite pointed spaces to pointed spaces; with the appropriate model structure these functors model spectra, see, e.g., [**1, 9, 14**] or [**15**]). Monoids with respect to the smash product are called ring spectra or **S**-algebras. The accompanying homotopy notions are never used.

If k is a natural number, then we let $\mathbf{k}$ be the set $\{1, \dots, k\}$ and k_+ be the pointed set $\{0, 1, \dots, k\}$ with base point 0, and $C_{k+1} = \mathbf{Z}/(k+1)$ is the cyclic group of order $k+1$.

1.2. Side results

On our way we (re)prove the following results (where HP is periodic cyclic homology); cf. [**5, 11, 12, 19**]:

Proposition 1.3.

1. *If $A \to B$ is a surjection of $\mathbf{Q}$-algebras with nilpotent kernel, then the induced map $HP_n(A) \to HP_n(B)$ is an isomorphism for every n.*
2. *Periodic cyclic homology has the Mayer-Vietoris property, in the sense that for a cartesian square $\mathcal{A}$ of $\mathbf{Q}$-algebras and surjections, there is a long exact sequence*
$$\cdots \to HP_n(A^0) \to HP_n(A^1) \oplus HP_n(A^2) \to HP_n(A^{12}) \to HP_{n-1}(A^0) \to \cdots.$$

The proofs are very hands-on, filtering cyclic modules through filtrations where the subquotients are built out of retracts — up to multiplication by concrete integers — of free cyclic objects (on which periodic homology vanishes). The good thing about this is that the proofs are combinatorial enough to work directly to show vanishing results for $\mathbf{T}$-Tate homology of $THH(-)^{\frown}_{(0)}$, where THH is topological Hochschild homology. For instance,

Proposition 1.4. *If $\mathcal{A}$ is a cartesian square of connective ring spectra and 0-connected maps, then the square $\left(THH(\mathcal{A})^{\frown}_{(0)}\right)^{t\mathbf{T}}$ is cartesian.*

Remark 1.5. The problem of showing the main result 1.1 with a connectivity hypothesis on only one of the maps essentially boils down to the fact that we are not able to prove that $\left(THH(A)^{\frown}_{(0)}\right)^{t\mathbf{T}} \to \left(THH(A_0)^{\frown}_{(0)}\right)^{t\mathbf{T}}$ is an equivalence for a graded ring $A = A_0 \oplus A_1 \oplus \cdots$ without some finiteness hypothesis; cf. also Remark 1.2(2).

1.3. The core of the proof of Theorem 1.1

Consider the arithmetic square

$$\begin{array}{ccc}
\operatorname{ifib}\operatorname{hofib}_{trc}(\mathcal{A}) & \longrightarrow & \operatorname{ifib}\operatorname{hofib}_{trc}(\mathcal{A})_{(0)} \\
\downarrow & & \downarrow \\
\operatorname{ifib}\operatorname{hofib}_{trc}(\mathcal{A})^{\frown} & \longrightarrow & \operatorname{ifib}\operatorname{hofib}_{trc}(\mathcal{A})^{\frown}_{(0)}
\end{array}$$

(by the characterization of rationalization and profinite completion as, e.g., in [**2**] we need not be concerned about whether the (iterated) fiber is taken before or after these processes, and so we will be sloppy about providing parentheses). Theorem 1.1 claims that $\mathrm{ifib}\,\mathrm{hofib}_{trc}(\mathcal{A}) \simeq *$, and so it clearly suffices to show that $\mathrm{ifib}\,\mathrm{hofib}_{trc}(\mathcal{A})_{(0)} \simeq \mathrm{ifib}\,\mathrm{hofib}_{trc}(\mathcal{A})^{\frown} \simeq *$.

The profinite completion part, namely that $\mathrm{ifib}\,\mathrm{hofib}_{trc}(\mathcal{A})^{\frown}$ is contractible, is the main result of [**8**], which relied heavily on the work of Geisser and Hesselholt [**10**] in the discrete ring case, which again used ideas from Cortiñas' rational paper [**4**].

In [**4**], Cortiñas proved that if $\mathcal{A}$ is a cartesian square of discrete rings with f^1 surjective, then the the Goodwillie-Jones lift of the Dennis trace map from the "birelative rational K-groups" to the "birelative negative cyclic homology groups" (see 2.10 below for a discussion of negative cyclic homology) of $\mathcal{A} \otimes \mathbf{Q}$ is an equivalence, or in our terminology, that the trace gives rise to a cartesian cube

$$K(\mathcal{A})_{(0)} \to (H(HH(\mathcal{A} \otimes \mathbf{Q})))^{h\mathbf{T}}$$

(where H denotes the Eilenberg-MacLane construction). In view of the equivalence $THH(A)_{(0)} \simeq H(HH(A) \otimes \mathbf{Q})$ of Lemma 2.20, Cortiñas' result states that the composite

$$K(\mathcal{A})_{(0)} \to TC(\mathcal{A})_{(0)} \to \left(THH(\mathcal{A})_{(0)}\right)^{h\mathbf{T}}$$

is cartesian. Just as we did in [**8**], this extends to the case where $\mathcal{A}$ is a homotopy cartesian square of connective ring spectra with f^1 0-connected (though there are also other and even simpler alternatives since we are only concerned with rational results).

Hence, to conclude the main theorem, all we have to do is to prove that

Lemma 1.6. *Let $\mathcal{A}$ be a homotopy cartesian square of connective ring spectra and 0-connected maps. Then the resulting cube*

$$TC(\mathcal{A})_{(0)} \to \left(THH(\mathcal{A})_{(0)}\right)^{h\mathbf{T}}$$

is homotopy cartesian.

This follows from the results in Section 3.

Acknowledgements

The authors want to thank an anonymous referee for helpful suggestions.

2. Excision and Tate homology

That rational periodic homology is excisive is well known and follows from Cuntz and Quillen's models [**5**]. However, we need a proof that is generalizable to a slightly more involved situation.

In this section we give such a proof. A very similar argument gives a simpler proof of Goodwillie's result that rational periodic homology is insensitive to nilpotent extensions. As a matter of fact, the way we present it, the results are logically intertwined.

2.1. Free cyclic objects

Let Δ^o and Λ^o be the simplicial and cyclic categories, and let $j\colon \Delta^o \to \Lambda^o$ be the inclusion. If X is a simplicial object in a category with finite coproducts, then we let j_*X be the "free cyclic object" on X (i.e., the left Kan extension associated to the inclusion $j\colon \Delta^o \to \Lambda^o$, which exists if the category in question has finite coproducts). Explicitly, the factorization properties of Λ^o (see, e.g., [**13**, 6.1.8]) give that the q-simplices are given by $(j_*X)_q = \coprod_{C_{q+1}} X_q$, the coproduct indexed over the cyclic group $C_{q+1} = \{1, t, t^2, \dots, t^q\}$ with structure maps

$$\begin{aligned} d_r(t^s, a) &= \begin{cases} (t^s, d_{r-s}a) & \text{if } 0 \leqslant s \leqslant r \leqslant q \\ (t^{s-1}, d_{q+1+r-s}a) & \text{if } 0 \leqslant r < s \leqslant q, \end{cases} \\ s_r(t^s, a) &= \begin{cases} (t^s, s_{r-s}a) & \text{if } 0 \leqslant s \leqslant r \leqslant q \\ (t^{s+1}, s_{q+1+r-s}a) & \text{if } 0 \leqslant r < s \leqslant q \end{cases} \\ t(t^s, a) &= (t^{s+1}, a), \end{aligned}$$

where we have written (t^s, a) to signify an "element" $a \in X_q$ in the t^sth summand of $(j_*X)_q$.

If Y is a cyclic object, then the adjoint of the identity is the map $j_*Y \to Y$ given by $(s, y) \mapsto t^s y$.

Example 2.1. A pointed symmetric monoid N is a symmetric monoid in the symmetric monoidal category of pointed sets and smash products. The smash product becomes the coproduct in the category of pointed symmetric monoids. Considering N as a constant simplicial object, the free cyclic object j_*N is the cyclic nerve: $(j_*N)_q = N^{\wedge q+1}$ (this is true in general for symmetric monoids in any symmetric monoidal category).

The following example of a symmetric pointed monoid will be important to us shortly: $Q = \{*, 0, 1\}$ pointed at $*$, with $0+0=0$, $0+1=1$ and $1+1=*$. We see that $j_*Q \cong \bigvee_{k=0}^{\infty} Q(k)$, where $Q(k)$ is the cyclic subset of j_*Q whose q-simplices are either the base point or of the form $n_0 \wedge \dots \wedge n_q$ where the sum of the n's is k (so that we have a bijection $Q(k)_q \cong \left\{(n_0, \dots, n_q) \in \{0,1\}^{\times(q+1)} | \sum n_i = k\right\}_+$).

2.2. Rational retracts of free cyclic objects

We will need a result (Lemma 2.6 below) about variants of Hochschild homology which naturally are rational retracts of free cyclic objects. However, we start with a simpler version since in many situations this is all what is needed and it is easier to encode. In order to highlight certain phenomena we choose an indexation in the simple example which is not the same as the one we fall back on in the general case.

Definition 2.2. A cyclic spectrum or simplicial abelian group Y is said to be an *almost free cyclic object* if there is a simplicial object X and maps $Y \to j^*X \to Y$ such that the composite induces multiplication by some integer $k \neq 0$ on homotopy $\pi_*Y \to \pi_*Y$.

If A is a discrete ring, then the Hochschild homology $HH(A)$ of A is the cyclic abelian group $[q] \mapsto A^{\otimes q+1}$ (with tensor products over the integers unless otherwise noted). If A is a simplicial ring, then $HH(A)$ is the associated cyclic simplicial abelian group. Flatness is always assumed (so really one should take free resolutions, and we are considering what some people call Shukla homology. Since all the applications in this section will be rational and applied to rings that already may have a simplicial direction, we do not bother making the resolutions explicit).

For a ring B and B-bimodule M, let $B \ltimes M$ be the square zero extension of B by M. We have a natural decomposition

$$HH(B \ltimes M) \cong \oplus_{k \geqslant 0} H(k)(B, M)$$

of cyclic abelian groups, where $H(k)(B,M)$ consists of the tensors with exactly k factors of M in each dimension.

If we set $M(*) = 0$, $M(0) = B$, $M(1) = M$, and $M(n) = \bigotimes_{j=0}^{q} M(n_j)$ for $n = n_0 \wedge \cdots \wedge n_q \in (j_*Q)_q$, where $Q = \{*, 0, 1\}$ is the pointed symmetric monoid of example 2.1, then the group of q-simplices of $H(k)(B,M)$ is isomorphic to

$$\bigoplus_{n \in (Q(k))_q} M(n),$$

where $Q(k)$ is the cyclic subcomplex of j_*Q defined in 2.1. We will use the notation a/n to specify an element $a = a_0 \otimes \cdots \otimes a_q$ in the $n = n_0 \wedge \cdots \wedge n_q$ summand.

The summands with $n_0 = 1$ (i.e., the zeroth factor in the tensor product $M(n)$ is $M(1) = M$) assemble to a *simplicial* subcomplex $G(k)(B,M) \subseteq H(k)(B,M)$.

If H is a simplicial abelian group, then the free cyclic abelian group j_*H has q-simplices $\bigoplus_{C_{q+1}} H_q$, and we write an element h in the t^jth summand as (t^j, h).

Lemma 2.3. *There is a cyclic map*

$$H(k)(B,M) \to j_*G(k)(B,M)$$

which is given by sending $a = a_0 \otimes \cdots \otimes a_q$ in the $n = n_0 \wedge \cdots \wedge n_q$'th summand of $H(k)(B,M)_q$ to

$$\sum_{n_j = 1} (t^j, t^{-j}a/t^{-j}n) = \sum_{n_j=1} (t^j, a_j \otimes \cdots \otimes a_{j-1}/n_j \wedge \cdots \wedge n_{j-1}),$$

where the sums are over all j such that $n_j = 1$.

Proof. To check that this is a well-defined cyclic map, let $\phi \in \Delta$, use the definition of the structure maps in the free cyclic object and unique factorization $\phi^* t^j = t^{(\phi,j)} \phi_j^*$ to see that the map commutes with ϕ^*, basically because the index sets of the two resulting sums, $\{i | (\phi^* n)_i = 1\}$ and $\{(\phi, j) | n_j = 1\}$ are equal. □

For future reference we note

Lemma 2.4. *The composite*

$$H(k)(B,M) \to j_*G(k)(B,M) \to j_*H(k)(B,M) \to H(k)(B,M)$$

is multiplication by k, where the first map is defined in Lemma 2.3, the second is induced by the inclusion $G(k)(B,M) \subseteq H(k)(B,M)$ and the third is the adjoint of the inclusion. Hence $H(k)(B,M)$ is an almost free cyclic abelian group.

As an immediate corollary (since rationalization commutes with infinite coproducts) we get

Corollary 2.5. *The fiber of the induced map $HH(B \ltimes M) \to HH(B)$ is isomorphic to $\bigoplus_{k \geqslant 1} H(k)(B, M)$ and so rationally a retract of a free cyclic object.*

However, our applications are more delicate in that they need to navigate rather carefully through functors that are not particularly well behaved with respect to (co)limits, and we will need the formulation in Lemma 2.4 and the slightly more general Lemma 2.6 below.

Let $A = A_0$ be a ring and let $A_1, \dots, A_l$ be A-bimodules. Let $A \ltimes (A_1 \oplus \cdots \oplus A_l)$ be the square zero extension of A. It is convenient to grade this ring, so that A_j is in degree j.

Consider the partitions of $k \geqslant 0$, i.e., sequences $P = (k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r)$ of positive integers such that their sum $k_1 + k_2 + \cdots + k_r$ is k (the empty partition is a partition of 0). The *length* of P is r and its *norm* is $|P| = k_1 k^{k-1} + k_2 k^{k-2} + \cdots + k_r k^{k-r}$. We also write $P = (k_1 + k_2 + \cdots + k_r)$ where notationally convenient.

Partitions of k are ordered according to their norm; if $k = 4$ we get that $(4) > (3+1) > (2+2) > (2+1+1) > (1+1+1+1)$. If l is a natural number, then we say that a partition is *bounded by l* if all the numbers in the partition are less than or equal to l. For instance, the partitions of $k = 4$ bounded by $l = 2$ are the three partitions $(2+2) > (2+1+1) > (1+1+1+1)$.

Let $P = (k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r)$ be a partition of k bounded by l and let q a natural number. Consider the sum $H(P)_q = H(P)(A_0; A_1, \dots, A_l)_q$ of all the tensor products you get by tensoring together (in any order) $A_{k_1}, \dots, A_{k_r}$ and $q + 1 - r$ copies of $A = A_0$ (if $q + 1 - r < 0$ we just get the trivial group). So, $A_0 \otimes A_1 \otimes A_2 \otimes A_0 \otimes A_1$ is a summand in $H(2+1+1)_4$ where l is any number greater than or equal to 2. Explicitly,

$$H(P)_q = \bigoplus_f \bigotimes_{j=0}^{q} A_{f(j)},$$

where f varies over the set $S_q(P)$ of functions $\mathbf{Z}/(q+1) \to \mathbf{Z}/(l+1)$ such that the nonzero values of f correspond to (a permutation of) P; i.e., such that there is a bijection $\sigma \colon \mathbf{r} \to Supp(f)$ with $f(\sigma(j)) = k_j$. Varying q and inserting Hochschild-style face, degeneracy and cyclic operators we get a cyclic abelian group (note that only when at least one of the factors getting multiplied in a face operation is A_0 do we get a nonzero map).

This structure is uniquely characterized by the statement that distributivity gives a natural isomorphism

$$HH(A \ltimes (A_1 \oplus \cdots \oplus A_l)) \cong \bigoplus_{k \geqslant 0} \bigoplus_P H(P)$$

of cyclic abelian groups, where the second summand is over all partitions P of k bounded by l.

Let $G(P)$ be the subsimplicial object of $H(P)$ consisting of the summands corresponding to the $f \in S_q(P)$ with $f(0) \neq 0$, and let $H(P) \to j_* G(P)$ be the cyclic map which sends a in the $f \in S_q(P)$ summand to $\sum_{j \in Supp(f)} (t^{f(j)}, t^{-f(j)} a)$.

We note that in the case $B = A$, $M = A_1$, $r = k$, $l = 1$, we are in the situation of Lemma 2.4. The conclusion holds in the more general context:

Lemma 2.6. *Let A, $A_1, \dots, A_l$ and $P = (k_1 \geqslant \dots \geqslant k_r)$ a partition of $k > 0$. The map $H(P) \to j_* G(P)$ is well defined, and the composite*

$$H(P) \to j_* G(P) \to j_* H(P) \to H(P)$$

is multiplication by the length r of P, and so $H(P) = H(P)(A; A_1, \dots, A_l)$ is an almost free cyclic object.

Eventually this leads to the lemma that decomposes relative Hochschild homology in terms of almost free cyclic objects.

If $A \twoheadrightarrow A/I$ is a surjection of flat (= flat in every degree) simplical rings, let $F^k(A, I) = F^k$ be the cyclic subobject of $HH(A)$ which in degree q is given by

$$F^k_q = \sum_{\sum n_j \geqslant k} \otimes_{j=0}^q I^{n_j}.$$

We get that $F^0 = HH(A)$ and $F^0/F^1 = HH(A/I)$.

Lemma 2.7. *Let $A \twoheadrightarrow A/I$ be a surjection of flat simplical rings. Then, for each $k > 0$, there is a sequence of surjections*

$$F^k/F^{k+1} \twoheadrightarrow X^k(1) \twoheadrightarrow \dots \twoheadrightarrow X^k(p(k)) = 0,$$

where $p(k)$ is the number of partitions of k and such that the kernel of each surjection is an almost free cyclic object.

Proof. There is a natural isomorphism $F^k/F^{k+1}(A, I) \cong F^k/F^{k+1}(gr(A, I))$, where $gr(A, I)$ is the associated graded pair $\left(\bigoplus_{j=0}^\infty I^j/I^{j+1}, \bigoplus_{j=1}^\infty I^j/I^{j+1}\right)$, and so we only need to worry about the graded situation, where $A = \bigoplus_{n=0}^\infty A_n$ and $I = \bigoplus_{n=1}^\infty A_n$. We may assume that for each $n \geqslant 0$ the n'th homogeneous piece A_n is (degreewise) flat ($A_0 = A/I$ is flat by assumption, and if A_n is not, then we may replace it with a free simplicial resolution as an A_0-bimodule). Then $HH(A)$ splits as a sum according to total degree. The piece of total degree 0 is simply $HH(A_0)$. The group of q-simplices in F^k/F^{k+1} is isomorphic to $\bigoplus \bigotimes_{j=0}^q A_{n_j}$, where the sum is over sequences of non-negative integers $n_0, \dots, n_q$ such that $\sum n_j = k$.

Given a partition $P = (k_1 \geqslant k_2 \geqslant \dots \geqslant k_r)$ of k, the group of q-simplices in the cyclic abelian group $H(P)(A_0; A_1, \dots, A_k)$ discussed before Lemma 2.6 is a subgroup of the group of q-simplices in F^k/F^{k+1}, but does not usually form a simplicial subgroup as q varies. Actually, the group of q-simplices in F^k/F^{k+1} is isomorphic to $\bigoplus H(P)(A_0; A_1, \dots, A_k)_q$, where the sum runs over all partitions P of k, but the face maps can take summands belonging to a certain partition to a summand belonging to a smaller partition.

However, if $P_1 > P_2 > \dots > P_{p(k)}$ are all the partitions of k, then we get that

$$H(P_1)(A_0; A_1, \dots, A_k) = H(k)(A_0, A_k)$$

(in the notation of Lemma 2.4) is a cyclic subobject of F^k/F^{k+1}. Let $X^k(1)$ be the quotient of $H(k)(A_0, A_k) \to F^k/F^{k+1}$, and notice that $H(P_2)(A_0; A_1, \dots, A_k)$ is a cyclic subobject. Calling the quotient of this inclusion $X^k(2)$, we notice that $H(P_3)(A_0; A_1, \dots, A_k)$ is a cyclic subobject, and so on, until we reach $X^k(p(k)) = 0$.

By Lemma 2.6, all the kernels in the sequence of surjections

$$F^k/F^{k+1} \twoheadrightarrow X^k(1) \twoheadrightarrow \cdots \twoheadrightarrow X^k(p(k)) = 0$$

are almost free cyclic abelian groups. □

2.3. Homology and free cyclic objects

There is another view on free cyclic objects in a category $\mathcal{C}$ with coproducts, which is useful for some purposes. For convenience, if X is an object in $\mathcal{C}$ and S is a set, then we write $X \otimes S$ for the S-fold coproduct of X with itself.

Recall that if I is a small category, $\mathcal{C}$ a category with coproducts and $M\colon I^o \times I \to \mathcal{C}$, then we can define the (Hochschild) homology $H(I, M)$ as the simplicial object in $\mathcal{C}$ whose n-simplices are given by $\coprod_{i_0,\dots,i_n\in I} M(i_0, i_n) \otimes I(i_1, i_0) \otimes \cdots \otimes I(i_n, i_{n-1})$ with face maps given by composition and the functoriality of M and degeneracies by inserting identity maps. If $M\colon J^o \times J \to \mathcal{C}$, then $f\colon I \to J$ induces an obvious map $f\colon H(I, f^*M) \to H(J,M)$. If M factors as $N \circ pr$ where pr is the projection $I^o \times I \to I$ one most frequently refers to $H(I,M)$ as the (simplicial replacement of the) homotopy colimit of N.

If $\mathcal{C}$ has coequalizers we let $H_0(I, M)$ be the coequalizer of the two face maps from the 1-simplices to the 0-simplices.

If $f\colon I \to J$ and $X\colon I \to \mathcal{C}$ are functors, then we can identify the left Kan extension $(f_*X)(j)$ with the homology $H_0(I, X(-) \otimes J(f(-), j))$, and

$$ho(f_*)X(j) = H(I, X(-) \otimes J(f(-), j))$$

is a "homotopy left Kan extension".

In the particular case where $f = id\colon I = I$, the map

$$ho(id_*)X(i) = H(I, X(-) \otimes I(-, i)) \to X(i)$$

given by composition has a simplicial contraction given by inserting identities, and so we have a homotopy version of the dual Yoneda lemma (which says that $(id)_*X \cong X$).

Recall the inclusion $j\colon \Delta^o \subseteq \Lambda^o$.

Lemma 2.8. *Let M be a simplicial object in a category with finite colimits. Then $ho(j_*)M \to j_*M$ is an objectwise simplicial homotopy equivalence, in the sense that for each $[q] \in \Lambda^o$, the map of simplicial objects (the target is constant) $ho(j_*)M([q]) = H(\Delta^o, M \otimes \Lambda^o(j(-), [q]))) \to (j_*M)_q$ is a simplicial homotopy equivalence.*

Proof. Identifying Δ with its image under $j\colon \Delta \to \Lambda$, composition in Λ defines a bijection $\Delta([q], [n]) \times Aut_\Lambda([q]) \to \Lambda([q], [n])$.

Hence $ho(j_*)M([q]) = H(\Delta^o, X(-) \otimes \Lambda^o(j(-), [q]))$ is naturally isomorphic to

$$\begin{aligned} &H(\Delta^o, X(-) \otimes (\Delta^o(-, [q]) \times Aut_\Lambda([q])^o)) \\ &\quad\cong H(\Delta^o, X(-) \otimes (\Delta^o(-, [q]) \otimes Aut_\Lambda([q])^o)) \\ &\quad\cong H(\Delta^o, X(-) \otimes \Delta^o(-, [q])) \otimes Aut_\Lambda([q])^o, \end{aligned}$$

which by the homotopical dual Yoneda lemma maps contracts to

$$X(q) \otimes Aut_\Lambda([q])^o = (j_*X)([q]).$$

□

As an example, if M is a cyclic abelian group, then $HC(M) = H(\Lambda^o, M)$ and $HH(M) = H(\Delta^o, j^*M) \simeq j^*M$, and $j\colon \Delta \to \Lambda$ induces a map $HH(M) \to HC(M)$. In the free cyclic case one has

Lemma 2.9. *Let M be a simplicial abelian group and j_*M the associated free cyclic object. Then the map $HH(j_*M) \to HC(j_*M)$ is a split surjection in the homotopy category.*

Proof. We will prove that the corresponding statement is always true for the homotopy Kan extension. As we have seen, the homotopy and categorical notions coincide up to homotopy for $j\colon \Delta^o \to \Lambda^o$, so this proves the result.

Consider the general situation $f\colon I \to J$ and $X\colon I \to \mathcal{C}$. We prove that the map

$$H(I, f^*ho(f_*)X) \to H(J, ho(f_*)X)$$

induced by f is a split epimorphism modulo simplicial homotopy.

Consider the inclusion

$$X(i) \to f^*ho(f_*)X(i)_n = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n, f(i_n) \leftarrow f(i)} X(i_n)$$

onto the $i = \cdots = i, f(i) = f(i)$ summand. This gives a natural transformation $X \to f^*ho(f_*)X$. Precomposing the map we want to show is a split epimorphism with $H(I, X) \to H(I, f^*ho(f_*)X)$ gives us a map $F\colon H(I, X) \to H(J, ho(f_*)X)$. The claim will therefore follow once we show that F is simplicially homotopic to a simplicial homotopy equivalence G.

Now, F sends $a = x \otimes (i_0 \leftarrow \cdots \leftarrow i_n)$ to

$$F(a) = ((x \otimes 1) \otimes (i_n = \cdots = i_n)) \otimes (f(i_0) \leftarrow \cdots \leftarrow f(i_n)).$$

Letting k vary from 0 to n, the assignments sending a to $((x \otimes 1) \otimes (i_k = \cdots = i_k \leftarrow \cdots \leftarrow i_n)) \otimes (f(i_0) \leftarrow \cdots \leftarrow f(i_k) = \cdots = f(i_k))$ assemble to a simplicial homotopy between F and G, where $G(a) = ((x \otimes 1) \otimes (i_0 \leftarrow \cdots \leftarrow i_n)) \otimes (f(i_0) = \cdots = f(i_0))$.

The inclusion $X(i) \to H(J, X(i) \otimes J(f(i'), -))_n = \coprod_{j_0 \leftarrow \cdots \leftarrow j_n, j_n \leftarrow f(i')} X(i)$ onto the $f(i') = \cdots = f(i')$, $f(i') = f(i')$ summand gives a natural transformation. The map G is a composite

$$H(I, X) \to H(I, (i', i) \mapsto H(J, X(i) \otimes J(f(i'), -))) \cong H(J, H(I, X \otimes J(f(-), -))),$$

where the first map is induced by the degeneracy $X(i) \to H(J, X(i) \otimes J(f(i'), -))$ (which is a simplicial homotopy equivalence) and the isomorphism is simply reversal of priorities.

The lemma is the special case where $I = \Delta^o$, $J = \Lambda^o$, $X = M$ and $f = j\colon I \to J$. □

2.4. Periodic cyclic homology

In order to fix notation and for reference we recall the construction of (periodic) cyclic homology; see, for instance, [**13**] for more details. Let $M\colon \Lambda^o \to \mathcal{A}b$ be a cyclic abelian group, and define the periodic bicomplex $CP(M)$

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
\cdots & M_3 \arrow[l, "1+t"'] \arrow[d, "-d_2+d_1-d_0"'] & M_3 \arrow[l, "1-t+t^2-t^3"'] \arrow[d, "d_0-d_1+d_2-d_3"'] & M_2 \arrow[l, "1+t"'] \arrow[d, "-d_2+d_1-d_0"'] & \cdots \arrow[l, "1-t+t^2-t^3"'] \\
\cdots & M_2 \arrow[l, "1-t"'] \arrow[d, "d_1-d_0"'] & M_2 \arrow[l, "1+t+t^2"'] \arrow[d, "d_0-d_1+d_2"'] & M_2 \arrow[l, "1-t"'] \arrow[d, "d_1-d_0"'] & \cdots \arrow[l, "1+t+t^2"'] \\
\cdots & M_1 \arrow[l, "1+t"'] \arrow[d, "-d_0"'] & M_1 \arrow[l, "1-t"'] \arrow[d, "d_0-d_1"'] & M_1 \arrow[l, "1+t"'] \arrow[d, "-d_0"'] & \cdots \arrow[l, "1-t"'] \\
\cdots & M_0 \arrow[l, "1-t=0"'] \arrow[r, equal] & M_0 & M_0 \arrow[l, "1-t=0"'] \arrow[r, equal] & \cdots
\end{tikzcd}$$

repeated indefinitely in both horizontal directions, with the middle column (which is the Moore complex of the simplicial abelian group underlying M) in degree 0. The odd columns are acyclic. Notice that the rows are acyclic when M is rational.

The homology groups of the zeroth column are referred to as *Hochschild homology* $HH_*(M)$ and are naturally isomorphic to the homotopy groups $\pi_*(j^*M)$ where j^* is precomposition with $j\colon \Delta \to \Lambda$; see the previous section.

The homology of the total complex consisting of only the non-negative columns is referred to as *cyclic homology*, $HC_*(M)$, and can alternatively be calculated as the homotopy groups of $\operatorname{holim}_{\overrightarrow{\Lambda^o}} M = H(\Lambda^o, M)$.

Definition 2.10. The *periodic homology* $HP_*(M)$ of M is the homology of the total complex $\{n \mapsto \prod_{r+s} CP_{(r,s)=n}(M)\}$. *Negative cyclic homology* $HC^-(M)$ is the homology of the total complex of the sub bicomplex $CC^-(M) \subseteq CP(M)$ concentrated in non-positive degrees.

We get canonical isomorphisms $HC_{*-2}(M) \cong H_*(CP(M)/CC^-(M))$ and long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & HC_{n-1}(M) \arrow[r] \arrow[d, equal] & HC^-_n(M) \arrow[r] \arrow[d] & HP_n(M) \arrow[r] \arrow[d] & HC_{n-2}(M) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & HC_{n-1}(M) \arrow[r, "B"] & HH_n(M) \arrow[r] & HC_n(M) \arrow[r, "S"] & HC_{n-2}(M) \arrow[r] & \cdots .
\end{tikzcd}$$

Lemma 2.11. *Let N be a simplicial abelian group. Then the periodic homology of the associated free cyclic object j_*N vanishes, and so the map*

$$HC_{n-1}(j_*N) \to HC^-_n(j_*N)$$

is an isomorphism for all n.

Proof. The map $HH_n(j_*N) \to HC_n(j_*N)$ is split surjective by Lemma 2.9. Hence the map $S\colon HC_n(j_*N) \to HC_{n-2}(j_*N)$ is zero. Filtering $CP(M)$ by columns, we get the short exact sequence

$$0 \to \lim_{\overleftarrow{S}}{}^1 HC_{n-2k+1}(M) \to HP_n(M) \to \lim_{\overleftarrow{S}} HC_{n-2k}(M) \to 0,$$

and so $HP_*(j_*N) = 0$. □

2.5. Consequences for functors vanishing on almost free cyclic objects

The fact 2.11 that periodic homology vanishes on free cyclic objects and the retracts of Lemma 2.4 lead to a sequence of important results.

Recall the following result by Goodwillie from [**12**, p. 356]. We repeat it here since we need extra information which is obvious from Goodwillie's proof, but not stated as part of his result.

Lemma 2.12. *Suppose $I \subseteq A$ is a (k-1)-connected ideal in a simplicial ring. Then there exists a degreewise free simplicial ring F and a k-reduced (i.e., $J_q = 0$ for $q < k$) ideal $J \subseteq F$ generated in each degree by generators of F, and an equivalence of surjections of simplicial rings*

$$\begin{array}{ccc} F & \longrightarrow & F/J \\ \simeq\downarrow & & \simeq\downarrow \\ A & \longrightarrow & A/I. \end{array}$$

As we will see in Section 2.6 below, the conditions on the functor V in the following proposition are satisfied for the Eilenberg-MacLane spectrum associated with $M \mapsto HP(M \otimes \mathbf{Q})$, and so the statement 1 in Proposition 1.3 about nilpotent extensions follows:

Proposition 2.13. *Let V be a pointed homotopy functor from cyclic simplicial abelian groups to spectra satisfying the homotopy properties*

1. *V preserves finite products and homotopy fibers of 0-connected maps up to weak equivalence.*
2. *If $\cdots \to F^3 \to F^2 \to F^1$ is a sequence of cyclic simplicial abelian groups such that the connectivity of F^n goes to infinity with n, then $\operatorname{holim}_{\overleftarrow{n}} V(F^n) \simeq *$.*
3. *V vanishes on almost free cyclic objects.*

Assume that $A \to B$ is a map of simplicial rings and (at least) one of the following conditions are met:

1. *$A \to B$ is a surjection of flat simplicial rings with nilpotent kernel.*
2. *$A \to B$ is a 1-connected map of simplicial rings.*

Then

$$V\,HH(A) \to V\,HH(B)$$

is an equivalence.

Proof. First, assume that $A \to B$ is a surjection of flat rings with kernel I satisfying $I^n = 0$. Recall the filtration of $HH(A)$ given just before Lemma 2.7. Let $F^k(A, I) = F^k$ be the simplicial subcomplex of $HH(A)$ which in degree q is given by $F^k_q = \sum_{\sum n_j \geqslant k} \otimes_{j=0}^q I^{n_j} i$. From Lemma 2.7 and the conditions on V we get that

$V(F^k/F^{k+1}) \simeq *$ for all $k > 0$, and so $V(F^1) \simeq V(F^2) \simeq \cdots \simeq \operatorname{holim}_{\overleftarrow{k}} V(F^k)$. Hence, in order to prove that $V\,HH(A) \to V\,HH(B)$ is an equivalence, we only need to show that the connectivity of F^k grows to infinity with k, which follows since $F^k(A, I)_q = 0$ for $k \geqslant n(q+1)$.

Now, let $A \to B$ be a 1-connected map. Since V is a homotopy functor one may assume that the map is a surjection of flat simplicial rings and by Lemma 2.12 that the kernel I is 1-reduced (that is, the group of zero simplices is trivial: $I_0 = 0$). Let $A(1) = A$ and $I(1) = I$. We will construct a sequence of ring-ideal pairs

$$\cdots \to (A(n), I(n)) \to \cdots \to (A(2), I(2)) \to (A(1), I(1))$$

such that for each n the following is true:

1. For each $[q] \in \Delta^o$ the ring $A(n)_q$ is free and the ideal $I(n)_q$ is generated as an ideal by generators of $A(n)_q$.
2. The map $A(n+1) \to A(n)$ is an equivalence and $I(n+1) \to I(n)$ factors as $I(n+1) \to I(n)^2 \subseteq I(n)$ with the first map an equivalence.
3. $I(n)$ is n-reduced.

Assuming that for given n the pair $(A(n), I(n))$ is already constructed, we consider $I(n)^2$. Since $I(n)_q$ is generated by generators of $A(n)_q$, both $A(n)/I(n)$ and $A(n)/I(n)^2$ are degreewise flat. Since $I(n)$ is n-reduced, the short exact sequence

$$0 \to \ker\{\text{mult.}\} \to I(n) \otimes I(n) \xrightarrow{\text{mult.}} I(n)^2 \to 0$$

gives that $I(n)^2$ is n-connected, and we let the equivalence $(A(n+1), I(n+1)) \to (A(n), I(n)^2)$ be the result of Lemma 2.12, replacing an n-connected ideal by an $n+1$-reduced ones.

Since Hochschild homology preserves connectivity and $I(n)$ is n-reduced, the homotopy fiber $F(n)$ of

$$HH(A(n)) \to HH(A(n)/I(n))$$

is $(n-1)$-connected. Letting $G(n)$ be the homotopy fiber of

$$HH(A(n)) \to HH(A(n)/I(n)^2),$$

we see that $F(n+1) \to F(n)$ factors as $F(n+1) \xrightarrow{\sim} G(n) \to F(n)$. By the first part of the proposition (regarding nilpotent extensions), the map $V(G(n)) \to V(F(n))$ is an equivalence. Consequently the homotopy fiber $V(F(1))$ of $V\,HH(A) \to V\,HH(A/I)$ is equivalent to $\operatorname{holim}_{\overleftarrow{n}} V\,F(n)$, and as the connectivity of $F(n)$ grows to infinity with n, our assumptions about the functor V implies that $\operatorname{holim}_{\overleftarrow{n}} V\,F(n)$ is contractible. □

Definition 2.14. A *split square* of simplicial rings is a categorically cartesian square of simplicial flat rings, where all maps are split surjective.

If $\mathcal{A}$ is a commutative square of simplicial flat rings and split surjections, then set

$$\begin{aligned} I(0) &= A^{12}, \\ I(1) &= \ker\{A^0 \to \mathcal{A}^2\} \cong \ker\{f^1\} \text{ and} \\ I(2) &= \ker\{A^0 \to \mathcal{A}^1\} \cong \ker\{f^1\}. \end{aligned}$$

That the square is categorically cartesian is then the same as the condition that the intersection $I(1)\cap I(2)$ is trivial, considered as an ideal in A^0.

In this situation, the iterated fiber of $HH(\mathcal{A})$ is, via distributivity, isomorphic to the cyclic abelian group with q-simplices

$$\bigoplus_f \bigotimes_{i=0}^{q} I(f(i)),$$

where the sum is over all functions $f\colon \mathbf{Z}/(q+1)\to\mathbf{Z}/3$ (not necessarily linear) with both $f^{-1}(1)$ and $f^{-1}(2)$ non-empty.

Definition 2.15. Given a function $f\colon \mathbf{Z}/(q+1)\to\mathbf{Z}/3$, let A_f be the set consisting of the j in $\mathbf{Z}/(q+1)$ such that $f(j)=2$ and such that there is an i with $f(i)=1$ and such that all intermediate values of f (in cyclic ordering from i to j) are 0.

Example 2.16. If $f,g\colon \mathbf{Z}/11\to\mathbf{Z}/3$ have values

n	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	2	2	0	1	2	1	1	0	2	0	1
$g(n)$	2	2	0	1	2	1	1	0	2	0	2

then $A_f=\{0,4,8\}$ and $A_g=\{4,8\}$.

Remark 2.17. Perhaps it is appropriate that we spend a few words on the rationale behind our choices. The set A_f should be thought of as “marks on the circle”, where one switches from label 1 to label 2. In the following lemma we use this to “turn the dial back” to display the iterated fiber of $HH(\mathcal{A})$ as a sum of almost free cyclic objects. That this works relies on the fact that the factors marked 1 and 2 will multiply trivially. Just noting where there is a factor 2 (for instance, if we chose $f^{-1}(2)$ instead of our A_f) would not have worked since the resulting splitting would not have been closed under the simplicial operations.

Lemma 2.18. *For a simplicial ring A, let $P(A)=HH(A)_{(0)}$ or $P(A)=HH(A)\hat{}_{(0)}$. Let V be a homotopy functor from cyclic groups to spectra, preserving products and homotopy fibers of 0-connected maps and vanishing on free cyclic objects. Then $VP(\mathcal{A})$ is cartesian, where $\mathcal{A}$ is a cartesian square of simplicial rings and 0-connected maps.*

Proof. Let $\mathcal{A}$ be a split square. Note that, since $I(1)\cdot I(2)\subseteq I(1)\cap I(2)=0$, we have a decomposition of the iterated fiber of $HH(\mathcal{A})$ into a sum $\bigoplus_{k=1}^{\infty}H(k)$ where $H(k)$ is the cyclic abelian group with q-simplices

$$H(k)_q=\bigoplus_{\substack{f\\ |A_f|=k}} \bigotimes_{i=0}^{q} I(f(i)).$$

Analogous to the argument in Lemma 2.4, there is an interesting subsimplicial abelian group $G(k)\subseteq H(k)$ given as the sum over only those f with $|A_f|=k$ and $0\in A_f$ and a map

$$H(k)\to j_*G(k)$$

sending $a\in H(k)_q$ in the fth summand to $\sum_{r\in A_f}(r,t^{-r}a)$. Notice that the composite $H(k)\to j_*G(k)\to H(k)$ is multiplication by k, and so $H(k)$ is almost free cyclic. This proves the lemma in the case where the square $\mathcal{A}$ is split since the connectivity of $H(k)$ goes to infinity with k, and so $\bigoplus_{k>0} H(k) \simeq \prod_{k>0} H(k)$ is a retract of a free cyclic object both under rationalization and under profinite completion followed by rationalization.

We reduce the general case to the split case. For simplicity of notation let

$$\mathcal{A} = \left\{ \begin{array}{ccc} B \times_D C & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \xrightarrow{g} & D \end{array} \right\}$$

with $B \to D$ and $C \to D$ surjective on π_0. We may assume that these maps are fibrations, and so surjections (since a map $B \to D$ of simplicial abelian groups is a fibration if and only if $B \to D \times_{\pi_0 D} \pi_0 B$ is a surjection) and that the square is categorically cartesian.

Consider the (bi)simplicial resolution of D

$$B^D = \{r \mapsto B \times_D \cdots \times_D B\}$$

($r+1$ factors of B in degree r and multiplication componentwise), where d_i projects away from the i'th factor and s_i repeats the i'th factor. That $B^D \to D$ is an equivalence is fairly general, but in this context it can be seen directly by noting that the normalized chain complex of B^D is simply the inclusion of $0 \times_D B$ into B.

By taking pullback, we get a resolution of $\mathcal{A}$ with r-simplices

$$\begin{array}{ccc} B \times_D B^D_r \times_D C & \longrightarrow & B \times_D B^D_r \\ \big\downarrow & & \big\downarrow \\ B^D_r \times_D C & \longrightarrow & B^D_r . \end{array}$$

Note that $B \times_D B^D$ and $B \times_D B^D \times_D C$ have an "extra degeneracy" given by duplicating the first factor: $(b, b_0, \dots, b_r, c) \mapsto (b, b, b_0, \dots, b_r, c)$.

If $i\colon \{1,2\} \to \{0,\dots,s\}$ is an injection and $t \in \{0,\dots,s\}$, then let $I(i,t)$ equal $B \times_D B^D \times_D C$ if $t \notin im(i)$ and $I(i,i(1))$ (resp. $I(i,i(2))$) be the ideal $0 \times_D C$ (resp. $B \times_D 0$) in $B \times_D B^D \times_D C$.

Applying Hochschild homology to the square in each dimension and taking the iterated kernel gives us a simplicial cyclic object which in dimension (r,s) is

$$I_{rs} = \sum_i \bigotimes_{t=0}^{s} I_r(i,t) \subseteq (B \times_D B^D_r \times_D C)^{\otimes s+1}.$$

Note that the extra degeneracy $B \times_D B^D_r \times_D C \to B \times_D B^D_{r+1} \times_D C$ induces a map on all the $I_r(i,t)$'s compatible with the structure map in the Hochschild direction, giving us a simplicial cyclic object $I = \{[r] \mapsto I_r = \{[s] \mapsto I_{rs}\}\}$ and a *simplicial* homotopy equivalence $I \xrightarrow{\sim} I_{-1} = \operatorname{ifib} HH(\mathcal{A})$.

Simplicial homotopy equivalences are preserved when functors are applied degreewise to them, and so we get a simplicial homotopy equivalence

$$\{[r] \mapsto V(I_r)\} \xrightarrow{\sim} V(I_{-1}).$$

But since V preserves products and homotopy fibers of 0-connected maps, $V(I_r)$ is the iterated fiber of $V \circ HH$ applied to the r-simplices of our resolution of $\mathcal{A}$. In dimension r this splits in the vertical direction, so it is enough to show excision in cartesian squares with vertical (or horizontal) splittings.

We may repeat the argument above, starting this time with a square with horizontal splitting we reduce to the case where both the vertical and the horizontal maps split. $\square$

Note that we did not assume that V could be "calculated degreewise" (which is false in the applications we have in mind), but we got around this by considering simplicial homotopy equivalences, where we could apply V degreewise to our resolution without destroying the homotopy type in our special case.

2.6. Proofs of Propositions 1.3 and 1.4

Proof of Proposition 1.3. Notice that Proposition 1.3 follows from Proposition 2.13 and Lemma 2.18 as soon as we establish that if V is the functor which to the cyclic abelian group M assigns (the Eilenberg-MacLane spectrum associated with) $HP(M \otimes \mathbf{Q})$, then V satisfies the conditions of Proposition 2.13 (when the input is rational, an extra rationalization will not matter). Firstly, rationalization and taking the Eilenberg-MacLane spectrum will not cause any trouble, so the question is really only about periodic cyclic homology. As the total of a double complex with M in each column, we see that $M \mapsto HP(M)$ preserves products and homotopy fibers of 0-connected maps (the 0-connectivity is needed at one point since we are translating from simplicial abelian groups to chain complexes). Let $\cdots \to F^3 \to F^2 \to F^1$ be a sequence of cyclic simplicial abelian groups such that the connectivity of F^n goes to infinity with n. Filtering periodic cyclic homology by columns, we may write $\operatorname{holim}_{\overleftarrow{n}} HP(F^n)$ as a double homotopy limit $\operatorname{holim}_{\overleftarrow{n}} \operatorname{holim}_{\overleftarrow{k}} \Sigma^{-2k} HC(F^n) \simeq \operatorname{holim}_{\overleftarrow{k}} \operatorname{holim}_{\overleftarrow{n}} \Sigma^{-2k} HC(F^n)$. Since cyclic homology preserves connectivity, we have that $\operatorname{holim}_{\overleftarrow{n}} \Sigma^{-2k} HC(F^n) \simeq *$ and we get that $\operatorname{holim}_{\overleftarrow{n}} HP(F^n) \simeq *$. Given a rational input, vanishing on almost free cyclic objects is the same as vanishing on free cyclic objects, which is true for periodic cyclic homology by Lemma 2.11. $\square$

Proof of Proposition 1.4. By resolving connective $\mathbf{S}$-algebras by simplicial rings as in [**6**], we see that it is enough to establish 1.4 for $\mathcal{A}$ a cartesian square of simplicial rings, with all maps 0-connected. In Lemma 2.18, let $P(A) = HH(A)\hat{}_{(0)}$. By Lemma 2.20 below, the Eilenberg-MacLane spectrum associated with $P(A)$ is equivalent to $THH(A)\hat{}_{(0)}$. Let $V(M) = (H(M))^{t\mathbf{T}}$ be the $\mathbf{T}$-Tate homology of the Eilenberg-MacLane spectrum, and observe that, by Lemma 2.22 below, V satisfies the conditions of Lemma 2.18, showing that $(THH(\mathcal{A})\hat{}_{(0)})^{t\mathbf{T}}$ is cartesian. $\square$

Definition 2.19. Let X be a spectrum and let $N\colon \mathbf{Z} \to \mathbf{Z}_+$ be a function from the integers to the positive integers. We say that X is *N-annihilated* if for each k the group $\pi_k X$ is annihilated by $N(k)$. A map $X \to Y$ is an *N-equivalence* if its homotopy fiber is N-annihilated and a *torsion equivalence* if it is an M-equivalence for some unspecified $M\colon \mathbf{Z} \to \mathbf{Z}_+$.

Note that there are no finiteness requirements in this definition, just a statement about the torsion.

Lemma 2.20. *Let A be a simplicial ring. Then the linearization map*

$$THH(HA) \to H(HH(A))$$

is a torsion equivalence. Consequently, there are natural equivalences of cyclic spectra

$$THH(A)_{(0)} \xrightarrow{\sim} H\left(HH(A)\right)_{(0)},$$
$$THH(A)\hat{}_{(0)} \xrightarrow{\sim} H\left(HH(A)\right)\hat{}_{(0)}.$$

Proof. Let $\{[s] \mapsto E_s\}$ be a simplicial spectrum such that, for each s, the spectrum E_s is connective and N_s-annihilated. Then the first quadrant spectral sequence $\pi_r E_s \Rightarrow \pi_{r+s} \operatorname{diag}^* E$ shows that $\operatorname{diag}^* E$ is N-annihilated, where $N(t) = \sum_{r+s=t} N_s(r)$. Hence, if a map of simplicial connective spectra is a degreewise torsion equivalence then its diagonal is a torsion equivalence. The topological Hochschild homology of HA is a simplicial spectrum which in dimension q is equivalent to $HA \wedge^L_{\mathbf{S}} \cdots \wedge^L_{\mathbf{S}} HA$ and maps to $HA \wedge^L_{H\mathbf{Z}} \cdots \wedge^L_{H\mathbf{Z}} HA$ which is equivalent to the q-simplices of $H(HH(A))$. Hence, it is enough to show that for simplicial abelian groups M and N the map $HM \wedge^L_{\mathbf{S}} HN \to HM \wedge^L_{H\mathbf{Z}} HN$ is a torsion equivalence. There is an associated map of first quadrant spectral sequences with E^2-sheet

$$\operatorname{Tor}^{\pi_*\mathbf{S}}_*(\pi_* M, \pi_* N) \to \operatorname{Tor}^{\mathbf{Z}}_*(\pi_* M, \pi_* N)$$

converging to $\pi_*(HM \wedge^L_{\mathbf{S}} HN) \to \pi_*(HM \wedge^L_{H\mathbf{Z}} HN)$. Now, the map of E^2-sheets has kernel and cokernel annihilated by integers depending on position since $\mathbf{S} \to H\mathbf{Z}$ is a torsion equivalence. The numbers annihilating the kernel and cokernel do not change as we move to the E^∞-sheets, and moving to $\pi_k(HM \wedge^L_{\mathbf{S}} HN) \to \pi_k(HM \wedge^L_{H\mathbf{Z}} HN)$ the kernel and cokernel are annihilated by the product of the numbers needed for the $E^\infty_{s,k-s}$ as s runs from 0 to k. □

Corollary 2.21. *There is a function $L\colon \mathbf{Z} \to \mathbf{Z}_+$ such that, for any subgroup C of the circle, the map*

$$|THH(HA)|_{hC} \to |H(HH(A))|_{hC}$$

is an L-equivalence.

The point of this corollary is that L does not depend on C.

Proof. Consider the spectral sequence calculating the C-homotopy orbits of the homotopy fiber F of $|THH(HA)| \to |H(HH(A))|$. Lemma 2.20 gives that F is N-annihilated by some N. Hence $E^1_{s,r} = \pi_s F$ and $E^\infty_{r,s}$ are annihilated by $N(s)$ and $\pi_n F_{hC}$ is annihilated by $L(n) = N(0) \cdot N(1) \cdots N(n)$. □

Lemma 2.22. *Let Y be a simplicial spectrum. Then the $\mathbf{T}$-Tate homology of $|j_* Y|$ vanishes.*

Proof. This follows since $|j_* Y| \cong \mathbf{T}_+ \wedge |Y|$, and Tate homology vanishes on free objects. □

Corollary 2.23. *Let X be an almost free cyclic spectrum. Then the natural map $(X^{h\mathbf{T}})_{(0)} \to (X_{(0)})^{h\mathbf{T}}$ is an equivalence.*

Proof. By Lemma 2.22, both the source and the target of $(X^{t\mathbf{T}})_{(0)} \to (X_{(0)})^{t\mathbf{T}}$ are contractible, so the $\mathbf{T}$-norm maps $S^1 \wedge (X_{h\mathbf{T}})_{(0)} \to (X^{h\mathbf{T}})_{(0)}$ and $S^1 \wedge (X_{(0)})_{h\mathbf{T}} \to (X_{(0)})^{h\mathbf{T}}$ are both equivalences. Homotopy orbits commute with rationalization, so we are done. □

3. Relations between TC and homotopy T-fixed points

Topological cyclic homology $TC(A)$ of a connective $\mathbf{S}$-algebra A is most effectively defined integrally, as in [**7**], by a cartesian square

$$\begin{array}{ccc} TC(A) & \longrightarrow & THH(A)^{h\mathbf{T}} \\ \downarrow & & \downarrow \\ \left(\operatorname{holim}_{\overleftarrow{R,F}} THH(A)^{C_n}\right)^{\wedge} & \longrightarrow & \left(\operatorname{holim}_{\overleftarrow{F}} THH(A)^{hC_n}\right)^{\wedge}. \end{array}$$

Here R and F are maps $THH(A)^{C_{mn}} \to THH(A)^{C_n}$ called, respectively, the restriction and Frobenius (the latter is just the inclusion of fixed points), where m and n are positive integers. The homotopy limit in the lower left corner is over the category whose objects are the positive integers and where the morphisms are freely generated by commuting morphisms $R\colon mn \to n$ and $F\colon mn \to m$.

The lower horizontal map in the defining square for TC is a composite

$$\left(\operatorname{holim}_{\overleftarrow{R,F}} THH(A)^{C_n}\right)^{\wedge} \to \left(\operatorname{holim}_{\overleftarrow{F}} THH(A)^{C_n}\right)^{\wedge} \to \left(\operatorname{holim}_{\overleftarrow{F}} THH(A)^{hC_n}\right)^{\wedge},$$

where the first map is projection to the homotopy limit of the subcategory generated by the F's only, and the second map is the map from fixed points to homotopy fixed points. The rightmost vertical map in the defining square for TC is given by the restriction from the homotopy fixed points of all of $\mathbf{T}$ to its finite subgroups.

This definition is equivalent to Goodwillie's original definition in terms of an enriched homotopy limit involving a mix of the restriction, Frobenius and the entire circle action, but it is better suited for our purposes.

Note that profinite completion commutes with homotopy fixed points, whereas rationalization usually does not.

Lemma 3.1 (Goodwillie)**.** *For any connective* $\mathbf{S}$*-algebra* A*, both the squares in*

$$\begin{array}{ccccc} TC(A)_{(0)} & \longrightarrow & \left(THH(A)^{h\mathbf{T}}\right)_{(0)} & \longrightarrow & \left(THH(A)_{(0)}\right)^{h\mathbf{T}} \\ \downarrow & & \downarrow & & \downarrow \\ TC(A)^{\wedge}{}_{(0)} & \longrightarrow & \left(THH(A)^{h\mathbf{T}}\right)^{\wedge}{}_{(0)} & \longrightarrow & \left(THH(A)^{\wedge}{}_{(0)}\right)^{h\mathbf{T}} \end{array}$$

are homotopy cartesian.

Proof. The right vertical map $THH(A)^{h\mathbf{T}} \to (\operatorname{holim}_{\overleftarrow{F}} THH(A)^{hC_n})^{\wedge}$ in the defining square for TC is an equivalence after profinite completion (essentially because

$\lim_{\overrightarrow{n}} BC_n \to B\mathbf{T}$ is a profinite equivalence), and so the square

$$\begin{array}{ccc} TC(A) & \longrightarrow & THH(A)^{h\mathbf{T}} \\ \downarrow & & \downarrow \\ TC(A)^{\widehat{}} & \longrightarrow & \left(THH(A)^{h\mathbf{T}}\right)^{\widehat{}} \end{array}$$

is homotopy cartesian even before rationalization. Both the left and outer square in

$$\begin{array}{ccccc} THH(A)^{h\mathbf{T}} & \longrightarrow & \left(THH(A)^{h\mathbf{T}}\right)_{(0)} & \longrightarrow & \left(THH(A)_{(0)}\right)^{h\mathbf{T}} \\ \downarrow & & \downarrow & & \downarrow \\ \left(THH(A)^{h\mathbf{T}}\right)^{\widehat{}} & \longrightarrow & \left(THH(A)^{h\mathbf{T}}\right)^{\widehat{}}{}_{(0)} & \longrightarrow & \left(THH(A)^{\widehat{}}{}_{(0)}\right)^{h\mathbf{T}} \end{array}$$

are homotopy cartesian (they both come from arithmetic squares), and so the right square is homotopy cartesian. □

A technical issue we are faced with in proving Theorem 1.1 is commuting homotopy limits and rationalization. Apart from connectivity arguments we need to be able to commute homotopy $\mathbf{T}$-fixed points and rationalization in the almost free cyclic case.

Lemma 3.2. *Given an almost free cyclic spectrum X, the map*

$$(\operatorname{holim}_{\overleftarrow{F}} X^{hC_n})^{\widehat{}}{}_{(0)} \to (X^{\widehat{}}{}_{(0)})^{h\mathbf{T}}$$

is an equivalence.

Proof. Not using anything about free cyclic spectra, we have that both the maps $(\operatorname{holim}_{\overleftarrow{F}} X^{hC_n})^{\widehat{}}{}_{(0)} \to \left(X^{h\mathbf{T}}\right)^{\widehat{}}{}_{(0)} \to \left((X^{\widehat{}})^{h\mathbf{T}}\right)_{(0)}$ are weak equivalences. Since the Tate spectrum vanishes for free cyclic spectra, we have that both the horizontal $\mathbf{T}$-transfers in

$$\begin{array}{ccc} \Sigma\left((X^{\widehat{}})_{h\mathbf{T}}\right)_{(0)} & \longrightarrow & \left((X^{\widehat{}})^{h\mathbf{T}}\right)_{(0)} \\ \downarrow & & \downarrow \\ \Sigma(X^{\widehat{}}{}_{(0)})_{h\mathbf{T}} & \longrightarrow & (X^{\widehat{}}{}_{(0)})^{h\mathbf{T}} \end{array}$$

are equivalences, and the lemma follows since the left vertical map is an equivalence since homotopy orbits commute with rationalization. □

Let us recall some more or less standard notation. The category of finite sets of the form $\mathbf{n} = \{1, \dots, n\}$ and injections is denoted $\mathcal{I}$. We write $S^{\mathbf{n}}$ for S^1 smashed with itself n times (so that $S^{\mathbf{0}} = S^0$). Our $\mathbf{S}$-algebras A are either Γ-spaces or connective symmetric spectra, according to taste, but ultimately they give rise to simplicial functors, and it is as such they are input to the machinery, and so we write $A(S^{\mathbf{n}})$ for the n-th level. In particular, when A is the Eilenberg-MacLane spectrum of a simplicial ring R, $A(S^{\mathbf{n}}) = U(R \otimes \tilde{\mathbf{Z}}[S^{\mathbf{n}}])$, where $(\tilde{\mathbf{Z}}, U)$ is the free/forgetful pair between abelian groups and pointed sets.

In this notation, the q-simplices of Bökstedt's $THH(A)$ is the homotopy colimit over $(\mathbf{x}_0, \dots, \mathbf{x}_q) \in \mathcal{I}^{q+1}$ of $Map_*(\bigwedge_{i=0}^q S^{\mathbf{x}_i}, \bigwedge_{i=0}^q A(S^{\mathbf{x}_i}))$ with Hochschild-style cyclic operators.

Let $\mathcal{A}$ be a square arising as the Eilenberg-MacLane spectra of a split square of simplicial rings (see Definition 2.14), and let as before $I(0) = A^{12}$, $I(1) = \ker\{A^0 \to A^2\} \cong \ker\{f^1\}$ and $I(2) = \ker\{A^0 \to A^1\} \cong \ker\{f^2\}$. For $\mathbf{x} = (x_0, \dots, x_q) \in \mathcal{I}^{q+1}$, let

$$V^{(k)}(\mathcal{A})(\mathbf{x}) = \bigvee_f \bigwedge_{i=0}^{q} I(f(i))(S^{\mathbf{x}_i}),$$

where the wedge runs over the $f\colon \mathbf{Z}/(q+1) \to \mathbf{Z}/3$ such that $|A_f| = k$, where A_f is the set given by Definition 2.15. Observe that if $\mathbf{x} \in \mathcal{I}^{q+1}$ and $\mathbf{x}^n = (\mathbf{x}, \dots, \mathbf{x}) \in \mathcal{I}^{n(q+1)}$ is the diagonal, then

$$V^{(k)}(\mathcal{A})(\mathbf{x}^n)^{C_n} \cong \begin{cases} V^{(k/n)}(\mathcal{A})(\mathbf{x}) & \text{if } k = 0 \mod n \\ * & \text{otherwise.} \end{cases}$$

In analogy with the cyclic objects $H(k)$ defined in the proof of Lemma 2.18, let $T(k)$ be the cyclic object which in degree q is the homotopy colimit over $\mathbf{x} \in \mathcal{I}^{q+1}$ of $Map_*(\bigwedge_{i=0}^{q} S^{\mathbf{x}_i}, V^{(k)}(\mathcal{A})(\mathbf{x}))$. We get equivalences of cyclic objects

$$\bigvee_{k>0} T(k) \xrightarrow{\sim} \operatorname{ifib} THH(\mathcal{A}) \xrightarrow{\sim} \prod_{k>0} T(k),$$

where the infinite wedge and product are weakly equivalent as the connectivity of $T(k)$ goes to infinity with k (since $V^{(k)}(\mathcal{A})(\mathbf{x})$ is trivial for $2k > q+1$).

For positive integers n and k, let $T(n,k) = sd_n T(k)^{C_n}$, and extend to rational n and k by setting $T(n,k) = *$ if n or k is not integral.

Restriction induces maps $T(n,k) \to T(n/m, k/m)$ which are interesting only when m divides both n and k.

Lemma 3.3. *The homotopy fiber of the restriction map*

$$T(n,k) \to \operatorname*{holim}_{\overleftarrow{m>1}} T(n/m, k/m) \cong \operatorname*{holim}_{\overleftarrow{1 \neq m \mid \gcd(n,k)}} T(n/m, k/m)$$

is equivalent to $T(k)_{hC_n}$. In particular, if $1 = \gcd(n,k)$ then we have an equivalence $T(k)_{hC_n} \simeq T(n,k)$

Proof. This follows by the standard arguments proving the "fundamental cofibration sequence" for fixed points of topological Hochschild homology, as in [**7**, VI.1.3.8]. For a published account see [**3**, 5.2.5], but remove the intricacies which are present in the commutative situation where non-cyclic group actions are allowed. □

Consider the homotopy limit of the fixed points of $\prod_{k>0} T(k)$ under the restriction and Frobenius maps. By prioritizing the restriction map, we write this as

$$\left(\operatorname*{holim}_{\overleftarrow{R}} \prod_{k>0} T(n,k)\right)^{hF}.$$

The homotopy limit of the restriction maps gives the homotopy limit of the diagram

(extended to infinity in both directions)

$$
\begin{tikzcd}
T(1,1) & T(2,1) & T(3,1) & T(4,1) & T(5,1) & T(6,1) \\
T(1,2) & T(2,2) \arrow[ul] & T(3,2) & T(4,2) \arrow[ull] & T(5,2) & T(6,2) \arrow[ulll] \\
T(1,3) & T(2,3) & T(3,3) \arrow[uull, bend left] & T(4,3) & T(5,3) & T(6,3) \arrow[uullll, bend right] \\
T(1,4) & T(2,4) \arrow[uul] & T(3,4) & T(4,4) \arrow[uull, bend right] & T(5,4) & T(6,4) \arrow[uulll] \\
T(1,5) & T(2,5) & T(3,5) & T(4,5) & T(5,5) \arrow[uuuullll, bend right] & T(6,5) \\
T(1,6) & T(2,6) \arrow[uuul] & T(3,6) \arrow[uuuull, bend left] & T(4,6) \arrow[uuull] & T(5,6) & T(6,6), \arrow[uuulll, bend left] \arrow[uuuullll, bend right]
\end{tikzcd}
$$

which, by reversal of priorities, is the same as $\operatorname{holim}_{\overleftarrow{R}} \prod_{n>0} T(n,k)$:

$$
\begin{aligned}
\operatorname*{holim}_{\overleftarrow{n}} \prod_{k>0} T(n,k) &\cong \operatorname*{holim}_{\overleftarrow{n}} \prod_{t\in\mathbf{Q}^*} T(n,tn) \\
&\cong \operatorname*{holim}_{\overleftarrow{k}} \prod_{t\in\mathbf{Q}^*} T(k/t,k) \cong \operatorname*{holim}_{\overleftarrow{k}} \prod_{n>0} T(n,k).
\end{aligned}
$$

Lemma 3.4. *Let $\mathcal{A}$ be the square of $\mathbf{S}$-algebras associated with a split square. Then the map*

$$
\begin{aligned}
\operatorname{ifib} TC(\mathcal{A})\hat{}_{(0)} &\simeq \left(\left(\operatorname*{holim}_{\overleftarrow{R}} \prod_{k>0} T(n,k)\right)^{hF}\right)\hat{}_{(0)} \\
&\to \left(\operatorname*{holim}_{\overleftarrow{R}} \left(\left(\prod_{n>0} T(n,k)\hat{}\right)_{(0)}\right)\right)^{hF}
\end{aligned}
$$

is an equivalence.

Proof. If in a tower of spectra the connectivity of the maps grows to infinity, then the rationalization of the homotopy limit is equivalent to the homotopy limits of the rationalized tower. Since the connectivity of $\prod_{n>0} T(n,k)$ grows to infinity with k (and the category of natural numbers and factorizations has cofinal directed subcategories), we have the claimed equivalence. $\square$

Lemma 3.5. *The restriction map*

$$\left(\prod_{n>0} T(n,k)\hat{}\right)_{(0)} \to \operatorname*{holim}_{\overleftarrow{1\neq l|k}} \left(\prod_{n>0} T(n,k/l)\hat{}\right)_{(0)}$$

is split surjective up to homotopy.

Proof. We have seen that the homotopy fiber of the restriction map may be identified with $\left(\prod_{n>0} T(k)_{hC_n}\hat{}\right)_{(0)}$, and the lemma follows once we know that the left and lower arrows in the commutative diagram

$$\begin{tikzcd}
\left(\prod\limits_{n>0} T(k)_{hC_n}\right)\hat{}_{(0)} \arrow[r] \arrow[d] & \left(\prod\limits_{n>0} T(n,k)\right)\hat{}_{(0)} \arrow[r] & \left(\prod\limits_{n>0} T(k)^{hC_n}\right)\hat{}_{(0)} \arrow[d] \\
\left(\prod\limits_{n>0} H(H(k))_{hC_n}\right)\hat{}_{(0)} \arrow[rr] & & \left(\prod\limits_{n>0} H(H(k))^{hC_n}\right)\hat{}_{(0)}
\end{tikzcd}$$

are equivalences. Here the vertical maps are induced by the linearization maps $T(k) \to H(H(k))$, where $H(k)$ is the cyclic module introduced in the proof of the split part of Lemma 2.18. Exactly as in Corollary 2.21, there is a function $L\colon \mathbf{Z} \to \mathbf{Z}_+$ (not depending on the cyclic group C) such that $T(k)_{hC} \to H(H(k))_{hC}$ is an L-equivalence. Consequently, the infinite product $\prod_{n>0} T(n,k) \to \prod_{n>0} H(H(k))_{hC_n}$ is also an L-equivalence, which shows that the left map in the displayed diagram is an equivalence. The lower map is an equivalence, since the cofiber is $\left(\prod_{n>0} H(H(k))^{tC_n}\right)\hat{}_{(0)}$, and each Tate homology is k-torsion (since – up to multiplication by k – $H(k)$ is a retract of a free cyclic object). □

Corollary 3.6. *The map*

$$\operatorname*{holim}_{\overleftarrow{R}} \left(\prod_{n>0} T(n,k)\hat{}\right)_{(0)} \to \prod_{k} \left(\prod_{n} H(H(k))^{hC_n}\hat{}\right)_{(0)}$$

is an equivalence. On the right-hand side the action by the Frobenius is represented by the product of the maps $F\colon H((k))^{hC_{nm}} \to H(H(k))^{hC_m}$ associated to $C_m \subseteq C_{nm}$.

Proof. Lemma 3.5 shows that the restriction maps split, and so there is an equivalence between $\left(\prod_{n>0} T(n,k)\hat{}\right)_{(0)}$ and the product of the fibers up to that stage. We saw in the proof of Lemma 3.5 that the map from the fiber $\left(\prod\limits_{n>0} T(k)_{hC_n}\right)\hat{}_{(0)}$ to $\left(\prod\limits_{n>0} H(H(k))^{hC_n}\right)\hat{}_{(0)}$ is a weak equivalence. Hence the map

$$\begin{aligned}
\left(\prod_{n>0} T(n,k)\hat{}\right)_{(0)} &\to \prod_{d|k} \left(\prod_{n} H(H(k/d))^{hC_{n/d}}\hat{}\right)_{(0)} \\
&= \left(\prod_{n} \prod_{d|\gcd(k,n)} H(H(k/d))^{hC_{n/d}}\hat{}\right)_{(0)}
\end{aligned}$$

is a weak equivalence, and the homotopy limit over R just adds successively new factors. □

Corollary 3.7. *All maps in the commuting diagram*

$$\begin{tikzcd}
\operatorname{ifib} TC(\mathcal{A})\hat{}_{(0)} \arrow[r] \arrow[d] & \operatorname{ifib}(THH(\mathcal{A})\hat{}_{(0)})^{h\mathbf{T}} \arrow[d] \\
\prod_k \left(\left(\prod_n H(H(k))^{hC_n}\right)^{hF}\right)\hat{}_{(0)} \arrow[d] & \operatorname{ifib}(H(HH(\mathcal{A}))\hat{}_{(0)})^{h\mathbf{T}} \arrow[d] \\
\prod_k \left(\left(\operatorname{holim}_{\overleftarrow{F}} H(H(k))^{hC_n}\right)\hat{}_{(0)}\right) \arrow[r] & \prod_k \left(H(H(k))\hat{}_{(0)}\right)^{h\mathbf{T}}
\end{tikzcd}$$

are equivalences.

Proof. The upper left vertical map is an equivalence by the definition of TC, Lemma 3.4, Corollary 3.6 and the fact that profinite completion commutes with taking F-homotopy fixed points. The lower left vertical map is simply rewriting the homotopy limit of a directed system as homotopy fixed points of a product. The upper right vertical map is an equivalence by Lemma 2.20. The lower right vertical map is the decomposition of the Hochschild homology of a split square. The horizontal lower map is an equivalence by Lemma 3.2 since $H(k)$ is almost free cyclic. □

Proof of Theorem 1.1. As observed in Section 1.3, Theorem 1.1 follows from Lemma 1.6, which claims that the cube $TC(\mathcal{A})_{(0)} \to \left(THH(\mathcal{A})_{(0)}\right)^{h\mathbf{T}}$ is homotopy cartesian. Lemma 3.1 reduces the problem to showing that the cube

$$TC(\mathcal{A})\hat{}_{(0)} \to \left(THH(\mathcal{A})\hat{}_{(0)}\right)^{h\mathbf{T}}$$

is homotopy cartesian.

Recall from [**6**] that we may resolve connective $\mathbf{S}$-algebras by simplicial rings. More precisely, if A is an $\mathbf{S}$-algebra, then $U\tilde{\mathbf{Z}}A$ is the $\mathbf{S}$-algebra obtained by applying the free/forgetful pair $(\tilde{\mathbf{Z}}, U)$. This gives rise to a cosimplicial resolution $A \to \{[q] \to (U\tilde{\mathbf{Z}})^{q+1}A\}$, and the connectivity of $A \to \operatorname{holim}_{\overleftarrow{q<r}}(U\tilde{\mathbf{Z}})^{q+1}A$ goes to infinity with r.

For our purposes, it is important to note that if $\mathcal{A}$ is a homotopy cartesian square, then its underlying cube of spectra is homotopy cocartesian, and so the cube of "spectrum homologies" $U\tilde{\mathbf{Z}}\mathcal{A}$ is again homotopy cartesian. If the maps in $\mathcal{A}$ are 0-connected, then so are the maps in $U\tilde{\mathbf{Z}}\mathcal{A}$.

Furthermore, $U\tilde{\mathbf{Z}}A$ is naturally equivalent to the Eilenberg-MacLane spectrum $H(R_A)$, where R_A is a simplicial ring, and so if $\mathcal{A}$ is a homotopy cartesian square of $\mathbf{S}$-algebras, then $R_{\mathcal{A}}$ is a homotopy cartesian diagram of simplicial rings.

Now, exactly the same set of arguments used in [**6**] to reduce the profinite Goodwillie conjecture to McCarthy's theorem [**16**] can now be used to see that it is enough to prove Lemma 1.6 in the case where $\mathcal{A}$ is the result of applying the Eilenberg-MacLane functor to a homotopy cartesian square of simplicial rings and 0-connected maps.

By the reduction performed in the proof of Lemma 2.18, it is enough to consider squares $\mathcal{A}$ associated with split squares of simplicial rings, and we assume in the rest of the proof that $\mathcal{A}$ has this form (although all the results used could be generalized to the more general case using the reductions above).

In this special case the cube $TC(\mathcal{A})\hat{}_{(0)} \to \left(THH(\mathcal{A})\hat{}_{(0)}\right)^{h\mathbf{T}}$ is homotopy cartesian by Corollary 3.7. □

References

[1] G. Biedermann, B. Chorny and O. Röndigs, Calculus of functors and model categories, *Adv. Math.* **214** (2007), no. 1, 92–115. MR 2348024 (2008k:55036)

[2] A.K. Bousfield, The localization of spectra with respect to homology, *Topology* **18** (1979), no. 4, 257–281. MR 0551009 (80m:55006)

[3] M. Brun, G. Carlsson and B.I. Dundas, Covering homology, *Adv. Math.* **225** (2010), no. 6, 3166–3213. MR 2729005 (2012c:19003)

[4] G. Cortiñas, The obstruction to excision in K-theory and in cyclic homology, *Invent. Math.* **164** (2006), no. 1, 143–173. MR 2207785 (2006k:19006)

[5] J. Cuntz and D. Quillen, On excision in periodic cyclic cohomology. II. The general case, *C. R. Acad. Sci. Paris Sér. I Math.* **318** (1994), no. 1, 11–12. MR 1260526 (94m:19001)

[6] B.I. Dundas, Relative K-theory and topological cyclic homology, *Acta Math.* **179** (1997), no. 2, 223–242. MR 1607556 (99e:19007)

[7] B.I. Dundas, T.G. Goodwillie and R. McCarthy, *The local structure of algebraic K-theory*, Algebra and Applications, Springer-Verlag, New York, 2012.

[8] B.I. Dundas and H.O. Kittang, Excision for K-theory of connective ring spectra, *Homology, Homotopy Appl.* **10** (2008), no. 1, 29–39. MR 2369021 (2008j:19010)

[9] B.I. Dundas, O. Röndigs and P.A. Østvær, Enriched functors and stable homotopy theory, *Doc. Math.* **8** (2003), 409–488 (electronic). MR 2029170 (2005a:55005)

[10] T. Geisser and L. Hesselholt, Bi-relative algebraic K-theory and topological cyclic homology, *Invent. Math.* **166** (2006), no. 2, 359–395. MR 2249803 (2008a:19003)

[11] T.G. Goodwillie, Cyclic homology, derivations, and the free loopspace, *Topology* **24** (1985), no. 2, 187–215. MR 0793184 (87c:18009)

[12] T.G. Goodwillie, Relative algebraic K-theory and cyclic homology, *Ann. of Math. (2)* **124** (1986), no. 2, 347–402. MR 0855300 (88b:18008)

[13] J.-L. Loday, *Cyclic homology*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **301**, Springer-Verlag, New York, 1998, Appendix E by M.O. Ronco, Chapter 13 by the author in collaboration with T. Pirashvili. MR 1600246 (98h:16014)

[14] M. Lydakis, Simplicial functors and stable homotopy theory, Preprint 98-049, SFB 343, Bielefeld, June 1998.

[15] M.A. Mandell, J.P. May, S. Schwede and B. Shipley, Model categories of diagram spectra, *Proc. London Math. Soc. (3)* **82** (2001), no. 2, 441–512. MR 1806878 (2001k:55025)

[16] R. McCarthy, Relative algebraic K-theory and topological cyclic homology, *Acta Math.* **179** (1997), no. 2, 197–222. MR 1607555 (99e:19006)

[17] K. Schwede, *Gluing schemes and a scheme without closed points*, Recent progress in arithmetic and algebraic geometry, Contemp. Math. **386** Amer. Math. Soc., Providence, RI, 2005, pp. 157–172. MR 2182775 (2006j:14003)

[18] R.W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math. **88**, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435. MR 1106918 (92f:19001)

[19] M. Wodzicki, Excision in cyclic homology and in rational algebraic K-theory, *Ann. of Math. (2)* **129** (1989), no. 3, 591–639. MR 0997314 (91h:19008)

Bjørn Ian Dundas `dundas@math.uib.no`

Department of Mathematics, University of Bergen, Postbox 7803, N-5020 Bergen, Norway

Harald Øyen Kittang `harald.kittang@cappelendamm.no`

Cappelen Damm AS, 0055 Oslo, Norway