*Homology, Homotopy and Applications, vol. 13(1), 2011, pp.349–379*

# RATIONAL VISIBILITY OF A LIE GROUP IN THE MONOID OF SELF-HOMOTOPY EQUIVALENCES OF A HOMOGENEOUS SPACE

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(*communicated by Charles A. Weibel*)

*Abstract*

Let $M$ be a homogeneous space admitting a left translation by a connected Lie group $G$. The adjoint to the action gives rise to a map from $G$ to the monoid of self-homotopy equivalences of $M$. The purpose of this paper is to investigate the injectivity of the homomorphism which is induced by the adjoint map on the rational homotopy group. In particular, the *visibility degrees* are determined explicitly for all the cases of simple Lie groups and their associated homogeneous spaces of rank one which are classified by Oniscik.

## 1. Introduction

The study of rational visibility problems which we consider here is motivated by work due to Kedra and McDuff [**16**] in which symplectic topological methods are effectively used. In this paper, we deal with such problems relying upon algebraic models for spaces and maps which are complements of models developed and used in recent work on rational homotopy of function spaces [**4, 6, 7, 8, 18, 19, 20**].

Let $f\colon X \to Y$ be a map between connected spaces whose fundamental groups are abelian. We say that $X$ is *rationally visible* in $Y$ with respect to the map $f$ if the induced map $f_* \otimes 1\colon \pi_i(X)\otimes\mathbb{Q} \to \pi_i(Y)\otimes\mathbb{Q}$ is injective for any $i \geqslant 1$. Let $\mathrm{aut}_1(M)$ denote the identity component of the monoid of self-homotopy equivalences of a space $M$. Let $G$ be a connected Lie group and $M$ an appropriate homogeneous space $M$ admitting a left translation by $G$. We then define a map of monoids

$$\lambda_{G,M}\colon G \to \mathrm{aut}_1(M)$$

by $\lambda_{G,M}(g)(x) = gx$ for $g \in G$ and $x \in M$. The aim of this paper is to discuss the rational visibility of $G$ in $\mathrm{aut}_1(M)$ with respect to the map $\lambda_{G,M}$.

The monoid map $\lambda_{G,M}$ factors through the identity component $\mathrm{Homeo}_1(M)$ of the monoid of homeomorphisms of $M$ and the identity component $\mathrm{Diff}_1(M)$ of the space of diffeomorphisms of $M$. Therefore, the rational visibility of $G$ in $\mathrm{aut}_1(M)$

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This research was partially supported by a Grant-in-Aid for Scientific Research (C)20540070 from Japan Society for the Promotion of Science.
Received January 28, 2011, revised March 21, 2011; published on June 7, 2011.
2000 Mathematics Subject Classification: 55P62, 57R19, 57R20, 57T35.
Key words and phrases: self-homotopy equivalence, homogeneous space, Sullivan model.
Article available at `http://intlpress.com/HHA/v13/n1/a13` and `doi:10.4310/HHA.2011.v13.n1.a13`

implies that of $G$ in $\mathrm{Homeo}_1(M)$ as well as in $\mathrm{Diff}_1(M)$. We moreover expect that non-trivial characteristic classes of the classifying spaces $B\mathrm{aut}_1(M)$, $B\mathrm{Homeo}_1(M)$ and $B\mathrm{Diff}_1(M)$ can be obtained through the study of rational visibility. Very little is known about the (rational) homotopy of the groups $\mathrm{Homeo}_1(M)$ and $\mathrm{Diff}_1(M)$ for a general manifold $M$; see [**10**] for the calculation of $\pi_i(\mathrm{Diff}_1(S^n)) \otimes \mathbb{Q}$ for $i$ in some range. Then such implication and expectation inspire us to consider the visibility problems of Lie groups. We refer the reader to papers [**12**] and [**30**] for the study of rational homotopy types of $\mathrm{aut}_1(M)$ itself and related function spaces.

The key device for the study of rational visibility is the function space model due to Brown and Szczarba [**5**] and Haefliger [**13**], which is regarded as Lannes' division functor. Especially, an explicit rational model for the map $\lambda_{G,M}$ is constructed by using a model for the evaluation map described in [**7**] and [**17**]; see Theorem 4.3. We mention that a model for the left translation $G \times M \to M$ provided in Section 5 completes the construction.

By analyzing such elaborate models, a recognition principle for rational visibility is obtained in Theorem 4.1 below. We emphasize that not only does our machinery in this paper allow us to give other proofs of results in [**16**], [**25**], [**27**] and [**31**] concerning rational visibility, but it also leads us to an unifying way of looking at the visibility problem explicitly. Some answers to such problems are described in Tables 1 and 2 below. Moreover, we have non-trivial characteristic classes of the classifying spaces $B\mathrm{aut}_1(M)$ and hence of $B\mathrm{Homeo}_1(M)$ and $B\mathrm{Diff}_1(M)$ for an appropriate homogeneous space $M$; see Remark 8.1.

It is important to remark that for considering the rational visibility problem the derivation argument on a Sullivan model $\wedge W$ for $M$ may be useful. Indeed, the rational homotopy group of $\mathrm{aut}_1(M)$ is isomorphic to the homology of the complex $\mathrm{Der}(\wedge W)$ of derivations on $\wedge W$. Then the map $\lambda_{G,M}$ is modeled by a map

$$\lambda_{G,M\sharp} \colon V_G \to H_*(\mathrm{Der}(\wedge W)) \cong \pi_*(\mathrm{aut}_1(M))_{\mathbb{Q}},$$

where $\wedge V_G$ is a minimal model for $G$. In particular, one might obtain other recognition principles for the rational visibility problem along the lines of the derivation arguments developed in [**4, 8, 18**]. We leave such considerations to the reader, focusing here on the rational visibility problem in the Lannes' division functor argument.

## Acknowledgements

I am particularly grateful to Kojin Abe, Kohhei Yamaguchi and Masaki Kameko for valuable comments on this work and to Hiroo Shiga for drawing my attention to this subject. I thank the anonymous referee of a previous version of this paper for showing me a very simple proof of the latter half of Theorem 2.2, which is described in Remark 6.1. I also thank the referee of this version for valuable advice.

## 2. Results

We retain the notations in the introduction. Our results are described more precisely in this section. The first one is a generalization of the result [**25**, Proposition 2.4].

**Theorem 2.1.** *Let $G$ be a simply-connected Lie group and $T$ a torus in $G$ which is not necessarily maximal. Then $G$ is rationally visible in* $\mathrm{aut}_1(G/T)$ *with respect to the map $\lambda_{G,G/T}$ defined by the left translation of $G/T$ by $G$.*

In [**25**, Proposition 2.4], it is assumed that $T$ is a maximal torus of $G$. We mention that the result due to Notbohm and Smith plays an important role in the proof of the uniqueness of fake Lie groups with a maximal torus; see [**24**, Section 1]. Theorem 2.1 is deduced directly from Theorem 2.2 below, which gives a tractable criterion for the rational visibility.

In order to describe Theorem 2.2, we fix notation. Let $G$ be a connected Lie group and $U$ a closed connected subgroup of $G$. Let $B\iota\colon BU \to BG$ be the map induced by the inclusion $\iota\colon U \to G$. We can assume that the rational cohomology of $BG$ is a polynomial algebra, say $H^*(BG;\mathbb{Q}) \cong \mathbb{Q}[c_1, \dots, c_k]$. In what follows, we write $H^*(X)$ for the cohomology of a space $X$ with coefficients in the rational field.

Consider the Lannes' division functor $(H^*(BU){:}H^*(G/U))$ in the category of differential graded algebras (DGA's). The functor is regarded as the quotient $\wedge(H^*(BU) \otimes H_*(G/U))/I$ of the free algebra $\wedge(H^*(BU) \otimes H_*(G/U))$ by the ideal $I$ generated by $1 \otimes 1_* - 1$ and all elements of the form

$$a_1a_2 \otimes \beta - \sum_i (-1)^{|a_2||\beta_i'|}(a_1 \otimes \beta_i')(a_2 \otimes \beta_i''),$$

where $a_1, a_2 \in \wedge V$, $\beta \in B_*$ and $D(\beta) = \sum_i \beta_i' \otimes \beta_i''$ with the coproduct $D$ on $H_*(G/U)$ which is the dual to the product on $H^*(G/U)$. The quotient algebra in turn is isomorphic to an algebra of the form $\wedge(QH^*(BU) \otimes H_*(G/U))$, where $QH^*(BU)$ denotes the vector space of indecomposable elements. More precisely, the composite of the natural inclusion $i$ and the projection $p$

$$\wedge(QH^*(BU) \otimes H_*(G/U)) \xrightarrow{i} \wedge(H^*(BU) \otimes H_*(G/U)) \xrightarrow{p} (H^*(BU){:}H^*(G/U))$$

give rise to the isomorphism; see Section 3. Under the isomorphism $p \circ i$, we can define an algebra map $u\colon (H^*(BU){:}H^*(G/U)) \to \mathbb{Q}$ by $u(h \otimes b_*) = \langle j^*(h), b_* \rangle$, where $j\colon G/U \to BU$ is the fibre inclusion of the fibration $G/U \xrightarrow{j} BU \xrightarrow{B\iota} BG$.

Let $M_u$ be the ideal of $(H^*(BU){:}H^*(G/U))$ generated by the set

$$\{\eta \mid \deg \eta < 0\} \cup \{\eta - u(\eta) \mid \deg \eta = 0\}.$$

Let $\pi\colon H^*(BU) \otimes H_*(G/U) \to (H^*(BU){:}H^*(G/U))$ denote the composite of the inclusion $H^*(BU) \otimes H_*(G/U) \to \wedge(H^*(BU) \otimes H_*(G/U))$ and the projection $p$.

A recognition principle for rational visibility, Theorem 4.1 mentioned below, enables one to deduce the following result.

**Theorem 2.2.** *With the above notation, assume that for $c_{i_1}, \dots, c_{i_s} \in \{c_1, \dots, c_k\}$, there are elements $c_{j_1}, \dots, c_{j_s} \in H^*(BG)$ and $u_{1*}, \dots, u_{s*} \in H^{\geqslant 1}(G/U)$ such that*

$$\pi((B\iota)^*(c_{i_t}) \otimes 1_*) \equiv \pi((B\iota)^*(c_{j_t}) \otimes u_{t*})$$

*for $t = 1, \dots, s$ modulo decomposable elements in $(H^*(BG){:}H^*(G/U))/M_u$. Then there exists a map $\rho\colon \times_{t=1}^s S^{\deg c_{i_t t} - 1} \to G$ such that $\times_{t=1}^s S^{\deg c_{i_t} - 1}$ is rationally visible in* $\mathrm{aut}_1(G/U)$ *with respect to the map $(\lambda_{G,G/U}) \circ \rho$. In particular, if the elements $(B\iota)^*(c_{i_1}), \dots, (B\iota)^*(c_{i_s})$ are decomposable, then $\pi((B\iota)^*(c_{i_t}) \otimes 1_*) \equiv 0$ in $(H^*(BG){:}H^*(G/U))/M_u$ for $t = 1, \dots, s$, and hence one obtains the same conclusion.*

For a Lie group $G$ and a homogeneous space $M$ which admits a left translation by $G$, put $n(G) := \{i \in \mathbb{N} \mid \pi_i(G) \otimes \mathbb{Q} \neq 0\}$ and define the set $\mathrm{vd}(G, M)$ of *visibility degrees* by

$$\mathrm{vd}(G, M) = \{i \in n(G) \mid (\lambda_{G,M})_* \colon \pi_i(G) \otimes \mathbb{Q} \to \pi_i(\mathrm{aut}_1(M)) \otimes \mathbb{Q} \text{ is injective}\}.$$

As for monoids of homeomorphisms and of diffeomorphisms, we benefit by the study of rational visibility. In fact, we have an immediate but very important corollary.

**Corollary 2.3.** *If $l \in \mathrm{vd}(G, M)$, then there exists an element with infinite order in $\pi_l(\mathrm{Diff}_1(M))$ and $\pi_l(\mathrm{Homeo}_1(M))$.*

*Example* 2.4. Since $SO(d+1)/SO(d)$ is homeomorphic to the sphere $S^d$, we can define the maps $\lambda_{SO(d+1),S^d} \colon SO(d+1) \to \mathrm{aut}_1(S^d)$ by left translations. The Haefliger, Brown and Szczarba model for the function space $\mathrm{aut}_1(S^d)$ allows us to deduce that $\mathrm{aut}_1(S^{2m+1}) \simeq_{\mathbb{Q}} S^{2m+1}$ and $\mathrm{aut}_1(S^{2m}) \simeq_{\mathbb{Q}} S^{4m-1}$; see Example 3.4 below. Therefore, $\lambda_{SO(d+1),S^d}$ is not injective on the rational homotopy in general. However, it follows that the induced maps

$$(\lambda_{SO(2m+2),S^{2m+1}})_* \colon \pi_{2m+1}(SO(2m+2)) \otimes \mathbb{Q} \to \pi_{2m+1}(\mathrm{aut}_1(S^{2m+1})) \otimes \mathbb{Q},$$

$$(\lambda_{SO(2m+1),S^{2m}})_* \colon \pi_{4m-1}(SO(2m+1)) \otimes \mathbb{Q} \to \pi_{4m-1}(\mathrm{aut}_1(S^{2m})) \otimes \mathbb{Q}$$

are injective. In fact it is well-known that $H^*(BSO(2m+1)) \cong \mathbb{Q}[p_1, \dots, p_m]$ and $H^*(BSO(2m+2)) \cong \mathbb{Q}[p_1, \dots, p_m, \chi]$ as algebras, where $\deg p_j = 4j$ for $j = 1, \dots, m$ and $\deg \chi = 2m+2$. Moreover, for the inclusions $\iota_1 \colon SO(2m+1) \to SO(2m+2)$ and $\iota_2 \colon SO(2m) \to SO(2m+1)$, we see that $(B\iota_1)^*(\chi) = 0$ and $(B\iota_2)^*(p_m) = \chi^2$; see [**23**]. Thus the latter half of Theorem 2.2 enables us to conclude that

$$\mathrm{vd}(SO(2m+2), S^{2m+1}) = \{2m+1\} \quad \text{and} \quad \mathrm{vd}(SO(2m+1), S^{2m}) = \{4m-1\}.$$

The result [**1**, 1.1.5 Lemma] allows one to conclude that the map $SO(d+1) \to \mathrm{Diff}_1(S^d)$, induced by the left translations, is injective on the homotopy group. This implies that the inclusion $\mathrm{Diff}_1(S^d) \to \mathrm{aut}_1(S^d)$ is surjective on the rational homotopy group.

Theorem 4.1, which deduces Theorem 2.2, also yields another proof of a result due to Kedra and McDuff [**16**] and Sasao [**27**].

**Theorem 2.5** ([**16**, Proposition 4.8], [**27**])**.** *Assume that $M$ is a flag manifold of the form $U(m)/U(m_1) \times \cdots \times U(m_l)$. Then $SU(m)$ is rationally visible in $\mathrm{aut}_1(M)$ with respect to the map $\lambda_{SU(m),M}$ given by the left translations; that is, $\mathrm{vd}(SU(m), M) = n(SU(m)) = \{3, 5, \dots, 2m-1\}$. In particular, the localized map*

$$(\lambda_{SU(m),U(m)/U(m-1)\times U(1)})_{\mathbb{Q}} \colon SU(m)_{\mathbb{Q}} \to \mathrm{aut}_1(\mathbb{C}P^{m-1})_{\mathbb{Q}}$$

*is a homotopy equivalence.*

Furthermore, the same argument as in the proof of Theorem 2.5 allows one to establish the following result.

**Theorem 2.6.** *Let $M$ be the flag manifold of the form $Sp(m)/Sp(m_1) \times \cdots \times Sp(m_l)$. Then $\mathrm{vd}(Sp(m), M) = \{7, 11, \dots, 4m-1\}$. In particular, we see that the 3-connected cover $Sp(m)\langle 3\rangle$ is rationally visible in $\mathrm{aut}_1(M)$ with respect to $\lambda_{Sp(m),M} \circ \pi$, where $\pi \colon Sp(m)\langle 3\rangle \to Sp(m)$ is the projection.*

Let $G$ be a compact connected simple Lie group and $U$ a closed connected subgroup for which $G/U$ is a simply-connected homogeneous space of rank one; that is, its rational cohomology is generated by a single element. In order to illustrate usefulness of Theorems 2.2 and 4.1, we determine visibility degrees of $G$ in $\mathrm{aut}_1(G/U)$ for each couple $(G, U)$ classified by Oniscik in [**26**, Theorems 2 and 4] by applying the results.

In the following table, we first list such homogeneous spaces of the form $G/U$ with a simple Lie group $G$ and its subgroup $U$, which is not diffeomorphic to spheres or projective spaces, together with the sets $\mathrm{vd}(G, G/U)$:

| $(G, U, \text{index})$ | $(G/U)_{\mathbb{Q}}$ | $\mathrm{vd}(G, G/U)$ | $n(G)$ |
|---|---|---|---|
| (1) $(SO(2n+1), SO(2n-1)\times SO(2), 1)$ | $\mathbb{C}P^{2n-1}$ | $\{3, \ldots, 4n-1\}$ | $\{3, \ldots, 4n-1\}$ |
| (2) $(SO(2n+1), SO(2n-1), 1)$ | $S^{4n-1}$ | $\{4n-1\}$ | $\{3, \ldots, 4n-1\}$ |
| (3) $(SU(3), SO(3), 4)$ | $S^5$ | $\{5\}$ | $\{3, 5\}$ |
| (4) $(Sp(2), SU(2), 10)$ | $S^7$ | $\{7\}$ | $\{3, 7\}$ |
| (5) $(G_2, SO(4), (1,3))$ | $\mathbb{H}P^2$ | $\{11\}$ | $\{3, 11\}$ |
| (6) $(G_2, U(2), 3)$ | $\mathbb{C}P^5$ | $\{3, 11\}$ | $\{3, 11\}$ |
| (7) $(G_2, SU(2), 3)$ | $S^{11}$ | $\{11\}$ | $\{3, 11\}$ |
| (6)$'$ $(G_2, U(2), 1)$ | $\mathbb{C}P^5$ | $\{3, 11\}$ | $\{3, 11\}$ |
| (7)$'$ $(G_2, SU(2), 1)$ | $S^{11}$ | $\{11\}$ | $\{3, 11\}$ |
| (8) $(G_2, SO(3), 4)$ | $S^{11}$ | $\{11\}$ | $\{3, 11\}$ |
| (9) $(G_2, SO(3), 28)$ | $S^{11}$ | $\{11\}$ | $\{3, 11\}$ |

Table 1

Here the value of the index of the inclusion $j\colon U \to G$ is regarded as the integer $i$ by which the induced map $j_*\colon H_3(U;\mathbb{Z}) \to H_3(G;\mathbb{Z}) = \mathbb{Z}$ is a multiplication; see the proof of [**26**, Lemma 4]. The second column denotes the rational homotopy type of $G/U$ corresponding to a triple $(G, U, i)$. The homogeneous spaces $G/U$ for the cases (6)$'$ and (7)$'$ are diffeomorphic to those for the cases (1) and (2) with $n = 3$, respectively. Moreover, the homogeneous spaces are not diffeomorphic to each other except for the cases (6)$'$ and (7)$'$.

The following table describes visibility degrees of a simple Lie group $G$ in $\mathrm{aut}_1(G/U)$ for which $G/U$ is of rank one and is diffeomorphic to the sphere or the projective space, where the second column denotes the diffeomorphism type of the homogeneous space $G/U$ for the triple $(G, U, i)$:

| $(G, U, \text{index})$ | $G/U$ | $\mathrm{vd}(G, G/U)$ | $n(G)$ |
|---|---|---|---|
| (10) $(SU(n+1), SU(n), 1)$ | $S^{2n+1}$ | $\{2n+1\}$ | $\{3, \ldots, 2n+1\}$ |
| (11) $(SU(n+1), S(U(n)\times U(1)), 1)$ | $\mathbb{C}P^n$ | $\{3, \ldots, 2n+1\}$ | $\{3, \ldots, 2n+1\}$ |
| (12) $(SO(2n+1), SO(2n), 1)$ | $S^{2n}$ | $\{4n-1\}$ | $\{3, \ldots, 4n-1\}$ |
| (13) $(SO(9), SO(7), 1)$ | $S^{15}$ | $\{15\}$ | $\{3, 7, 11, 15\}$ |
| (14) $(Spin(7), G_2, 1)$ | $S^7$ | $\{7\}$ | $\{3, 7, 11\}$ |
| (15) $(Sp(n), Sp(n-1), 1)$ | $S^{4n-1}$ | $\{4n-1\}$ | $\{3, \ldots, 4n-1\}$ |
| (16) $(Sp(n), Sp(n-1)\times S^1, 1)$ | $\mathbb{C}P^{2n-1}$ | $\{3, \ldots, 4n-1\}$ | $\{3, \ldots, 4n-1\}$ |
| (17) $(Sp(n), Sp(n-1)\times Sp(1), 1)$ | $\mathbb{H}P^{n-1}$ | $\{7, \ldots, 4n-1\}$ | $\{3, \ldots, 4n-1\}$ |
| (18) $(SO(2n), SO(2n-1), 1)$ | $S^{2n-1}$ | $\{2n-1\}$ | $\{3, \ldots, 4n-5, 2n-1\}$ |
| (19) $(F_4, Spin(9), 1)$ | $\mathcal{L}P^2$ | $\{23\}$ | $\{3, 11, 15, 23\}$ |
| (20) $(G_2, SU(3), 1)$ | $S^6$ | $\{11\}$ | $\{3, 11\}$ |

Table 2

Here $\mathcal{L}P^2$ stands for the Cayley plane.

The former half of Theorem 2.2, namely the Lannes' functor argument, does work well enough when determining the set $\mathrm{vd}(G_2, G_2/U(2))$ of visibility degrees in case (6) in Table 1; see Section 8. Observe that for the cases (12) and (18) the results follow from those in Example 2.4. We are aware that in the above tables $G$ is rationally visible in $\mathrm{aut}_1(G/U)$ if and only if $G/U$ has the rational homotopy type of the complex projective space. It should be mentioned that for the map $\lambda_* \colon \pi_*(F_4)\otimes\mathbb{Q} \to \pi_*(\mathrm{aut}_1(\mathcal{L}P^2))\otimes\mathbb{Q}$, the restriction $(\lambda_*)_{15}$ is not injective though the vector space $\pi_{15}(\mathrm{aut}_1(\mathcal{L}P^2))\otimes\mathbb{Q}$ and $\pi_{15}(F_4)\otimes\mathbb{Q}$ are non-trivial; see Section 8. Moreover, Corollary 2.3 enables us to obtain non-trivial elements with infinite order in $\pi_l(\mathrm{Diff}_1(M))$ and $\pi_l(\mathrm{Homeo}_1(M))$ for each homogeneous space $M$ described in Tables 1 and 2 if $l \in \mathrm{vd}(G, M)$.

Let $X$ be a space and $\mathcal{H}_{H,X}$ the monoid of all homotopy equivalences from $X$ to itself that act trivially on the rational homology of $X$. The result [**16**, Proposition 4.8] asserts that if $X$ is the generalized flag manifold $U(m)\,/U(m_1)\times\cdots\times U(m_l)$, then the map $B\psi_{SU(m)} \colon BSU(m) \to B\mathcal{H}_{H,X}$ arising from the left translations is injective on the rational homotopy. Let $\iota\colon \mathrm{aut}_1(X) \to \mathcal{H}_{H,X}$ be the inclusion. Since $B\psi_{SU(m)} = B\iota \circ B\lambda_{SU(m),X}$, the result [**16**, Proposition 4.8] yields Theorem 2.5. Theorem 2.7 below guarantees that the converse also holds; that is, the result due to Kedra and McDuff is deduced from Theorem 2.5; see Section 8.

Before describing Theorem 2.7, we recall an $F_0$-space, which is a simply-connected finite complex with finite-dimensional rational homotopy and trivial rational cohomology in odd degree. For example, a homogeneous space $G/U$ for which $G$ is a connected Lie group and $U$ is a maximal rank subgroup of $G$ is an $F_0$-space.

**Theorem 2.7.** *Let $X$ be an $F_0$-space or a space having the rational homotopy type of the product of odd-dimensional spheres and $G$ a connected topological group which acts on $X$. Then $(B\lambda_{G,X})_* \colon H_*(BG) \to H_*(B\mathrm{aut}_1(X))$ is injective if and only if $(B\psi)_* \colon H_*(BG) \to H_*(B\mathcal{H}_{H,X})$ is injective. Here $\psi\colon G \to \mathcal{H}_{H,X}$ denotes the morphism of monoids induced by the action of $G$ on $X$.*

We now provide an overview of the rest of the paper. In Section 3, we recall a model for the evaluation map of a function space from [**7**], [**15**] and [**17**]. In Section 4, a rational model for the map $\lambda_{G,M}$ mentioned above is constructed. Section 5 is devoted to the study of a model for the left translation of a Lie group on a homogeneous space. In Section 6, Theorem 2.2 is proved. By using Theorems 2.2 and 4.1, we prove Theorem 2.5 in Section 7. In Section 8, we prove Theorem 2.7. The results on visibility degrees in Tables 1 and 2 are verified in Section 9.

## 3. Preliminaries

The tool for the study of the rational visibility problem is a rational model for the evaluation map $\mathrm{ev}\colon \mathrm{aut}_1(M)\times M \to M$, which is described in terms of the rational model due to Brown and Szczarba [**5**] and Haefliger [**13**]. For the convenience of the reader and to make notation more precise, we recall from [**7**] and [**17**] the model for the evaluation map. We shall use the same terminology as in [**3**] and [**11**].

Throughout the paper, for an augmented algebra $A$, we write $QA$ for the space $\overline{A}/\overline{A}\cdot\overline{A}$ of indecomposable elements, where $\overline{A}$ denotes the augmentation ideal. For a DGA $(A, d)$, let $d_0$ denote the linear part of the differential.

In what follows, we assume that a space is nilpotent and has the homotopy type of a connected CW-complex with rational homology of finite type unless otherwise explicitly stated. We denote by $X_{\mathbb{Q}}$ the localization of a nilpotent space $X$.

Let $A_{PL}$ be the simplicial commutative cochain algebra of polynomial differential forms with coefficients in $\mathbb{Q}$; see [**3**] and [**11**, Section 10]. Let $\mathcal{A}$ and $\Delta\mathcal{S}$ be the category of DGA's and that of simplicial sets, respectively. Let $\mathrm{DGA}(A,B)$ and $\mathrm{Simpl}(K,L)$ denote the hom-sets of the categories $\mathcal{A}$ and $\Delta\mathcal{S}$, respectively. Following Bousfield and Gugenheim [**3**], we define functors $\Delta\colon \mathcal{A}\to\Delta\mathcal{S}$ and $\Omega\colon\Delta\mathcal{S}\to\mathcal{A}$ by $\Delta(A)=\mathrm{DGA}(A,A_{PL})$ and by $\Omega(K)=\mathrm{Simpl}(K,A_{PL})$.

Let $(B,d_B)$ be a connected, locally finite DGA and let $B_*$ denote the differential graded coalgebra defined by $B_q=\mathrm{Hom}(B^{-q},\mathbb{Q})$ for $q\leqslant 0$ together with the coproduct $D$ and the differential $d_{B*}$, which are dual to the multiplication of $B$ and to the differential $d_B$, respectively. We denote by $I$ the ideal of the free algebra $\wedge(\wedge V\otimes B_*)$ generated by $1\otimes 1_*-1$ and all elements of the form

$$a_1a_2\otimes\beta-\sum_i(-1)^{|a_2||\beta_i'|}(a_1\otimes\beta_i')(a_2\otimes\beta_i''),$$

where $a_1,a_2\in\wedge V$, $\beta\in B_*$ and $D(\beta)=\sum_i\beta_i'\otimes\beta_i''$. Observe that $\wedge(\wedge V\otimes B_*)$ is a DGA with the differential $d:=d_A\otimes 1\pm 1\otimes d_{B*}$. The result [**5**, Theorem 3.5] implies that the composite $\rho\colon \wedge(V\otimes B_*)\hookrightarrow\wedge(\wedge V\otimes B_*)\to\wedge(\wedge V\otimes B_*)/I$ is an isomorphism of graded algebras. Moreover, it follows [**5**, Theorem 3.3] that $dI\subset I$. Thus $(\wedge(V\otimes B_*),\delta=\rho^{-1}d\rho)$ is a DGA. Observe that, for an element $v\in V$ and a cycle $e\in B_*$, if $d(v)=v_1\cdots v_m$ with $v_i\in V$ and $D^{(m-1)}(e_j)=\sum_j e_{j_1}\otimes\cdots\otimes e_{j_m}$, then

$$\delta(v\otimes e)=\sum_j\pm(v_1\otimes e_{j_1})\cdots(v_m\otimes e_{j_m}).\tag{3.1}$$

Here the sign is determined by the Koszul rule; that is, $ab=(-1)^{\deg a\deg b}ba$ in a graded algebra. Let $F$ be the ideal of $\widetilde{E}:=\wedge(V\otimes B_*)$ generated by $\oplus_{i<0}\widetilde{E}^i$ and $\delta(\widetilde{E}^{-1})$. Then $\widetilde{E}/F$ is a free algebra and $(\widetilde{E}/F,\delta)$ is a Sullivan algebra (not necessarily connected); see the proofs of [**5**, Theorem 6.1] and of [**7**, Proposition 19].

*Remark* 3.1. The result [**5**, Corollary 3.4] implies that there exists a natural isomorphism $\mathrm{DGA}(\wedge(\wedge V\otimes B_*)/I,C)\cong\mathrm{DGA}(\wedge V,B\otimes C)$ for any DGA $C$. Then the DGA $\wedge(\wedge V\otimes B_*)/I$ is regarded as Lannes' division functor $(\wedge V:B)$ by definition.

The singular simplicial set of a topological space $U$ is denoted by $\Delta U$ and let $|X|$ be the geometrical realization of a simplicial set $X$. By definition, $A_{PL}(U)$ the DGA of polynomial differential forms on $U$ is given by $A_{PL}(U)=\Omega\Delta U$. Given spaces $X$ and $Y$, we denote by $\mathcal{F}(X,Y)$ the space of continuous maps from $X$ to $Y$. The connected component of $\mathcal{F}(X,Y)$ containing a map $f\colon X\to Y$ is denoted by $\mathcal{F}(X,Y;f)$.

Let $\alpha\colon A=(\wedge V,d)\xrightarrow{\simeq}A_{PL}(Y)=\Omega\Delta Y$ be a Sullivan model (not necessarily minimal) for $Y$ and $\beta\colon(B,d)\xrightarrow{\simeq}A_{PL}(X)$ a Sullivan model for $X$ for which $B$ is connected and locally finite. For the function space $\mathcal{F}(X,Y)$ which is considered below, we assume that

$$\dim\oplus_{q\geqslant 0}H^q(X;\mathbb{Q})<\infty\quad\text{or}\quad\dim\oplus_{i\geqslant 2}\pi_i(Y)\otimes\mathbb{Q}<\infty.\tag{3.2}$$

Then the proof of [**17**, Proposition 4.3] enables us to deduce the following lemma; see also [**7**].

**Lemma 3.2.**

*(i) Let $\{b_j\}$ and $\{b_{j*}\}$ be a basis of $B$ and its dual basis of $B_*$, respectively and let $\widetilde{\pi} : \wedge(A \otimes B_*) \to (\wedge(A \otimes B_*)/I)\big/F \cong \widetilde{E}/F$ denote the projection. Define a map $m(\mathrm{ev})\colon A \to \widetilde{E}/F \otimes B$ by*

$$m(\mathrm{ev})(x) = \sum_j (-1)^{\tau(|b_j|)}\widetilde{\pi}(x \otimes b_{j*}) \otimes b_j,$$

*for $x \in A$, where $\tau(n) = [(n+1)/2]$, the greatest integer in $(n+1)/2$. Then $m(\mathrm{ev})$ is a well-defined DGA map.*

*(ii) There exists a commutative diagram*

$$\begin{tikzcd} \mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \times X_{\mathbb{Q}} \arrow[r, "\mathrm{ev}"] & Y_{\mathbb{Q}} \arrow[d, equal] \\ |\Delta(\widetilde{E}/F)| \times |\Delta(B)| \arrow[u, "\Theta\times 1"] \arrow[r, "|\Delta m(\mathrm{ev})|"'] & |\Delta(A)| \end{tikzcd}$$

*in which $\Theta$ is the homotopy equivalence described in [**5**, Sections 2 and 3]; see also [**17**, (3.1)].*

We next recall a Sullivan model for a connected component of a function space. Choose a basis $\{a'_k, b'_k, c'_j\}_{k,j}$ for $B_*$ so that $d_{B_*}(a'_k) = b'_k$, $d_{B_*}(c'_j) = 0$ and $c'_0 = 1$. Moreover, we take a basis $\{v_i\}_{i\geqslant 1}$ for $V$ such that $\deg v_i \leqslant \deg v_{i+1}$ and $d(v_{i+1}) \in \wedge V_i$, where $V_i$ is the subvector space spanned by the elements $v_1, \dots, v_i$. The result [**5**, Lemma 5.1] ensures that there exist free algebra generators $w_{ij}$, $u_{ik}$ and $v_{ik}$ such that

$$w_{i0} = v_i \otimes 1 \text{ and } w_{ij} = v_i \otimes c'_j + x_{ij}, \text{ where } x_{ij} \in \wedge(V_{i-1} \otimes B_*), \tag{3.3}$$

$$\delta w_{ij} \text{ is in } \wedge(\{w_{sl}; s < i\}), \tag{3.4}$$

$$u_{ik} = v_i \otimes a'_k \text{ and } \delta u_{ik} = v_{ik}. \tag{3.5}$$

We then have an inclusion

$$\gamma\colon E := (\wedge(w_{ij}), \delta) \hookrightarrow (\wedge(V \otimes B_*), \delta) = \widetilde{E}, \tag{3.6}$$

which is a homotopy equivalence with a retract

$$r\colon \widetilde{E} = (\wedge(V \otimes B_*), \delta) \to E; \tag{3.7}$$

see [**5**, Lemma 5.2] for more details. Let $q$ be a Sullivan representative for a map $f\colon X \to Y$; that is, $q$ fits into the homotopy commutative diagram

$$\begin{tikzcd} \wedge W \arrow[r, "\simeq"] & A_{PL}(X) \\ \wedge V \arrow[u, "q"] \arrow[r, "\simeq"'] & A_{PL}(Y). \arrow[u, "A_{PL}(f)"'] \end{tikzcd}$$

Moreover, we define a 0-simplex $\widetilde{u} \in \Delta(\wedge(\wedge V \otimes B_*)/I)_0$ by

$$\widetilde{u}(a \otimes b) = (-1)^{\tau(|a|)} b(q(a)), \tag{3.8}$$

where $a \in \wedge V$ and $b \in B_*$. Put $u = \Delta(\gamma)\widetilde{u}$. Let $M_u$ be the ideal of $E$ generated by the set $\{\eta \mid \deg \eta < 0\} \cup \{\delta\eta \mid \deg \eta = 0\} \cup \{\eta - u(\eta) \mid \deg \eta = 0\}$. Then we see that $(E/M_u, \delta)$ is an explicit model for the connected component $F(X, Y; f)$; see [**5**, Theorem 6.1] and [**15**, Section 3]. The proof of [**17**, Proposition 4.3] and [**15**, Remark 3.4] allow us to deduce the following proposition; see also [**7**].

**Proposition 3.3.** *With the same notation as in Lemma 3.2, we define a map*

$$m(\mathrm{ev})\colon A = (\wedge V, d) \to (E/M_u, \delta) \otimes B$$

*by*

$$m(\mathrm{ev})(x) = \sum_j (-1)^{\tau(|b_j|)} \pi \circ r(x \otimes b_{j*}) \otimes b_j,$$

*for $x \in A$, where $\pi\colon E \to E/M_u$ denotes the natural projection. Then $m(\mathrm{ev})$ is a model for the evaluation map $\mathrm{ev}\colon \mathcal{F}(X, Y; f) \times X \to Y$; that is, there exists a homotopy commutative diagram*

$$\begin{tikzcd}
A_{PL}(Y) \arrow[r, "A_{PL}(\mathrm{ev})"] & A_{PL}(\mathcal{F}(X,Y;f) \times X) \\
 & A_{PL}(\mathcal{F}(X,Y;f)) \otimes A_{PL}(X) \arrow[u, "\simeq"'] \\
A \arrow[uu, "\alpha", "\simeq"'] \arrow[r, "m(\mathrm{ev})"'] & (E/M_u, \delta) \otimes B, \arrow[u, "\simeq", "\xi \otimes \beta"']
\end{tikzcd}$$

*in which $\xi\colon (E/M_u, \delta) \xrightarrow{\simeq} A_{PL}(\mathcal{F}(X, Y; f))$ is the Sullivan model for $\mathcal{F}(X, Y; f)$ due to Brown and Szczarba [**5**].*

We call the DGA $(E/M_u, \delta)$ the Haefliger-Brown-Szczarba model (HBS-model for short) for the function space $\mathcal{F}(X, Y; f)$.

*Example* 3.4. Let $M$ be a space whose rational cohomology is isomorphic to the truncated algebra $\mathbb{Q}[x]/(x^m)$, where $\deg x = l$. Recall the model $(E/M_u, \delta)$ for $\mathrm{aut}_1(M)$ mentioned in [**15**, Example 3.6]. Since the minimal model for $M$ has the form $(\wedge(x, y), d)$ with $dy = x^m$, it follows that

$$E/M_u = \wedge(x \otimes 1_*, y \otimes (x^s)_*; 0 \leqslant s \leqslant m-1)$$

with $\delta(x \otimes 1_*) = 0$ and $\delta(y \otimes (x^s)_*) = (-1)^s \begin{pmatrix} m \\ s \end{pmatrix} (x \otimes 1_*)^{m-s}$, where $\deg x \otimes 1_* = l$ and $\deg(y \otimes (x^s)_*) = lm - ls - 1$. Then the rational model $m(\mathrm{ev})$ for the evaluation map $\mathrm{ev}\colon \mathrm{aut}_1(M) \times M \to M$ is given by $m(\mathrm{ev})(x) = (x \otimes 1_*) \otimes 1 + 1 \otimes x$ and

$$m(\mathrm{ev})(y) = \sum_{s=0}^{m-1} (-1)^s (y \otimes (x^s)_*) \otimes x^s + 1 \otimes y.$$

*Remark* 3.5. We describe here variants of the HBS-model for a function space.

(i) Let $\wedge\widetilde{V} \xrightarrow{\simeq} A_{PL}(Y)$ be a Sullivan model (not necessarily minimal) and $B \xrightarrow{\simeq} A_{PL}(X)$ a Sullivan model of finite type. We recall the homotopy equivalence $\gamma\colon E \to \widetilde{E} = \wedge(\wedge V \otimes B_*)/I$ mentioned in (3.6). Let $\widetilde{u} \in \Delta(\widetilde{E})_0$ be a 0-simplex and $u$ a 0-simplex of $E$ defined by composing $\widetilde{u}$ with the quasi-isomorphism $\gamma$. Then the induced map $\overline{\gamma}\colon E/M_u \to \widetilde{E}/M_{\widetilde{u}}$ is a quasi-isomorphism. In fact, the results [**5**, Theorem 6.1] and [**7**, Proposition 19] imply that the projections onto the quotient DGA's $E/M_u$ and $\widetilde{E}/M_{\widetilde{u}}$ induce homotopy equivalences $\Delta(p)\colon \Delta(E/M_u) \to \Delta(E)_u$ and $\Delta(\widetilde{p})\colon \Delta(\widetilde{E}/M_{\widetilde{u}}) \to \Delta(\widetilde{E})_{\widetilde{u}}$, respectively. Here $K_v$ denotes the connected component containing the vertex $v$ for a simplicial set $K$, namely, the set of simplices all of whose faces are at $v$. Then we have a commutative diagram

$$\begin{tikzcd}
\pi_*(|\Delta(E/M_u)|) \arrow[r, "|\Delta(p)|", "\cong"'] & \pi_*(|\Delta(E)|, |u|) \\
\pi_*(|\Delta(\widetilde{E}/M_{\widetilde{u}})|) \arrow[r, "\cong", "|\Delta(\widetilde{p})|"'] \arrow[u, "|\Delta(\overline{\gamma})|_*"] & \pi_*(|\Delta(\widetilde{E})|, |\widetilde{u}|). \arrow[u, "|\Delta(\gamma)|_*"']
\end{tikzcd}$$

Since $\gamma$ is a homotopy equivalence, it follows that $|\Delta(\gamma)|_*$ is an isomorphism and hence so is $|\Delta(\overline{\gamma})|_*$. This yields that $|\Delta(\overline{\gamma})|$ is a homotopy equivalence. By virtue of the Sullivan-de Rham equivalence Theorem [**3**, 9.4], we see that $\overline{\gamma}$ is a quasi-isomorphism.

As in Lemma 3.2, we define a DGA map $\widetilde{m(\mathrm{ev})}\colon (\wedge V, d) \to \widetilde{E}/F \otimes B$ and let $m(\mathrm{ev})\colon (\wedge V, d) \to \widetilde{E}/M_{\widetilde{u}} \otimes B$ be a DGA map defined by $m(\mathrm{ev}) = \pi \otimes 1 \circ \widetilde{m(\mathrm{ev})}$. We then have a homotopy commutative diagram

$$\begin{tikzcd}
 & E/M_u \otimes B \arrow[d, "\overline{\gamma}\otimes 1", "\simeq"'] \\
\wedge V \arrow[ur, "m(\mathrm{ev})"] \arrow[r, "m(\mathrm{ev})"'] & \widetilde{E}/M_{\widetilde{u}} \otimes B.
\end{tikzcd}$$

In fact, the homotopy between $\mathrm{id}_{\widetilde{E}}$ and $\gamma \circ r$ defined in [**5**, Lemma 5.2] induces a homotopy between $\mathrm{id}_{\widetilde{E}/\widetilde{F}}$ and $\gamma \circ r\colon \widetilde{E}/F \to E/F' \to \widetilde{E}/F$. Here $F'$ denotes the ideal of $E$ generated by $\oplus_{i<0} E^i$ and $\delta(E^{-1})$. It is immediate that $r \circ \gamma = \mathrm{id}_{E/F}$. Let $m(\mathrm{ev})'\colon \wedge V \to E/F' \otimes B$ be the DGA defined as in Proposition 3.3. Then it follows that

$$\begin{aligned}
\overline{\gamma} \otimes 1 \circ m(\mathrm{ev}) &= \overline{\gamma} \otimes 1 \circ \pi \otimes 1 \circ m(\mathrm{ev})' \\
&= \pi \otimes 1 \circ \gamma \otimes 1 \circ r \otimes 1 \circ \widetilde{m(\mathrm{ev})} \\
&\simeq \pi \otimes 1 \circ \widetilde{m(\mathrm{ev})} = m(\mathrm{ev}).
\end{aligned}$$

(ii) In the case where $X$ is formal, we have a more tractable model for $\mathcal{F}(X, Y; f)$. Suppose that $X$ is a formal space with a minimal model $(B, d_B) = (\wedge W', d)$. Then there exists a quasi-isomorphism $k\colon (\wedge W', d) \to H^*(B)$ which is surjective; see [**9**, Theorem 4.1]. With the notation mentioned above, let $\{e_j\}_j$ be a basis for the homology $H(B_*)$ of the differential graded coalgebra $B_* = (\wedge W')_*$ and $\{v_i\}_i$ a basis for $V$. Then it follows from the proof of [**5**, Theorem 1.9] that the subalgebra $\mathbb{Q}\{v_i \otimes e_j\}$ is closed for the differential $\delta$ and that the inclusion

$\mathbb{Q}\{v_i \otimes e_j\} \to \wedge(W \otimes B_*) = \widetilde{E}$ gives rise to a homotopy equivalence

$$\gamma\colon E' := (\wedge(v_i \otimes e_j), \delta) \to (\wedge(W \otimes B_*), \delta) = \widetilde{E}.$$

In fact, the elements $w_{ij}$ in (3.3) can be chosen so that $w_{i0} = v_i \otimes 1_*$ and $w_{ij} = v_i \otimes e_j$ for $j \geqslant 1$. Moreover, we see that there exists a retraction $r\colon \wedge(W \otimes B_*) \to E'$ which is the homotopy inverse of $\gamma$. Thus Proposition 3.3 remains true after replacing $E$ by $E'$. Here the 0-simplex $\widetilde{u} \in \Delta(\wedge(W \otimes B_*))_0$ needed in the construction of the model for $\mathcal{F}(X, Y; f)$ has the same form as in (3.8).

We conclude this section with some comments on models for a connected component of a function space and related maps.

In the original construction in [**7**] and [**13**] of a model for a function space $\mathcal{F}(X, Y)$, it is assumed that the source space $X$ admits a finite-dimensional model. Indeed the construction of a model for the evaluation map in [**7**, Theorem 1] requires existence of such a model for the space $X$. As described in Lemma 3.2 and Proposition 3.3, our construction only needs the assumption (3.2). Thus our model for a function space endowed with a model for evaluation map is viewed as a generalization of that in [**7**].

The arguments in [**5**, Section 7] and [**7**, Section 4] on a model for a connected component of $\mathcal{F}(X, Y)$ begin with a 0-simplex. That is, the considered component is that containing a map $f$ which corresponds to the given 0-simplex via a sequence of weak equivalences between the singular simplicial set of $\mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}})$ and the simplicial set $\Delta(E/F)$; see [**7**, Theorem 6] and also [**15**, (2.3)]. On the other hand, for any given map $f\colon X \to Y$, an explicit form of a 0-simplex corresponding to $f$ is clarified in [**15**, Remark 3.4] with (3.8). Thus our constructions *complement* the basic constructions in rational homotopy theory of function spaces due to Buijs and Murillo [**7**]. This point is mentioned once again in the next section with more explicit notations after describing Theorem 4.1.

## 4. A rational model for the map $\lambda$ induced by left translation

We first observe that $\mathrm{aut}_1(X)$ is nothing but the function space $\mathcal{F}(X, X; \mathrm{id}_M)$. Moreover, for a manifold $M$, the function space $\mathrm{aut}_1(M)$ satisfies assumption (3.2). Thus we can obtain explicit models for $\mathrm{aut}_1(X)$ and for the evaluation map according to the construction in the previous section. Using such the models, we have an elaborate model for the map $\lambda_{G,M}$ mentioned in the introduction.

Let $M$ be a space admitting an action of Lie group $G$ on the left. We define the map $\lambda\colon G \to \mathrm{aut}_1(M)$ by $\lambda(g)(x) = gx$. The subjective in this section is to construct an algebraic model for the map

$$\mathrm{in} \circ \lambda\colon G \to \mathrm{aut}_1(M) \to \mathcal{F}(M, M),$$

where $\mathrm{in}\colon \mathrm{aut}_1(M) \to \mathcal{F}(M, M)$ denotes the inclusion. To this end we use a model for the evaluation map

$$\mathrm{ev}\colon \mathcal{F}(X, Y) \times X \to Y$$

defined by $\mathrm{ev}(f)(x) = f(x)$ for $f \in \mathcal{F}(X, Y)$ and $x \in X$, which is considered in [**7**] and [**17**].

Let $G$ be a connected Lie group, $U$ a closed subgroup of $G$ and $K$ a closed subgroup which contains $U$. Let $(\wedge V_G, d)$ and $(\wedge W, d)$ denote a minimal model for $G$ and a Sullivan model for the homogeneous space $G/U$, respectively. Let $\lambda\colon G \to \mathcal{F}(G/U, G/K)$ be the adjoint of the composite of the left translation $G \times G/U \to G/U$ and projection $p\colon G/U \to G/K$. Observe that the map $\lambda$ coincides with the composite

$$p_* \circ \mathrm{in} \circ \lambda_{G,G/U}\colon G \to \mathrm{aut}_1(G/U) \to \mathcal{F}(G/U, G/U) \to \mathcal{F}(G/U, G/K).$$

We construct a model for $\lambda$ by using the HBS-model for $\mathcal{F}(G/U, G/K; p)$ mentioned in Remark 3.5(i). To this end, we first take a Sullivan representative

$$\zeta'\colon \wedge W \to \wedge V_G \otimes \wedge W'$$

for the composite $G \times G/U \to G/K$ of the left translation $G \times G/U \to G/U$ and the projection $p\colon G/U \to G/K$.

Let $A$, $B$ and $C$ be connected DGA's. Recall from [**5**, Section 3] the bijection $\Psi\colon (A \otimes B_*, C)_{DG} \overset{\cong}{\to} (A, C \otimes B)_{DG}$ defined by

$$\Psi(w)(a) = \sum_j (-1)^{\tau(|b_j|)} w(a \otimes b_{j*}) \otimes b_j.$$

Consider the case where $A = (\wedge W, d)$, $B = (\wedge W', d)$ and $C = (\wedge V_G, d)$. Moreover, define a map $\widetilde{\mu}\colon \wedge(A \otimes B_*) \to \wedge V_G$ by

$$\widetilde{\mu}(y \otimes b_{j*}) = (-1)^{\tau(|b_j|)} \langle \zeta'(y), b_{j*} \rangle, \tag{4.1}$$

where $\langle\ \ , b_{j*}\rangle\colon \wedge V_G \otimes \wedge W' \to \wedge V_G$ is a map defined by $\langle x \otimes a, b_{j*}\rangle = x \cdot \langle a, b_{j*}\rangle$. Then we see that $\Psi(\widetilde{\mu}) = \zeta'$. Hence it follows from [**5**, Theorem 3.3] that

$$\widetilde{\mu}\colon \widetilde{E} := \wedge(A \otimes B_*)/I \to \wedge V_G$$

is a well-defined DGA map. We define an augmentation $\widetilde{u}\colon \widetilde{E} \to \mathbb{Q}$ by $\widetilde{u} = \varepsilon \circ \widetilde{\mu}$, where $\varepsilon\colon \wedge V_G \to \mathbb{Q}$ is the augmentation. Recall the ideal $M_{\widetilde{u}}$ of $\widetilde{E}$ generated by

$$\{\eta \mid \deg \eta < 0\} \cup \{\delta\eta \mid \deg \eta = 0\} \cup \{\eta - \widetilde{u}(\eta) \mid \deg \eta = 0\}.$$

It is readily seen that $\widetilde{\mu}(M_{\widetilde{u}}) = 0$. Thus we see that $\widetilde{\mu}$ induces a DGA map

$$\widetilde{\widetilde{\mu}}\colon \widetilde{E}/M_{\widetilde{u}} \to \wedge V_G.$$

The result [**14**, Theorem 3.11] asserts that the map

$$e_\sharp\colon \mathcal{F}(G/U, (G/K); p) \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$$

is a localization. Thus we have a map $\lambda_{\mathbb{Q}}\colon G_{\mathbb{Q}} \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ which fits into the homotopy commutative diagram

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, "\lambda_{\mathbb{Q}}"] & \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p) \\
G \arrow[u, "e"] \arrow[r, "\lambda"'] & \mathcal{F}(G/U, (G/K); p), \arrow[u, "e_\sharp"']
\end{tikzcd}$$

where $e$ denotes the localization map. We then have a recognition principle for rational visibility.

**Theorem 4.1.** *Let $\{x_i\}_i$ be a basis for the image of the induced map*

$$H^*(Q(\widetilde{\widetilde{\mu}}))\colon H^*(Q(\widetilde{E}/M_{\widetilde{u}}), \delta_0) \to H^*(Q(\wedge V_G), d_0) = V_G.$$

*Then there exists a map $\rho\colon \times_{i=1}^{s} S^{\deg x_i} \to G$ such that the map*

$$(\lambda_{\mathbb{Q}} \circ \rho_{\mathbb{Q}})_*\colon \pi_*((\times_{i=1}^{s} S^{\deg x_i})_{\mathbb{Q}}) \to \pi_*(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}), e \circ p)$$

*is injective for $* > 1$. Moreover, $(\lambda_{\mathbb{Q}})_*\colon \pi_j(G_{\mathbb{Q}}) \to \pi_j(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}, e \circ p))$ is injective if and only if $H^j(Q(\widetilde{\widetilde{\mu}}))$ is surjective.*

We stress here that $\widetilde{E}/M_{\widetilde{u}}$ and $\widetilde{\widetilde{\mu}}$ in Theorem 4.1 are explicit and computable models for the function space $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}, e \circ p)$ and for the map $\lambda$, respectively; see Theorem 4.3.

Following the basic construction described in [**7**, Section 4], we can take a 0-simplex $\widetilde{u} \in (\Delta\widetilde{E})_0$, which corresponds to $e \circ p$ through equivalences between simplicial sets $\Delta(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}))$ and $\Delta\widetilde{E}$ in order to construct a Sullivan model for the component $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}, e \circ p)$. Here the map $e \circ p$ is considered an element in $\Delta(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}))_0$. However, this does not exhibit how to precisely describe $\widetilde{u}$ in terms of a Sullivan representative for the projection $p\colon G/U \to G/K$. The novelty of Theorem 4.1 is that, in the construction of the model $\widetilde{E}/M_{\widetilde{u}}$, we can use the 0-simplex $\widetilde{u}$, which is constructed explicitly with a Sullivan representative for the left translation $G \times G/U \to G/U \xrightarrow{p} G/K$; see (4.1). This fact is an important thread in proving Theorems 2.2 and 2.5.

In order to prove Theorem 4.1, we first observe that the diagram

$$\begin{tikzcd} \wedge V_G \otimes \wedge W' & & (\wedge(A \otimes B_*)/I)/F \otimes \wedge W' = \widetilde{E}/F \otimes \wedge W' \arrow[ll, "\widetilde{\mu}\otimes 1"'] \\ & \wedge W \arrow[ul, "\zeta'"] \arrow[ur, "m(\mathrm{ev})"'] & \end{tikzcd} \tag{4.2}$$

is commutative, where $F$ is the ideal of $\widetilde{E}$ defined in Section 3. Thus Lemma 3.2 enables us to obtain a commutative diagram

$$\begin{tikzcd} {|\Delta \wedge V_G| \times |\Delta \wedge W'|} \arrow[rr, "(\Theta\circ|\Delta\widetilde{\mu}|)\times 1"] \arrow[dr, "|\Delta\zeta'| = \mathrm{action}_{\mathbb{Q}}"'] & & \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}) \times (G/U)_{\mathbb{Q}} \arrow[dl, "\mathrm{ev}"] \\ & {|\Delta \wedge W| = (G/K)_{\mathbb{Q}}.} & \end{tikzcd} \tag{4.3}$$

Observe that assumption (3.2) is now satisfied.

Since the restriction $|\Delta\zeta'|_{|_{*\times|\Delta\wedge W|}}$ is homotopic to $p_{\mathbb{Q}}$, it follows from the commutativity of diagram (4.3) that $p_{\mathbb{Q}} \simeq \Theta \circ |\Delta\widetilde{\mu}|(*)$. This implies that $\Theta \circ |\Delta\widetilde{\mu}|$ maps $G_{\mathbb{Q}}$ into the function space $\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; p_{\mathbb{Q}})$.

**Lemma 4.2.** *Let $\lambda_{\mathbb{Q}}\colon G_{\mathbb{Q}} \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ be the localized map of $\lambda$ mentioned above and let $e^\sharp\colon \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; p_{\mathbb{Q}}) \to \mathcal{F}((G/U), (G/K)_{\mathbb{Q}}; e \circ p)$ the map induced by the localization $e\colon (G/U) \to (G/U)_{\mathbb{Q}}$. Then*

$$e^\sharp \circ \Theta \circ |\Delta\widetilde{\mu}| \simeq \lambda_{\mathbb{Q}}\colon G_{\mathbb{Q}} \to \mathcal{F}((G/U), (G/K)_{\mathbb{Q}}; e \circ p).$$

*Proof.* Consider the commutative diagram

$$\begin{tikzcd}
{[G \times G/U, G/K]} \arrow[r, "\theta", "\approx"'] \arrow[d, "e_*"'] & {[G, \mathcal{F}(G/U, G/K)]} \arrow[d, "(e_\sharp)_*"] \\
{[G \times G/U, (G/K)_{\mathbb{Q}}]} \arrow[r, "\theta", "\approx"'] & {[G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}})]} \\
{[G_{\mathbb{Q}} \times (G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}]} \arrow[u, "(e\times e)^*", "\approx"'] \arrow[dr, "\theta", "\approx"'] & {[G_{\mathbb{Q}}, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}]} \arrow[u, "e^*"'] \\
 & {[G_{\mathbb{Q}}, \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}})]} \arrow[u, "\approx", "(e^\sharp)_*"']
\end{tikzcd} \tag{4.4}$$

in which $\theta$ is the adjoint map and $e$ stands for the localization map. It follows from diagram (4.3) that $\theta(\text{action}_{\mathbb{Q}}) = \Theta \circ |\Delta\widetilde{\mu}|$. Moreover, we have $\theta(\text{action}) = e_\sharp \circ \lambda = \lambda_K \circ e$. Thus the commutativity of diagram (4.3) implies that $e^*([e^\sharp \circ \Theta \circ |\Delta\widetilde{\mu}|]) = e^*([\lambda_{\mathbb{Q}}])$ in $[G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}})]$. Since $G$ is connected, it follows that

$$(e^\sharp) \circ \Theta \circ |\Delta\widetilde{\mu}| \circ e \simeq \lambda_{\mathbb{Q}} \circ e \colon G \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p).$$

The fact that $e_\sharp \colon \mathcal{F}(G/U, (G/K); p) \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ is the localization yields that the induced map

$$e^* \colon [G_{\mathbb{Q}}, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)] \to [G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)]$$

is bijective. This completes the proof. □

Before proving Theorem 4.1, we recall some maps. For a simplicial set $K$, there exists a natural homotopy equivalence $\xi_K \colon K \to \Delta|K|$, which is defined by $\xi_K(\sigma) = t_\sigma \colon \Delta^n \to \{\sigma\} \times \Delta \to |K|$. This gives rise to a quasi-isomorphism $\xi_A \colon \Omega\Delta|\Delta A| \xrightarrow{\simeq} \Omega\Delta A$. Moreover, we can define a bijection $\eta \colon \mathrm{DGA}(A, \Omega K) \xrightarrow{\cong} \mathrm{Simp}(K, \Delta A)$ by $\eta \colon \phi \mapsto f; f(\sigma)(a) = \phi(a)(\sigma)$, where $a \in A$ and $\sigma \in K$. We observe that $\eta^{-1}(\mathrm{id}) \colon A \to \Omega\Delta A$ is a quasi-isomorphism if $A$ is a connected Sullivan algebra; see [**3**, 10.1. Theorem].

*Proof of Theorem 4.1.* Let $\pi \colon \widetilde{E} \to \widetilde{E}/M_{\widetilde{u}}$ be the projection. With the same notation as above, we have a commutative diagram

$$\begin{tikzcd}
 & {|\Delta(\wedge(W \otimes B_*)/F)|} \arrow[rr, "\Theta", "\simeq"'] & & {\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}})} \\
{|\Delta(\wedge V_G)|} \arrow[ur, "|\Delta(\widetilde{\mu})|"] \arrow[r, "|\Delta(\widetilde{\widetilde{\mu}})|"'] & {|\Delta(\widetilde{E}/M_{\widetilde{u}})|} \arrow[u, "|\Delta\pi|"'] \arrow[r, "\simeq", "|\Delta\pi|"'] & {|(\Delta\widetilde{E})_{\widetilde{u}}|} \arrow[r, "\simeq", "\Theta"'] & {\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; \Theta([(1, \widetilde{u})]))}, \arrow[u, hook]
\end{tikzcd}$$

where $[(1, \widetilde{u})] \in |\Delta\widetilde{E}|$ is the element whose representative is $(1, \widetilde{u}) \in \Delta^0 \times (\Delta\widetilde{E})_0$. Lemma 4.2 yields that

$$e^\sharp \circ \Theta \circ |\Delta\pi| \circ |\Delta\widetilde{\widetilde{\mu}}| \simeq e^\sharp \circ \Theta \circ |\Delta\widetilde{\mu}| \simeq \lambda_{\mathbb{Q}}.$$

Thus we see that $e^\sharp$ maps $\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; \Theta([(1, \widetilde{u})]))$ to $\mathcal{F}((G/U), (G/K)_{\mathbb{Q}}; e^\sharp \circ \Theta([(1, \widetilde{u})]))$, which is the connected component containing $\mathrm{Im}(\lambda_{\mathbb{Q}})$. This implies that

$\mathcal{F}((G/U),(G/K)_{\mathbb{Q}};e^{\sharp}\circ\Theta([(1,\widetilde{u})]))=\mathcal{F}((G/U),(G/K)_{\mathbb{Q}};e\circ p)$. Therefore, by the naturality of maps $\eta$ and $\xi_A$, we have a diagram

$$\begin{tikzcd}
A_{PL}(G_{\mathbb{Q}}) \arrow[dd, equal] & A_{PL}(\mathcal{F}(G/U,(G/K)_{\mathbb{Q}};e\circ p)) \arrow[l, "A_{PL}(\lambda_{\mathbb{Q}})"'] \arrow[d, "((e^{\sharp}))^*"] \\
 & A_{PL}(\mathcal{F}((G/U)_{\mathbb{Q}},(G/K)_{\mathbb{Q}};\Theta([(1,\widetilde{u})]))) \arrow[d, "\Theta^*"] \\
A_{PL}(|\Delta\wedge V_G|) \arrow[d, "\xi_{\wedge V_G}"'] & A_{PL}(|\Delta(\widetilde{E}/M_{\widetilde{u}})|)=\Omega\Delta(|\Delta\widetilde{E}/M_{\widetilde{u}}|) \arrow[l, "|\Delta\widetilde{\widetilde{\mu}}|^*"'] \arrow[d, "\xi_{\widetilde{E}/M_{\widetilde{u}}}"] \\
\Omega\Delta(\wedge V_G) & \Omega\Delta(\widetilde{E}/M_{\widetilde{u}}) \arrow[l, "\Omega\Delta\widetilde{\widetilde{\mu}}"] \\
\wedge V_G \arrow[u, "\eta^{-1}(\mathrm{id})", "\simeq"'] & \widetilde{E}/M_{\widetilde{u}} \arrow[l, "\widetilde{\widetilde{\mu}}"'] \arrow[u, "\simeq", "\eta^{-1}(\mathrm{id})"']
\end{tikzcd}$$

in which the upper square is homotopy commutative and the lower two squares are strictly commutative. The Lifting Lemma allows us to obtain a DGA map

$$\varphi\colon \widetilde{E}/M_{\widetilde{u}}\to A_{PL}(\mathcal{F}(G/U,(G/K)_{\mathbb{Q}}))$$

such that $\xi_{\widetilde{E}/M_{\widetilde{u}}}\circ\Theta^*\circ((e^{\sharp})_*)^*\circ\varphi\simeq\eta^{-1}(\mathrm{id})$. We then see that $\xi_{\wedge V_G}\circ A_{PL}(\lambda_{\mathbb{Q}})\circ\varphi\simeq\eta^{-1}(\mathrm{id})\circ\widetilde{\widetilde{\mu}}$. This implies that $\widetilde{\widetilde{\mu}}$ is a Sullivan representative for the map $\lambda$.

Given a space $X$, let $u\colon A\to A_{PL}(X)$ be a DGA map from a Sullivan algebra $A$. Let $[f]$ be an element of $\pi_n(X)$ and let $\iota\colon(\wedge Z,d)\xrightarrow{\simeq}A_{PL}(S^n)$ the minimal model. By taking a Sullivan representative $\widetilde{f}\colon A\to\wedge Z$ with respect to $u$, namely a DGA map satisfying the condition that $\iota\circ\widetilde{f}\simeq A_{PL}(f)\circ u$, we define a map $\nu_u\colon\pi_n(X)\to\mathrm{Hom}(H^nQ(A),\mathbb{Q})$ by $\nu_u([f])=H^nQ(\widetilde{f})\colon H^nQ(A)\to H^nQ(\wedge Z)=\mathbb{Q}$. By virtue of [**3**, 6.4 Proposition], in particular, we have a commutative diagram

$$\begin{tikzcd}
\pi_n(G_{\mathbb{Q}}) \arrow[r, "\lambda_{\mathbb{Q}}"] \arrow[d, "\nu_{t'}"', "\cong"] & \pi_n(\mathcal{F}(G/U,(G/K)_{\mathbb{Q}});e\circ p) \arrow[d, "\cong"', "\nu_{\varphi}"] \\
\mathrm{Hom}((V_G)^n,\mathbb{Q}) \arrow[r, "HQ(\widetilde{\widetilde{\mu}})^*"'] & \mathrm{Hom}(H^nQ(\widetilde{E}/M_{\widetilde{u}}),\mathbb{Q}),
\end{tikzcd}$$

in which $\nu_{t'}$and $\nu_{\varphi}$ are isomorphisms; see [**3**, 8.13 Proposition]. There exists an element $[f_i]\otimes q$ in $\pi_*(G)\otimes\mathbb{Q}$ which corresponds to the dual element $x_i^*$ via the isomorphism $\pi_*(G)\otimes\mathbb{Q}\cong\pi_*(G_{\mathbb{Q}})\xrightarrow{\nu_{t'}}\mathrm{Hom}((V_G)^n,\mathbb{Q})$ for any $i=1,\dots,s$. The required map $\rho\colon\times_{i=1}^{s}S^{\deg x_i}\to G$ is defined by the composite of the map $\times_{i=1}^{s}f_i$ and the product $\times_{i=1}^{s}G\to G$. $\square$

The proof of Theorem 4.1 yields the following result.

**Theorem 4.3.** *The DGA map $\widetilde{\widetilde{\mu}}\colon\widetilde{E}/M_{\widetilde{u}}\to\wedge V_G$ is a model for the map $\lambda\colon G\to\mathcal{F}(G/U,G/K;p)$, namely a Sullivan representative in the sense of [**11**, Definition, page 154].*

## 5. A model for the left translation

In order to prove Theorems 2.2 and 2.5, a more explicit model for the map $\lambda_{G,M}\colon G \to \mathrm{aut}_1(M)$ is required. To this end, we refine the model of the left translation described in the proof of Theorem 4.1.

We first observe that the cohomology $H^*(BU;\mathbb{Q})$ is isomorphic to a polynomial algebra with finite generators, say $H^*(BU;\mathbb{Q}) \cong \mathbb{Q}[h_1,\dots,h_l]$. Consider a commutative diagram of fibrations

$$\begin{tikzcd}
 & G \arrow[r, equal] \arrow[d, "i"'] & G \arrow[d] \\
G/U & G \times_U E_U \arrow[l, "h", "\simeq"'] \arrow[r] \arrow[d, "\overline{\pi}"'] & E_G \arrow[d, "\pi"] \\
 & BU \arrow[r, "B\iota"'] & BG
\end{tikzcd}$$

in which $h\colon G \times_U E_U \to G/U$ is a homotopy equivalence defined by $h([g,e]) = [g]$. This diagram yields a Sullivan model $(\wedge W, d)$ for $G/U$ which has the form $(\wedge W, d) = (\wedge(h_1,\dots,h_l,x_1,\dots,x_k),d)$ with $dx_j = (B\iota)^* c_j$; see [**11**, Proposition 15.16] for the details. Moreover, we have a model $(\wedge V_G, d)$ for $G$ of the form $(\wedge(x_1,\dots,x_k),0)$. Since $h \circ i$ is nothing but the projection $\pi\colon G \to G/U$, it follows that the natural projection

$$\eta\colon (\wedge(h_1,\dots,h_l,x_1,\dots,x_k),d) \to (\wedge(x_1,\dots,x_k),0) \tag{5.1}$$

is a Sullivan model for the map $\pi$.

Let $\beta\colon G \times (G \times_U E_U) \to G \times_U E_G$ be the action of $G$ on $G \times_U E_U$. Then the left translation $\mathrm{tr}\colon G \times G/U \to G/U$ coincides with $\beta$ up to the homotopy equivalence $h\colon (G \times_U E_U) \to G/U$ mentioned above. Thus, in order to obtain a model for the linear action, it suffices to construct a model for $\beta$. Recall the fibration $G \to G \times_U E_U \xrightarrow{\overline{\pi}} BU$ and the universal fibration $G \to E_G \xrightarrow{\pi} BG$. We consider here a commutative diagram

$$\begin{tikzcd}
 & G \times (G \times_U E_U) \arrow[rr, "1\times f"] \arrow[dl, "\beta"'] \arrow[dd, "\overline{\pi}'"' near start] & & G \times E_G \arrow[dl, "\alpha"'] \arrow[dd, "\pi'"] \\
G \times_U E_U \arrow[rr, crossing over, "f" near end] \arrow[dd, "\overline{\pi}"'] & & E_G & \\
 & BU \arrow[rr, "B\iota"' near start] \arrow[dl, "="'] & & BG \arrow[dl, "="] \\
BU \arrow[rr, "B\iota"'] & & BG \arrow[from=uu, crossing over, "\pi" near start] &
\end{tikzcd} \tag{5.2}$$

in which $\pi'$ and $\overline{\pi}'$ are fibrations with the same fibre $G \times G$, and the restrictions $\alpha|_{\mathrm{fibre}}\colon G \times G \to G$ and $\beta|_{\mathrm{fibre}}\colon G \times (G \times_U E_U) \to (G \times_U E_U)$ are the multiplication on $G$ and the action of $G$, respectively. Let $i\colon (\wedge V_{BG},0) \rightarrowtail \wedge(\widetilde{V_{BU}},d)$ be a Sullivan model for $B\iota$. In particular, we can choose such a model so that

$$\wedge\widetilde{V_{BU}} = \wedge(c_1,\dots,c_m) \otimes \wedge(h_1,\dots,h_l) \otimes \wedge(\tau_1,\dots,\tau_m)$$

and $d(\tau_i) = B\iota(c_i) - c_i$. By the construction of a model for pullback fibration mentioned in [**11**, page 205], we obtain a diagram

$$
\begin{tikzcd}
& \wedge Z & & \wedge W' \arrow[ll, "v'"'] \\
\wedge V \arrow[ur, "\widetilde{\beta}"] & & \wedge V' \arrow[ll, "v"'] \arrow[ur, "\widetilde{\alpha}"'] & \\
& \wedge \widetilde{V_{BU}} \arrow[uu, "u'"] & & \wedge V_{BG} \arrow[ll, "i"'] \arrow[uu] \\
\wedge \widetilde{V_{BU}} \arrow[uu, "u"] \arrow[ur, "="] & & \wedge V_{BG} \arrow[ll, "i"] \arrow[uu] \arrow[ur, "="'] &
\end{tikzcd}
\tag{5.3}
$$

in which vertical arrows are Sullivan models for the fibrations in diagram (5.2). Observe that the squares are commutative except for the top square. Let $\Psi\colon \wedge Z \to A_{PL}(G \times (G \times_U E_U))$ be the Sullivan model with which Sullivan representatives in (5.3) are constructed. The argument in [**11**, page 205] allows us to choose homotopies, which make the maps $v$, $\widetilde{\beta}$, $v'$ and $\widetilde{\alpha}$ Sullivan representatives for the corresponding maps, so that all of them are relative with respect to $\wedge V_{BG}$. This implies that $\Psi \circ \widetilde{\beta} \circ v \simeq \Psi \circ v' \circ \widetilde{\alpha}$ rel $\wedge V_{BG}$. By virtue of the Lifting Lemma [**11**, Proposition 14.6], we have a homotopy $H\colon \widetilde{\beta} \circ v \simeq v' \circ \widetilde{\alpha}$ rel $\wedge V_{BG}$. This yields a homotopy commutative diagram

$$
\begin{tikzcd}
\wedge V' \otimes_{\wedge V_{BG}} \wedge \widetilde{V_{BU}} \arrow[r, "u \cdot v"] \arrow[d, "\widetilde{\alpha} \otimes 1"'] & \wedge V \arrow[d, "\widetilde{\beta}"] \\
\wedge W' \otimes_{\wedge V_{BG}} \wedge \widetilde{V_{BU}} \arrow[r, "u' \cdot v'"'] & \wedge Z
\end{tikzcd}
$$

in which horizontal arrows are quasi-isomorphisms; see [**11**, (15.9), page 204]. In fact, the homotopy $K\colon \wedge \widetilde{V_{BU}} \otimes_{\wedge V_{BG}} \wedge V' \to \wedge W \otimes \wedge(t, dt)$ is given by $K = (\widetilde{\beta} \circ u) \cdot H$. Observe that $\widetilde{\beta} \circ u = u'$. Thus we have a model $\widetilde{\alpha} \otimes 1$ for $\widetilde{\beta}$ and hence for the left translation.

The model $\widetilde{\alpha} \otimes 1$ can be replaced by a more tractable one. In fact, by recalling the model $(\wedge \widetilde{V_{BU}}, d)$ for $BU$ mentioned above, it is readily seen that the map $s\colon \wedge \widetilde{V_{BU}} \to \wedge V_{BU} = \wedge(h_1, \ldots, h_l)$, which is defined by $s(c_i) = (B\iota)^*(c_i)$, $s(h_i) = h_i$ and $s(\tau_j) = 0$, is a quasi-isomorphism and is compatible with $\wedge V_{BG}$-action. Here the Sullivan representative for $B\iota\colon BU \to BG$ is also denoted by $(B\iota)^*$. Thus we have a commutative diagram

$$
\begin{tikzcd}
\wedge V' \otimes_{\wedge V_{BG}} \wedge V_{BU} \arrow[d, "\zeta := \widetilde{\alpha} \otimes 1"'] & \wedge V' \otimes_{\wedge V_{BG}} \wedge \widetilde{V_{BU}} \arrow[l, "1 \otimes s"'] \arrow[d, "\widetilde{\alpha} \otimes 1"] \\
\wedge W' \otimes_{\wedge V_{BG}} \wedge V_{BU} & \wedge W' \otimes_{\wedge V_{BG}} \wedge \widetilde{V_{BU}} \arrow[l, "1 \otimes s"]
\end{tikzcd}
$$

in which the DGA maps $1 \otimes s$ are quasi-isomorphisms. As usual, the Lifting Lemma enables us to deduce the following lemma.

**Lemma 5.1.** *The DGA map* $\zeta := \widetilde{\alpha} \otimes 1\colon \wedge V' \otimes_{\wedge V_{BG}} \wedge V_{BU} \to \wedge W' \otimes_{\wedge V_{BG}} \wedge V_{BU}$ *is a Sullivan representative for the left translation* $\mathrm{tr}\colon G \times G/U \to G/U$.

In order to construct a model for tr more explicitly, we proceed to construct an appropriate model for $\alpha\colon G \times E_G \to E_G$.

**Lemma 5.2.** *There exists a Sullivan representative $\psi$ for $\alpha$ such that a diagram*

$$\begin{array}{ccc} & & \wedge(x_1,\dots,x_l)\otimes\wedge V_{BG} = \wedge V' \\ & \nearrow^{i_1} & \downarrow \psi \\ \wedge V_{BG} & & \\ & \searrow_{i_2} & \\ & & \wedge(x_1,\dots,x_l)\otimes\wedge(x_1,\dots,x_l)\otimes\wedge V_{BG} = \wedge W' \end{array}$$

*is commutative and $\psi(x_i) = x_i\otimes 1\otimes 1 + 1\otimes x_i\otimes 1 + \sum_n X_n\otimes X'_n C_n$ for some monomials $X_n \in \wedge(x_1,\dots,x_l)$, $X'_n\in\wedge^+(x_1,\dots,x_l)$ and monomials $C_n\in\wedge^+V_{BG}$. Here $i_1$ and $i_2$ denote Sullivan models for $\pi$ and $\pi'$, respectively.*

*Proof.* We first observe that $d(x_i\otimes 1)=0$ and $d(1\otimes x_i) = c_i \in \wedge(c_1,\dots,c_l) = \wedge V_{BG}$ in $\wedge W'$. It follows from [**11**, 15.9] that there exists a Sullivan representative $\psi$ for $\alpha$ which makes the diagram commutative. We write

$$\psi(x_i) = x_i\otimes 1\otimes 1 + 1\otimes x_i\otimes 1 + \sum_n X_n\otimes X'_nC_n + \sum_n \tilde{X}_n\otimes\tilde{X}'_n + \sum_n X''_n\otimes C''_n$$

with monomial bases, where $C_n, C''_n\in\wedge^+V_{BG}$, $X_n, X''_n\in\wedge(x_1,\dots,x_l)\otimes 1\otimes 1$, $X'_n\in 1\otimes\wedge^+(x_1,\dots,x_l)\otimes 1$ and $\tilde{X}_n\otimes\tilde{X}'_n\in\wedge(x_1,\dots,x_l)\otimes\wedge(x_1,\dots,x_l)\otimes 1$.

The map $\widetilde{\psi}\colon \wedge(x_1,\dots,x_l)\to\wedge(x_1,\dots,x_l)\otimes\wedge(x_1,\dots,x_l)$ induced by $\psi$ is a Sullivan representative for the product of $G$. This allows us to conclude that $\tilde{X}_n$ and $\tilde{X}'_n$ are in $\wedge^+(x_1,\dots,x_l)$. Since $\psi$ is a DGA map, it follows that

$$dx_i = \psi(dx_i) = dx_i + \sum_n X_n\otimes d(X'_n)C_n + \sum_n\tilde{X}_n\otimes d(\tilde{X}'_n).$$

This implies that $\sum_n X_n\otimes d(X'_n)C_n = 0$ and $\sum_n\tilde{X}_n\otimes d(\tilde{X}'_n)=0$. Since the map $d\colon \wedge^+(x_1,\dots,x_l)\to\wedge(x_1,\dots,x_l)\otimes\wedge V_{BG}$ is a monomorphism, it follows that $\sum_n\tilde{X}_n\otimes\tilde{X}'_n = 0$. We write $C''_n = c_{i_n}^{k_n}\tilde{C}_n$, where $k_n\geqslant 1$. Define a homotopy

$$H\colon \wedge(x_1,\dots,x_l)\otimes\wedge V_{BG}\to\wedge(x_1,\dots,x_l)\otimes\wedge(x_1,\dots,x_l)\otimes\wedge V_{BG}\otimes\wedge(t,dt)$$

by $H(c_i) = c_i\otimes 1$ and

$$\begin{aligned} H(x_i) &= x_i\otimes 1\otimes 1 + 1\otimes x_i\otimes 1 + \sum_n X_n\otimes X'_nC_n \\ &\quad - \sum_n X''_n\otimes x_{i_n}\otimes c_{i_n}^{k_n-1}\tilde{C}_n\otimes dt + \sum_n X''_n\otimes 1\otimes c_{i_n}^{k_n}\tilde{C}_n\otimes t. \end{aligned}$$

Put $\widetilde{\psi} = (\varepsilon_0\otimes 1)\circ\psi$. We see that $\widetilde{\psi}\simeq\psi$ rel $\wedge V_{BG}$. This completes the proof. $\square$

## 6. Proof of Theorem 2.2

We prove Theorem 2.2 by means of the model for the left translation described in the previous section.

*Proof of Theorem 2.2.* We adapt Theorem 4.1. We recall the Sullivan model $(\wedge W, d)$ for $G/U$ mentioned in Section 5. Observe that $(\wedge W, d)$ has the form

$$(\wedge W, d) = (\wedge(h_1, \dots, h_l, x_1, \dots, x_k), d)$$

with $dx_j = (B\iota)^* c_j$. Let $l\colon (H^*(BU), 0) \to (\wedge W, d)$ be the inclusion and

$$k\colon (\wedge W, d) \longrightarrow (\wedge(h_1, \dots, h_l)/(dx_1, \dots, dx_l), 0) \rightarrowtail (H^*(G/U), 0)$$

the DGA map defined by $k(h_i) = (-1)^{\tau(|h_i|)} h_i$ and $k(x_i) = 0$. Recall the DGA $\widetilde{E} = \wedge(\wedge W \otimes (\wedge W)_*)/I$ and the DGA map $\widetilde{\mu}\colon \widetilde{E} \to \wedge V_G$ mentioned in Section 4, where we use the model $\zeta\colon \wedge W \to \wedge V_G \otimes \wedge W$ for the action $G \times G/U \to G/U$ constructed in Lemmas 5.1 and 5.2 in order to define $\widetilde{\mu}$; see (4.1). Consider the composite

$$\begin{aligned}\theta\colon (H^*(BU){:}H^*(G/U)) &= \wedge(H^*(BU) \otimes H_*(G/U))/I \\ &\xrightarrow{l\otimes 1} \wedge(\wedge W \otimes H_*(G/U))/I \xrightarrow{1\otimes k^\sharp} \wedge(\wedge W \otimes (\wedge W)_*)/I = \widetilde{E}.\end{aligned}$$

Let $\widetilde{u}\colon \widetilde{E} \to \mathbb{Q}$ be an augmentation defined by $\widetilde{u} = \varepsilon \circ \widetilde{\mu}$, where $\varepsilon\colon \wedge V_G \to \mathbb{Q}$ is the augmentation. Then we have $\theta(M_u) \subset M_{\widetilde{u}}$, where $M_u$ is the ideal of the Lannes' division functor $(H^*(BU){:}H^*(G/U))$ defined before describing Theorem 2.2 in Section 2, and $M_{\widetilde{u}}$ denotes the ideal of $\widetilde{E}$ defined in Section 4. In fact, since $i^*(h_i) = (-1)^{\tau(|h_i|)} k \circ l(h_i)$ and $\langle h_i, k^\sharp b_* \rangle = \langle \zeta h_i, b_* \rangle$ for $h_i \in H^*(BU)$, it follows that

$$\begin{aligned}\theta(h_i \otimes b_* - u(h_i \otimes b_*)) &= h_i \otimes k^\sharp b_* - \langle i^* h_i, b_* \rangle \\ &= h_i \otimes k^\sharp b_* - (-1)^{\tau(|h|)} \langle k h_i, b_* \rangle \\ &= h_i \otimes k^\sharp b_* - (-1)^{\tau(|h|)} \langle \zeta h_i, b_* \rangle \\ &= h_i \otimes k^\sharp b_* - \widetilde{u}(h_i \otimes k^\sharp b_*).\end{aligned}$$

Consider an element $z := x_{i_t} \otimes 1_* - (-1)^{\tau(|u_{t*}|)} x_{j_t} \otimes k^\sharp(u_{t*}) \in Q(\widetilde{E}/M_{\widetilde{u}})$. For any $\alpha \in \wedge W$, $\langle \alpha, d^\sharp k^\sharp u_{t*} \rangle = \langle k d\alpha, u_{t*} \rangle = 0$. Therefore we see that, in $Q(\widetilde{E}/M_{\widetilde{u}})$,

$$\begin{aligned}\delta_0(z) &= dx_{i_t} \otimes 1_* - (-1)^{\tau(|u_{t*}|)} dx_{j_t} \otimes k^\sharp(u_{t*}) \\ &= \theta((B\iota)^*(c_{i_t}) \otimes 1_* - (B\iota)^*(c_{j_t}) \otimes u_{t*}) = 0.\end{aligned}$$

The last equality follows from the assumption that $(B\iota)^*(c_{i_t}) \otimes 1_* \equiv (B\iota)^*(c_{j_t}) \otimes u_{t*}$ modulo decomposable elements in $(H^*(BU){:}\ H^*(G/U))/M_u$. By using the notation in Lemma 5.2, we see that

$$\begin{aligned}H^*Q(\widetilde{\widetilde{\mu}})(z) &= \langle \zeta x_{i_t}, 1_* \rangle - \langle \zeta x_{j_t}, k^\sharp u_{t*} \rangle \\ &= \langle x_{i_t} \otimes 1, 1_* \rangle - \langle \sum X_n \otimes X'_n C_n, k^\sharp u_{t*} \rangle \\ &= x_{i_t} - \sum X_n \langle k(X'_n) C_n, u_{t*} \rangle = x_{i_t}.\end{aligned}$$

Observe that $k(X'_n) = 0$. By virtue of Theorem 4.1, we have the result. $\square$

*Remark* 6.1. As for the latter half of Theorem 2.2, we have a very simple proof of the assertion. In fact, the composite of the evaluation map $\mathrm{ev}_0\colon \mathrm{aut}_1(G/U) \to G/U$ and the map $\lambda\colon G \to \mathrm{aut}_1(G/U)$ is nothing but the projection $\pi\colon G \to G/U$. Suppose that $(B\iota)^*(c_{i_1}),\dots,\ (B\iota)^*(c_{i_s})$ are decomposable. We consider the model $\eta\colon (\wedge W, d) \to (\wedge V_G, 0)$ for $\pi$ mentioned in (5.1). Then we see that $HQ(\eta)(x_{i_t}) = x_{i_t}$ for the map

$HQ(\eta)\colon HQ(\wedge W) \to HQ(\wedge V_G) = V_G$. Observe that $x_{i_t} \in HQ(\wedge W)$ since $(B\iota)^*(c_{i_t})$ is decomposable. The same argument as the proof of Theorem 4.1 enables us to conclude that there is a map $\rho\colon \times_{t=1}^{s} S^{\deg c_{i_t}-1} \to G$ such that

$$\pi_* \circ \rho_* \colon \pi_*(\times_{t=1}^{s} S_{\mathbb{Q}}^{\deg c_{i_t}-1}) \to \pi_*(G_{\mathbb{Q}})$$

is injective. Thus $\lambda_* \circ \rho_*$ is injective in the rational homotopy.

*Remark* 6.2. In the proof of Theorem 2.2, we construct a model for $G$ of the form $(\wedge(x_1, \dots, x_k), 0)$. By virtue of [**11**, Proposition 15.13], we can choose the elements $x_j$ so that $\sigma^*(c_j) = x_j$, where $\sigma^*\colon H^*(BG) \xrightarrow{\pi^*} H^*(E_G, G) \underset{\cong}{\xleftarrow{\delta}} H^*(G)$ denotes the cohomology suspension.

In the rest of this section, we describe a suitable model for $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ for proving Theorems 2.5 and 2.6.

Let $G$ be a connected Lie group, $U$ a connected maximal rank subgroup and $K$ another connected maximal rank subgroup which contains $U$. We recall from Section 3 a Sullivan model for the connected component $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$. Let $\iota_1\colon K \to G$ and $\iota_2\colon U \to K$ be the inclusions and put $\iota = \iota_1 \circ \iota_2$. Let $\varphi_U\colon (\wedge W', d) \overset{\simeq}{\to} \Omega\Delta(G/U)$ and $\varphi_K\colon (\wedge \widetilde{W}, d) \overset{\simeq}{\to} \Omega\Delta(G/K)$ be the Sullivan models for the homogeneous spaces $G/U$ and $G/K$, respectively, mentioned in the proof of Theorem 2.2; that is,

$$(\wedge W', d) = (\wedge(h_1, \dots, h_l, x_1, \dots, x_k), d) \text{ with } d(x_i) \ = (B\iota)^*(c_i)$$

and

$$(\wedge \widetilde{W}, d) = (\wedge(e_1, \dots, e_s, x_1, \dots, x_k), d) \text{ with } d(x_i) = (B\iota_1)^*(c_i).$$

By applying the Lifting Lemma to the commutative diagram

$$\begin{tikzcd} \wedge V_{BK} \arrow[r, "(B\iota_2)^*"] \arrow[d, tail] & \wedge V_{BU} \arrow[r, tail] & \wedge W' \arrow[d, "\varphi_U"] \\ \wedge \widetilde{W} \arrow[r, "\varphi_K"'] & \Omega\Delta(G/K) \arrow[r, "\Omega\Delta(p)"'] & \Omega\Delta(G/U), \end{tikzcd}$$

we have a diagram

$$\begin{tikzcd} H^*(G/U) & \wedge W' \arrow[l, "k", "\simeq"'] \arrow[r, "\simeq"'] & \Omega\Delta(G/U) \\ H^*(G/K) \arrow[u, "p^*"] & \wedge \widetilde{W} \arrow[l, "\simeq", "l"'] \arrow[r, "\simeq"] \arrow[u, "\varphi"] & \Omega\Delta(G/K) \arrow[u, "\Omega\Delta(p)"'] \end{tikzcd} \tag{6.1}$$

in which the right square is homotopy commutative and the left that is strictly commutative. In particular, $k(x_i) = 0$, $l(x_i) = 0$ and $\varphi(e_i) = (B\iota_2)^* e_i$.

Let $w\colon \wedge W \to \wedge \widetilde{W}$ be a minimal model for $(\wedge \widetilde{W}, d)$ and $k^\sharp\colon (H^*(G/U))^\sharp \to (\wedge W')^\sharp$ the dual to the map $k$. As in Remark 3.5(ii), we construct a DGA $E'$ by using $(\wedge W', d) = (B, d_B)$ and $(\wedge W, d)$. We then have a sequence of quasi-isomorphisms

$$E' \underset{\simeq}{\xrightarrow{\gamma := 1 \otimes k^\sharp}} \wedge(\wedge W \otimes (\wedge W')_*)/I \underset{\simeq}{\xrightarrow{w \otimes 1}} \wedge(\wedge \widetilde{W} \otimes (\wedge W')_*)/I = \widetilde{E}.$$

Moreover, we choose a model $\zeta'$ for the action $G \times G/U \overset{\mathrm{tr}}{\to} G/U \overset{p}{\to} G/K$ which is defined by the composite $\wedge\widetilde{W} \xrightarrow{\zeta} \wedge V_G \otimes \wedge\widetilde{W} \xrightarrow{1\otimes\varphi} \wedge V_G \otimes \wedge W'$, where $\zeta$ is the Sullivan representative for the left translation tr mentioned in Lemmas 5.1 and 5.2. Then the map $\zeta'$ deduces a model

$$\widetilde{\widetilde{\mu}}\colon E'/M_u \to \wedge V_G \tag{6.2}$$

for $\lambda\colon G \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ as in Theorem 4.1. Observe that

$$\widetilde{\mu}(v_i \otimes e_j) = (-1)^{\tau(|e_j|)}\langle (1\otimes\varphi)\zeta w(v_i), k^{\sharp}e_j\rangle \quad \text{and} \quad u = \varepsilon \circ \widetilde{\mu}, \tag{6.3}$$

where $\varepsilon\colon \wedge V_G \to \mathbb{Q}$ denotes the augmentation. In the next section, we shall prove Theorem 2.5 by using the model $\widetilde{\widetilde{\mu}}\colon E'/M_u \to \wedge V_G$.

## 7. Proof of Theorem 2.5

Let $G$ and $U$ be the Lie group $U(m+k)$ and a maximal rank subgroup of the form $U(m_1) \times \cdots \times U(m_s) \times U(k)$, respectively. Without loss of generality, we can assume that $m_1 \geqslant \cdots \geqslant m_s \geqslant k$. Let $K$ denote the subgroup $U(m) \times U(k)$ of $U(m+k)$, where $m = m_1 + \cdots + m_s$. Then the Leray-Serre spectral sequence, with coefficients in the rational field for the fibration $p\colon G/U \to G/K$ with fibre $K/U$, collapses at the $E_2$-term because the cohomologies of $G/K$ and of $K/U$ are algebras generated by elements with even degree. Therefore, it follows that the induced map $p^*\colon H^*(G/K) \to H^*(G/U)$ is a monomorphism. In order to prove Theorem 2.5, we apply Theorem 4.1 to the function space $\mathcal{F}(G/U, G/K, p)$.

Let $P = \{S_1, \ldots, S_n\}$ be a family consisting of subsets of the finite ordered set $\{1, \ldots, s\}$, which satisfies the condition that $x < y$ whenever $x \in S_i$ and $y \in S_{i+1}$. Define $\sharp^l P$ to be the number of elements of the set $\{S_j \in P \mid |S_j| = l\}$. Let $k$ be a fixed integer. We call the family $P$ a $(i_1, \ldots, i_k)$-type block partition of $\{1, \ldots, s\}$ if $\sharp^l P = i_l$ for $1 \leqslant l \leqslant k$. Let $Q^{(s)}_{i_1,\ldots,i_k}$ denote the number of $(i_1, \ldots, i_k)$-type block partitions of $\{1, \ldots, s\}$.

We construct a minimal model explicitly for the Grassmann manifold $G/K = U(m+k)/U(m) \times U(k)$. Assume that $m \geqslant k$. As in the proof of Theorem 2.2, we have a Sullivan model for $U(m+k)/U(m) \times U(k)$ of the form

$$(\wedge\widetilde{W}, d) = (\wedge(\tau_1, \ldots, \tau_{m+k}, c_1, \ldots, c_k, c'_1, \ldots, c'_m), d)$$

with $d\tau_l = \sum_{i+j=l} c'_i c_j$.

**Lemma 7.1.** *There exists a sequence of quasi-isomorphisms*

$$\wedge\widetilde{W} \xleftarrow{\simeq} \wedge W_{(1)} \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} \wedge W_{(s)} \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} \wedge W_{(m)}$$

*in which, for any $s$, $(\wedge W_{(s)}, d_{(s)})$ is a DGA of the form*

$$\wedge W_{(s)} = \wedge(\tau_{s+1}, \ldots, \tau_{m+k}, c_1, \ldots, c_k, c'_{s+1}, \ldots, c'_m) \quad \textit{with}$$

$$\begin{aligned} d_{(s)}\tau_l = {} & c'_l + c'_{l-1}c_1 + \cdots + c'_{s+1}c_{l-(s+1)} \\ & + \sum_{i_1+2i_2+\cdots+ki_k=s} (-1)^{i_1+\cdots+i_k} Q^{(s)}_{i_1,\dots,i_k} c_1^{i_1}\cdots c_k^{i_k} c_{l-s} \\ & + \sum_{i_1+2i_2+\cdots+ki_k=s-1} (-1)^{i_1+\cdots+i_k} Q^{(s-1)}_{i_1,\dots,i_k} c_1^{i_1}\cdots c_k^{i_k} c_{l-(s-1)} \\ & + \cdots + (-c_1)c_{l-1} + c_l \end{aligned}$$

*for $s+1 \leqslant l \leqslant m+k$, where $c_i = 0$ for $i < 0$ or $i > k$.*

*Proof.* We shall prove this lemma by induction on the integer $s$. We first observe that $d\tau_2 = c'_2 - c_1c_1 + c_2$ in $\wedge W_{(1)}$ because $Q^{(1)}_{i_1} = 1$. Define a map $\varphi\colon \wedge W_{(1)} \to \wedge\widetilde{W}$ by $\varphi(c_i) = c_i$, $\varphi(c'_j) = c'_j$ and $\varphi(\tau_2) = \tau_2 - \tau_1c_1$. Since $d\tau_1 = c'_1 + c_1$ in $\wedge W$, it follows that $\varphi$ is a well-defined quasi-isomorphism. Suppose that $(\wedge W_{(s)}, d_{(s)})$ in the lemma can be constructed for some $s \leqslant m-1$. In particular, we have

$$d_{(s)}\tau_{s+1} = c'_{s+1} + \sum_{0\leqslant j\leqslant s}\ \sum_{i_1+2i_2+\cdots+ki_k=j} (-1)^{i_1+\cdots+i_k} Q^{(j)}_{i_1,\dots,i_k} c_1^{i_1}\cdots + c_k^{i_k} c_{s+1-j}.$$

*Claim 1.*

$$Q^{(s+1)}_{i_1,\dots,i_k} = Q^{(s)}_{i_1-1,i_2,\dots,i_k} + Q^{(s-1)}_{i_1,i_2-1,\dots,i_k} + \cdots + Q^{(s+1-k)}_{i_1,\dots,i_k-1}.$$

Claim 1 implies that

$$d_{(s)}\tau_{s+1} = c'_{s+1} - \sum_{i_1+2i_2+\cdots+ki_k=s+1} (-1)^{i_1+\cdots+i_k} Q^{(s+1)}_{i_1,\dots,i_k} c_1^{i_1}\cdots c_k^{i_k}.$$

We define $d_{(s+1)}\tau_{l+1}$ in $\wedge W_{(s+1)}$ by replacing the factor $c'_{s+1}$ which appears in $d_{(s)}\tau_{l+1}$ with $c'_{s+1} - d_{(s)}\tau_{s+1}$, namely,

$$\begin{aligned} d_{(s+1)}\tau_{l+1} = {} & c'_{l+1} + c'_l c_1 + \cdots + c'_{s+2}c_{(l+1)-(s+2)} \\ & + \sum_{i_1+2i_2+\cdots+ki_k=s+1} (-1)^{i_1+\cdots+i_k} Q^{(s+1)}_{i_1,\dots,i_k} c_1^{i_1}\cdots c_k^{i_k} c_{l-s} \\ & + \cdots + (-c_1)c_l + c_{l+1}. \end{aligned}$$

Moreover, define a map $\varphi\colon \wedge W_{(s+1)} \to \wedge W_{(s)}$ by $\varphi(c_i) = c_i$, $\varphi(c'_j) = c'_j$ and $\varphi(\tau_{l+1}) = \tau_{l+1} - \tau_{s+1}c_{l+1-(s+1)}$. It is readily seen that $\varphi$ is a well-defined DGA map. The usual spectral sequence argument enables us to deduce that $\varphi$ is a quasi-isomorphism. This finishes the proof. □

*Proof of Claim 1.* Let $\{P_l\}$ denote the family of all $(i_1,\dots,i_k)$-type block partitions of $\{1,\dots,s+1\}$. We write $P_l = \{S^{(l)}_1,\dots,S^{(l)}_{n(l)}\}$. Then $\{P_l\}$ is represented as the disjoint union of the families of $(i_1,\dots,i_k)$-type block partitions whose last sets $S^{(l)}_{n(l)}$ consist of $j$ elements, namely, $\{P_l\} = \amalg_{1\leqslant j\leqslant k}\{P_l \mid |S_{n(l)}| = j\}$. It follows that

$$\bigl|\{P_l \mid |S_{n(l)}| = j\}\bigr| = Q^{(s+1-j)}_{i_1,\dots,i_{j-1},i_j-1,i_{j+1},\dots,i_k}.$$

We have the result. □

Recall the minimal model $(\wedge W_{(m)}, d)$ for $G/K$ mentioned in Lemma 7.1. We see that $\deg d\tau_{m+1} = \deg c_1^m c_1 = 2(m+1)$ and that $d\alpha = 0$ for any element $\alpha$ with

$\deg \alpha \leqslant 2m+1$. This yields that $c_1^m \neq 0$ in $H^*(G/K;\mathbb{Q})$. As was mentioned before Lemma 7.1, the induced map $p^*\colon H^*(G/K) \to H^*(G/U)$ is injective. Therefore, we have $(p^*c_1)^s \neq 0$ for $s \leqslant m$.

Let $\widetilde{\widetilde{\mu}}\colon E'/M_u \to \wedge V_G$ be the model for the map $\lambda\colon G \to \mathcal{F}(G/U,(G/K)_{\mathbb{Q}}; e\circ p)$ mentioned in the previous section; see (6.2) and (6.3). The following four lemmas are keys to proving Theorem 2.5. The proofs are deferred to the end of this section.

**Lemma 7.2.** *$\delta_0(\tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_*) = (-1)^m c_{m-s+1}$ if $m \neq s$.*

**Lemma 7.3.** *$\widetilde{\widetilde{\mu}}(\tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_*) = 0$ if $m \neq s$.*

**Lemma 7.4.** *$\delta_0(\tau_{m+1} \otimes ((p^*c_1)^s)_*) = (-1)^s s c_{m-s+1}$.*

**Lemma 7.5.** *$\widetilde{\widetilde{\mu}}(\tau_{m+1} \otimes ((p^*c_1)^s)_*) = \tau_{m-s+1}$.*

*Proof of Theorem 2.5.* By virtue of Lemmas 7.2, 7.3, 7.4 and 7.5, we have

$$\begin{aligned}\delta_0((-1)^m\tau_{m+(m-s+1)} &\otimes ((p^*c_1)^m)_* - \frac{(-1)^s}{s}\tau_{m+1}\otimes((p^*c_1)^s)_*)\\ &= (-1)^m(-1)^m c_{m-s+1} - \frac{(-1)^s}{s}(-1)^s s c_{m+s-1} = 0\end{aligned}$$

and

$$\begin{aligned}\widetilde{\widetilde{\mu}}((-1)^m\tau_{m+(m-s+1)} &\otimes ((p^*c_1)^m)_* - \frac{(-1)^s}{s}\tau_{m+1}\otimes((p^*c_1)^s)_*)\\ &= -\frac{(-1)^s}{s}\tau_{m-s+1},\end{aligned}$$

where $s \leqslant m-1$. Theorem 4.1 implies that

$$(\lambda_{\mathbb{Q}})_i\colon \pi_i(G_{\mathbb{Q}}) \to \pi_i(\mathcal{F}(G/U,(G/K)_{\mathbb{Q}}, e\circ p))$$

is injective for $i = \deg\tau_2,\dots,\deg\tau_m$. We see that the map $(\lambda_{\mathbb{Q}})_i$ factors through the map $(\lambda_{G,G/U})_{\mathbb{Q}\ i}\colon \pi_i(G_{\mathbb{Q}}) \to \pi_i(\mathrm{aut}_1(G/U)_{\mathbb{Q}})$ and that the inclusion $SU(m+k)\to G$ induces an injective map $\pi_*(SU(m+k)_{\mathbb{Q}}) \to \pi_*(G_{\mathbb{Q}})$. This implies that $\{3,\dots,2m-1\} \subset \mathrm{vd}(SU(m+k), G/U)$. Since $d\tau_l = \sum_{i+j} c_i' c_j$ in $(\wedge W)$, it follows that $d\tau_l$ is decomposable for $l \geqslant m+1$. Therefore, Theorem 2.2 yields that $(\lambda_{G,G/U})_{\mathbb{Q}\ i}$ is also injective for $i = \deg\tau_{m+1},\dots,\deg\tau_{m+k}$. We have

$$\mathrm{vd}(SU(m+k), G/U) = \{3,\dots,2m-1, 2m+1,\dots,2(m+k)-1\} = n(SU(m+k)).$$

The latter half of Theorem 2.5 is obtained by comparing the dimension of rational homotopy groups. In fact, it follows from the rational model for $\mathrm{aut}_1(\mathbb{C}P^{m-1})$ mentioned in Example 3.4 that

$$\begin{aligned}\pi_*(\mathrm{out}_1(\mathbb{C}P^{m-1})\otimes\mathbb{Q})^\sharp &\cong H_*(Q(\widetilde{E}/M_u),\delta_0)\\ &\cong \mathbb{Q}\{y\otimes 1_*, y\otimes(x^1)_*,\dots,y\otimes(x^{m-2})_*\}.\end{aligned}$$

This implies that

$$\dim\pi_i(\mathrm{aut}_1(\mathbb{C}P^{m-1}))\otimes\mathbb{Q} = 1 = \dim\pi_i(SU(m))\otimes\mathbb{Q}$$

for $i = 3,\dots,2m-1$. The result follows from the first assertion. This completes the proof. $\square$

We conclude this section with proofs of Lemmas 7.2, 7.3, 7.4 and 7.5.

*Proof of Lemma 7.2.* We regard the free algebra $\wedge(c_1,\dots,c_l)$ as a primitively generated Hopf algebra. Observe that $(c_i^s)_* = \frac{1}{s!}((c_i)_*)^s$. Recall the 0-simplex $u$ in $\Delta E'$ mentioned in (6.3). We have $u(c_j\otimes(p^*c_1)_*)=0$ if $j\neq 1$ and

$$\begin{aligned}u(c_1\otimes(p^*c_1)_*) &= (-1)^{\tau(|p^*(c_1)|)}k^\sharp(p^*(c_1)_*)(\varphi\circ w(c_1))\\ &= (-1)((p^*(c_1)_*)k\circ\varphi\circ w(c_1) = (-1)((p^*(c_1)_*)p^*c_1) = -1.\end{aligned}$$

For the map $k$, see diagram (6.1) and the ensuing paragraph. Thus it follows that

$$\begin{aligned}&\delta_0(\tau_{m+(m-s+1)}\otimes((p^*c_1)^m)_*)\\ &\quad= c_1^m c_{m-s+1}\cdot D^{(m)}(p^*c_1^m)_* = c_1^m c_{m-s+1}\cdot\frac{1}{m!}D^{(m)}(p^*c_1)_*^m\\ &\quad= \frac{1}{m!}c_1^m c_{m-s+1}\cdot\Big((p^*c_1)_*\otimes 1\otimes\cdots\otimes 1 + 1\otimes(p^*c_1)_*\otimes 1\otimes\cdots\otimes 1\\ &\qquad\qquad + \cdots + 1\otimes\cdots\otimes 1\otimes(p^*c_1)_*\Big)^m\\ &\quad= \frac{1}{m!}c_1^m c_{m-s+1}\cdot(\cdots + m!(p^*c_1)_*\otimes\cdots\otimes(p^*c_1)_*\otimes 1+\cdots)\\ &\quad= u(c_1\otimes(p^*c_1)_*)\cdots u(c_1\otimes(p^*c_1)_*)c_{m-s+1} = (-1)^m c_{m-s+1}.\end{aligned}$$

□

*Proof of Lemma 7.3.* Recall the quasi-isomorphism $\varphi_{s+1}\colon \wedge W_{(s+1)}\to\wedge W_{(s)}$ in the proof of Lemma 7.1, which is defined by $\varphi(\tau_{l+1}) = \tau_{s+1}-\tau_{l+1}c_{l+1-(s+1)}$. Let $w$ be the composite $\varphi_1\circ\cdots\circ\varphi_m\colon \wedge W = \wedge W_{(m)}\to\wedge\widetilde{W}$. It is readily seen that $w(\tau_{m+(m-s+1)})$ does not have the element $c_1^m$ as a factor if $s\neq m$. By using the DGA map $\zeta$ in Lemma 5.1, we have

$$\begin{aligned}&\widetilde{\widetilde{\mu}}(\tau_{m+(m-s+1)}\otimes((p^*c_1)^m)_*)\\ &\qquad\qquad= (-1)^{\tau(|p^*c_1^m|)}\langle(1\otimes\varphi)\zeta w(\tau_{m+(m-s+1)}),k^\sharp(p^*c_1^m)_*\rangle = 0.\end{aligned}$$

See (6.1) for the notations. Observe that $H^*(G/K)\cong H^*(\wedge W)\cong\mathbb{Q}[c_1,\dots,c_k]$ for $*\leqslant 2m$. This completes the proof. □

*Proof of Lemma 7.4.* From Lemma 7.1, we see that in $\wedge W_{(m)}$,

$$\begin{aligned}d\tau_{m+1} = &\sum_{i_1+2i_2+\cdots+ki_k=m}(-1)^{i_1+\cdots+i_k}Q^{(m)}_{i_1,\dots,i_k}c_1^{i_1}\cdots c_k^{i_k}c_1\\ &+\sum_{i_1+2i_2+\cdots+ki_k=m-1}(-1)^{i_1+\cdots+i_k}Q^{(m-1)}_{i_1,\dots,i_k}c_1^{i_1}\cdots c_k^{i_k}c_2\\ &+\cdots+\sum_{i_1+2i_2+\cdots+ki_k=l}(-1)^{i_1+\cdots+i_k}Q^{(l)}_{i_1,\dots,i_k}c_1^{i_1}\cdots c_k^{i_k}c_{m-l+1}\\ &+\cdots.\end{aligned}$$

Suppose that $c_1^{i_1}\cdots c_k^{i_k}c_{m-l+1}\otimes((p^*c_1)^s)_*\neq 0$ in $Q(\widetilde{E}/M_u)$, where $i_1+2i_2+\cdots ki_k = l$. Then we have

(1) $l=m$ and $c_1^{i_1}\cdots c_k^{i_k} = c_1^{s-1}c_{m-s+1}$, or

(2) $l\neq m$, $l=s$ and $c_1^{i_1}\cdots c_k^{i_k} = c_1^s$.

It follows that

$$(-1)^{i_1+\cdots+i_k} Q^{(m)}_{i_1,\ldots,i_k} c_1^{s-1} c_{m-s+1} c_1 = (-1)^{s-1+1}(s-1) c_1^s c_{m-s+1}$$

if $(i_1,\ldots,i_k) = (s-1,0,\ldots,0,1,0,\ldots,0)$ with $i_{m-s+1} = 1$ and that

$$Q^{(s)}_{i_1,\ldots,i_k} c_1^s c_{m-s+1} = (-1)^s \cdot 1 \cdot c_1^s c_{m-s+1} \quad \text{if} \quad (i_1,\ldots,i_k) = (s,0,\ldots,0).$$

This fact allows us to conclude that

$$\delta_0(\tau_{m+1} \otimes ((p^* c_1)^s)_*) = (-1)^s (s-1) c_{m-s+1} + (-1)^s c_{m-s+1} = (-1)^s s c_{m-s+1}.$$

We have the result. □

*Proof of Lemma 7.5.* In order to compute $\widetilde{\widetilde{\mu}}$, we determine

$$\langle (1 \otimes \varphi)\zeta w(\tau_{m+1}), k^\sharp (p^* c_1^s)_* \rangle.$$

With the the same notation as in the proof of Lemma 7.3, we have $w(\tau_{m+1}) = \cdots + (-1)^s \tau_{m-s+1} c_1^s + \cdots$. Lemmas 5.1 and 5.2 imply that

$$\begin{aligned}\zeta(\tau_{m-s+1} c_1^s) &= \psi \otimes 1(\tau_{m-s+1} \otimes c_1^s) \\ &= (\tau_{m-s+1} \otimes 1 \otimes 1 + 1 \otimes \tau_{m-s+1} \otimes 1 + \sum_n X_n \otimes X_n' C_n) c_1^s.\end{aligned}$$

Thus it follows that

$$\begin{aligned}\widetilde{\widetilde{\mu}}(\tau_{m+1} \otimes ((p^* c_1)^s)_*) &= (-1)^{\tau(|p^* c_1^s|)} \langle (1 \otimes \varphi)\zeta w(\tau_{m+1}), k^\sharp (p^* c_1^s)_* \rangle \\ &= (-1)^{s+s} \langle (1 \otimes \varphi)\zeta(\tau_{m-s+1} c_1^s), k^\sharp (p^* c_1^s)_* \rangle \\ &= \tau_{m-s+1} \langle \varphi(c_1^s), k^\sharp (p^* c_1^s)_* \rangle + \langle \varphi(\tau_{m-s+1} c_1^s), k^\sharp (p^* c_1^s)_* \rangle \\ &\quad + \sum_n X_n \langle \varphi(X_n' C_n c_1^s), k^\sharp (p^* c_1^s)_* \rangle \\ &= \tau_{m-s+1} \langle k\varphi(c_1^s), (p^* c_1^s)_* \rangle + \langle k\varphi(\tau_{m-s+1} c_1^s), (p^* c_1^s)_* \rangle \\ &\quad + \sum_n X_n \langle k\varphi(X_n' C_n c_1^s), (p^* c_1^s)_* \rangle \\ &= \tau_{m-s+1}.\end{aligned}$$

The last equality is extracted from the commutativity of diagram (6.1). This completes the proof. □

## 8. Proof of Theorem 2.7

This section is devoted to proving Theorem 2.7. The inclusion $\iota\colon \mathrm{aut}_1(X) \to \mathcal{H}_{H,X}$ induces the map $B\iota\colon B\mathrm{aut}_1(X) \to B\mathcal{H}_{H,X}$ with $B\iota \circ B\lambda_{G,X} = B\psi$. Therefore, if $B\psi$ is injective on homology, then so is $B\lambda_{G,X}$.

We shall prove the "only if" part by using the general categorical construction of a classifying space due to May [**21**, Section 12] and by applying a part of the argument in the proof of [**22**, Theorem 3.2] to our case.

We recall here the notion of a $\mathcal{O}$-graph briefly; see [**22**, page 68] for more detail. Let $\mathcal{O}$ be a discrete topological space. Define a $\mathcal{O}$-graph to be a space $\mathcal{A}$ together with the maps $S\colon \mathcal{A} \to \mathcal{O}$ and $T\colon \mathcal{A} \to \mathcal{O}$. The space $\mathcal{O}$ itself is regarded as a $\mathcal{O}$-graph with the arrows $S$ and $T$ the identity map. Let $\mathcal{O}Gr$ be the category of $\mathcal{O}$-graphs whose morphisms are maps $h\colon \mathcal{A} \to \mathcal{A}'$ compatible with the maps $S$ and $T$. Observe that the pullback construction with respect to $S$ and $T$ makes $\mathcal{O}Gr$ a monoidal category. In fact, for $\mathcal{O}$-graphs $\mathcal{A}$ and $\mathcal{A}'$, $\mathcal{A}\Box\mathcal{A}'$ is defined by $\{(a, a') \in \mathcal{A} \times \mathcal{A}' \mid Sa = Ta'\}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be a left $\mathcal{O}$-graph and a right $\mathcal{O}$-graph, respectively; that is, $\mathcal{X}$ is a space with a map $T\colon \mathcal{X} \to \mathcal{O}$ and the space $\mathcal{Y}$ admits only a map $S\colon \mathcal{Y} \to \mathcal{O}$.

Let $\mathcal{M}$ be a monoid in $\mathcal{O}Gr$ the category of $\mathcal{O}$-graphs and $B(\mathcal{Y}, \mathcal{M}, \mathcal{X})$ denote the two-sided bar construction in the sense of May [**21**, Section 12], which is the geometric realization of the simplicial space $B_*$ with $B_j = \mathcal{Y}\Box\mathcal{M}^{\Box j}\Box\mathcal{X}$. We regard a topological monoid $G$ as that in $\mathcal{O}Gr$ with $\mathcal{O} = \{x\}$ the space of a point. Then the classifying space $BG$ we consider here is regarded as the bar construction $B(x, G, x)$.

*Proof of the "only if" part of Theorem 2.7.* Let $\iota'\colon \mathcal{H}_{H,X} \to \mathcal{F}(X, X)$ be the inclusion and $e_*\colon \mathcal{F}(X, X) \to \mathcal{F}(X, X_{\mathbb{Q}})$ the map induced by the localization $e\colon X \to X_{\mathbb{Q}}$. Since $X$ is an $F_0$-space or a space having the rational homotopy type of the product of odd-dimensional spheres by assumption, it follows from [**2**, 3.6 Corollary] and [**11**, Proposition 32.16] that the natural map $[X, X_{\mathbb{Q}}] \to \mathrm{Hom}(H^*(X_{\mathbb{Q}}; \mathbb{Q}), H^*(X; \mathbb{Q}))$ is bijective. We see that $e \circ \varphi \simeq e$ for any $\varphi \in \mathcal{H}_{H,X}$. Therefore, the composite $e_* \circ \iota'$ factors through the connected component $\mathcal{F}(X, X_{\mathbb{Q}}; e)$ of $\mathcal{F}(X, X_{\mathbb{Q}})$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{H}_{H,X} \arrow[dr, "e_*\circ\iota'"] & & \\ & \mathcal{F}(X, X_{\mathbb{Q}}; e) & \mathrm{aut}_1(X_{\mathbb{Q}}) \arrow[l, "e^*", "\simeq"'] \\ \mathrm{aut}_1(X) \arrow[uu, "\iota"] \arrow[ur, "e_*"'] & & \end{tikzcd}$$

in which the induced map $e^*$ is a homotopy equivalence.

Define $\mathcal{O}$ to be the discrete space with two points $x$ and $y$. Let $\mathcal{M}$ be the monoid in $\mathcal{O}Gr$ defined by $\mathcal{M}(x, x) = \mathrm{aut}_1(X)$, $\mathcal{M}(y, y) = \mathrm{aut}_1(X_{\mathbb{Q}})$ and $\mathcal{M}(x, y) = \mathcal{F}(X, X_{\mathbb{Q}}; e)$ with $\mathcal{M}(y, x)$ empty. Arrows $S, T\colon \mathcal{M}(a, b) \to \mathcal{O}$ are defined by $S(z) = a$ and $T(z) = b$ for $z \in \mathcal{M}(a, b)$. Moreover, we define another monoid $\mathcal{M}'$ in $\mathcal{O}Gr$ by $\mathcal{M}'(x, x) = \mathcal{H}_{H,X}$, $\mathcal{M}'(y, y) = \mathrm{aut}_1(X_{\mathbb{Q}})$, $\mathcal{M}'(x, y) = \mathcal{F}(X, X_{\mathbb{Q}}; e)$ and $\mathcal{M}'(y, x) = \phi$ with arrows defined immediately as mentioned above.

Consider the inclusions $i\colon \mathrm{aut}_1(X) \to \mathcal{M}$, $j\colon \mathrm{aut}_1(X_{\mathbb{Q}}) \to \mathcal{M}$, $i'\colon \mathcal{H}_{H,X} \to \mathcal{M}'$ and $j'\colon \mathrm{aut}_1(X_{\mathbb{Q}}) \to \mathcal{M}'$. They induce the maps between classifying spaces which fit into the commutative diagram

$$\begin{tikzcd} & B\mathcal{H}_{H,X} \arrow[r, "Bi'"] & B(\mathcal{O}, \mathcal{M}', \mathcal{O}) & \\ BG \arrow[ur, "B\psi"] \arrow[dr, "B\lambda_{G,X}"'] & & & B\mathrm{aut}_1(X_{\mathbb{Q}}), \arrow[ul, "Bj'"', "\simeq"] \arrow[dl, "Bj", "\simeq"'] \\ & B\mathrm{aut}_1(X) \arrow[uu, "B\iota"] \arrow[r, "Bi"'] & B(\mathcal{O}, \mathcal{M}, \mathcal{O}) \arrow[uu, "B\widetilde{\iota}"] & \end{tikzcd}$$

where $\widetilde{\iota}\colon \mathcal{M} \to \mathcal{M}'$ is the morphism of monoids in $\mathcal{O}Gr$ induced by the inclusion $\iota\colon \mathrm{aut}_1(X) \to \mathcal{H}_{H,X}$. The proof of [**22**, Theorem 3.2] enables us to conclude that the maps $Bj$ and $Bj'$ are homotopy equivalences. The map $\Omega((Bj)^{-1} \circ (Bi))$ coincides with the composite $(e^*)^{-1} \circ e_*\colon \mathrm{aut}_1(X) \to \mathcal{F}(X, X_{\mathbb{Q}}; e) \to \mathrm{aut}_1(X_{\mathbb{Q}})$ up to weak equivalence; see [**22**, Theorem 3.2(i)]. Moreover, the map $e_*\colon \mathrm{aut}_1(X) \to \mathcal{F}(X, X_{\mathbb{Q}}; e)$

is a localization; see [**14**]. These facts yield that $\pi_*(\Omega Bi) \otimes \mathbb{Q}$ is an isomorphism and hence so is $\pi_*(Bi) \otimes \mathbb{Q}$. Thus the localized map $(Bi)_{\mathbb{Q}}$ is a weak equivalence. This implies that $(Bi)_* \colon H_*(B\mathrm{aut}_1(X); \mathbb{Q}) \to H_*(B(\mathcal{O}, \mathcal{M}, \mathcal{O}); \mathbb{Q})$ is an isomorphism. The commutative diagram (7.1) enables us to conclude that $H_*(B\psi; \mathbb{Q})$ is injective if so is $H_*(B\lambda_{G,X}; \mathbb{Q})$. This completes the proof. $\square$

As we pointed out in the introduction, [**16**, Proposition 4.8] follows from Theorems 2.5 and 2.7. In fact, suppose that $M$ is the flag manifold $U(m)/U(m_1) \times \cdots \times U(m_l)$, and $G = SU(m)$. Then as is seen in Remark 8.1 below $(\lambda_{G,M})_* \colon \pi_*(BG) \otimes \mathbb{Q} \to \pi_*(B\mathrm{aut}_1(M)) \otimes \mathbb{Q}$ is injective if and only if

$$(B\lambda_{G,M})^* \colon H^*(BG) \to H^*(B\mathrm{aut}_1(M))$$

is surjective.

*Remark* 8.1. Suppose that $M$ is a homogeneous space of the form $G/H$ for which rank $G =$ rank $H$. The main theorem in [**28**] due to Shiga and Tezuka implies that $\pi_{2i}(\mathrm{aut}_1(M)) \otimes \mathbb{Q} = 0$ for any $i$. Thus $H^*(B\mathrm{aut}_1(M); \mathbb{Q})$ is a polynomial algebra generated by the graded vector space $(sV)^\sharp$, where $(sV)_l = \pi_{l-1}(\mathrm{aut}_1(M))$. Therefore, the dual map to the Hurewicz homomorphism

$$\Xi^\sharp \colon H^*(B\mathrm{aut}_1(M); \mathbb{Q}) \to \mathrm{Hom}(\pi_*(B\mathrm{aut}_1(M)), \mathbb{Q})$$

induces an isomorphism on the vector space of indecomposable elements; see [**11**, page 173] for example. Thus the commutative diagram

$$\begin{tikzcd}
H^*(BG;\mathbb{Q}) \arrow[d, "\Xi^\sharp"'] & H^*(B\mathrm{aut}_1(M);\mathbb{Q}) \arrow[l, "(B\lambda_{G,M})^*"'] \arrow[d, "\Xi^\sharp"] \\
\mathrm{Hom}(\pi_*(BG),\mathbb{Q}) & \mathrm{Hom}(\pi_*(B\mathrm{aut}_1(M)),\mathbb{Q}) \arrow[l, "((B\lambda_{G,M})_*)^\sharp"]
\end{tikzcd}$$

yields that the map $(B\lambda_{G,M})^*$ is surjective if $G$ is rationally visible in $\mathrm{aut}_1(M)$. We also see that the induced map $(B\psi)_* \colon H_j(BG) \to H_j(B\mathcal{H}_{H,G/U})$ is injective for each triple $(G, U, i)$ in Tables 1 and 2 if $j \in \mathrm{vd}(G, G/U)$.

## 9. The sets $\mathrm{vd}(G, G/U)$ of visibility degrees in Tables 1 and 2

In this section, we deal with the visibility degrees described in Tables 1 and 2 in the introduction.

For the case where the homogeneous space $G/U$ has the rational homotopy type of the sphere, the assertions on the visibility degrees follow from the latter half of Theorem 2.2. In fact, the argument in Example 2.4 does work well to obtain such results. The details are left to the reader. The results for (11) and for (17) follow from Theorems 2.5 and 2.6, respectively. We are left to verify the visibility degrees for the cases (1), (5), (6), (6)$'$ (16) and (19).

*Case* (1). It is well-known that $(B\iota)^*(p_i) = (-1)^i(\chi^2 p'_{i-1} + p'_i)$ for the induced map $(B\iota)^* \colon H^*(BSO(2m+1)) \to H^*(B(SO(2) \times SO(2m-1))$, where $p'_i$ is the $i$th Pontrjagin class in $H^*(B(SO(2m-1)) \cong \mathbb{Q}[p'_1, \ldots, p'_{m-1}]$; see [**23**].

We can construct a Sullivan model $(\wedge W, d)$ for the Grassmann manifold $M := SO(2m+1)/SO(2) \times SO(2m-1)$ for which $\wedge W = \wedge(\chi, p'_1, \ldots, p'_{m-1}, \tau_2, \tau_4, \ldots, \tau_{2m})$

and $d(\tau_{2i}) = (-1)^i(\chi^2 p'_{i-1} + p'_i)$ for $1 \leqslant i \leqslant m$. We see that there exists a quasi-isomorphism $w\colon (\wedge(\chi, \tau_{2m}), d\tau_{2m} = -\chi^{2m}) \to (\wedge W, d)$ such that $w(\chi) = \chi$ and

$$w(\tau_{2m}) = \chi^{2(m-1)}\tau_2 + \cdots + \chi^2\tau_{2(m-1)} + \tau_{2m}.$$

In view of the rational model $\widetilde{\widetilde{\mu}}\colon E'/M_u \to \wedge V_G$ for $\lambda_{G,M}\colon SO(2m+1) \to \mathrm{aut}_1(M)$ mentioned in (6.2) and Theorem 4.3, it follows from Lemma 5.2 that

$$\begin{aligned}\widetilde{\widetilde{\mu}}(\tau_{2m} \otimes (\chi^{2l})_*)) &= (-1)^{\tau(|\chi^{2l}|)}\langle \zeta \circ w(\tau_{2m}), (\chi^{2l})_*\rangle \\ &= \langle \chi^{2(m-1)}\tau_2 + \cdots + \chi^2\tau_{2(m-1)} + \tau_{2m},\ (\chi^{2l})_*\rangle \\ &= \tau_{2(m-l)},\end{aligned}$$

where $\zeta$ is the Sullivan representative for the action $SO(2m-1) \times M \to M$ described in Lemma 5.1. We have the result.

The same argument does work well to prove the result for the case (16).

*Case* (19). Let $\iota\colon Spin(9) \to F_4$ be the inclusion map. Without loss of generality, we can assume that the induced map

$$(B\iota)^*\colon H^*(BF_4;\mathbb{Q}) = \mathbb{Q}[y_4, y_{12}, y_{16}, y_{24}] \to H^*(BSpin(9);\mathbb{Q}) = \mathbb{Q}[y_4, y_8, y_{12}, y_{16}]$$

satisfies the condition that $(B\iota)^*(y_i) = y_i$ for $i = 4, 12, 16$ and $(B\iota)^*(y_{24}) = y_8^3$, where $\deg y_i = i$. This fact follows from a usual argument with the Eilenberg-Moore spectral sequence for the fibration $\mathcal{L}P^2 \to BSpin(9) \xrightarrow{B\iota} BF_4$. By virtue of Lemmas 5.1 and 5.2, we see that there exists a model for the linear action $F_4 \times \mathcal{L}P^2 \to \mathcal{L}P^2$ of the form

$$\zeta\colon (\wedge(x'_{23}) \otimes \wedge(y_8), d) \to (\wedge(x_3, x_{11}, x_{15}, x_{23}) \otimes \wedge(x'_{23} \otimes \wedge(y_8), d')$$

with $\zeta(x'_{23}) = x_{23} \otimes 1 \otimes 1 + 1 \otimes x'_{23} \otimes 1$, where $d(x'_{23}) = y_8^3$, $d'(x_j) = 0$ for $j = 3, 11, 15, 23$. In fact, for dimensional reasons, we write $\zeta(x'_{23}) = 1 \otimes x'_{23} \otimes 1 + x_{23} \otimes 1 \otimes 1 + cx_{15} \otimes 1 \otimes y_8$ with a rational number $c$. By definition, we see that $\zeta = \psi \otimes 1$, where $\psi$ denotes the DGA map in Lemma 5.2. Since the image of each element with degree less than 24 by $(B\iota)^*$ does not have the element $y_8$ as a factor, it follows that $c = 0$. Observe that $\wedge V_{BF_4}$-action on $\wedge V_{BSpin(9)}$ is induced by the map $(B\iota)^*$. The dual to the map $(\lambda_*)_i\colon \pi_i(F_4) \otimes \mathbb{Q} \to \pi_i(\mathrm{aut}_1(F_4/Spin(9))) \otimes \mathbb{Q}$ is regarded as the induced map $H(Q(\widetilde{\widetilde{\mu}}))\colon H^*(Q(\widetilde{E}/M_u), \delta_0) \to V_G = \mathbb{Q}\{x_3, x_{11}, x_{15}, x_{23}\}$ in Theorem 4.1. We see that $Q(\widetilde{E}/M_u) = \mathbb{Q}\{y_8 \otimes 1_*, x'_{23} \otimes 1_*, x'_{23} \otimes (y_8)_*, x'_{23} \otimes (y_8^2)_*\}$, $\delta_0(x'_{23} \otimes (y_8^2)_*) = 3y_8 \otimes 1_*$, $\delta_0(x'_{23} \otimes 1_*) = \delta_0(x'_{23} \otimes (y_8^1)_*) = 0$; see Example 3.4. Furthermore the direct computation with (6.2) shows that $Q(\widetilde{\widetilde{\mu}})(x'_{23} \otimes 1_*) = \pm x_{23}$ and $Q(\widetilde{\widetilde{\mu}})(x'_{23} \otimes (y_8)_*) = 0$. This implies that $\mathrm{vd}(F_4, \mathcal{L}P^2) = \{23\}$.

*Case* (5). The inclusion $\iota\colon SO(4) \to G_2$ induces the ring homomorphism

$$(B\iota)^*\colon H^*(BG_2) \cong \mathbb{Q}[y_4, y_{12}] \to H^*(BSO(4)) \cong \mathbb{Q}[p_1, \chi],$$

where $\deg p_1 = 4$ and $\deg \chi = 4$. It is immediate that $(B\iota)^*(y_{12})$ is decomposable for dimensional reasons. From Example 3.4, we see that $\pi_*(\mathrm{aut}_1(\mathbb{H}P^2)) \cong \mathbb{Q}\{y \otimes 1_*, y \otimes (x^1)_*\}$, where $\deg y \otimes 1_* = 11$ and $\deg y \otimes (x^1)_* = 7$. It follows from Theorem 2.2 that $\mathrm{vd}(G_2, G_2/SO(4)) = \{11\}$.

*Case* (6). Let $T^2$ be the standard maximal torus of $U(2)$. We assume that $G_2 \supset U(2) \supset T^2$ without loss of generality. Then the inclusion $W(G_2) \supset W(U(2))$ of Weyl groups gives the inclusions

$$\begin{tikzcd} \mathbb{Q}[t_1,t_2]^{W(G_2)} \arrow[r, tail] & \mathbb{Q}[t_1,t_2]^{W(U(2))} \arrow[r, tail] & \mathbb{Q}[t_1,t_2] \\ H^*(BG_2) \arrow[u, "\cong"] & H^*(BU(2)) \arrow[u, "\cong"] & H^*(BT^2). \arrow[u, "\cong"] \end{tikzcd}$$

The result [**29**, page 212, Example 3] implies that there exist generators $y_4$, $y_{12}$ of $H(BG_2)$ such that $H(BG_2) \cong \mathbb{Q}[y_4, y_{12}]$ and $y_4 = t_1^2 - t_1t_2 + t_2^2$, $y_{12} = (t_1t_2^2 - t_1^2t_2)^2$ in $\mathbb{Q}[t_1,t_2]^{W(G_2)}$. Since the Chern classes $c_1, c_2 \in H^*(BU(2))$ are regarded as $t_1 + t_2$ and $t_1t_2$, respectively in $\mathbb{Q}[t_1,t_2]^{W(U(2))}$, it follows that

$$(B\iota)^*(y_4) = c_1^2 - 3c_2 \quad \text{and} \quad (B\iota)^*(y_{12}) = c_1^2c_2^2 - 4c_2^3,$$

where $\iota\colon U(2) \to G_2$ is the inclusion. Put $\tilde{c}_2 = -\frac{1}{3}c_1^2 + c_2$. Then it is readily seen that $(B\iota)^*(-\frac{1}{3}y_4) = \tilde{c}_2$ and

$$(B\iota)^*(y_{12}) = -\frac{1}{27}c_1^6 - \frac{2}{3}c_1^4\tilde{c}_2 - 3c_1^2\tilde{c}_2^2 - 4\tilde{c}_2^3.$$

By the direct computation implies that

$$\begin{aligned} &(B\iota)^*(-\frac{1}{3}y_4) \otimes 1_* - (B\iota)^*(y_{12}) \otimes (-\frac{3}{2})(c_1^4)_* \\ &= \tilde{c}_2 \otimes 1_* + \frac{3}{2}\big(-\frac{1}{27}c_1^6 - \frac{2}{3}c_1^4\tilde{c}_2 - 3c_1^2\tilde{c}_2^2 - 4\tilde{c}_2^3\big) \otimes (c_1^4)_* \\ &\equiv \tilde{c}_2 \otimes 1_* - \tilde{c}_2 \otimes 1_* \equiv 0 \end{aligned}$$

modulo decomposable elements in $(H^*(BU(2)){:}H^*(G_2/U(2)))/M_u$. It is immediate that the element $(B\iota)^*(y_{12})$ is decomposable. By virtue of Theorem 2.2, we have $\mathrm{vd}(G_2, G_2/U(2)) = \{3, 11\}$. The same argument works well to deduce the result for the case $(6)'$.

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