

ON THE 3-ARROW CALCULUS FOR HOMOTOPY CATEGORIES

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Abstract

We develop a localisation theory for certain categories, yielding a 3-arrow calculus: Every morphism in the localisation is represented by a diagram of length 3, and two such diagrams represent the same morphism if and only if they can be embedded in a 3-by-3 diagram in an appropriate way. Applications include the localisation of an arbitrary Quillen model category with respect to its weak equivalences as well as the localisation of its full subcategories of cofibrant, fibrant and bifibrant objects, giving the homotopy category in all four cases. In contrast to the approach of DWYER, HIRSCHHORN, KAN and SMITH, the Quillen model category under consideration does not need to admit functorial factorisations.

1. Introduction

The construction of the homotopy category of a Quillen model category, that is, the localisation with respect to its set of weak equivalences, is usually done by a construction that works for arbitrary subsets of morphisms to be formally inverted, called Gabriel-Zisman localisation. However, the morphisms in the Gabriel-Zisman localisation are, in general, represented by zigzags

$$\longrightarrow \longleftarrow \approx \longleftarrow \longrightarrow \cdots \longleftarrow \approx \longleftarrow \longrightarrow$$

of finite but arbitrary length, where the “backward” arrows are in the set of those morphisms to be formally inverted. Furthermore, in the Gabriel-Zisman localisation one has, in general, no convenient criterion to decide whether two zigzags represent the same morphism in the localisation.

For a Quillen model category \mathcal{M} , one can do better: Recently, DWYER, HIRSCHHORN, KAN and SMITH developed in [4, sec. 10, sec. 36] a 3-arrow calculus for the homotopy category of \mathcal{M} , provided \mathcal{M} admits functorial factorisations (cf. [4, sec. 9.1, ax. MC5]). That is, they showed that each morphism in $\text{Ho } \mathcal{M}$ is represented by a

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diagram

$$\leftarrow \approx \text{---} \longrightarrow \leftarrow \approx \text{---} \ ,$$

and, moreover, that two of these diagrams represent the same morphism if and only if they can be embedded as the top and the bottom row in a commutative diagram of the following form.

$$\begin{array}{ccccc} \leftarrow \approx \text{---} & \longrightarrow & \leftarrow \approx \text{---} & & \\ \parallel & \uparrow \wr & \uparrow \wr & \parallel & \\ \leftarrow \approx \text{---} & \longrightarrow & \leftarrow \approx \text{---} & & \\ \parallel & \downarrow \wr & \downarrow \wr & \parallel & \\ \leftarrow \approx \text{---} & \longrightarrow & \leftarrow \approx \text{---} & & \\ \parallel & \uparrow \wr & \uparrow \wr & \parallel & \\ \leftarrow \approx \text{---} & \longrightarrow & \leftarrow \approx \text{---} & & \end{array}$$

To do this, they introduced the notion of a *homotopical category admitting a 3-arrow calculus* [4, sec. 33.1, 36.1] and developed a 3-arrow calculus in this context [4, sec. 36.3].

In this article, we introduce the concept of a uni-fractionable category, see definition 3.1. Our main result is the construction of a localisation of a uni-fractionable category (with respect to its set of denominators) that satisfies a 3-arrow calculus in the sense described above, see theorem 5.13. In contrast to [4], we will not make use of the Gabriel-Zisman localisation. Instead, we will give an elementary ad hoc construction of a localisation of a uni-fractionable category, in the spirit of the Ore localisation for a 2-arrow calculus. ⁽¹⁾

Both in the approach of [4, sec. 36.1] and in our uni-fractionable categories, one has three distinguished kinds of morphisms, which, in our terminology, are called denominators, S-denominators and T-denominators. The denominators are the morphisms to be formally inverted, while the S- and T-denominators are particular denominators. The essential stipulations in [4, sec. 36.1] are that every denominator factors functorially into an S-denominator followed by a T-denominator ⁽²⁾ and that one has functorial Ore completions along S-denominators resp. T-denominators. For uni-fractionable categories, we omit the stipulations of functoriality; instead, we require the existence of weakly universal Ore completions along S-denominators resp. T-denominators.

The advantage of uni-fractionable categories is that functoriality of factorisations is not needed. On the one hand, this is convenient for applications. On the other hand, the theory developed here can be applied to arbitrary Quillen model categories. Moreover, it can also be applied to the full subcategories of the cofibrant, fibrant resp. bifibrant objects of a Quillen model category. As a consequence, all of them admit a 3-arrow calculus.

¹It is easy to show that every morphism in the Gabriel-Zisman localisation of a uni-fractionable category can be represented by a diagram of length 3 (cf. the definition of the composition in proposition 5.2). However, the author does not know how to prove in that context that two of these diagrams represent the same morphism if and only if they can be embedded in a 3-by-3 diagram as above.

²The S resp. the T should remind us of the fact that the S-denominator resp. the T-denominator in a factorisation has the same source resp. the same target as the factorised morphism.

Furthermore, a derivable category in the sense of CISINSKI [2, sec. 2.25] ⁽³⁾, which is a self-dual generalisation of a category of fibrant objects in the sense of K. BROWN [1, sec. 1], admits a 3-arrow calculus, provided stronger variants of the factorisation axioms and the axioms which ensure stability of acyclic cofibrations under pushouts resp. of acyclic fibrations under pullbacks hold. For the relationship of CISINSKI's approach with other axiom systems, see [14, sec. 2].

Outline

We recall in section 2 some notions of localisation theory and indicate how quotients of (ordered) graphs with respect to so-called graph congruences can be constructed. In section 3, uni-fractionable categories are introduced. Recall that the aim of this article is to construct a localisation of a uni-fractionable category with respect to its set of denominators. To this end, we proceed in two steps: In section 4, we assign to a uni-fractionable category a certain graph, its 3-arrow graph, and introduce a graph congruence on this graph. Then, in section 5, it turns out that the quotient graph has a canonically given category structure, and we will show that this category is a localisation of the uni-fractionable category we started with. Our main theorem 5.13 then gives a criterion on when two 3-arrows represent the same morphism in the localisation. Finally, in section 6, we show how Quillen model categories and derivable categories (under additional conditions) fit into this framework.

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Conventions and notations

We use the following conventions and notations.

- The composite of morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is usually denoted by $fg: X \rightarrow Z$. The composite of functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ is usually denoted by $G \circ F: \mathcal{C} \rightarrow \mathcal{E}$.

- Given a coproduct C of X_1 and X_2 , the embedding $X_k \rightarrow C$ is denoted by $\text{emb}_k = \text{emb}_k^C$ for $k \in \{1, 2\}$. Given morphisms $f_k: X_k \rightarrow Y$ for $k \in \{1, 2\}$, the induced morphism $C \rightarrow Y$ is denoted by $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^C$.

- Given an initial object I , the unique morphism $I \rightarrow X$ to an object X will be denoted by $\text{ini} = \text{ini}_X = \text{ini}_X^I$.

- Given a category admitting finite coproducts and objects X_1, X_2 , we denote by $X_1 \amalg X_2$ a chosen coproduct and by \mathfrak{i} a chosen initial object. Analogously, given morphisms $f_k: X_k \rightarrow Y_k$ for $k \in \{1, 2\}$, the coproduct of f_1 and f_2 is denoted by $f_1 \amalg f_2$.

- Given a category admitting finite coproducts \mathcal{C} and a category \mathcal{D} , we say that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ preserves finite coproducts if $F\mathfrak{i}$ is an initial object in \mathcal{D} , and if, given $X_1, X_2 \in \text{Ob}\mathcal{C}$, the object $F(X_1 \amalg X_2)$ is a coproduct of FX_1 and FX_2 , where the embeddings are given by $\text{emb}_k^{F(X_1 \amalg X_2)} = F(\text{emb}_k^{X_1 \amalg X_2})$ for $k \in \{1, 2\}$.

³Also called an Anderson-Brown-Cisinski premodel category by RĂDULESCU-BANU [14, def. 1.1.3].

- By a weak pushout rectangle (resp. weak pullback rectangle) we understand a quadrangle having the universal property of a pushout rectangle (resp. pullback rectangle) except for the uniqueness of the induced morphism.
- Given integers $a, b \in \mathbb{Z}$, we write $[a, b] := \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ for the set of integers lying between a and b .

2. Preliminaries

In this section, we give some preliminaries on localisations of categories and quotient graphs with respect to graph congruences.

Localisations of categories

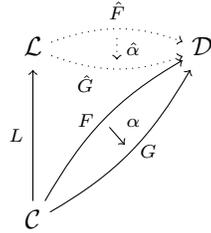
We suppose given a category \mathcal{C} . A *denominator set* in \mathcal{C} is a subset $D \subseteq \text{Mor } \mathcal{C}$. We will consider denominator sets with special properties later in this article, but at the moment, a denominator set D is just an arbitrary subset of $\text{Mor } \mathcal{C}$. Informally, it is a subset singled out with the “intention of localising with respect to it”, in the following sense.

A *localisation* of \mathcal{C} with respect to a denominator set D in \mathcal{C} consists of a category \mathcal{L} and a functor $L: \mathcal{C} \rightarrow \mathcal{L}$ such that the following axioms hold.

(Inv) *Invertibility.* For all $d \in D$, the morphism Ld is invertible.

(1-uni) *1-universality.* Given a category \mathcal{D} and a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that Fd is invertible for all $d \in D$, there exists a unique functor $\hat{F}: \mathcal{L} \rightarrow \mathcal{D}$ with $F = \hat{F} \circ L$.

(2-uni) *2-universality.* We suppose given a category \mathcal{D} and functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ such that Fd and Gd are invertible for all $d \in D$, and we denote by $\hat{F}: \mathcal{L} \rightarrow \mathcal{D}$ resp. $\hat{G}: \mathcal{L} \rightarrow \mathcal{D}$ the unique functor with $F = \hat{F} \circ L$ resp. $G = \hat{G} \circ L$. Given a transformation $\alpha: F \rightarrow G$, there exists a unique transformation $\hat{\alpha}: \hat{F} \rightarrow \hat{G}$ such that $\hat{\alpha}_{LX} = \alpha_X$ for all $X \in \text{Ob } \mathcal{C}$.



By abuse of notation, we refer to the localisation as well as to its underlying category just by \mathcal{L} . The functor L is said to be the *localisation functor* of the localisation \mathcal{L} . Given a localisation \mathcal{L} of \mathcal{C} with respect to D with localisation functor $L: \mathcal{C} \rightarrow \mathcal{L}$, we write $\text{loc} = \text{loc}^{\mathcal{L}} := L$.

GABRIEL and ZISMAN have shown in [7, sec. 1.1] that there exists a localisation of every category \mathcal{C} with respect to an arbitrary denominator set D in \mathcal{C} . We will not make use of this result. Rather, given a uni-fractionable category, see definition 3.1, we construct a localisation directly, cf. propositions 5.2 and 5.5.

Saturatedness

We suppose given a category \mathcal{C} , a denominator set D in \mathcal{C} , and a localisation \mathcal{L} of \mathcal{C} with respect to D . By definition of a localisation, $\text{loc}(d)$ is invertible for every $d \in D$. But in general, not every morphism f in \mathcal{C} for which $\text{loc}(f)$ is invertible in \mathcal{L} has to be an element of D . The denominator set D is said to be *saturated* if $f \in D$ for all $f \in \text{Mor } \mathcal{C}$ with $\text{loc}(f)$ invertible in \mathcal{L} . We use the following notions to indicate how far D is away from this property.

The denominator set D is said to be *multiplicative* if it fulfills:

(Cat) *Multiplicativity*. For all $d, e \in D$ with $\text{Target } d = \text{Source } e$, their composite de is in D , and for every object X in \mathcal{C} , the identity 1_X is in D .

The denominator set D is said to be *semi-saturated* if it is multiplicative and fulfills:

(2 of 3) *2 out of 3 axiom*. We suppose given morphisms f and g in \mathcal{C} with $\text{Target } f = \text{Source } g$. If two out of the morphisms f, g, fg are in D , then so is the third.

Finally, the denominator set D is said to be *weakly saturated* if it is multiplicative and fulfills:

(2 of 6) *2 out of 6 axiom*. We suppose given morphisms f, g, h in \mathcal{C} with $\text{Target } f = \text{Source } g$ and $\text{Target } g = \text{Source } h$. If $fg, gh \in D$, then $f, g, h, fgh \in D$.

Saturatedness implies weak saturatedness, weak saturatedness implies semi-saturatedness, and semi-saturatedness implies multiplicativity (the last implication holds by definition).

Graph congruences and quotient graphs

We suppose given an (oriented) graph \mathcal{G} . An equivalence relation \equiv on $\text{Arr } \mathcal{G}$ is said to be a *graph congruence* on \mathcal{G} if $\text{Source } a = \text{Source } \tilde{a}$ and $\text{Target } a = \text{Target } \tilde{a}$ for all $a, \tilde{a} \in \text{Arr } \mathcal{G}$ with $a \equiv \tilde{a}$. Given a graph congruence \equiv on \mathcal{G} , the *quotient graph* of \mathcal{G} with respect to \equiv is the graph \mathcal{G}/\equiv with $\text{Ob } \mathcal{G}/\equiv := \text{Ob } \mathcal{G}$, $\text{Arr } \mathcal{G}/\equiv := (\text{Arr } \mathcal{G})/\equiv$ and $\text{Source } [a]_{\equiv} := \text{Source } a$, $\text{Target } [a]_{\equiv} := \text{Target } a$ for $a \in \text{Arr } \mathcal{G}$. The graph morphism $\text{quo} = \text{quo}^{\mathcal{G}/\equiv}: \mathcal{G} \rightarrow \mathcal{G}/\equiv$ given by $\text{quo}(X) := X$ and $\text{quo}(a) := [a]_{\equiv}$ is called the *quotient graph morphism*.

The quotient graph of \mathcal{G} with respect to a graph congruence \equiv fulfills the following universal property. Given $a, \tilde{a} \in \text{Arr } \mathcal{G}$ with $a \equiv \tilde{a}$, we have $\text{quo}(a) = \text{quo}(\tilde{a})$. For every graph \mathcal{H} and every graph morphism $F: \mathcal{G} \rightarrow \mathcal{H}$ with $Fa = F\tilde{a}$ for $a, \tilde{a} \in \text{Arr } \mathcal{G}$ with $a \equiv \tilde{a}$, there exists a unique graph morphism $\overline{F}: \mathcal{G}/\equiv \rightarrow \mathcal{H}$ with $F = \overline{F} \circ \text{quo}$.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{F} & \mathcal{H} \\ \text{quo} \downarrow & \searrow \overline{F} & \\ \mathcal{G}/\equiv & & \end{array}$$

3. Uni-fractionable categories

Definition 3.1 (uni-fractionable category). A *uni-fractionable category* ⁽⁴⁾ consists of a category \mathcal{C} together with a semi-saturated denominator set D in \mathcal{C} and multiplicative subsets $S, T \subseteq D$, such that the following axioms hold.

⁴There exists also the notion of a *fractionable category*, cf. the author's forthcoming doctoral thesis.

(WU) *Weakly universal Ore completions.* Given morphisms i and f in \mathcal{C} with $i \in S$ and $\text{Source } i = \text{Source } f$, there exists a weak pushout rectangle in \mathcal{C} as displayed below on the left, such that $i' \in S$. Dually, given morphisms p and f in \mathcal{C} with $p \in T$ and $\text{Target } p = \text{Target } f$, there exists a weak pullback rectangle in \mathcal{C} as displayed below on the right, such that $p' \in T$.

$$\begin{array}{ccc} & \xrightarrow{f'} & \\ i \uparrow & \dashrightarrow & i' \\ & \xrightarrow{f} & \end{array} \qquad \begin{array}{ccc} & \xrightarrow{f'} & \\ p' \downarrow & \dashrightarrow & p \\ & \xrightarrow{f} & \end{array}$$

(Fac) *Factorisations.* For every $d \in D$, there exist $i \in S$ and $p \in T$ with $d = ip$.

$$\begin{array}{ccc} & \nearrow i & \\ & d & \\ & \searrow p & \end{array}$$

By abuse of notation, we refer to the uni-fractionable category as well as to its underlying category just by \mathcal{C} . The elements of D resp. S resp. T are called *denominators* resp. *S-denominators* resp. *T-denominators* in \mathcal{C} .

Given a uni-fractionable category \mathcal{C} with set of denominators D , set of S-denominators S and set of T-denominators T , we write $\text{Den } \mathcal{C} := D$, $\text{SDen } \mathcal{C} := S$, $\text{TDen } \mathcal{C} := T$. In diagrams, a denominator d resp. an S-denominator i resp. a T-denominator p in \mathcal{C} will usually be depicted as

$$\xrightarrow{\approx d} \quad \text{resp.} \quad \xrightarrow{\circ i} \quad \text{resp.} \quad \xrightarrow{\square p} .$$

Some examples of uni-fractionable categories can be found in section 6.

Definition 3.2 (denominator preserving functor). Given uni-fractionable categories \mathcal{C} and \mathcal{D} , a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to *preserve denominators* if Fd is a denominator in \mathcal{D} for every denominator d in \mathcal{C} .

4. The 3-arrow graph

We want to construct a localisation $\text{Frac } \mathcal{C}$ of a uni-fractionable category \mathcal{C} with respect to its set of denominators $\text{Den } \mathcal{C}$. To this end, we begin in this section by introducing its 3-arrow graph $\text{AG } \mathcal{C}$ and a graph congruence \equiv on $\text{AG } \mathcal{C}$.

In this section, we suppose given a uni-fractionable category \mathcal{C} .

Definition 4.1 (3-arrow shape). The graph

$$0 \xleftarrow{\tau} 1 \xrightarrow{\nu} 2 \xleftarrow{\sigma} 3$$

is said to be the *3-arrow shape* and will be denoted by Θ .

Recall that a *diagram* of shape Θ in \mathcal{C} is just a graph morphism $A: \Theta \rightarrow \mathcal{C}$. Given a diagram D of shape Θ in \mathcal{C} , we write $D_i := D(i)$ for $i \in \text{Ob } \Theta$ and $D_a := D(a)$ for $a \in \text{Arr } \Theta$. Given diagrams D and E , a *diagram morphism* from D to E is a family $f = (f_i)_{i \in \text{Ob } \Theta}$ in $\text{Mor } \mathcal{C}$ with $D_a f_j = f_i E_a$ for all arrows $a: i \rightarrow j$ in Θ . The category consisting of diagrams of shape Θ in \mathcal{C} as objects and diagram morphisms between those diagrams as morphisms will be denoted by \mathcal{C}^Θ .

Definition 4.2 (3-arrow graph). The 3-arrow graph of \mathcal{C} is defined to be the graph $\text{AG}\mathcal{C}$ with object set $\text{Ob}\text{AG}\mathcal{C} := \text{Ob}\mathcal{C}$ and arrow set $\text{Arr}\text{AG}\mathcal{C} := \{A \in \text{Ob}\mathcal{C}^{\ominus} \mid A_{\sigma}, A_{\tau} \in \text{Den}\mathcal{C}\}$. The source resp. the target of $A \in \text{Arr}\text{AG}\mathcal{C}$ are defined by $\text{Source } A := A_0$ resp. $\text{Target } A := A_3$.

An arrow A in $\text{AG}\mathcal{C}$ is called a 3-arrow in \mathcal{C} . Given a denominator $b: \tilde{X} \rightarrow X$, a morphism $f: \tilde{X} \rightarrow \tilde{Y}$ and a denominator $a: Y \rightarrow \tilde{Y}$ in \mathcal{C} , we abuse notation and denote the unique 3-arrow A with $A_{\tau} = b, A_{\nu} = f, A_{\sigma} = a$ by $(b, f, a) := A$. Moreover, we use the notation $(b, f, a): X \leftarrow \tilde{X} \rightarrow \tilde{Y} \leftarrow Y$.

$$X \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y$$

Our next step will be the introduction of an equivalence relation on the arrow set of the 3-arrow graph.

Definition 4.3 (fraction equality). The equivalence relation \equiv on $\text{Arr}\text{AG}\mathcal{C}$ is defined to be generated by the following relation on $\text{Arr}\text{AG}\mathcal{C}$: Given $(b, f, a) \in \text{Arr}\text{AG}\mathcal{C}$ and $c \in \text{Mor}\mathcal{C}$ with $ac \in \text{Den}\mathcal{C}$, the 3-arrow (b, f, a) is in relation to the 3-arrow (b, fc, ac) ; and given $(b, f, a) \in \text{Arr}\text{AG}\mathcal{C}$ and $c \in \text{Mor}\mathcal{C}$ with $cb \in \text{Den}\mathcal{C}$, the 3-arrow (b, f, a) is in relation to the 3-arrow (cb, cf, a) .

$$\begin{array}{ccc} \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{fc} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} ac & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} a \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \end{array}$$

Given $(b, f, a), (\tilde{b}, \tilde{f}, \tilde{a}) \in \text{Arr}\text{AG}\mathcal{C}$ with $(b, f, a) \equiv (\tilde{b}, \tilde{f}, \tilde{a})$, we say that (b, f, a) and $(\tilde{b}, \tilde{f}, \tilde{a})$ are *fraction equal*.

In practice, it is sometimes convenient to work with different generating sets for fraction equality.

Remark 4.4. (a) *The fraction equality relation \equiv on $\text{Arr}\text{AG}\mathcal{C}$ is generated by the following relation: Given $(b, f, a) \in \text{Arr}\text{AG}\mathcal{C}$ and $c, c' \in \text{Mor}\mathcal{C}$ with $ac, c'b \in \text{Den}\mathcal{C}$, the 3-arrow (b, f, a) is in relation to the 3-arrow $(c'b, c'fc, ac)$.*

(b) *The fraction equality relation \equiv on $\text{Arr}\text{AG}\mathcal{C}$ is generated by the following relation: Given $(b, f, a), (\tilde{b}, \tilde{f}, \tilde{a}) \in \text{Arr}\text{AG}\mathcal{C}$, the 3-arrow (b, f, a) is in relation to the 3-arrow $(\tilde{b}, \tilde{f}, \tilde{a})$ if there exist $c, c' \in \text{Mor}\mathcal{C}$ with $b = c'\tilde{b}, fc = c'\tilde{f}, ac = \tilde{a}$.*

$$\begin{array}{ccc} \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{c'fc} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} ac & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} a \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \\ \parallel & & \parallel \\ \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y & & \leftarrow \underset{\approx}{\leftarrow} \tilde{X} \xrightarrow{f} \tilde{Y} \leftarrow \underset{\approx}{\leftarrow} Y \end{array}$$

As $\text{Den}\mathcal{C}$ is semi-saturated, the morphisms c and c' in definition 4.3 and remark 4.4 are automatically denominators in \mathcal{C} .

Remark 4.5. *We suppose given 3-arrows (b, f, a) and $(\tilde{b}, \tilde{f}, \tilde{a})$ in \mathcal{C} . If $(b, f, a) \equiv (\tilde{b}, \tilde{f}, \tilde{a})$, then f is a denominator in \mathcal{C} if and only if \tilde{f} is a denominator in \mathcal{C} .*

Proof. This follows by the definition of fraction equality 4.3 and by the semi-saturatedness of $\text{Den}\mathcal{C}$. \square

Remark 4.6. *The fraction equality relation \equiv on $\text{Arr AG } \mathcal{C}$ defines a graph congruence on $\text{AG } \mathcal{C}$. In particular, the quotient graph $(\text{AG } \mathcal{C})/\equiv$ is defined.*

Proof. For $(b, f, a) \in \text{Arr AG } \mathcal{C}$, $c, c' \in \text{Mor } \mathcal{C}$ with $ac, c'b \in \text{Den } \mathcal{C}$, we have

$$\text{Source}(c'b, c'fc, ac) = \text{Target}(c'b) = \text{Target } b = \text{Source}(b, f, a)$$

and analogously $\text{Target}(c'b, c'fc, ac) = \text{Target}(b, f, a)$. Thus the assertion follows from remark 4.4(a). \square

Definition 4.7 (double fraction). Given a 3-arrow (b, f, a) in \mathcal{C} , its equivalence class in the quotient graph $(\text{AG } \mathcal{C})/\equiv$ is denoted by $b \setminus f / a := [(b, f, a)]_{\equiv}$ and is said to be the *double fraction* of (b, f, a) .

Now we will present a certain reduced form for 3-arrows. We will see that every 3-arrow is fraction equal to such a reduced form.

Definition 4.8 (normal 3-arrows). A 3-arrow (p, f, i) in \mathcal{C} is said to be *normal* if i is an S-denominator and p is a T-denominator in \mathcal{C} .

$$\begin{array}{ccc} \leftarrow \frac{p}{\square} & \xrightarrow{f} & \leftarrow \frac{i}{\circ} \end{array}$$

The following lemma and its proof is (essentially) taken from [4, sec. 36.5].

Lemma 4.9 (normalisation lemma). *Every 3-arrow in \mathcal{C} is fraction equal to a normal 3-arrow in \mathcal{C} .*

Proof. We suppose given an arbitrary 3-arrow (b, f, a) in \mathcal{C} . There exist an S-denominator i and a T-denominator p in \mathcal{C} with $b = ip$, and there exist an S-denominator i' and a morphism f' in \mathcal{C} with $if' = fi'$. By multiplicativity, ai' is a denominator in \mathcal{C} . Thus there exist an S-denominator j and a T-denominator q in \mathcal{C} with $ai' = jq$, and there exist a T-denominator q' and a morphism f'' in \mathcal{C} with $f''q = q'f'$. By multiplicativity, $q'p$ is a T-denominator.

$$\begin{array}{ccccc} \leftarrow \frac{b}{\approx} & & \xrightarrow{f} & & \leftarrow \frac{a}{\approx} \\ \parallel & \downarrow i & & \downarrow i' & \parallel \\ \leftarrow \frac{p}{\square} & \xrightarrow{f'} & & \xrightarrow{ai'} & \\ \parallel & \uparrow q' & & \uparrow q & \parallel \\ \leftarrow \frac{q'p}{\square} & \xrightarrow{f''} & & \xrightarrow{j} & \\ \leftarrow \frac{q'p}{\square} & & & & \leftarrow \frac{j}{\circ} \end{array}$$

Altogether, $(b, f, a) \equiv (p, f', ai') \equiv (q'p, f'', j)$, and since j is an S-denominator and $q'p$ is a T-denominator, the 3-arrow $(q'p, f'', j)$ is normal. \square

5. The fraction category

In this section, our main theorem 5.13 will be proven. We begin by constructing a localisation of a uni-fractionable category \mathcal{C} with respect to its set of denominators $\text{Den } \mathcal{C}$, see proposition 5.2 and proposition 5.5. To this end, we consider the quotient graph $(\text{AG } \mathcal{C})/\equiv$ of its 3-arrow graph $\text{AG } \mathcal{C}$ with respect to fraction equality \equiv . The crucial point in the construction will be the following lemma.

Troughout this section, we suppose given a uni-fractionable category \mathcal{C} .

We get $q'b_1 = c'r'\tilde{b}_1$, $f'_1f'_2c'' = c'g_1g_2$, $a_2j'c'' = \tilde{a}_2k'$, $\tilde{q}'\tilde{b}_1 = \tilde{c}'r'\tilde{b}_1$, $\tilde{f}'_1\tilde{f}'_2\tilde{c}'' = \tilde{c}'g_1g_2$, $\tilde{a}_2\tilde{j}'\tilde{c}'' = \tilde{a}_2k'$ and therefore $(q'b_1, f'_1f'_2, a_2j') \equiv (r'\tilde{b}_1, g_1g_2, \tilde{a}_2k') \equiv (\tilde{q}'\tilde{b}_1, \tilde{f}'_1\tilde{f}'_2, \tilde{a}_2\tilde{j}')$.

$$\begin{array}{ccccc} & \xleftarrow{q'b_1} & \xrightarrow{f'_1f'_2} & \xleftarrow{a_2j'} & \\ \parallel & \downarrow c' & \downarrow c'' & \downarrow c'' & \parallel \\ \xleftarrow{r'\tilde{b}_1} & \xrightarrow{g_1g_2} & \xrightarrow{\tilde{a}_2k'} & \xleftarrow{\tilde{a}_2k'} & \\ \parallel & \uparrow \tilde{c}' & \uparrow \tilde{c}'' & \uparrow \tilde{c}'' & \parallel \\ \xleftarrow{\tilde{q}'\tilde{b}_1} & \xrightarrow{\tilde{f}'_1\tilde{f}'_2} & \xrightarrow{\tilde{a}_2\tilde{j}'} & \xleftarrow{\tilde{a}_2\tilde{j}'} & \end{array}$$

Thus we have $q'b_1 \setminus f'_1f'_2 / a_2j' = \tilde{q}'\tilde{b}_1 \setminus \tilde{f}'_1\tilde{f}'_2 / \tilde{a}_2\tilde{j}'$ in $(\text{AGC})/\equiv$.

In the special case where $c_1 = 1$, $c'_1 = 1$, $c_2 = 1$, $c'_2 = 1$, we see that different choices of constructions via weak pullback and weak pushout rectangles lead to the same double fraction $q'b_1 \setminus f'_1f'_2 / a_2j' = \tilde{q}'\tilde{b}_1 \setminus \tilde{f}'_1\tilde{f}'_2 / \tilde{a}_2\tilde{j}'$. Hence we obtain a well-defined map

$$\begin{aligned} c: \text{Arr AGC}_{\text{Target} \times \text{Source}} \text{Arr AGC} &\rightarrow \text{Arr}(\text{AGC})/\equiv, \\ (b_1, f_1, a_1), (b_2, f_2, a_2) &\mapsto q'b_1 \setminus f'_1f'_2 / a_2j', \end{aligned}$$

where q' , f'_1 , f'_2 , j' are constructed as described above. Now the general case shows that c is independent of the choice of the representatives in the equivalence classes with respect to \equiv , and thus we obtain an induced map

$$\bar{c}: \text{Arr}(\text{AGC})/\equiv_{\text{Target} \times \text{Source}} \text{Arr}(\text{AGC})/\equiv \rightarrow \text{Arr}(\text{AGC})/\equiv$$

given by $\bar{c}(b_1 \setminus f_1 / a_1, b_2 \setminus f_2 / a_2) = c((b_1, f_1, a_1), (b_2, f_2, a_2)) = q'b_1 \setminus f'_1f'_2 / a_2j'$.

We claim that arbitrary commutative quadrangles may be used instead of weak pullback and weak pushout rectangles to compute \bar{c} . Indeed, given a weak pullback rectangle and a weak pushout rectangle

$$\begin{array}{ccc} & \xrightarrow{f'_1} & \\ q' \downarrow & & \downarrow q \\ & \xrightarrow{f_1} & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \xrightarrow{f'_2} & \\ j \uparrow & & \uparrow j' \\ & \xrightarrow{f_2} & \end{array}$$

and arbitrary commutative quadrangles

$$\begin{array}{ccc} & \xrightarrow{\tilde{f}'_1} & \\ \tilde{q}' \downarrow & & \downarrow q \\ & \xrightarrow{f_1} & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \xrightarrow{\tilde{f}'_2} & \\ j \uparrow & & \uparrow \tilde{j}' \\ & \xrightarrow{f_2} & \end{array}$$

such that q' , \tilde{q}' are T-denominators and j' , \tilde{j}' are S-denominators in \mathcal{C} , we obtain induced morphisms c and c' such that $\tilde{q}' = cq'$, $\tilde{f}'_1 = cf'_1$, $\tilde{f}'_2 = f'_2c'$, $\tilde{j}' = j'c'$.

$$\begin{array}{ccccc} & \xrightarrow{\tilde{f}'_1} & & \xrightarrow{\tilde{f}'_2} & \\ \tilde{q}' \downarrow & \searrow c & \xrightarrow{f'_1} & \xrightarrow{f'_2} & \downarrow c' \\ & \xrightarrow{f_1} & & \xrightarrow{f_2} & \\ q' \downarrow & & \downarrow q & & \downarrow j' \\ & \xrightarrow{f_1} & & \xrightarrow{f_2} & \\ b_1 \swarrow & & a_1 \swarrow & & \swarrow a_2 \end{array}$$

Thus we have $(\tilde{q}'b_1, \tilde{f}'_1\tilde{f}'_2, a_2\tilde{j}') = (cq'b_1, cf'_1f'_2c', a_2j'c') \equiv (q'b_1, f'_1f'_2, a_2j')$ and therefore $\bar{c}(b_1 \setminus f_1/a_1, b_2 \setminus f_2/a_2) = q'b_1 \setminus f'_1f'_2/a_2j' = \tilde{q}'b_1 \setminus \tilde{f}'_1\tilde{f}'_2/a_2\tilde{j}'$. This proves the claim.

In addition to \bar{c} , we define the map

$$e: \text{Ob}(\text{AGC})/\equiv \rightarrow \text{Arr}(\text{AGC})/\equiv, X \mapsto 1_X \setminus 1_X/1_X.$$

To show that $(\text{AGC})/\equiv$ is a category with composition \bar{c} and identity map e , it remains to verify the category axioms. By the definitions of \bar{c} and e , we have

$$\begin{aligned} \text{Source } \bar{c}(b_1 \setminus f_1/a_1, b_2 \setminus f_2/a_2) &= \text{Source } q'b_1 \setminus f'_1f'_2/a_2j' = \text{Target}(q'b_1) = \text{Target } b_1 \\ &= \text{Source } b_1 \setminus f_1/a_1 \end{aligned}$$

and analogously $\text{Target } \bar{c}(b_1 \setminus f_1/a_1, b_2 \setminus f_2/a_2) = \text{Target } b_2 \setminus f_2/a_2$ for all $(b_1, f_1, a_1), (b_2, f_2, a_2) \in \text{Arr AGC}$ with $\text{Target } b_1 \setminus f_1/a_1 = \text{Source } b_2 \setminus f_2/a_2$, as well as

$$\text{Source } e(X) = \text{Source } 1_X \setminus 1_X/1_X = \text{Target } 1_X = X$$

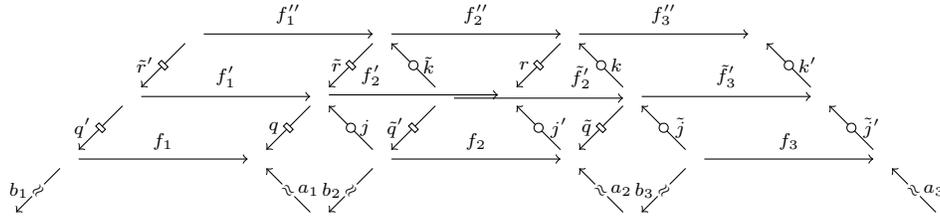
and analogously $\text{Target } e(X) = X$ for all $X \in \text{Ob}(\text{AGC})/\equiv$.

For the associativity of \bar{c} , we suppose given $(b_l, f_l, a_l) \in \text{Arr AGC}$ for $l \in \{1, 2, 3\}$ such that $\text{Target } b_1 \setminus f_1/a_1 = \text{Source } b_2 \setminus f_2/a_2$ and $\text{Target } b_2 \setminus f_2/a_2 = \text{Source } b_3 \setminus f_3/a_3$. We choose S-denominators j, \tilde{j} and T-denominators q, \tilde{q} with $b_2a_1 = jq$ and $b_3a_2 = \tilde{j}\tilde{q}$. Then we choose T-denominators q', \tilde{q}' and morphisms f'_1, \tilde{f}'_1 in \mathcal{C} with $f'_1q = q'f_1$ and $\tilde{f}'_1\tilde{q} = \tilde{q}'f_2$, and we choose S-denominators j', \tilde{j}' and morphisms f'_2, \tilde{f}'_2 in \mathcal{C} with $jf'_2 = f_2j'$ and $\tilde{j}\tilde{f}'_2 = f_3\tilde{j}'$. By definition of \bar{c} , we obtain $\bar{c}(b_1 \setminus f_1/a_1, b_2 \setminus f_2/a_2) = q'b_1 \setminus f'_1f'_2/a_2j'$ and $\bar{c}(b_2 \setminus f_2/a_2, b_3 \setminus f_3/a_3) = \tilde{q}'b_2 \setminus \tilde{f}'_2\tilde{f}'_3/a_3\tilde{j}'$.

Moreover, we have $\tilde{q}'j'f'_2 = \tilde{f}'_2\tilde{q}j'$, and thus by the factorisation axiom and the factorisation lemma 5.1 there exist S-denominators k, \tilde{k} , T-denominators r, \tilde{r} and a morphism f''_2 in \mathcal{C} with $\tilde{q}j' = kr$, $\tilde{q}'j' = \tilde{k}\tilde{r}$, $\tilde{r}f''_2 = f'_2r$, $\tilde{f}'_2k = \tilde{k}f''_2$. We choose a T-denominator \tilde{r}' and a morphism f''_1 in \mathcal{C} with $f''_1\tilde{r}' = \tilde{r}'f'_1$, and we choose an S-denominator k' and a morphism f''_3 in \mathcal{C} with $k'f''_3 = \tilde{f}'_3k'$. Then we obtain $\tilde{r}'f'_1f'_2 = f''_1f''_2r$, $\tilde{j}k'f''_3 = f_3\tilde{j}'k'$, $\tilde{f}'_2\tilde{f}'_3k' = \tilde{k}f''_2f''_3$, $f''_1\tilde{r}'q = \tilde{r}'q'f_1$, and therefore

$$\begin{aligned} \bar{c}(\bar{c}(b_1 \setminus f_1/a_1, b_2 \setminus f_2/a_2), b_3 \setminus f_3/a_3) &= \bar{c}(q'b_1 \setminus f'_1f'_2/a_2j', b_3 \setminus f_3/a_3) \\ &= \tilde{r}'q'b_1 \setminus f''_1f''_2f''_3/a_3\tilde{j}'k' = \bar{c}(b_1 \setminus f_1/a_1, \tilde{q}'b_2 \setminus \tilde{f}'_2\tilde{f}'_3/a_3\tilde{j}') \\ &= \bar{c}(b_1 \setminus f_1/a_1, \bar{c}(b_2 \setminus f_2/a_2, b_3 \setminus f_3/a_3)). \end{aligned}$$

Thus \bar{c} is associative.



Finally, we suppose given $(b, f, a) \in \text{Arr AGC}$. We want to show that we have $\bar{c}(b \setminus f/a, e(\text{Target } b \setminus f/a)) = b \setminus f/a$. By the normalisation lemma 4.9, there exists a normal arrow $(p, g, i) \in \text{Arr AGC}$ with $(b, f, a) \equiv (p, g, i)$. We obtain

$$\bar{c}(b \setminus f/a, e(\text{Target } b \setminus f/a)) = \bar{c}(p \setminus g/i, 1 \setminus 1/1) = 1p \setminus g/1i = p \setminus g/i = b \setminus f/a.$$

the unique inverse of Ld , that is, $(Ld)^{-1} = d \setminus 1 / 1 = 1 \setminus 1 / d$.

(1-uni) We let \mathcal{D} be a category and $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor such that Fd is invertible for all $d \in \text{Den } \mathcal{C}$. For $(b, f, a) \in \text{Arr } \text{AG } \mathcal{C}$, we have

$$\begin{aligned} \text{Source}((Fb)^{-1}(Ff)(Fa)^{-1}) &= \text{Source}(Fb)^{-1} = \text{Target}(Fb) = F(\text{Target } b) \\ &= F(\text{Source}(b, f, a)), \end{aligned}$$

and analogously $\text{Target}((Fb)^{-1}(Ff)(Fa)^{-1}) = F(\text{Target}(b, f, a))$. Thus there exists a graph morphism $F': \text{AG } \mathcal{C} \rightarrow \mathcal{D}$ given on the objects by $F'X = FX$ for $X \in \text{Ob } \text{AG } \mathcal{C}$ and on the arrows by $F'(b, f, a) = (Fb)^{-1}(Ff)(Fa)^{-1}$ for $(b, f, a) \in \text{Arr } \text{AG } \mathcal{C}$. Moreover, given $(b, f, a) \in \text{Arr } \text{AG } \mathcal{C}$ and $c, c' \in \text{Den } \mathcal{C}$ with $\text{Target } c' = \text{Source } b$, $\text{Source } c = \text{Target } a$, we obtain

$$\begin{aligned} F'(c'b, c'fc, ac) &= (F(c'b))^{-1}(F(c'fc))(F(ac))^{-1} \\ &= ((Fc')(Fb))^{-1}((Fc')(Ff)(Fc))((Fa)(Fc))^{-1} \\ &= (Fb)^{-1}(Fc')^{-1}(Fc')(Ff)(Fc)(Fc)^{-1}(Fa)^{-1} \\ &= (Fb)^{-1}(Ff)(Fa)^{-1} = F'(b, f, a). \end{aligned}$$

Hence F' maps fraction equal 3-arrows to the same morphism and we obtain an induced graph morphism $\hat{F}: (\text{AG } \mathcal{C}) / \equiv \rightarrow \mathcal{D}$ with $F' = \hat{F} \circ \text{quo}$.

We want to show that \hat{F} is a functor. Given $(b_1, f_1, a_1), (b_2, f_2, a_2) \in \text{Arr } \text{AG } \mathcal{C}$ with $\text{Target}(b_1, f_1, a_1) = \text{Source}(b_2, f_2, a_2)$, we have

$$\begin{aligned} \hat{F}((b_1 \setminus f_1 / a_1)(b_2 \setminus f_2 / a_2)) &= \hat{F}(q'b_1 \setminus f'_1 f'_2 / a_2 j') \\ &= (F(q'b_1))^{-1}(F(f'_1 f'_2))(F(a_2 j'))^{-1} = (Fb_1)^{-1}(Fq')^{-1}(Ff'_1)(Ff'_2)(Fj')^{-1}(Fa_2)^{-1} \\ &= (Fb_1)^{-1}(Ff_1)(Fq)^{-1}(Fj)^{-1}(Ff_2)(Fa_2)^{-1} \\ &= (Fb_1)^{-1}(Ff_1)(Fa_1)^{-1}(Fb_2)^{-1}(Ff_2)(Fa_2)^{-1} = \hat{F}(b_1 \setminus f_1 / a_1) \hat{F}(b_2 \setminus f_2 / a_2), \end{aligned}$$

where j, j', q, q', f'_1, f'_2 are supposed to be constructed as in proposition 5.2.

$$\begin{array}{ccccc} & & f'_1 & \xrightarrow{\quad} & f'_2 & & \\ & q' \swarrow & & \searrow q & & \swarrow q & \searrow q' \\ & f_1 & & & f_2 & & \\ b_1 \swarrow & & a_1 \swarrow & & b_2 & & \swarrow a_2 \\ & & & & & & \end{array}$$

Moreover, we have

$$\hat{F}(1_X) = \hat{F}(1_X \setminus 1_X / 1_X) = (F1_X)^{-1}(F1_X)(F1_X)^{-1} = 1_{FX} = 1_{\hat{F}X}$$

for $X \in \text{Ob } \text{Frac } \mathcal{C}$. This implies that $\hat{F}: \text{Frac } \mathcal{C} \rightarrow \mathcal{D}$ is a functor, given by $\hat{F}(b \setminus f / a) = F'(b, f, a) = (Fb)^{-1}(Ff)(Fa)^{-1}$ for $(b, f, a) \in \text{Arr } \text{AG } \mathcal{C}$. In particular,

$$\hat{F}Lf = \hat{F}(1 \setminus f / 1) = (F1)^{-1}(Ff)(F1)^{-1} = Ff$$

for all $f \in \text{Mor } \mathcal{C}$, that is, $\hat{F} \circ L = F$.

Conversely, given an arbitrary functor $G: \text{Frac } \mathcal{C} \rightarrow \mathcal{D}$ with $F = G \circ L$, we conclude by remark 5.4 that

$$\begin{aligned} G(b \setminus f / a) &= G((b \setminus 1 / 1)(1 \setminus f / 1)(1 \setminus 1 / a)) = G((Lb)^{-1}(Lf)(La)^{-1}) \\ &= (GLb)^{-1}(GLf)(GLa)^{-1} = (Fb)^{-1}(Ff)(Fa)^{-1} \end{aligned}$$

for $(b, f, a) \in \text{Arr } \text{AG } \mathcal{C}$.

(2-uni) We suppose given a category \mathcal{D} and functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ such that Fd and Gd are invertible for all $d \in \text{Den } \mathcal{C}$, and we let $\hat{F}, \hat{G}: \text{Frac } \mathcal{C} \rightarrow \mathcal{D}$ be the unique functors with $F = \hat{F} \circ L$ resp. $G = \hat{G} \circ L$. Moreover, we suppose given a transformation $\alpha: F \rightarrow G$. We define a family $\hat{\alpha} := (\hat{\alpha}_X)_{X \in \text{Ob } \text{Frac } \mathcal{C}}$ by $\hat{\alpha}_X := \alpha_X$ for $X \in \text{Ob } \text{Frac } \mathcal{C} = \text{Ob } \mathcal{C}$. Then $\hat{\alpha}_{LX} = \hat{\alpha}_X = \alpha_X$ for $X \in \text{Ob } \mathcal{C}$. Moreover, $\hat{\alpha}$ is a transformation from \hat{F} to \hat{G} as for every 3-arrow $(b, f, a): X \leftarrow \tilde{X} \rightarrow \tilde{Y} \leftarrow Y$ in \mathcal{C} , we have

$$\begin{aligned} \hat{\alpha}_X(\hat{G}(b \setminus f / a)) &= \alpha_X(Gb)^{-1}(Gf)(Ga)^{-1} = (Fb)^{-1}\alpha_{\tilde{X}}(Gf)(Ga)^{-1} \\ &= (Fb)^{-1}(Ff)\alpha_{\tilde{Y}}(Ga)^{-1} = (Fb)^{-1}(Ff)(Fa)^{-1}\alpha_Y = (\hat{F}(b \setminus f / a))\hat{\alpha}_Y. \end{aligned}$$

Conversely, given an arbitrary transformation $\beta: \hat{F} \rightarrow \hat{G}$ such that $\beta_{LX} = \alpha_X$ for all $X \in \text{Ob } \mathcal{C}$, we necessarily have $\beta_X = \beta_{LX} = \alpha_X$ for all $X \in \text{Ob } \text{Frac } \mathcal{C} = \text{Ob } \mathcal{C}$. \square

Corollary 5.6 (splitting double fractions). *For each 3-arrow (b, f, a) in \mathcal{C} , we have*

$$b \setminus f / a = (\text{loc}(b))^{-1} \text{loc}(f) (\text{loc}(a))^{-1}.$$

Proof. By proposition 5.5, the fraction category $\text{Frac } \mathcal{C}$ is a localisation of \mathcal{C} with respect to $\text{Den } \mathcal{C}$. In particular, $\text{loc}(d)$ is invertible for all $d \in \text{Den } \mathcal{C}$, and hence there exists a unique functor $\hat{L}: \text{Frac } \mathcal{C} \rightarrow \text{Frac } \mathcal{C}$ with $\text{loc} = \hat{L} \circ \text{loc}$, which is given by $\hat{L}(b \setminus f / a) = (\text{loc}(b))^{-1} \text{loc}(f) (\text{loc}(a))^{-1}$ for $(b, f, a) \in \text{Arr } \text{AG } \mathcal{C}$. But since $\text{loc} = \text{id}_{\text{Frac } \mathcal{C}} \circ \text{loc}$, we must have $\hat{L} = \text{id}_{\text{Frac } \mathcal{C}}$ and therefore the assertion holds. \square

In the construction of the composition of the fraction category in proposition 5.2, the occurring morphisms j, j' were S-denominators, and q, q' were T-denominators. We shall now show that it suffices to have a diagram with arbitrary denominators at their places to get the correct composite.

Proposition 5.7. (a) *We suppose given 3-arrows $(b_1, f_1, a_1), (b_2, f_2, a_2)$ in \mathcal{C} with $\text{Target}(b_1, f_1, a_1) = \text{Source}(b_2, f_2, a_2)$. Moreover, we suppose given denominators d, d', e, e' and morphisms g_1, g_2 in \mathcal{C} with $b_2 a_1 = de, g_1 e = e' f_1, dg_2 = f_2 d'$. Then we have $(b_1 \setminus f_1 / a_1)(b_2 \setminus f_2 / a_2) = e' b_1 \setminus g_1 g_2 / a_2 d'$.*

(b) *Given a 3-arrow (b, d, a) in \mathcal{C} with a denominator d , the double fraction $b \setminus d / a$ is invertible in $\text{Frac } \mathcal{C}$, and the inverse of $b \setminus d / a$ can be constructed as follows. We choose denominators $d_1, d'_1, d_2, d'_2, a', b'$ in \mathcal{C} with $d = d_1 d_2, d_1 b' = b d'_1, a' d_2 = d'_2 a$. Then we have $(b \setminus d / a)^{-1} = d'_2 \setminus a' b' / d'_1$.*

$$\begin{array}{ccc} & \xrightarrow{g_1} & \xrightarrow{g_2} \\ e' \swarrow & f_1 & e \swarrow & \swarrow d & f_2 & \swarrow d' \\ b_1 \swarrow & & a_1 \swarrow & & \swarrow b_2 & & \swarrow a_2 \end{array} \qquad \begin{array}{ccccccc} & & \xleftarrow{b'} & & \xleftarrow{a'} & & \\ & & \approx & & \approx & & \\ d'_1 \swarrow & & b & & d_1 \swarrow & & d_2 & & a & & \swarrow d'_2 \\ & & \approx & & \approx & & \approx & & \approx & & \end{array}$$

Proof. (a) We compute

$$\begin{aligned} & (b_1 \setminus f_1 / a_1)(b_2 \setminus f_2 / a_2) \\ &= (\text{loc}(b_1))^{-1} \text{loc}(f_1) (\text{loc}(a_1))^{-1} (\text{loc}(b_2))^{-1} \text{loc}(f_2) (\text{loc}(a_2))^{-1} \\ &= (\text{loc}(b_1))^{-1} \text{loc}(f_1) (\text{loc}(e))^{-1} (\text{loc}(d))^{-1} \text{loc}(f_2) (\text{loc}(a_2))^{-1} \\ &= (\text{loc}(b_1))^{-1} (\text{loc}(e'))^{-1} \text{loc}(g_1) \text{loc}(g_2) (\text{loc}(d'))^{-1} (\text{loc}(a_2))^{-1} \\ &= (\text{loc}(e' b_1))^{-1} \text{loc}(g_1 g_2) (\text{loc}(a_2 d'))^{-1} = e' b_1 \setminus g_1 g_2 / a_2 d'. \end{aligned}$$

(b) The double fraction $b \setminus d / a = (\text{loc}(b))^{-1} \text{loc}(d) (\text{loc}(a))^{-1}$ is invertible in $\text{Frac } \mathcal{C}$ since the localisation functor $\text{loc}: \mathcal{C} \rightarrow \text{Frac } \mathcal{C}$ maps denominators in \mathcal{C} to isomorphisms in $\text{Frac } \mathcal{C}$.

Given denominators $d_1, d'_1, d_2, d'_2, a', b'$ in \mathcal{C} with $d = d_1 d_2, d_1 b' = b d'_1, a' d_2 = d'_2 a$, we obtain

$$\begin{aligned} (b \setminus d / a)^{-1} &= ((\text{loc}(b))^{-1} \text{loc}(d) (\text{loc}(a))^{-1})^{-1} = \text{loc}(a) (\text{loc}(d))^{-1} \text{loc}(b) \\ &= \text{loc}(a) (\text{loc}(d_1 d_2))^{-1} \text{loc}(b) = \text{loc}(a) (\text{loc}(d_2))^{-1} (\text{loc}(d_1))^{-1} \text{loc}(b) \\ &= (\text{loc}(d'_2))^{-1} \text{loc}(a') \text{loc}(b') (\text{loc}(d'_1))^{-1} \\ &= (\text{loc}(d'_2))^{-1} \text{loc}(a' b') (\text{loc}(d'_1))^{-1} = d'_2 \setminus a' b' / d'_1. \quad \square \end{aligned}$$

Proposition 5.8. *Given a uni-fractionable category \mathcal{D} and a denominator preserving functor $F: \mathcal{C} \rightarrow \mathcal{D}$, there exists a unique induced functor*

$$\text{Frac } F: \text{Frac } \mathcal{C} \rightarrow \text{Frac } \mathcal{D}$$

with $\text{loc}^{\text{Frac } \mathcal{D}} \circ F = (\text{Frac } F) \circ \text{loc}^{\text{Frac } \mathcal{C}}$. It is given on the objects by $(\text{Frac } F)X = FX$ for $X \in \text{Ob } \text{Frac } \mathcal{C}$ and on the morphisms by $(\text{Frac } F)(b \setminus f / a) = Fb \setminus Ff / Fa$ for $(b, f, a) \in \text{Arr } \text{AG } \mathcal{C}$.

Proof. Since F preserves denominators and $\text{loc}^{\text{Frac } \mathcal{D}}$ maps denominators in \mathcal{D} to isomorphisms in $\text{Frac } \mathcal{D}$, the composite $\text{loc}^{\text{Frac } \mathcal{D}} \circ F$ maps denominators in \mathcal{C} to isomorphisms in $\text{Frac } \mathcal{D}$. By the universal property of $\text{Frac } \mathcal{C}$, there exists a unique functor $\text{Frac } F: \text{Frac } \mathcal{C} \rightarrow \text{Frac } \mathcal{D}$ with $\text{loc}^{\text{Frac } \mathcal{D}} \circ F = (\text{Frac } F) \circ \text{loc}^{\text{Frac } \mathcal{C}}$. It follows that

$$(\text{Frac } F)X = (\text{Frac } F)\text{loc}(X) = \text{loc}(FX) = FX$$

for $X \in \text{Ob } \mathcal{C}$ as well as

$$\begin{aligned} (\text{Frac } F)(b \setminus f / a) &= (\text{Frac } F)((\text{loc}(b))^{-1} \text{loc}(f) (\text{loc}(a))^{-1}) \\ &= ((\text{Frac } F)\text{loc}(b))^{-1} ((\text{Frac } F)\text{loc}(f)) ((\text{Frac } F)\text{loc}(a))^{-1} \\ &= (\text{loc}(Fb))^{-1} \text{loc}(Ff) (\text{loc}(Fa))^{-1} = Fb \setminus Ff / Fa \end{aligned}$$

for $(b, f, a) \in \text{Arr } \text{AG } \mathcal{C}$. □

Here is another elementary property of the fraction category, which will be needed in proposition 5.15, where we deal with coproducts.

Proposition 5.9. *We suppose given morphisms φ_1 and φ_2 in $\text{Frac } \mathcal{C}$.*

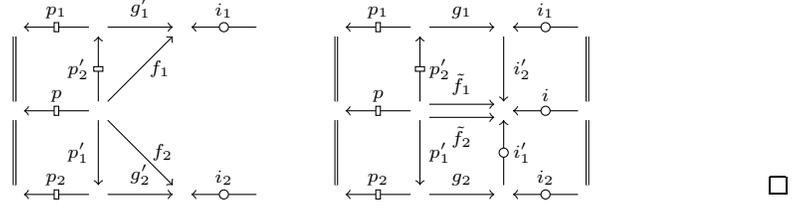
(a) *If $\text{Source } \varphi_1 = \text{Source } \varphi_2$, then there exist normal 3-arrows $(p, f_1, i_1), (p, f_2, i_2)$ in \mathcal{C} with $\varphi_1 = p \setminus f_1 / i_1$ and $\varphi_2 = p \setminus f_2 / i_2$.*

(b) *If $\text{Source } \varphi_1 = \text{Source } \varphi_2$ and $\text{Target } \varphi_1 = \text{Target } \varphi_2$, then there exist normal 3-arrows $(p, f_1, i), (p, f_2, i)$ in \mathcal{C} with $\varphi_1 = p \setminus f_1 / i$ and $\varphi_2 = p \setminus f_2 / i$.*

Proof. (a) We choose 3-arrows (b_k, g_k, a_k) with $\varphi_k = b_k \setminus g_k / a_k$ for $k \in \{1, 2\}$. By the normalisation lemma 4.9, there exist normal 3-arrows (p_k, g'_k, i_k) in \mathcal{C} with $(b_k, g_k, a_k) \equiv (p_k, g'_k, i_k)$ for $k \in \{1, 2\}$, that is, with $\varphi_k = b_k \setminus g_k / a_k = p_k \setminus g'_k / i_k$ for $k \in \{1, 2\}$.

There exist a T-denominator p'_2 and a morphism p'_1 in \mathcal{C} with $p'_2 p_1 = p'_1 p_2$. We define $p := p'_2 p_1 = p'_1 p_2$, $f_1 := p'_2 g'_1$, $f_2 := p'_1 g'_2$. By multiplicativity, $p = p'_2 p_1$ is a T-denominator in \mathcal{C} , and we have $\varphi_1 = p_1 \setminus g'_1 / i_1 = p'_2 p_1 \setminus p'_2 g'_1 / i_1 = p \setminus f_1 / i_1$ and analogously $\varphi_2 = p \setminus f_2 / i_2$.

(b) This is proven similarly to (a).



Proposition 5.10 (cf. [4, sec. 36.4]). *The denominator set in \mathcal{C} is saturated if and only if it is weakly saturated.*

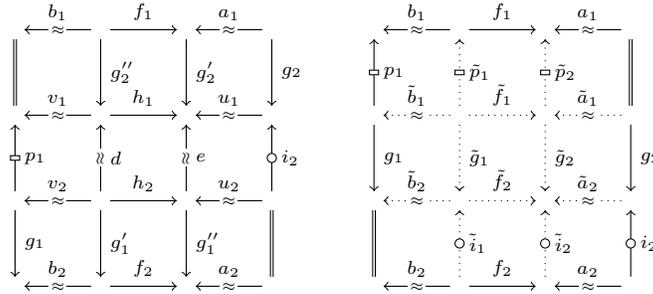
Proof. Since saturatedness always implies weak saturatedness, it suffices to show that if $\text{Den } \mathcal{C}$ is weakly saturated, then it is already saturated. So we suppose that $\text{Den } \mathcal{C}$ is weakly saturated and we suppose given a morphism f in \mathcal{C} such that $\text{loc}(f)$ is invertible in $\text{Frac } \mathcal{C}$. We let (p, g, i) be a normal 3-arrow in \mathcal{C} with $(\text{loc}(f))^{-1} = p \backslash g / i$. Moreover, we choose a T-denominator p' and a morphism f' in \mathcal{C} with $f'p = p'f$, and we choose an S-denominator i' and a morphism f'' in \mathcal{C} with $if'' = f'i'$.

$$\begin{array}{ccccc}
 & & f' & & g & & f'' & & \\
 p' \swarrow & \xrightarrow{f} & p \swarrow & \xrightarrow{g} & i & \xrightarrow{f} & i' & \\
 \swarrow & & \swarrow & \xrightarrow{g} & i' & \xrightarrow{f} & i' & \swarrow \\
 & & & & & & & &
 \end{array}$$

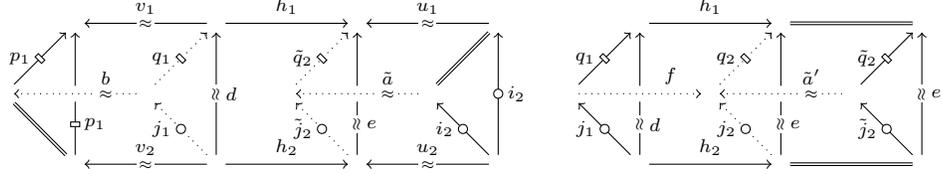
Then we have $1 \backslash 1 / 1 = (1 \backslash f / 1)(p \backslash g / i) = p' \backslash f' g / i$ and $1 \backslash 1 / 1 = (p \backslash g / i)(1 \backslash f / 1) = p \backslash g f'' / i'$. By remark 4.5, the morphisms $f'g$ and $g f''$ must be denominators. Now (2 of 6) implies that f' and thus f is a denominator. Hence $\text{Den } \mathcal{C}$ is saturated. \square

Now we come to the last part of the main theorem of this article, that is, we want to show that \mathcal{C} admits a 3-arrow calculus. It can be found in proposition 5.12. The key step of its proof is treated in the following lemma.

Lemma 5.11 (flipping lemma). *Given 3-arrows (b_1, f_1, a_1) , (b_2, f_2, a_2) , (v_1, h_1, u_1) , (v_2, h_2, u_2) , morphisms $g_1, g'_1, g''_1, g_2, g'_2, g''_2$, denominators d, e , an S-denominator i_2 and a T-denominator p_1 in \mathcal{C} , fitting into the commutative diagram in \mathcal{C} on the left, there exist 3-arrows $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ and normal 3-arrows $(\tilde{p}_1, \tilde{g}_1, \tilde{i}_1)$, $(\tilde{p}_2, \tilde{g}_2, \tilde{i}_2)$, fitting into the commutative diagram in \mathcal{C} on the right.*



Proof. By the factorisation axiom and the factorisation lemma 5.1, there exist S-denominators j_1, \tilde{j}_2 , T-denominators q_1, \tilde{q}_2 and morphisms b, \tilde{a} in \mathcal{C} with $d = j_1 q_1$, $e = \tilde{j}_2 \tilde{q}_2$, $q_1 v_1 = b p_1$, $v_2 = j_1 b$, $u_1 = \tilde{a} \tilde{q}_2$, $u_2 \tilde{j}_2 = i_2 \tilde{a}$. Moreover, by the factorisation lemma 5.1, there exist an S-denominator j_2 , a T-denominator q_2 , a morphism f and a denominator \tilde{a}' in \mathcal{C} with $e = j_2 q_2$, $q_1 h_1 = f q_2$, $j_1 f = h_2 j_2$, $\tilde{q}_2 = \tilde{a}' q_2$, $j_2 = \tilde{j}_2 \tilde{a}'$.



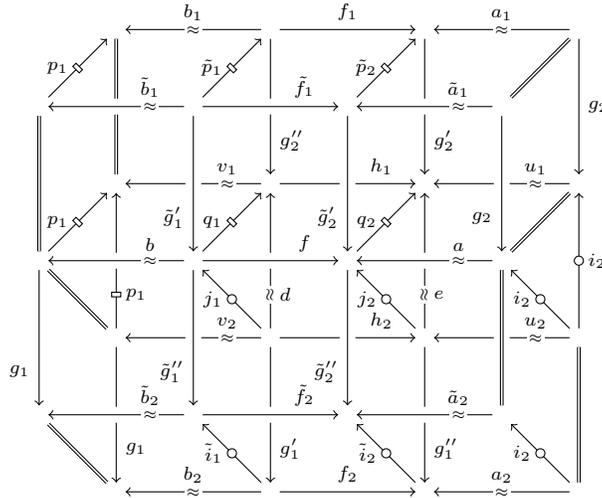
We set $a := \tilde{a} \tilde{a}'$ and obtain $u_1 = a q_2$ and $u_2 j_2 = i_2 a$. Next, we choose weak pullback rectangles

$$\begin{array}{ccc} & \tilde{g}'_1 & \\ \tilde{p}_1 \downarrow & \rightarrow & \downarrow q_1 \\ & g''_2 & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \tilde{g}'_2 & \\ \tilde{p}_2 \downarrow & \rightarrow & \downarrow q_2 \\ & g'_2 & \end{array}$$

in \mathcal{C} such that \tilde{p}_1 and \tilde{p}_2 are T-denominators, and we choose weak pushout rectangles

$$\begin{array}{ccc} & \tilde{g}''_1 & \\ j_1 \uparrow & \leftarrow & \uparrow \tilde{i}_1 \\ & g'_1 & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \tilde{g}''_2 & \\ j_2 \uparrow & \leftarrow & \uparrow \tilde{i}_2 \\ & g''_1 & \end{array}$$

in \mathcal{C} such that \tilde{i}_1 and \tilde{i}_2 are S-denominators. We obtain induced morphisms $\tilde{b}_1, \tilde{f}_1, \tilde{a}_1$ on the weak pullbacks, that is, with $\tilde{p}_1 b_1 = \tilde{b}_1 p_1$, $\tilde{b}_1 = \tilde{g}'_1 b$, $\tilde{p}_1 f_1 = \tilde{f}_1 \tilde{p}_2$, $\tilde{f}_1 \tilde{g}'_2 = \tilde{g}'_1 f$, $a_1 = \tilde{a}_1 \tilde{p}_2$, $\tilde{a}_1 \tilde{g}'_2 = g_2 a$, and induced morphisms $b_2, \tilde{f}_2, \tilde{a}_2$ on the weak pushouts, that is, with $b g_1 = \tilde{g}''_1 \tilde{b}_2$, $\tilde{i}_1 \tilde{b}_2 = b_2$, $f \tilde{g}''_2 = \tilde{g}''_1 \tilde{f}_2$, $\tilde{i}_1 \tilde{f}_2 = \tilde{f}_2 \tilde{i}_2$, $a \tilde{g}''_2 = \tilde{a}_2$, $i_2 \tilde{a}_2 = a_2 \tilde{i}_2$.



Setting $\tilde{g}_1 := \tilde{g}'_1 \tilde{g}''_1$ and $\tilde{g}_2 := \tilde{g}'_2 \tilde{g}''_2$ yields $\tilde{b}_1 g_1 = \tilde{g}_1 \tilde{b}_2$, $\tilde{f}_1 \tilde{g}_2 = \tilde{g}_1 \tilde{f}_2$, $\tilde{a}_1 \tilde{g}_2 = g_2 \tilde{a}_2$. Moreover, $\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2$ are denominators in \mathcal{C} by semi-saturatedness. \square

Proposition 5.12 (3-arrow calculus, cf. [4, sec. 36.3]). (a) *Given 3-arrows (b_1, f_1, a_1) and (b_2, f_2, a_2) in \mathcal{C} , we have $b_1 \setminus f_1 / a_1 = b_2 \setminus f_2 / a_2$ in $\text{Frac } \mathcal{C}$ if and only if there exist 3-arrows $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ and normal 3-arrows (p_1, d_1, i_1) , (p_2, d_2, i_2) with denominators d_1, d_2 , fitting into the commutative diagram in \mathcal{C} on the left.*

If (b_1, f_1, a_1) , (b_2, f_2, a_2) are normal 3-arrows, then $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ can be chosen to be normal, too.

(b) *Given 3-arrows (b_1, f_1, a_1) , (b_2, f_2, a_2) and normal 3-arrows (p_1, g_1, i_1) , (p_2, g_2, i_2) in \mathcal{C} , we have $(b_1 \setminus f_1 / a_1) (p_2 \setminus g_2 / i_2) = (p_1 \setminus g_1 / i_1) (b_2 \setminus f_2 / a_2)$ in $\text{Frac } \mathcal{C}$ if and only if there exist 3-arrows $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ and normal 3-arrows $(\tilde{p}_1, \tilde{g}_1, \tilde{i}_1)$, $(\tilde{p}_2, \tilde{g}_2, \tilde{i}_2)$, fitting into the commutative diagram in \mathcal{C} on the right.*

Proof. (a) If we have a commutative diagram as stated, then we have

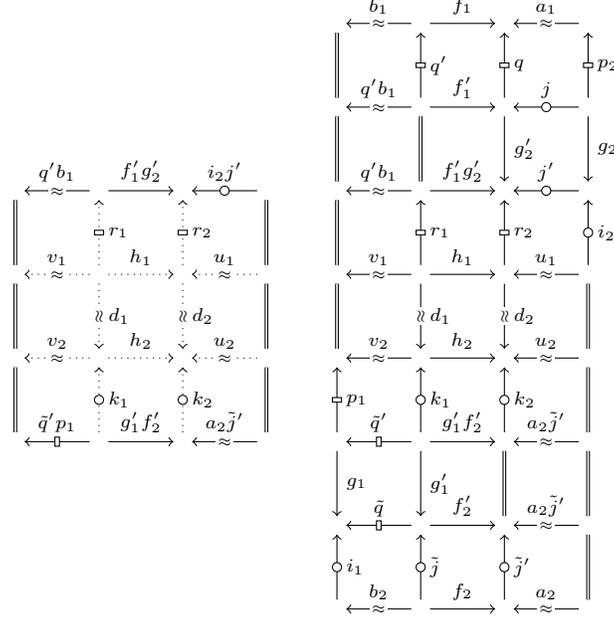
$$(b_1, f_1, a_1) \equiv (\tilde{b}_1, \tilde{f}_1, \tilde{a}_1) \equiv (\tilde{b}_2, \tilde{f}_2, \tilde{a}_2) \equiv (b_2, f_2, a_2)$$

and thus $b_1 \setminus f_1 / a_1 = b_2 \setminus f_2 / a_2$ in $\text{Frac } \mathcal{C}$.

So we suppose conversely that $b_1 \setminus f_1 / a_1 = b_2 \setminus f_2 / a_2$ in $\text{Frac } \mathcal{C}$, that is, we suppose that $(b_1, f_1, a_1) \equiv (b_2, f_2, a_2)$ in $\text{AG } \mathcal{C}$. By remark 4.4(b), there exist $n \in \mathbb{N}_0$, $(v_l, h_l, u_l) \in \text{Arr } \text{AG } \mathcal{C}$ for $l \in [0, 2n+1]$, $c_l, c'_l \in \text{Mor } \mathcal{C}$ for $l \in [0, n]$, w_l, w'_l for $l \in [0, n-1]$, with $(v_0, h_0, u_0) = (b_1, f_1, a_1)$, $(v_{2n+1}, h_{2n+1}, u_{2n+1}) = (b_2, f_2, a_2)$ and $v_{2l} = c_l v_{2l+1}$, $h_{2l} c'_l = c_l h_{2l+1}$, $u_{2l} c'_l = u_{2l+1}$ for $l \in [0, n]$ and $v_{2l+2} = w_l v_{2l+1}$, $w_l h_{2l+1} = h_{2l+2} w'_l$, $u_{2l+2} w'_l = u_{2l+1}$ for $l \in [0, n-1]$.

By semi-saturatedness, c_l and c'_l are denominators for all $l \in [0, n]$ and w_l, w'_l are denominators for all $l \in [0, n-1]$. Using the flipping lemma 5.11 and induction on $n \in \mathbb{N}_0$ yields the first assertion.

Now let us suppose that (b_1, f_1, a_1) and (b_2, f_2, a_2) are normal 3-arrows. By multiplicativity, $\tilde{b}_1 = p_1 b_1$ is a T-denominator and $\tilde{a}_2 = a_2 i_2$ is an S-denominator in \mathcal{C} . We



□

Altogether, we have proven the following main theorem of this article. Recall that \mathcal{C} is supposed to be a uni-fractionable category, see definition 3.1.

Theorem 5.13. *The fraction category $\text{Frac } \mathcal{C}$ fulfills the following properties.*

(a) *The object set of $\text{Frac } \mathcal{C}$ is the object set of \mathcal{C} . The morphism set of $\text{Frac } \mathcal{C}$ consists of double fractions, that is, equivalence classes of 3-arrows with respect to fraction equality, where a 3-arrow (b, f, a) is a diagram*

$$\begin{array}{ccc} & b & \\ \leftarrow \approx & & \\ & f & \\ \rightarrow \approx & & \\ & a & \end{array}$$

in \mathcal{C} with denominators a and b . For every 3-arrow (b, f, a) in \mathcal{C} , source and target of the double fraction $b \setminus f / a$ are given by $\text{Source } b \setminus f / a = \text{Target } b$ and $\text{Target } b \setminus f / a = \text{Source } a$. Given 3-arrows (b_1, f_1, a_1) and (b_2, f_2, a_2) in \mathcal{C} with $\text{Target } b_1 \setminus f_1 / a_1 = \text{Source } b_2 \setminus f_2 / a_2$, the composite of the double fractions can be constructed as follows: One chooses denominators d, d', e, e' and morphisms g_1, g_2 in \mathcal{C} with $b_2 a_1 = de$, $g_1 e = e' f_1$, $dg_2 = f_2 d'$. Then $(b_1 \setminus f_1 / a_1)(b_2 \setminus f_2 / a_2) = e' b_1 \setminus g_1 g_2 / a_2 d'$.

$$\begin{array}{ccccc} & & g_1 & & g_2 & \\ & & \rightarrow & & \rightarrow & \\ e' \swarrow & & f_1 & & e' \swarrow & \swarrow d & f_2 & \swarrow d' \\ b_1 \swarrow & & a_1 & \swarrow & b_2 & \swarrow & a_2 \end{array}$$

The identity of an object X in $\text{Frac } \mathcal{C}$ is given by $1_X = 1_X \setminus 1_X / 1_X$.

(b) *The fraction category $\text{Frac } \mathcal{C}$ is a localisation of \mathcal{C} with respect to $\text{Den } \mathcal{C}$, where the localisation functor $\text{loc}: \mathcal{C} \rightarrow \text{Frac } \mathcal{C}$ is given on the objects by $\text{loc}(X) = X$ for $X \in \text{Ob } \mathcal{C}$ and on the morphisms by $\text{loc}(f) = 1 \setminus f / 1$ for $f \in \text{Mor } \mathcal{C}$. The inverse of $\text{loc}(d)$ for $d \in \text{Den } \mathcal{C}$ is given by $(\text{loc}(d))^{-1} = d \setminus 1 / 1 = 1 \setminus 1 / d$.*

Given a category \mathcal{D} and a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that Fd is invertible for all $d \in \text{Den } \mathcal{C}$, the unique functor $\hat{F}: \text{Frac } \mathcal{C} \rightarrow \mathcal{D}$ with $F = \hat{F} \circ \text{loc}$ is given by $\hat{F}(b \setminus f / a) = (Fb)^{-1}(Ff)(Fa)^{-1}$.

(c) Given 3-arrows (b_1, f_1, a_1) , (b_2, f_2, a_2) and normal 3-arrows (p_1, g_1, i_1) , (p_2, g_2, i_2) in \mathcal{C} , we have $(b_1 \setminus f_1 / a_1)(p_2 \setminus g_2 / i_2) = (p_1 \setminus g_1 / i_1)(b_2 \setminus f_2 / a_2)$ in $\text{Frac } \mathcal{C}$ if and only if there exist 3-arrows $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ and normal 3-arrows $(\tilde{p}_1, \tilde{g}_1, \tilde{i}_1)$, $(\tilde{p}_2, \tilde{g}_2, \tilde{i}_2)$, fitting into the following commutative diagram in \mathcal{C} .

$$\begin{array}{ccccc}
 & \xleftarrow{\approx} & b_1 & \xrightarrow{f_1} & a_1 & \xleftarrow{\approx} & \\
 & \uparrow & \vdots & \uparrow & \vdots & \uparrow & \\
 p_1 & \square & \tilde{p}_1 & \square & \tilde{p}_2 & \square & p_2 \\
 & \downarrow & \vdots & \downarrow & \vdots & \downarrow & \\
 & \xleftarrow{\approx} & \tilde{b}_1 & \xrightarrow{\tilde{f}_1} & \tilde{a}_1 & \xleftarrow{\approx} & \\
 & \downarrow & \vdots & \downarrow & \vdots & \downarrow & \\
 g_1 & \square & \tilde{g}_1 & \square & \tilde{g}_2 & \square & g_2 \\
 & \downarrow & \vdots & \downarrow & \vdots & \downarrow & \\
 & \xleftarrow{\approx} & \tilde{b}_2 & \xrightarrow{\tilde{f}_2} & \tilde{a}_2 & \xleftarrow{\approx} & \\
 & \downarrow & \vdots & \downarrow & \vdots & \downarrow & \\
 i_1 & \square & \tilde{i}_1 & \square & \tilde{i}_2 & \square & i_2 \\
 & \downarrow & \vdots & \downarrow & \vdots & \downarrow & \\
 & \xleftarrow{\approx} & b_2 & \xrightarrow{f_2} & a_2 & \xleftarrow{\approx} &
 \end{array}$$

Proof. This follows from propositions 5.2 and 5.7(a), proposition 5.5 and proposition 5.12(b). \square

As a consequence of 3-arrow calculus, we get the following criterion. For a related 2-arrow version of this result, cf. [15, ch. 1, §2, th. 4-2] and [8, III.2.10].

Proposition 5.14. *We suppose given a uni-fractionable category \mathcal{U} such that \mathcal{U} is a full subcategory of \mathcal{C} and $\text{Den } \mathcal{U} = (\text{Den } \mathcal{C}) \cap (\text{Mor } \mathcal{U})$. We suppose that for every object X in \mathcal{C} , there exist an object \tilde{X} in \mathcal{U} and a denominator $d: \tilde{X} \rightarrow X$ in \mathcal{C} . We suppose that for every S -denominator $i: U \rightarrow \tilde{U}$ with U in \mathcal{U} , it follows that \tilde{U} is in \mathcal{U} . Then the inclusion functor $\text{inc}: \mathcal{U} \rightarrow \mathcal{C}$ induces an equivalence $\text{Frac inc}: \text{Frac } \mathcal{U} \rightarrow \text{Frac } \mathcal{C}$.*

Proof. To show that Frac inc is an equivalence of categories, we will verify that Frac inc is full, faithful and dense. Since for every $X \in \text{Ob } \mathcal{C}$ there exist $\tilde{X} \in \text{Ob } \mathcal{U}$ and a denominator $d: \tilde{X} \rightarrow X$ in \mathcal{C} , we have $X \cong \tilde{X} = (\text{Frac inc})\tilde{X}$ in $\text{Frac } \mathcal{C}$. Hence Frac inc is dense. To prove that Frac inc is full and faithful, we have to show that the map $\text{map}_{\text{Frac } \mathcal{U}}(U, V) \rightarrow \text{map}_{\text{Frac } \mathcal{C}}(U, V)$, $\varphi \mapsto (\text{Frac inc})\varphi$ is bijective for $U, V \in \text{Ob } \mathcal{U}$.

To show surjectivity, we suppose given a morphism $\psi \in \text{map}_{\text{Frac } \mathcal{C}}(U, V)$ and a normal 3-arrow $(p, f, i): U \leftarrow X \rightarrow Y \leftarrow V$ in \mathcal{C} with $\psi = p \setminus f / i$. Since i is an S -denominator and V is an object in \mathcal{U} , it follows that Y is an object in \mathcal{U} . Moreover, there exists an object \tilde{X} in \mathcal{U} and a denominator $d: \tilde{X} \rightarrow X$.

$$\begin{array}{ccccc}
 U & \xleftarrow{\approx} & \tilde{X} & \xrightarrow{df} & Y & \xleftarrow{\approx} & V \\
 \parallel & & \downarrow d & & \parallel & & \parallel \\
 U & \xleftarrow{p} & X & \xrightarrow{f} & Y & \xleftarrow{i} & V
 \end{array}$$

It follows that $(p, f, i) \equiv (dp, df, i)$, and as (dp, df, i) is a 3-arrow in \mathcal{U} , we have $\psi = p \setminus f / i = dp \setminus df / i = (\text{Frac inc})(dp \setminus df / i)$. Thus the map $\text{map}_{\text{Frac } \mathcal{U}}(U, V) \rightarrow \text{map}_{\text{Frac } \mathcal{C}}(U, V)$, $\varphi \mapsto (\text{Frac inc})\varphi$ is surjective.

To show injectivity, we suppose given $\varphi_1, \varphi_2 \in \text{Frac}\mathcal{U}(U, V)$ with $(\text{Frac inc})\varphi_1 = (\text{Frac inc})\varphi_2$. We choose normal 3-arrows $(p_1, f_1, i_1): U \leftarrow U_1 \rightarrow V_1 \leftarrow V$ and $(p_2, f_2, i_2): U \leftarrow U_2 \rightarrow V_2 \leftarrow V$ in \mathcal{U} with $\varphi_1 = p_1 \backslash f_1 / i_1$ and $\varphi_2 = p_2 \backslash f_2 / i_2$. By proposition 5.12(a), there exist normal 3-arrows $(\tilde{p}_1, \tilde{f}_1, \tilde{i}_1): U \leftarrow X_1 \rightarrow Y_1 \leftarrow V$, $(\tilde{p}_2, \tilde{f}_2, \tilde{i}_2): U \leftarrow X_2 \rightarrow Y_2 \leftarrow V$, $(q_1, d_1, j_1): U_1 \leftarrow X_1 \rightarrow X_2 \leftarrow U_2$, $(q_2, d_2, j_2): V_1 \leftarrow Y_1 \rightarrow Y_2 \leftarrow V_2$ in \mathcal{C} with denominators d_1, d_2 , fitting into a commutative diagram as displayed below on the left. Since \tilde{i}_1 resp. j_1 resp. j_2 is an S-denominator and V resp. U_2 resp. V_2 is an object in \mathcal{U} , it follows that Y_1 resp. X_2 resp. Y_2 is an object in \mathcal{U} . Moreover, there exists an object \tilde{X}_1 in \mathcal{U} and a denominator $d: \tilde{X}_1 \rightarrow X_1$ in \mathcal{C} . Thus we obtain the commutative diagram displayed below on the right, in which all objects – and hence all morphisms – are in \mathcal{U} , and where $d\tilde{p}_1$ is a denominator by multiplicativity. It follows that $\varphi_1 = p_1 \backslash f_1 / i_1 = p_2 \backslash f_2 / i_2 = \varphi_2$ in $\text{Frac}\mathcal{U}$. Thus the map $\text{Frac}\mathcal{U}(U, V) \rightarrow \text{Frac}\mathcal{C}(U, V)$, $\varphi \mapsto (\text{Frac inc})\varphi$ is injective.

$$\begin{array}{ccc}
\begin{array}{ccccc}
U & \xleftarrow{p_1} & U_1 & \xrightarrow{f_1} & V_1 & \xleftarrow{i_1} & V \\
\parallel & & \hat{\circ} q_1 & & \hat{\circ} q_2 & & \parallel \\
U & \xleftarrow{\tilde{p}_1} & X_1 & \xrightarrow{\tilde{f}_1} & Y_1 & \xleftarrow{\tilde{i}_1} & V \\
\parallel & & \hat{\circ} d_1 & & \hat{\circ} d_2 & & \parallel \\
U & \xleftarrow{\tilde{p}_2} & X_2 & \xrightarrow{\tilde{f}_2} & Y_2 & \xleftarrow{\tilde{i}_2} & V \\
\parallel & & \hat{\circ} j_1 & & \hat{\circ} j_2 & & \parallel \\
U & \xleftarrow{p_2} & U_2 & \xrightarrow{f_2} & V_2 & \xleftarrow{i_2} & V
\end{array} & &
\begin{array}{ccccc}
U & \xleftarrow{p_1} & U_1 & \xrightarrow{f_1} & V_1 & \xleftarrow{i_1} & V \\
\parallel & & \hat{\circ} dq_1 & & \hat{\circ} q_2 & & \parallel \\
U & \xleftarrow{d\tilde{p}_1} & \tilde{X}_1 & \xrightarrow{d\tilde{f}_1} & Y_1 & \xleftarrow{\tilde{i}_1} & V \\
\parallel & & \hat{\circ} dd_1 & & \hat{\circ} d_2 & & \parallel \\
U & \xleftarrow{\tilde{p}_2} & X_2 & \xrightarrow{\tilde{f}_2} & Y_2 & \xleftarrow{\tilde{i}_2} & V \\
\parallel & & \hat{\circ} j_1 & & \hat{\circ} j_2 & & \parallel \\
U & \xleftarrow{p_2} & U_2 & \xrightarrow{f_2} & V_2 & \xleftarrow{i_2} & V
\end{array} & & \square
\end{array}$$

Some of our examples of uni-fractionable categories in section 6 have finite coproducts, so it is a natural question to ask whether these are preserved when passing to the fraction category.

Proposition 5.15. *We suppose that \mathcal{C} admits finite coproducts.*

(a) *If $\text{Den}\mathcal{C}$ is closed under finite coproducts, then the fraction category $\text{Frac}\mathcal{C}$ admits finite coproducts and the localisation functor $\text{loc}: \mathcal{C} \rightarrow \text{Frac}\mathcal{C}$ preserves finite coproducts. In this case, we have $\text{ini}_{\text{loc}(X)}^{\text{loc}(i)} = \text{loc}(\text{ini}_X^i): \text{loc}(i) \rightarrow \text{loc}(X)$ for $X \in \text{Ob}\mathcal{C}$, and we have*

$$\left(\begin{array}{c} b_1 \backslash f_1 / a \\ b_2 \backslash f_2 / a \end{array} \right)^{\text{loc}(X_1 \amalg X_2)} = (b_1 \amalg b_2) \backslash \left(\begin{array}{c} f_1 \\ f_2 \end{array} \right)^{\tilde{X}_1 \amalg \tilde{X}_2} / a: \text{loc}(X_1 \amalg X_2) \rightarrow \text{loc}(Y)$$

for 3-arrows $(b_1, f_1, a): X_1 \leftarrow \tilde{X}_1 \rightarrow \tilde{Y} \leftarrow Y$, $(b_2, f_2, a): X_2 \leftarrow \tilde{X}_2 \rightarrow \tilde{Y} \leftarrow Y$ in \mathcal{C} .

(b) *If $\text{Den}\mathcal{C}$ is saturated and the localisation functor $\text{loc}: \mathcal{C} \rightarrow \text{Frac}\mathcal{C}$ preserves finite coproducts, then $\text{Den}\mathcal{C}$ is closed under finite coproducts.*

Proof. (a) We suppose that $\text{Den}\mathcal{C}$ is closed under finite coproducts. Moreover, we suppose given $X \in \text{Ob}\mathcal{C}$. Then $\text{loc}(\text{ini}_X^i)$ is a morphism from $\text{loc}(i)$ to $\text{loc}(X)$. So let us suppose given an arbitrary morphism $\varphi: \text{loc}(i) \rightarrow \text{loc}(X)$ in $\text{Frac}\mathcal{C}$, and we let $(b, f, a): i \leftarrow I \rightarrow \tilde{X} \leftarrow X$ be a 3-arrow in \mathcal{C} with $\varphi = b \backslash f / a$. By the universal property of i , we have $\text{ini}_I^i b = 1_i$ and $\text{ini}_I^i f = \text{ini}_{\tilde{X}}^i = \text{ini}_X^i a$, and therefore

$$\varphi = b \backslash f / a = \text{ini}_I^i b \backslash \text{ini}_I^i f / a = 1_i \backslash \text{ini}_X^i a / 1 = \text{loc}(\text{ini}_X^i).$$

Hence $\text{loc}(i)$ is an initial object in $\text{Frac } \mathcal{C}$ with $\text{ini}_{\text{loc}(X)}^{\text{loc}(i)} = \text{loc}(\text{ini}_X^i)$ for $X \in \text{Ob } \mathcal{C}$.

$$\begin{array}{ccccc} i & \longleftarrow & i & \xrightarrow{\text{ini}_X^i} & X & \longleftarrow & X \\ \parallel & & \downarrow \text{ini}_I^i & & \downarrow a & & \parallel \\ i & \xleftarrow{\approx} & I & \xrightarrow{f} & \tilde{X} & \xleftarrow{\approx} & X \end{array}$$

Next, we suppose given morphisms $\varphi_1: X_1 \rightarrow Y$ and $\varphi_2: X_2 \rightarrow Y$ in $\text{Frac } \mathcal{C}$. By proposition 5.9, there exist 3-arrows $(b_k, f_k, a): X_k \leftarrow \tilde{X}_k \rightarrow \tilde{Y} \leftarrow Y$ in \mathcal{C} with $\varphi_k = b_k \setminus f_k / a$ for $k \in \{1, 2\}$. As $b_1 \amalg b_2$ is a denominator in \mathcal{C} by assumption, we have the 3-arrow $(b_1 \amalg b_2, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, a)$ in \mathcal{C} . Moreover, since $\text{emb}_k^{\tilde{X}_1 \amalg \tilde{X}_2}(b_1 \amalg b_2) = b_k \text{emb}_k^{X_1 \amalg X_2}$, we have

$$\begin{aligned} \text{loc}(\text{emb}_k^{X_1 \amalg X_2})((b_1 \amalg b_2) \setminus \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2} / a) &= b_k \setminus \text{emb}_k^{\tilde{X}_1 \amalg \tilde{X}_2} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2} / a \\ &= b_k \setminus f_k / a = \varphi_k \end{aligned}$$

for $k \in \{1, 2\}$.

$$\begin{array}{ccccc} & \tilde{X}_k & \xrightarrow{\text{emb}_k^{\tilde{X}_1 \amalg \tilde{X}_2}} & \tilde{X}_1 \amalg \tilde{X}_2 & \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2}} & \tilde{Y} \\ & \swarrow \text{emb}_k^{X_1 \amalg X_2} & & \swarrow b_1 \amalg b_2 & & \swarrow \\ X_k & \xrightarrow{\text{emb}_k^{X_1 \amalg X_2}} & X_1 \amalg X_2 & \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2}} & \tilde{X}_1 \amalg \tilde{X}_2 & \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2}} & \tilde{Y} \\ & \swarrow & & \swarrow & & \swarrow \\ & X_k & & X_1 \amalg X_2 & & Y \end{array}$$

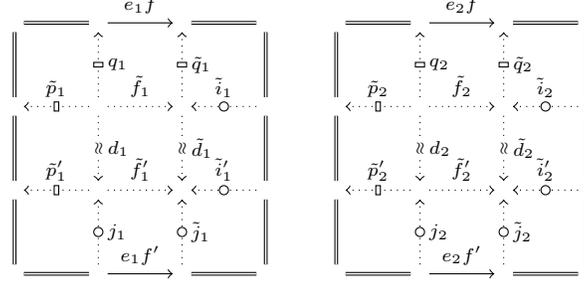
Conversely, we suppose given morphisms $\varphi, \varphi': \text{loc}(X_1 \amalg X_2) \rightarrow \text{loc}(Y)$ in $\text{Frac } \mathcal{C}$ such that $\text{loc}(\text{emb}_k^{X_1 \amalg X_2})\varphi = \text{loc}(\text{emb}_k^{X_1 \amalg X_2})\varphi' = \varphi_k$ for $k \in \{1, 2\}$. Then by proposition 5.9, there exist normal 3-arrows $(p, f, i), (p, f', i): X_1 \amalg X_2 \leftarrow \tilde{X} \rightarrow \tilde{Y} \leftarrow Y$ in \mathcal{C} with $\varphi = p \setminus f / i$ and $\varphi' = p \setminus f' / i$. For $k \in \{1, 2\}$, we choose a T-denominator $p_k: \tilde{X}_k \rightarrow X_k$ and a morphism $e_k: \tilde{X}_k \rightarrow \tilde{X}$ in \mathcal{C} with $p_k \text{emb}_k^{X_1 \amalg X_2} = e_k p$. We get

$$\varphi_k = \text{loc}(\text{emb}_k^{X_1 \amalg X_2})\varphi = \text{loc}(\text{emb}_k^{X_1 \amalg X_2})(p \setminus f / i) = p_k \setminus e_k f / i$$

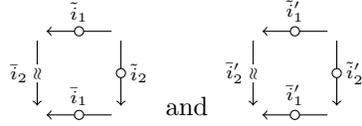
for $k \in \{1, 2\}$.

$$\begin{array}{ccccc} & \tilde{X}_k & \xrightarrow{e_k} & \tilde{X} & \xrightarrow{f} & \tilde{Y} \\ & \swarrow \text{emb}_k^{X_1 \amalg X_2} & & \swarrow p & & \swarrow \\ X_k & \xrightarrow{\text{emb}_k^{X_1 \amalg X_2}} & X_1 \amalg X_2 & \xrightarrow{p} & \tilde{X} & \xrightarrow{f} & \tilde{Y} \\ & \swarrow & & \swarrow & & \swarrow \\ & X_k & & X_1 \amalg X_2 & & Y \end{array}$$

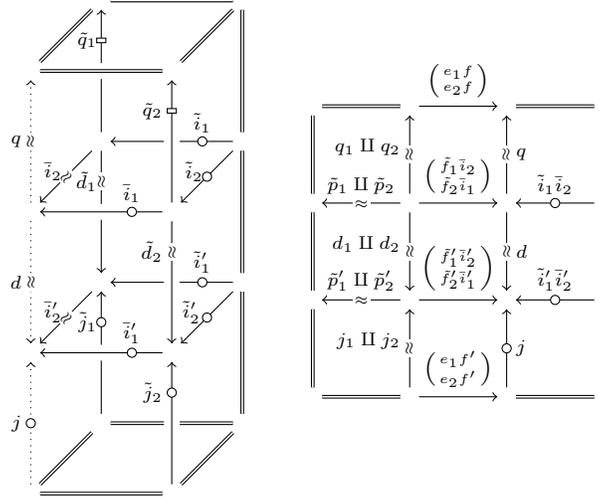
Analogously, we have $\varphi_k = p_k \setminus e_k f' / i$ and hence $\text{loc}(e_k f) = \text{loc}(e_k f')$ for $k \in \{1, 2\}$. By proposition 5.12(a), there exist normal 3-arrows $(\tilde{p}_k, f_k, \tilde{i}_k), (\tilde{p}'_k, f'_k, \tilde{i}'_k), (q_k, d_k, j_k), (\tilde{q}_k, \tilde{d}_k, \tilde{j}_k)$ in \mathcal{C} with denominators d_k, \tilde{d}_k for $k \in \{1, 2\}$, fitting into the following commutative diagrams in \mathcal{C} .



We let



be weak pushout rectangles in \mathcal{C} such that \bar{i}_1 and \bar{i}'_1 are S-denominators, so that we obtain morphisms q, d, j such that the diagram displayed below on the left commutes. Using coproducts, we obtain the commutative diagram displayed below on the right.



We finally have

$$\text{loc}\left(\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right)\text{loc}(f) = \text{loc}\left(\begin{pmatrix} e_1 f \\ e_2 f \end{pmatrix}\right) = \text{loc}\left(\begin{pmatrix} e_1 f' \\ e_2 f' \end{pmatrix}\right) = \text{loc}\left(\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\right)\text{loc}(f').$$

On the other hand,

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} p = \begin{pmatrix} e_1 p \\ e_2 p \end{pmatrix} = \begin{pmatrix} p_1 \text{emb}_1^{X_1 \amalg X_2} \\ p_2 \text{emb}_2^{X_1 \amalg X_2} \end{pmatrix} = p_1 \amalg p_2$$

implies that $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ is a denominator in \mathcal{C} by semi-saturatedness, so we have $\text{loc}(f) = \text{loc}(f')$ and therefore $\varphi = p \setminus f / i = p \setminus f' / i = \varphi'$.

Altogether, $\text{loc}(X_1 \amalg X_2)$ is a coproduct of $\text{loc}(X_1)$ and $\text{loc}(X_2)$ with embeddings $\text{emb}_k^{\text{loc}(X_1 \amalg X_2)} = \text{loc}(\text{emb}_k^{X_1 \amalg X_2})$ for $k \in \{1, 2\}$.

(b) We suppose that $\text{Den } \mathcal{C}$ is saturated and that loc preserves finite coproducts. Moreover, we suppose given denominators $d_1: X_1 \rightarrow Y_1$ and $d_2: X_2 \rightarrow Y_2$ in \mathcal{C} . Then we have

$$\begin{aligned} \text{loc}(d_k)\text{emb}_k^{\text{loc}(Y_1 \amalg Y_2)} &= \text{loc}(d_k)\text{loc}(\text{emb}_k^{Y_1 \amalg Y_2}) = \text{loc}(d_k\text{emb}_k^{Y_1 \amalg Y_2}) \\ &= \text{loc}(\text{emb}_k^{X_1 \amalg X_2}(d_1 \amalg d_2)) = \text{loc}(\text{emb}_k^{X_1 \amalg X_2})\text{loc}(d_1 \amalg d_2) \\ &= \text{emb}_k^{\text{loc}(X_1 \amalg X_2)}\text{loc}(d_1 \amalg d_2). \end{aligned}$$

Since d_1 and d_2 are denominators, $\text{loc}(d_1)$ and $\text{loc}(d_2)$ are isomorphisms. But then $\text{loc}(d_1 \amalg d_2)$ is also an isomorphism and hence $d_1 \amalg d_2$ is a denominator since $\text{Den } \mathcal{C}$ is saturated. \square

6. Applications

Quillen model categories

Given a Quillen model category \mathcal{M} [13, ch. I, §1, def. 1], we denote by $\mathbf{Cof}(\mathcal{M})$ resp. $\mathbf{Fib}(\mathcal{M})$ resp. $\mathbf{Bif}(\mathcal{M})$ the full subcategory of cofibrant resp. fibrant resp. bifibrant (that is, cofibrant and fibrant) objects.

Example 6.1. *Given a Quillen model category \mathcal{M} , the categories \mathcal{M} , $\mathbf{Cof}(\mathcal{M})$, $\mathbf{Fib}(\mathcal{M})$, $\mathbf{Bif}(\mathcal{M})$ carry the structure of uni-fractionable categories, where*

$$\begin{aligned} \text{Den } \mathcal{C} &= \{w \in \text{Mor } \mathcal{C} \mid w \text{ is a weak equivalence}\}, \\ \text{SDen } \mathcal{C} &= \{i \in \text{Mor } \mathcal{C} \mid i \text{ is an acyclic cofibration}\}, \\ \text{TDen } \mathcal{C} &= \{p \in \text{Mor } \mathcal{C} \mid p \text{ is an acyclic fibration}\} \end{aligned}$$

for $\mathcal{C} \in \{\mathcal{M}, \mathbf{Cof}(\mathcal{M}), \mathbf{Fib}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$. In particular, the homotopy category $\text{Ho } \mathcal{M}$ is isomorphic to $\text{Frac } \mathcal{M}$. If \mathcal{M} is a closed Quillen model category, then $\text{Den } \mathcal{C}$ is saturated in \mathcal{C} for $\mathcal{C} \in \{\mathcal{M}, \mathbf{Cof}(\mathcal{M}), \mathbf{Fib}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$. The localisation functor $\text{loc}: \mathcal{C} \rightarrow \text{Frac } \mathcal{C}$ preserves finite coproducts for $\mathcal{C} \in \{\mathbf{Cof}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$ and finite products for $\mathcal{C} \in \{\mathbf{Fib}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$. ⁽⁵⁾

Proof. (a) We consider \mathcal{M} and verify the axioms of a uni-fractionable category.

(Cat) By definition of a Quillen model category, weak equivalences, cofibrations and fibrations are closed under composition and contain all isomorphisms. Hence in particular weak equivalences, acyclic cofibrations and acyclic fibrations are closed under composition and contain all identities.

(2 of 3) This holds by definition of a Quillen model category.

⁵In general, the localisation functor $\text{loc}: \mathcal{M} \rightarrow \text{Frac } \mathcal{M}$ does not preserve finite coproducts or finite products since the set of denominators in a closed Quillen model category need not be closed under finite (co)products. A counterexample is provided by $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$, cf. [6, rem. 3.11], as considered in [5, ex.]: The coproduct of $2: (\mathbb{Z}/4, 1) \rightarrow (\mathbb{Z}/4, 2)$ with itself is given by $(2 \ 0): (\mathbb{Z}/4, 1) \rightarrow (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (2 \ 0))$; the former is a weak equivalence, but the latter is not since $\mathbb{Z}/4$ is a bijective object and $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ is not a bijective object in $\mathbf{mod}(\mathbb{Z}/4)$.

(WU) We suppose given an acyclic cofibration $i: X \rightarrow X'$ and a morphism $f: X \rightarrow Y$ in \mathcal{M} , and we let

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ i \uparrow & & \uparrow i' \\ X & \xrightarrow{f} & Y \end{array}$$

be a pushout rectangle in \mathcal{C} . Then i' is an acyclic cofibration.

The other assertion follows by duality.

(Fac) Since every morphism decomposes into a composite of a cofibration followed by an acyclic fibration, the assertion follows by semi-saturatedness.

Altogether, \mathcal{M} becomes a uni-fractionable category with

$$\begin{aligned} \text{Den } \mathcal{M} &= \{w \in \text{Mor } \mathcal{M} \mid w \text{ is a weak equivalence}\}, \\ \text{SDen } \mathcal{M} &= \{i \in \text{Mor } \mathcal{M} \mid i \text{ is an acyclic cofibration}\}, \\ \text{TDen } \mathcal{M} &= \{p \in \text{Mor } \mathcal{M} \mid p \text{ is an acyclic fibration}\}. \end{aligned}$$

The assertion on the saturatedness of $\text{Den } \mathcal{M}$ is proven in [13, ch. I, §5, prop. 1].

(b) We consider $\mathbf{Cof}(\mathcal{M})$ and have to verify the axioms of a uni-fractionable category. Since (Cat) and (2 of 3) hold for \mathcal{M} by (a), they hold in particular for $\mathbf{Cof}(\mathcal{M})$.

(WU) We suppose given an acyclic cofibration $i: X \rightarrow X'$ and a morphism $f: X \rightarrow Y$ in $\mathbf{Cof}(\mathcal{M})$, and we let

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ i \uparrow & & \uparrow i' \\ X & \xrightarrow{f} & Y \end{array}$$

be a pushout rectangle in \mathcal{C} . Then i' is an acyclic cofibration, and since Y is cofibrant and i' is in particular a cofibration, it follows that Y' is also cofibrant.

Now we suppose given an acyclic fibration $p: Y' \rightarrow Y$ and a morphism $f: X \rightarrow Y$ in $\mathbf{Cof}(\mathcal{M})$, and we let

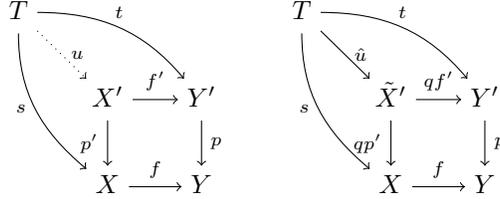
$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ p' \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

be a pullback rectangle in \mathcal{C} . Then p' is an acyclic fibration. We consider a strong cofibrant approximation of X' , that is, we let \tilde{X}' be a cofibrant object together with an acyclic fibration $q: \tilde{X}' \rightarrow X'$. The composite qp' is an acyclic fibration by multiplicativity. We will show that

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{qf'} & Y' \\ qp' \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

is a weak pullback of f along p . To this end, we suppose given an object $T \in$

Ob $\mathbf{Cof}(\mathcal{M})$ and morphisms $s: T \rightarrow X$, $t: T \rightarrow Y'$ with $sf = tp$. By the universal property of X' , there exists a (unique) morphism $u: T \rightarrow X'$ such that $up' = s$ and $uf' = t$. Moreover, as T is cofibrant and q is an acyclic fibration, there exists a lift $\hat{u}: T \rightarrow \tilde{X}'$ such that $u = \hat{u}q$. Now we have $\hat{u}qp' = up' = s$ and $\hat{u}qf' = uf' = t$.



(Fac) We let $w: X \rightarrow Y$ be a weak equivalence in $\mathbf{Cof}(\mathcal{M})$. Then there exists an acyclic cofibration $i: X \rightarrow Z$ and an acyclic fibration $p: Z \rightarrow Y$ in \mathcal{M} with $w = ip$. But since X is cofibrant and i is a cofibration, Z is cofibrant, too.

Altogether, $\mathbf{Cof}(\mathcal{M})$ becomes a uni-fractionable category with

$$\begin{aligned} \text{Den } \mathbf{Cof}(\mathcal{M}) &= \{w \in \text{Mor } \mathbf{Cof}(\mathcal{M}) \mid w \text{ is a weak equivalence}\}, \\ \text{SDen } \mathbf{Cof}(\mathcal{M}) &= \{i \in \text{Mor } \mathbf{Cof}(\mathcal{M}) \mid i \text{ is an acyclic cofibration}\}, \\ \text{TDen } \mathbf{Cof}(\mathcal{M}) &= \{p \in \text{Mor } \mathbf{Cof}(\mathcal{M}) \mid p \text{ is an acyclic fibration}\}. \end{aligned}$$

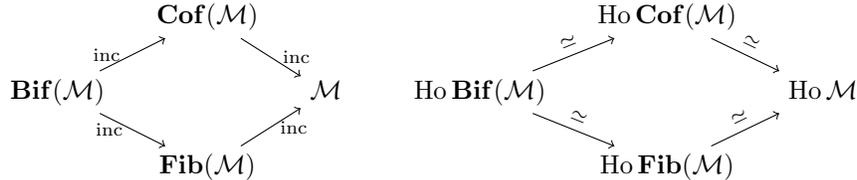
The assertion on the saturatedness of $\text{Den } \mathbf{Cof}(\mathcal{M})$ follows from (a) since if $\text{loc}^{\text{Frac } \mathbf{Cof}(\mathcal{M})}(f)$ is an isomorphism, then also $\text{loc}^{\text{Frac } \mathcal{M}}(f)$ is an isomorphism. The fact that the localisation functor $\text{loc}: \mathbf{Cof}(\mathcal{M}) \rightarrow \text{Frac } \mathbf{Cof}(\mathcal{M})$ preserves finite co-products follows from the gluing lemma [10, lem. 7.4], cf. also [9, ch. II, lem. 8.8], and proposition 5.15(a).

(c) The proof for $\mathbf{Fib}(\mathcal{M})$ is dual to (b).

(d) The proof for $\mathbf{Bif}(\mathcal{M})$ is a combination of (b) and (c). □

As an application of our abstract machinery, we obtain the following part of Quillen's homotopy category theorem [13, ch. I, §1, th. 1]. Given a Quillen model category \mathcal{M} , we (re-)define the *homotopy category* of $\mathcal{C} \in \{\mathcal{M}, \mathbf{Cof}(\mathcal{M}), \mathbf{Fib}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$ by $\text{Ho } \mathcal{C} := \text{Frac } \mathcal{C}$, using the uni-fractionable category structures from the preceding example.

Example 6.2. *We suppose given a Quillen model category \mathcal{M} . The commutative diagram of inclusion functors displayed below on the left induces the commutative diagram of equivalences displayed below on the right.*



In particular, $\text{Ho } \mathbf{Bif}(\mathcal{M}) \simeq \text{Ho } \mathcal{M}$.

Proof. This follows using proposition 5.14 for $\text{inc}: \mathbf{Cof}(\mathcal{M}) \rightarrow \mathcal{M}$ and $\text{inc}: \mathbf{Bif}(\mathcal{M}) \rightarrow \mathbf{Fib}(\mathcal{M})$, and its dual for $\text{inc}: \mathbf{Fib}(\mathcal{M}) \rightarrow \mathcal{M}$ and $\text{inc}: \mathbf{Bif}(\mathcal{M}) \rightarrow \mathbf{Cof}(\mathcal{M})$. □

The proof of Quillen’s homotopy category theorem, which states in particular that the homotopy category $\text{Ho } \mathcal{M}$ is equivalent to the quotient category $\mathbf{Bif}(\mathcal{M})/\sim$, where \sim denotes the homotopy congruence, can now be completed as in [11, cor. 1.2.9] by showing that $\mathbf{Bif}(\mathcal{M})/\sim$ fulfills the universal property of a localisation, which is essentially a corollary of Whitehead’s theorem [11, prop. 1.2.8].

Derivable categories

Recall that a *derivable category* in the sense of CISINSKI [2, sec. 2.25] consists of the same data as a Quillen model category, that is, a category \mathcal{C} together with three distinguished subsets of morphisms, called *cofibrations*, *fibrations* and *weak equivalences*, subject to the following axioms, where (co)fibrant objects and acyclic (co)fibrations are defined as in the Quillen model category case: The set of weak equivalences is supposed to be semi-saturated. The set of cofibrations is supposed to be closed under (binary) composition. There exists an initial object in \mathcal{C} , which is supposed to be cofibrant. The set of cofibrant objects is supposed to be closed under isomorphisms. The set of cofibrations between cofibrant objects and the subset of acyclic cofibrations therein are supposed to be stable under pushouts along morphisms between cofibrant objects. Every morphism with cofibrant source object factors into a cofibration followed by a weak equivalence. And dually for the fibrations and fibrant objects.

For homotopical algebra in derivable categories, cf. also the manuscript of RĂDULESCU-BANU [14], who uses the terminology *Anderson-Brown-Cisinski premodel category*.

Derivable categories are a natural generalisation of *categories of fibrant objects* in the sense of K. BROWN [1, sec. 1]. More precisely: Given a derivable category, then its full subcategory of fibrant objects is a category of fibrant objects in this sense, and its full subcategory of cofibrant objects fulfills the dual properties.

In the proof of example 6.1, we have not used the existence of general finite limits and colimits [13, ch. I, §1, def. 1, ax. M0]. Moreover, to show that a Quillen model category carries the structure of a uni-fractionable category, we also did not use the lifting axioms [13, ch. I, §1, def. 1, ax. M1]. Thus we obtain the following more general example.

Example 6.3. *We let \mathcal{C} be a derivable category such that the following properties hold.*

- *Every identity in \mathcal{C} is a cofibration and a fibration.*
- *Given an acyclic cofibration $i: X \rightarrow X'$ and a morphism $f: X \rightarrow Y$ in \mathcal{C} , there exists a pushout rectangle in \mathcal{C} as displayed below on the left, such that i' is an acyclic cofibration. Dually, given an acyclic fibration $p: Y' \rightarrow Y$ and a morphism $f: X \rightarrow Y$ in \mathcal{C} , there exists a pullback rectangle in \mathcal{C} as displayed below on the right, such that p' is an acyclic fibration.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ i \uparrow & & \uparrow i' \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ p' \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

- *For every weak equivalence $w: X \rightarrow Y$ in \mathcal{C} there exists an acyclic cofibration*

$i: X \rightarrow Z$ and an acyclic fibration $p: Z \rightarrow Y$ with $w = ip$.

$$\begin{array}{ccc} & Z & \\ \overset{i}{\curvearrowright} & & \overset{p}{\curvearrowleft} \\ X & \xrightarrow{w} & Y \end{array}$$

Then \mathcal{C} carries the structure of a uni-fractionable category, where

$$\begin{aligned} \text{Den } \mathcal{C} &= \{w \in \text{Mor } \mathcal{C} \mid w \text{ is a weak equivalence}\}, \\ \text{SDen } \mathcal{C} &= \{i \in \text{Mor } \mathcal{C} \mid i \text{ is an acyclic cofibration}\}, \\ \text{TDen } \mathcal{C} &= \{p \in \text{Mor } \mathcal{C} \mid p \text{ is an acyclic fibration}\}. \end{aligned}$$

Proof. This is the same proof as for a Quillen model category, see part (a) of the proof of example 6.1. \square

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