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COLOCALIZATION FUNCTORS IN DERIVED CATEGORIES AND TORSION THEORIES

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Abstract

Let R be a ring and let $\mathcal{A}$ be a hereditary torsion class of R-modules. The inclusion of the localizing subcategory generated by $\mathcal{A}$ into the derived category of R has a right adjoint, denoted $\mathrm{Cell}_{\mathcal{A}}$. Recently, Benson has shown how to compute $\mathrm{Cell}_{\mathcal{A}}R$ when R is a group ring of a finite group over a prime field and $\mathcal{A}$ is the hereditary torsion class generated by a simple module. We generalize Benson's construction to the case where $\mathcal{A}$ is any hereditary torsion class on R. It is shown that for every R-module M there exists an injective R-module E such that:

$$H^n(\mathrm{Cell}_{\mathcal{A}}M) \cong \mathrm{Ext}^{n-1}_{\mathrm{End}_R(E)}(\mathrm{Hom}_R(M,E),E) \text{ for } n \geqslant 2.$$

1. Introduction

Let R be a ring and let $\mathcal{D}_R$ be the (unbounded) derived category of chain complexes of left R-modules. Fix a class $\mathcal{A}$ of objects of $\mathcal{D}_R$. We recall some definitions of Dwyer and Greenlees from [**4**]. An object N of $\mathcal{D}_R$ is $\mathcal{A}$-*null* if $\mathrm{Ext}^*_R(A,N)=0$ for every $A\in\mathcal{A}$. An object C of $\mathcal{D}_R$ is $\mathcal{A}$-*cellular* if $\mathrm{Ext}^*_R(C,N)=0$ for every $\mathcal{A}$-null N. An $\mathcal{A}$-cellular object C is an $\mathcal{A}$-*cellular approximation* of $X\in\mathcal{D}_R$ if there is a map $\mu\colon C\to X$ such that $\mathrm{Ext}^*_R(A,\mu)$ is an isomorphism for all $A\in\mathcal{A}$. Finally, an $\mathcal{A}$-null object N is an $\mathcal{A}$-*nullification* of X if there is a map $\nu\colon X\to N$ which is universal among maps in $\mathcal{D}_R$ from X to $\mathcal{A}$-null objects. Denote an $\mathcal{A}$-cellular approximation of X by $\mathrm{Cell}_{\mathcal{A}}X$ and an A-nullification of X by $\mathrm{Null}_{\mathcal{A}}X$.

The following properties are easy to check: A map $\mu\colon C\to X$ is an $\mathcal{A}$-cellular approximation of X if and only if it is universal among all maps from $\mathcal{A}$-cellular objects to X. There is an exact triangle $\mathrm{Cell}_{\mathcal{A}}X\to X\to\mathrm{Null}_{\mathcal{A}}X$ whenever $\mathrm{Cell}_{\mathcal{A}}X$ or $\mathrm{Null}_{\mathcal{A}}X$ exists. An $\mathcal{A}$-cellular approximation of some object X is unique up to isomorphism, and the same goes for an $\mathcal{A}$-nullification of X.

Now suppose $\mathcal{A}$ is a set; then it turns out that the full subcategory of $\mathcal{A}$-cellular objects is the localizing subcategory generated by $\mathcal{A}$ (see [**5**] and [**7**, 5.1.5]). Moreover, when $\mathcal{A}$ is a set, the inclusion functor of the full subcategory of A-cellular objects into $\mathcal{D}_R$ has a right adjoint, which is $\mathrm{Cell}_{\mathcal{A}}X$ for every $X\in\mathcal{D}_R$; see [**6**] or [**7**]. Hence

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$\mathrm{Cell}_{\mathcal{A}}$ can be constructed as a *colocalization* functor (the right adjoint of an inclusion functor), and it follows that $\mathcal{A}$-cellular approximation and $\mathcal{A}$-nullification exist for any object of $\mathcal{D}_R$.

Similarly, when $\mathcal{A}$ is a set there exists a left adjoint to the inclusion of the full subcategory of $\mathcal{A}$-null objects; see, for example, Neeman's book [**10**, Section 9]. This functor is in fact $\mathcal{A}$-nullification and it is a *localization* functor (the left adjoint to an inclusion functor).

One method for calculating $\mathcal{A}$-cellular approximations is the formula given by Dwyer and Greenlees in [**4**], which holds whenever $\mathcal{A} = \{A\}$ and A is a perfect complex. This was later generalized by Dwyer, Greenlees and Iyengar in [**5**]. A new method for calculating the $\mathcal{A}$-cellular approximation for R-modules has been constructed by Benson in [**2**], dubbed *k-squeezed resolutions*. This method can be applied whenever $\mathcal{A}$ is a set of simple modules and R is an Artinian ring. One major benefit of Benson's construction is that it allows for explicit calculations.

As we will see, it is more natural to use Benson's method to construct the $\mathcal{A}$-nullification of a module rather than its $\mathcal{A}$-cellular approximation. We generalize Benson's construction so that it applies whenever $\mathcal{A}$ is a hereditary torsion class of modules. A *hereditary torsion class* of modules is a class of modules that is closed under submodules, quotient modules, coproducts and extensions. The main result of this paper is the following:

Theorem 1.1. *Let $\mathcal{T}$ be a hereditary torsion class on left R-modules. For every left R-module M there exists an injective left R-module E such that the complex*

$$\mathbf{R}\mathrm{Hom}_{\mathrm{End}_R(E)}(\mathrm{Hom}_R(M,E),E)$$

is a $\mathcal{T}$-nullification of M. In particular, the $\mathcal{T}$-nullification of R is the differential graded algebra $\mathbf{R}\mathrm{End}_{\mathrm{End}_R(E)}(E)$.

The formula given in the abstract follows immediately from the distinguished triangle $\mathrm{Cell}_{\mathcal{T}}M \to M \to \mathrm{Null}_{\mathcal{T}}M$ mentioned above.

The layout of this paper is as follows: The necessary background on hereditary torsion classes and the background on cellular approximations and nullifications is given in Section 2. In Section 3 we describe the construction of nullification with respect to a hereditary torsion class and prove Theorem 1.1. We study the case where R is an Artinian ring in Section 4. This section offers a different proof to a result of Benson ([**2**, Theorem 5.1]). Finally, Section 5 provides several examples.

1.1. Notation and terminology

By a ring we always mean an associative ring with a unit, not necessarily commutative. Unless otherwise noted all modules considered are left modules. A triangle always means an exact (distinguished) triangle in the unbounded derived category of left R-modules, denoted $\mathcal{D}_R$. A complex is always a chain-complex of R-modules. For complexes we use the standard convention that the subscript grading is the negative of the superscript grading; i.e., $\square_{-i} = \square^i$. It is taken for granted that every R-module is a complex concentrated in degree 0 and with zero differential. A complex X is *bounded-above* if, for some n and for all $i > n$, $H_i(X) = 0$. For complexes X and Y the notation $\mathrm{Hom}_R(X,Y)$ stands for the usual chain complex of homomorphisms. The notation $\mathbf{R}\mathrm{Hom}_R(-,-)$ stands for the derived functor of the $\mathrm{Hom}_R(-,-)$ functor.

By $\mathrm{End}_R(M)$ we mean the endomorphisms ring of an R-module M. The symbol $\simeq$ stands for quasi-isomorphism of complexes.

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2. Background on hereditary torsion theories and cellular-approximation, nullification and completion

2.1. Hereditary torsion theories

Below is a recollection of the definition and main properties of hereditary torsion theories. A thorough review of this material can be found in [**11**].

Definition 2.1. A *hereditary torsion class* $\mathcal{T}$ is a class of R-modules that is closed under submodules, quotient modules, coproducts and extensions. Closure under extensions means that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence with M_1 and M_3 in $\mathcal{T}$, then so is M_2. The modules in $\mathcal{T}$ will be called *$\mathcal{T}$-torsion modules* (or just torsion modules when the torsion theory is clear from the context). The class of *torsion-free* modules $\mathcal{F}$ is the class of all modules F satisfying $\mathrm{Hom}_R(C, F) = 0$ for every $C \in \mathcal{T}$. The pair $(\mathcal{T}, \mathcal{F})$ is referred to as a *hereditary torsion theory.* To every hereditary torsion theory there is an associated radical t, where $t(M)$ is the maximal torsion submodule of M. Note that $M/t(M)$ is therefore torsion-free.

Every hereditary torsion class $\mathcal{T}$ has an *injective cogenerator* (see [**11**, VI.3.7]). This means there exists an injective module E such that a module M is torsion if and only if $\mathrm{Hom}_R(M, E) = 0$. It is also important to note that in any hereditary torsion theory, the class of torsion-free modules is closed under injective hulls (see [**11**, VI.3.2]). Thus, if F is a torsion-free module then the injective hull of F is also torsion-free.

Definition 2.2. Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory and let t be the associated radical. An R-module M is called *$\mathcal{F}$-closed* if, for every left ideal $\mathfrak{a} \subset R$ such that $R/\mathfrak{a} \in \mathcal{T}$, the induced map $M = \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(\mathfrak{a}, M)$ is an isomorphism. The inclusion of the full subcategory of $\mathcal{F}$-closed modules has a left adjoint $M \mapsto M_{\mathcal{F}}$. The module $M_{\mathcal{F}}$ is called the *module of quotients* of M (see [**11**, IX.1]). The unit of this adjunction has the following properties: the kernel of the map $M \to M_{\mathcal{F}}$ is $t(M)$, $M_{\mathcal{F}}$ is torsion-free and the cokernel of this map is a torsion module.

2.2. Cellular-approximation, nullification and completion

The following recalls the basic properties of cellular approximation, as well as the definition of completion given by Dwyer and Greenlees in [**4**]:

Definition 2.3. Let R be a ring and let $\mathcal{A}$ be a class of R-complexes. We say that an R-complex X is *$\mathcal{A}$-complete* if $\mathrm{Ext}^*_R(N, X) = 0$ for any $\mathcal{A}$-null object N. An R-complex C is an *$\mathcal{A}$-completion* of X if C is $\mathcal{A}$-complete and there is an $\mathcal{A}$-equivalence $X \to C$. It is easy to see that an $\mathcal{A}$-completion of a complex X is unique up to an isomorphism in $\mathcal{D}_R$. As in [**4**], we denote an $\mathcal{A}$-completion of X by $X^\wedge_{\mathcal{A}}$.

The following criterion for nullification is usually easier to check than the original definition. Its proof is easy and therefore omitted.

Lemma 2.4. *Let R be a ring and let $\mathcal{A}$ be a class of R-complexes. A complex N is an* $\mathcal{A}$-nullification *of X if there is a triangle $C \to X \to N$ such that C is $\mathcal{A}$-cellular and N is $\mathcal{A}$-null. In this case it also follows that C is an A-cellular approximation of X.*

Recall that when $\mathcal{A}$ is a set, the full subcategory of $\mathcal{A}$-cellular objects in $\mathcal{D}_R$ is the localizing subcategory generated by A. The *localizing category generated by* $\mathcal{A}$, denoted $\langle\mathcal{A}\rangle$, is the smallest full triangulated subcategory of $\mathcal{D}_R$ that is closed under triangles, direct sums and retracts. Closure under triangles means that for every distinguished triangle in $\mathcal{D}_R$, if two of the objects are in the localizing subcategory, then so is the third. The proof of the following lemma is clear.

Lemma 2.5. *Let $\mathcal{A}$ be a class of R-complexes; then every object of $\langle\mathcal{A}\rangle$ is $\mathcal{A}$-cellular. If $\mathcal{B}$ is another class of R-complexes such that $\langle\mathcal{A}\rangle = \langle\mathcal{B}\rangle$, then $\mathcal{A}$-cellular approximation is the same as $\mathcal{B}$-cellular approximation.*

Remark 2.6. In [**4**] $\mathcal{A}$-cellular complexes were called $\mathcal{A}$-*torsion* while the term $\mathcal{A}$-*cellular* was reserved for complexes in $\langle\mathcal{A}\rangle$. When $\mathcal{T}$ is a hereditary torsion theory, the two terms agree (by Lemma 2.8 below).

2.3. Cellular-approximation with respect to a hereditary torsion theory

Let $\mathcal{T}$ be a hereditary torsion class. It is not immediately apparent that $\mathcal{T}$-cellular approximation exists. Below, in Lemma 2.8, we show that $\langle\mathcal{T}\rangle$ is the same as the localizing subcategory generated by a set $\mathcal{A}_{\mathcal{T}}$. This immediately implies that $\mathcal{T}$-cellular approximation and $\mathcal{T}$-nullification exist for any R-complex; see Corollary 2.9.

Definition 2.7. Given a hereditary torsion class of R-modules $\mathcal{T}$, we denote by $\mathcal{A}_{\mathcal{T}}$ the set of all cyclic $\mathcal{T}$-torsion modules.

Lemma 2.8. *Let $\mathcal{T}$ be a hereditary torsion class; then every $\mathcal{T}$-torsion module is $\mathcal{A}_{\mathcal{T}}$-cellular and hence $\langle\mathcal{T}\rangle = \langle\mathcal{A}_{\mathcal{T}}\rangle$.*

Proof. Clearly, every cyclic $\mathcal{T}$-torsion module is $\mathcal{A}_{\mathcal{T}}$-cellular. Therefore every direct sum of cyclic $\mathcal{T}$-torsion modules is $\mathcal{A}_{\mathcal{T}}$-cellular. Let M be a $\mathcal{T}$-torsion module; then there is a surjection $C(M) = \oplus_{m\in M} R/\mathrm{ann}(m) \twoheadrightarrow M$. Since every hereditary torsion theory is closed under submodules, $R/\mathrm{ann}(m)$ is $\mathcal{T}$-torsion for every $m \in M$. Clearly, $C(M)$ is $\mathcal{A}_{\mathcal{T}}$-cellular and $\mathcal{T}$-torsion. Next we build a resolution X of M using $\mathcal{A}_{\mathcal{T}}$-cellular modules. Let $X_0 = C(M)$ and let $d_0 \colon X_0 \twoheadrightarrow M$ be the map defined above. The kernel of d_0 is $\mathcal{T}$-torsion, so there is an epimorphism $C(\ker(d_0)) \twoheadrightarrow \ker(d_0)$. Let $X_1 = C(\ker(d_0))$ and let d_1 be the composition $X_1 \twoheadrightarrow \ker(d_0) \hookrightarrow X_0$. In this way X is built inductively, and it is clear that X is quasi-isomorphic to M. By construction, X is in the localizing subcategory generated by $\mathcal{A}_{\mathcal{T}}$. □

Corollary 2.9. *Let $\mathcal{T}$ be a hereditary torsion class; then $\mathcal{T}$-cellular approximation, $\mathcal{T}$-nullification and $\mathcal{T}$-completion exist for every complex. Moreover, a complex X is $\mathcal{T}$-cellular if and only if $X \in \langle\mathcal{T}\rangle$.*

Proof. Lemma 2.8 implies that $\mathcal{T}$-cellular approximation is the same as $\mathcal{A}_{\mathcal{T}}$-cellular approximation. As mentioned in Section 1, $\mathcal{A}_{\mathcal{T}}$-cellular approximation exists for every complex. The proof of the other claims is similar. □

Lemma 2.10. *Let $\mathcal{T}$ be a hereditary torsion class.*

1. *If X is a $\mathcal{T}$-cellular complex, then the homology groups of X are $\mathcal{T}$-torsion R-modules.*
2. *If X is a bounded-above complex such that the homology groups of X are $\mathcal{T}$-torsion, then X is $\mathcal{T}$-cellular.*

Proof. Let $\mathcal{C}$ be the full subcategory of $\mathcal{D}_R$ containing all objects whose homology groups are $\mathcal{T}$-torsion R-modules. The properties of a hereditary torsion theory show that $\mathcal{C}$ is a localizing subcategory. Since $\mathcal{C}$ contains $\mathcal{T}$, then $\mathcal{C}$ also contains $\langle \mathcal{T} \rangle$. This proves the first statement.

Now suppose X is a bounded-above complex and that $H_i X \in \mathcal{T}$ for all i. Because X is bounded-above, X belongs to the localizing subcategory generated by the homology groups of X (see for example [**4**, 5.2]). Since the homology groups of X all belong to $\langle \mathcal{T} \rangle$, so does X. □

Remark 2.11. If R is a commutative Noetherian ring, then a complex X is $\mathcal{T}$-cellular if and only if all the homology groups of X are $\mathcal{T}$-torsion. This easily follows from a result of Neeman [**9**, Theorem 2.8]. However, Example 5.2 shows a noncommutative ring R and a complex X such that $H_i(X)$ is $\mathcal{T}$-torsion for all i, but X is not $\mathcal{T}$-cellular.

3. Nullification construction

In [**2**] Benson gives a construction called *k-squeezed resolution* which yields k-cellular approximations over the ring kG, where k is a prime field and G is a finite group. In Construction 3.1 we generalize Benson's work so as to produce $\mathcal{T}$-cellular approximations over any ring R, where $\mathcal{T}$ is a hereditary torsion class. A second isomorphic construction of k-cellular approximations is given in Construction 3.2.

Nullification Construction 3.1. Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory with radical t. For an R-module M we construct the $\mathcal{T}$-nullification of M as a cochain complex $I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots$ inductively.

Let $M^0 = M$, let $F^0 = M^0/t(M^0)$ and let I^0 be the injective hull of F^0. Note that since F^0 is torsion-free, so is I^0. We proceed by induction; set

$$M^{n+1} = I^n/F^n, \qquad F^{n+1} = M^{n+1}/t(M^{n+1})$$

and let I^{n+1} be the injective hull of F^{n+1}. Again I^{n+1} is torsion-free because F^{n+1} is. The differential $d\colon I^n \to I^{n+1}$ is the composition $I^n \to M^{n+1} \to F^{n+1} \to I^{n+1}$. The image of $d\colon I^n \to I^{n+1}$ is F^{n+1}, and therefore $d \circ d = 0$. Denote the resulting complex by I. The natural map $M \to I^0$ extends to a map of complexes $M \to I$.

Nullification Construction 3.2. For an R-module M we construct a cochain complex $J^0 \xrightarrow{d^0} J^1 \xrightarrow{d^1} J^2 \xrightarrow{d^2} \cdots$ inductively.

Let $Q^0 = M$, let $N^0 = (Q^0)_{\mathcal{F}}$ and let J^0 be the injective hull of N^0. Denote by d^{-1} the map $M \to J^0$. Now proceed by induction; let

$$Q^{n+1} = J^n/\mathrm{im}(d^{n-1}), \qquad N^{n+1} = (Q^{n+1})_{\mathcal{F}}$$

and let J^{n+1} be the injective hull of N^{n+1}. The differential $d^n : J^n \to J^{n+1}$ is the composition $J^n \to Q^{n+1} \to N^{n+1} \to J^{n+1}$. Clearly, $d^{n+1} \circ d^n = 0$. Denote the resulting complex by J. The natural map $M \to J^0$ extends to a map of complexes $M \to J$. Note that for every n, J^n is torsion-free because N^n is.

Remark 3.3. For a fixed R-module M, the complex I of Construction 3.1 and the complex J of Construction 3.2 are isomorphic. To construct this isomorphism one needs the following property: for any R-module L the injective hull of $L_{\mathcal{F}}$ and the injective-hull of $L/t(L)$ are the same; this is because $(L/t(L))_{\mathcal{F}} = L_{\mathcal{F}}$ and $L/t(L)$ is an essential submodule of $(L/t(L))_{\mathcal{F}}$ (see [**11**, IX.2.4]). Using the aforementioned property, it is a simple exercise to construct the isomorphism inductively.

Lemma 3.4. *Let J be the complex constructed from M in 3.2; then $H_0(J) \cong M_{\mathcal{F}}$.*

Proof. It easily follows from the definition of an $\mathcal{F}$-closed module that any injective torsion-free module is $\mathcal{F}$-closed; therefore J^0 is $\mathcal{F}$-closed. For any $\mathcal{F}$-closed module K there is an isomorphism $K \cong K_{\mathcal{F}}$ (see [**11**, page 198]); therefore $(J^0)_{\mathcal{F}} \cong J^0$ and $(M_{\mathcal{F}})_{\mathcal{F}} \cong M_{\mathcal{F}}$.

The module of quotients functor is left exact (see [**11**, page 199]). Hence applying the module of quotients functor to the sequence $M_{\mathcal{F}} \to J^0 \to J^0/M_{\mathcal{F}}$ yields an exact sequence:

$$0 \to M_{\mathcal{F}} \to J^0 \to (J^0/M_{\mathcal{F}})_{\mathcal{F}}.$$

We see that $J^0/M_{\mathcal{F}}$ is torsion-free because it is isomorphic to a submodule of the torsion-free module $(J^0/M_{\mathcal{F}})_{\mathcal{F}}$.

Now consider the short exact sequence

$$M_{\mathcal{F}}/\mathrm{im}(M) \to Q^1 \to J^0/M_{\mathcal{F}}.$$

The module $M_{\mathcal{F}}/\mathrm{im}(M)$ is a torsion module (see Definition 2.2), while the module $J^0/M_{\mathcal{F}}$ is torsion free. From the definition of the radical t it follows that $M_{\mathcal{F}}/\mathrm{im}(M) \cong t(Q^1)$. Therefore $M_{\mathcal{F}}$ is the kernel of $J^0 \to N^1$ and the proof is complete. □

Lemma 3.5. *Let M be an R-module, let I be the complex constructed from M in 3.1 and let C be a complex such that $C \to M \to I$ is a distinguished triangle. Then C is a $\mathcal{T}$-cellular approximation of M, and I is a $\mathcal{T}$-nullification of M. In particular, $H_0(\mathrm{Null}_{\mathcal{T}} M) \cong M_{\mathcal{F}}$ for any R-module M.*

Proof. We can choose C to be the complex $M \to I^0 \to I^1 \to \cdots$ with M in degree 0. The homology of C is easy to compute: $H^n(C) = t(M^n)$, with M^n as defined in Construction 3.1 above. By Lemma 2.10, the complex C is $\mathcal{T}$-cellular. The complex I is $\mathcal{T}$-null, simply because I is composed of torsion-free injective modules. Thus, by Lemma 2.4, I is a $\mathcal{T}$-nullification of M and C is a $\mathcal{T}$-cellular approximation of M. Since I is isomorphic to the complex J of Construction 3.2, it follows that $H_0(\mathrm{Null}_{\mathcal{T}} M) = H_0(J) \cong M_{\mathcal{F}}$. □

Using the constructions above we can give a different description of $\mathcal{T}$-nullification, the one shown in Theorem 1.1. Before proving Theorem 1.1 it is necessary to note some properties of the functor $\mathrm{Hom}_R(-, E)$.

Let E be an R-module and $\mathcal{E}$ be the endomorphism ring $\mathrm{End}_R(E) = \mathrm{Hom}_R(E, E)$. The functor $\mathrm{Hom}_R(-, E)$ is a contravariant functor from left R-complexes to left $\mathcal{E}$-complexes. This left $\mathcal{E}$-action is simply composition on the left with the morphisms in $\mathcal{E}$. In other words, the left $\mathcal{E}$-action on $\mathrm{Hom}_R(-, E)$ is induced by the left $\mathcal{E}$-action on E itself. Moreover, the functor $\mathrm{Hom}_{\mathcal{E}}(-, E)$ is a contravariant functor, this time from left $\mathcal{E}$-complexes to left R-complexes. Here the left R-action on $\mathrm{Hom}_{\mathcal{E}}(-, E)$ comes from the left R-action on E (which commutes with the left $\mathcal{E}$-action on E). In particular, there is a derived version of this functor: $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(-, E) \colon \mathcal{D}_{\mathcal{E}} \to \mathcal{D}_R$.

Proof of Theorem 1.1. Given an R-module M, construct a $\mathcal{T}$-nullification of M in the way prescribed in 3.1. This construction results in a cochain complex I, with I^n being an injective torsion-free module. Let E be a torsion-free injective R-module such that for every n, I^n is a direct summand of a finite direct sum of copies of E. For example, one can take E to be the product $\prod_n I^n$. Denote by $\mathcal{E}$ the endomorphism ring $\mathrm{End}_R(E)$.

Now consider the triangle $C \to M \to I$. Since I is a $\mathcal{T}$-nullification of M, C is a $\mathcal{T}$-cellular approximation of M. Applying the functor $\mathrm{Hom}_R(-, E)$ to this triangle yields a triangle in $\mathcal{D}_{\mathcal{E}}$:

$$\mathrm{Hom}_R(I, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(C, E).$$

Since E is injective, $H_i(\mathrm{Hom}_R(C, E)) \cong \mathrm{Hom}_R(H_i(C), E)$. Since the homology groups of C are torsion, $\mathrm{Hom}_R(H_i(C), E) = 0$. Hence the map $\mathrm{Hom}_R(I, E) \to \mathrm{Hom}_R(M, E)$ is a quasi-isomorphism of $\mathcal{E}$-complexes.

Because I^n is a direct summand of a finite direct sum of copies of E, the $\mathcal{E}$-module $\mathrm{Hom}_R(I^n, E)$ is projective. Thus the map $\mathrm{Hom}_R(I, E) \to \mathrm{Hom}_R(M, E)$ is a projective resolution of $\mathrm{Hom}_R(M, E)$ in the category of $\mathcal{E}$-modules. We conclude that the complex $\mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(I, E), E)$ is the derived functor $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(M, E), E)$.

Because I^n is a direct summand of a finite direct sum of copies of E, one readily sees that the R-module

$$\mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(I^n, E), E)$$

is naturally isomorphic to I^n, and therefore $\mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(I, E), E) \cong I$. □

Remark 3.6. As noted in Theorem 1.1, $\mathrm{Null}_{\mathcal{T}} R \simeq \mathbf{R}\mathrm{End}_{\mathcal{E}}(E)$, and therefore

$$R_{\mathcal{F}} \cong H^0(\mathrm{Null}_{\mathcal{T}} R) \cong H^0(\mathbf{R}\mathrm{End}_{\mathcal{E}}(E)) = \mathrm{End}_{\mathcal{E}}(E).$$

This isomorphism recovers [**11**, IX.3.3], where it is stated that there is an injective R-module E such that $R_{\mathcal{F}} \cong \mathrm{End}_{\mathrm{End}_R(E)}(E)$. Also note that $\mathrm{Null}_{\mathcal{T}} R$ is quasi-isomorphic to a differential graded algebra. This also follows from a result of Dwyer [**3**, Proposition 2.5], where it is shown that for any set of complexes $\mathcal{A}$ the complex $\mathrm{Null}_{\mathcal{A}} R$ is quasi-isomorphic to a differential graded algebra.

Remark 3.7. Let $X = \cdots \to X_n \to X_{n-1} \to \cdots$ be a complex such that there exists some m for which $X_n = 0$ for all $n > m$. Then it is possible to generalize the Nullification Construction 3.1 to give the $\mathcal{T}$-nullification of X. Moreover, this generalized construction of $\mathrm{Null}_{\mathcal{T}}X$ can be done in such a way that for $n > m$ $(\mathrm{Null}_{\mathcal{T}}X)_n = 0$, while for $n \leqslant m$ $(\mathrm{Null}_{\mathcal{T}}X)_n$ is a finite direct sum of torsion-free injective modules. Therefore $(\mathrm{Null}_{\mathcal{T}}X)_n$ is itself a torsion-free injective for $n \leqslant m$. Now it is easy to see that the proof of Theorem 1.1 works for $\mathrm{Null}_{\mathcal{T}}X$ as well and yields the same result. Namely, there exists an injective R-module E such that

$$\mathrm{Null}_{\mathcal{T}}X \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{End}_R(E)}(\mathrm{Hom}_R(X,E),E).$$

Clearly, this result carries over to any bounded-above complex X.

Say an injective module E is *sufficient to compute the $\mathcal{T}$-nullification of M* if

$$\mathrm{Null}_{\mathcal{T}}M \simeq \mathbf{R}Hom_{\mathcal{E}}(\mathrm{Hom}_R(M,E),E),$$

where $\mathcal{E} = \mathrm{End}_R(M)$. Given an R-module M one can use the proof of Theorem 1.1 to construct an injective module E which is sufficient to compute the $\mathcal{T}$-nullification of M. However, there are other injective modules sufficient to compute the $\mathcal{T}$-nullification of M, as shown by the following proposition.

Proposition 3.8. *Let M be an R-module and let E be an injective cogenerator of $\mathcal{T}$. Denote by $\mathcal{E}$ the ring* $\mathrm{End}_R(E)$.

1. *If the $\mathcal{E}$-module* $\mathrm{Hom}_R(M,E)$ *has a resolution composed of finitely generated projective modules in each degree, then E is sufficient to compute the $\mathcal{T}$-nullification of M.*
2. *There exists an ordinal α such that the module $E' = \prod_{i<\alpha} E$ is sufficient to compute the $\mathcal{T}$-nullification of M.*

Proof. Let P be a finitely generated projective $\mathcal{E}$-module; then it is easy to see that $\mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{E}}(P,E),E)$ is naturally isomorphic to P. Now let F be a projective resolution of $\mathrm{Hom}_R(M,E)$ over $\mathcal{E}$ and assume F is composed of finitely generated projective modules in each degree. Then $\mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{E}}(F,E),E)$ is naturally isomorphic to F.

The quasi-isomorphism $\eta\colon F \to \mathrm{Hom}_R(M,E)$ induces a map

$$\mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(M,E),E) \to \mathrm{Hom}_{\mathcal{E}}(F,E).$$

Composing with the natural map $M \to \mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(M,E),E)$ yields a map $\mu\colon M \to \mathrm{Hom}_{\mathcal{E}}(F,E)$. It is easy to see that $\mathrm{Hom}_R(\mu,E)$ is the quasi-isomorphism η.

Consider the triangle $C \to M \xrightarrow{\mu} \mathrm{Hom}_{\mathcal{E}}(F,E)$. Clearly $\mathrm{Hom}_{\mathcal{E}}(F,E)$ is $\mathcal{T}$-null. Since $\mathrm{Hom}_R(\mu,E)$ is a quasi-isomorphism, $\mathrm{Hom}_R(C,E)$ is quasi-isomorphic to zero. This implies $\mathrm{Hom}_R(H_i(C),E) = 0$ for all i. Since E is an injective cogenerator for $\mathcal{T}$, $H_i(C)$ is torsion for all i. Clearly, C is bounded-above and so, by Lemma 2.10, C is $\mathcal{T}$-cellular. We conclude that $\mathrm{Hom}_{\mathcal{E}}(F,E)$ is a $\mathcal{T}$-nullification of M, and E is sufficient to compute the $\mathcal{T}$-nullification of M.

We now turn our attention to the second item in the proposition. By [**11**, VI.3.9], every torsion-free module has a monomorphism to some direct product of copies of E. In particular, every torsion-free injective is a isomorphic to a direct summand of some direct product of copies of E.

Let I be the complex described in the Nullification Construction 3.1. Let E' be a direct product of copies of E such that for every n, I^n is isomorphic to a direct summand of E'. Clearly, the $\mathrm{End}_R(E')$-complex $\mathrm{Hom}_R(I,E')$ is a projective resolution of

$\operatorname{Hom}_R(M, E')$, which is composed of finitely generated projective modules in every degree. Hence E' is sufficient to compute the $\mathcal{T}$-nullification of M. □

4. Torsion theories and cellular approximation in Artinian rings

Throughout this section R is an Artinian ring and $\mathbb{S}$ is a set of non-isomorphic simple modules of R. Define a class $\mathcal{F}$ of R-modules by

$$\mathcal{F} = \{F \mid \operatorname{Hom}_R(S, F) = 0 \text{ for all } S \in \mathbb{S}\}$$

and define a class $\mathcal{T}$ by setting

$$\mathcal{T} = \{M \mid \operatorname{Hom}_R(M, F) = 0 \text{ for all } F \in \mathcal{F}\}.$$

By [**11**, VIII.3], the pair $(\mathcal{T}, \mathcal{F})$ forms a hereditary torsion theory (alternatively, one can easily deduce this from Lemma 4.3 below). Because R is Artinian, every hereditary torsion theory of R-modules is generated by a set of simple modules (see [**11**, VIII]), so this context covers all hereditary torsion theories over R. In this section we give several results regarding $\mathcal{T}$-nullification. We also give a different proof for a result of Benson [**2**, Theorem 5.1] in Corollary 4.5.

Let Ω be the set of isomorphism classes of simple modules of R and let $\mathbb{S}'$ be the complement of $\mathbb{S}$ in Ω. We denote by E the product of the injective hulls of the simple modules in $\mathbb{S}'$ and denote by P the direct sum of the projective covers of those simple modules. We show that E is an injective cogenerator of $\mathcal{T}$ and that being $\mathcal{T}$-cellular is the same as being $\mathbb{S}$-cellular.

Lemma 4.1. *Let C be a cyclic R-module such that* $\operatorname{Hom}_R(C, E) = 0$*; then C is $\mathbb{S}$-cellular.*

Proof. Since R is Artinian, C admits a composition series

$$0 = C_0 \subset C_1 \subset \cdots \subset C_m = C,$$

where all the quotients C_i/C_{i-1} are simple modules. We next show that

$$C_i/C_{i-1} \in \mathbb{S} \quad \text{for all } i.$$

Suppose that for some i, $C_i/C_{i-1} \cong S'$ for some $S' \in \mathbb{S}'$. Let $x \in C_i \smallsetminus C_{i-1}$; then the cyclic module generated by x has S' as a quotient. This implies that the submodule Rx of C has a non-zero map to $E(S')$ — the injective hull of S'. Clearly, such a map can be lifted to a non-zero map $C \to E$, in contradiction. Therefore $C_i/C_{i-1} \cong S$ for some $S \in \mathbb{S}$. Now a simple inductive argument on i shows that $C_i \in \langle \mathbb{S} \rangle$ for every i, and hence C is $\mathbb{S}$-cellular. □

Corollary 4.2. *A complex X is $\mathcal{T}$-cellular if and only if X is $\mathbb{S}$-cellular.*

Proof. We need to show that $\langle \mathcal{T} \rangle = \langle \mathbb{S} \rangle$. Since $\mathbb{S} \subset \mathcal{T}$, we only need to show that $\mathcal{T} \subset \langle \mathbb{S} \rangle$. By Lemma 2.8 it is enough to show that every cyclic R-module is $\mathbb{S}$-cellular, but that is immediate from Lemma 4.1. □

Lemma 4.3. *The module E is an injective cogenerator for $\mathcal{T}$.*

Proof. Let $\mathcal{U}$ be the class of modules M such that $\mathrm{Hom}_R(M,E)=0$. Then $\mathcal{U}$ is a hereditary torsion theory. Because $\mathrm{Hom}_R(S,S')=0$ for every $S\in\mathbb{S}$ and $S'\in\mathbb{S}'$, we see that $\mathrm{Hom}_R(S,E(S'))=0$, where $E(S')$ is the injective envelope of S'. Hence $E\in\mathcal{F}$, and therefore $\mathcal{U}$ contains $\mathcal{T}$.

Next, let M be in $\mathcal{U}$. To show that M is a $\mathcal{T}$-torsion module it is enough to show that every cyclic submodule of M is a torsion module, because M is a quotient of the direct sum of its cyclic submodules. So let C be a cyclic submodule of M. Since E is injective, it follows that $\mathrm{Hom}_R(C,E)=0$. By Lemma 4.1 C is $\mathbb{S}$-cellular. Therefore C is $\mathcal{T}$-cellular, and by Lemma 2.10 C is $\mathcal{T}$-torsion. □

Lemma 4.4. *For any complex X, $\mathrm{Ext}^*_R(P,X)=0$ if and only if $\mathrm{Ext}^*_R(X,E)=0$.*

Proof. This is known when X is a finitely generated R-module; see Benson's book [**1**, 1.7.6 and 1.7.7]. Now suppose X is any R-module. Since P is a finitely generated projective module, $\mathrm{Hom}_R(P,X)=0$ if and only if $\mathrm{Hom}_R(P,X')=0$ for every finitely generated submodule X' of X. Similarly, because E is injective, $\mathrm{Hom}_R(X,E)=0$ if and only if $\mathrm{Hom}_R(X',E)=0$ for every finitely generated submodule X' of X. Hence the lemma holds for any R-module. Finally, let X be any complex; then $\mathrm{Ext}^*_R(P,X)=\mathrm{Hom}_R(P,H_*(X))$. Similarly, $\mathrm{Ext}^*_R(X,E)=\mathrm{Hom}_R(H^*(X),E)$. □

Corollary 4.5. *For any R-module M, a $\mathcal{T}$-nullification of M is also a P-completion of M and is therefore given by*

$$\mathrm{Null}_{\mathcal{T}}M\simeq \mathbf{R}\mathrm{Hom}_{\mathrm{End}_R(P)}(\mathrm{Hom}_R(P,R),\mathrm{Hom}_R(P,M)).$$

Proof. Consider the triangle $\mathrm{Cell}_{\mathcal{T}}M\to M\xrightarrow{\nu}\mathrm{Null}_{\mathcal{T}}M$. Lemma 4.3 implies that E is $\mathcal{T}$-null, and therefore $\mathrm{Ext}^*_R(\mathrm{Cell}_{\mathcal{T}}M,E)=0$. By Lemma 4.4, $\mathrm{Cell}_{\mathcal{T}}M$ is P-null and ν is a P-equivalence.

It remains to show that $\mathrm{Null}_{\mathcal{T}}M$ is P-complete. Let I be the $\mathcal{T}$-nullification of M described in 3.1. The full subcategory of P-complete objects in $\mathcal{D}_R$ is closed under isomorphisms, completion of triangles, products and retracts. Denote this subcategory by $\mathcal{C}$. From Lemma 4.4, we see that $E\in\mathcal{C}$, and therefore every product of E is also in $\mathcal{C}$. Lemma 4.3 and [**11**, VI.3.9] imply that every torsion-free module is a submodule of a product of copies of E. Since I^n is injective, it is a direct summand of some product of copies of E, hence I^n is also an object of $\mathcal{C}$.

Let $I(n)$ denote the cochain complex $I^0\to I^1\to\cdots\to I^n$. An inductive argument shows that $I(n)\in\mathcal{C}$. There is a triangle $I\to\prod_n I(n)\xrightarrow{\phi-1}\prod_n I(n)$, where the map ϕ is induced by the maps $I(n+1)\to I(n)$. Hence I is P-complete.

By Dwyer and Greenlees [**4**, Theorem 2.1], the P-completion of an R-module M is given by

$$M^\wedge_P\simeq \mathbf{R}\mathrm{Hom}_{\mathrm{End}_R(P)}(\mathrm{Hom}_R(P,R),\mathrm{Hom}_R(P,M)).$$ □

Corollary 4.5 above implies Benson's formula for $\mathcal{T}$-cellular approximation given in [**2**, Theorem 5.1]. This corollary also explains the connection between Benson's formula and Dwyer and Greenlees formula for P-completion from [**4**, Theorem 2.1].

5. Examples

Example 5.1. Let I be a two-sided ideal of R such that I is finitely generated as a left R-module. An R-module M will be called *I-torsion* if for every $m\in M$ there exists some n such that $I^n m = 0$. It is not difficult to show that the class of I-torsion modules forms a hereditary torsion class $\mathcal{T}$ (see [**11**, VI.6.10]). Using Lemma 2.8 it is easy to conclude that $\langle \mathcal{T} \rangle = \langle R/I \rangle$. Hence $\mathcal{T}$-cellular approximation is the same as R/I-cellular approximation and the same goes for nullification. Note that in this case the radical t associated with $\mathcal{T}$ has a simple description: for any R-module M

$$t(M) = \operatorname*{colim}_{n\to\infty} \mathrm{Hom}_R(R/I^n, M).$$

Now suppose R is a commutative Noetherian ring. Dwyer and Greenlees have shown in [**4**] that R/I-cellular approximation computes I-local cohomology, namely that there is a natural isomorphism $H^*_I(M) \cong H^*(\mathrm{Cell}_{R/I} M)$. Recall there is an isomorphism:

$$H^*_I(M) \cong \operatorname*{colim}_{n\to\infty} \mathrm{Ext}^*_R(R/I^n, M).$$

These facts show that $\mathcal{T}$-cellular approximation is the derived functor of the radical t. Moreover, in this case an object $X \in \mathcal{D}_R$ is $\mathcal{T}$-cellular if and only $H_n(X)$ is $\mathcal{T}$-torsion for all n; see [**4**, 6.12].

Example 5.2. Here is an example of a case where $\mathcal{T}$-cellular approximation is not the derived functor of the associated radical. Let G be the symmetric group on three elements, let k be the field $\mathbb{Z}/3\mathbb{Z}$ and let R be the group ring $k[G]$. There is an augmentation map $R \to k$, where k has the trivial G-action. Let I be the augmentation ideal. As before, denote the class of I-torsion modules by $\mathcal{T}$ and the associated radical by t. Since R is an Artinian ring, the sequence $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ stabilizes. So there is a fixed index m such that $t(M) = \mathrm{Hom}_R(R/I^m, M)$ for every R-module M. Therefore, the derived functors of the torsion radical t are the functors $\mathrm{Ext}^*_R(R/I^m, -)$. In particular, $\mathrm{Ext}^i_R(R/I^m, R) = 0$ for all $i > 0$, because R is injective. On the other hand, a calculation using Benson's methods from [**2**] shows that $H^n(\mathrm{Cell}_{\mathcal{T}} R)$ is nonzero for infinitely many values of n, thereby showing that $\mathrm{Cell}_{\mathcal{T}}$ is not the derived functor of t. We describe this calculation next.

From the surjection $G \to \mathbb{Z}/2$ one sees that R has two simple modules, the trivial module k and a one dimensional simple module ω. As a left module, $R \cong E_k \oplus E_\omega$ where E_k and E_ω are the injective hulls of k and ω respectively. The module E_k has a composition series

$$k \subset B \subset E_k, \quad \text{where } B/k \cong \omega \text{ and } E_k/B \cong k.$$

The composition series for E_ω is

$$\omega \subset B' \subset E_\omega, \quad \text{where } B'/\omega \cong k \text{ and } E_\omega/B' \cong \omega.$$

In addition, $E_\omega/\omega \cong B$ and $E_k/k \cong B'$. Since E_ω is k-null (see Lemma 4.3), then

$$\mathrm{Null}_{\mathcal{T}} R \simeq \mathrm{Null}_{\mathcal{T}} E_\omega \oplus \mathrm{Null}_{\mathcal{T}} E_k \cong E_\omega \oplus \mathrm{Null}_{\mathcal{T}} E_k.$$

So we need only compute $\mathrm{Null}_{\mathcal{T}} E_k$. Applying Construction 3.1 to the module E_k, we get the complex I which is $E_\omega \xrightarrow{d} E_\omega \xrightarrow{d} E_\omega \xrightarrow{d} \cdots$, where d is the composition

$E_\omega \twoheadrightarrow \omega \hookrightarrow E_\omega$. Hence $H^n(\mathrm{Null}_{\mathcal{T}} E_k) = k$ for $n > 1$, and therefore $H^n(\mathrm{Cell}_{\mathcal{T}} R)$ is non-zero for infinitely many values of n. In fact,

$$H^n(\mathrm{Cell}_{\mathcal{T}} R) = \begin{cases} k, & n = 0; \\ 0, & n = 1; \\ k, & n > 1. \end{cases}$$

It is important to note that, in this case, a complex X such that $H_n(X)$ is $\mathcal{T}$-torsion for all n need not be $\mathcal{T}$-cellular. Consider, for example, the complex $R^\wedge_{\mathcal{T}}$. As we explain below, the homology groups $H_n(R^\wedge_{\mathcal{T}})$ are $\mathcal{T}$-torsion for all n. On the other hand, the $\mathcal{T}$-equivalences $\mathrm{Cell}_{\mathcal{T}} R \to R$ and $R \to R^\wedge_{\mathcal{T}}$ show that $\mathrm{Cell}_{\mathcal{T}} R$ is $\mathcal{T}$-equivalent to $R^\wedge_{\mathcal{T}}$. If $R^\wedge_{\mathcal{T}}$ was $\mathcal{T}$-cellular, then $R^\wedge_{\mathcal{T}}$ would have been quasi-isomorphic to $\mathrm{Cell}_{\mathcal{T}} R$, because a $\mathcal{T}$-equivalence between $\mathcal{T}$-cellular complexes is a quasi-isomorphism. As we show next, $H^n(R^\wedge_{\mathcal{T}}) = 0$ for $n > 0$ and so $R^\wedge_{\mathcal{T}}$ cannot be quasi-isomorphic to $\mathrm{Cell}_{\mathcal{T}} R$.

It remains to explain the properties of $R^\wedge_{\mathcal{T}}$ used above. From Corollary 4.2 we learn that $R^\wedge_{\mathcal{T}} \simeq R^\wedge_k$ and $\mathrm{Cell}_{\mathcal{T}} R \simeq \mathrm{Cell}_k R$. Without going into details, combining [**5**, 5.9] with [**4**, 4.3] shows that

$$R^\wedge_k \simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{Cell}_k R, R).$$

This immediately implies that $H^n(R^\wedge_{\mathcal{T}}) = 0$ for $n > 0$. We next show that $H_n(R^\wedge_{\mathcal{T}})$ is $\mathcal{T}$-torsion for all n. Since E_ω is a $\mathcal{T}$-null module, $\mathrm{Ext}^*_R(E_\omega, R^\wedge_{\mathcal{T}}) = 0$. Recall that R is a group-algebra, and therefore E_ω is also the projective cover of ω. Because E_ω is projective we have

$$\mathrm{Ext}^{-n}_R(E_\omega, R^\wedge_{\mathcal{T}}) \cong \mathrm{Hom}_R(E_\omega, H_n(R^\wedge_{\mathcal{T}})).$$

Hence, by Lemma 4.4, $\mathrm{Ext}^*_R(H_n(R^\wedge_{\mathcal{T}}), E_\omega) = 0$. Lemma 4.3 shows E_ω is an injective cogenerator for $\mathcal{T}$; therefore $H_n(R^\wedge_{\mathcal{T}})$ is $\mathcal{T}$-torsion.

Example 5.3. This example relates $\mathcal{T}$-nullification with Cohn localization. We begin by recalling the definition of Cohn localization. Let $S = \{f_\alpha \colon P_\alpha \to Q_\alpha\}$ be a set of maps between finitely generated projective R-modules. Say a ring map $R \to R'$ is *S-inverting* if $\mathrm{Hom}_R(f, R')$ is an isomorphism for every $f \in S$. A *Cohn localization* of R with respect to S is a ring map $R \to S^{-1}R$, which is initial among all S-inverting ring maps. Note that the definition given here is not the standard definition (see e.g. [**3**]), but it is equivalent to the standard one.

Let $\mathcal{C}_S$ be the set of cones of the maps f_α. In [**3**], Dwyer considers $\mathcal{C}_S$-nullification and shows that $H_0(\mathrm{Null}_{\mathcal{C}_S} R) = S^{-1}R$ (see [**3**, 3.2]). Combining Dwyer's results with Theorem 1.1 yields the following proposition.

Proposition 5.4. *Let $\mathcal{T}$ be a hereditary torsion-class of R-modules. If $\langle \mathcal{T} \rangle = \langle \mathcal{C}_S \rangle$ for some set of maps S between finitely generated projective R-modules, then*

1. $\mathrm{Null}_{\mathcal{T}}(-) \simeq (-)_{\mathcal{F}}$,
2. *the module of quotients functor $(-)_{\mathcal{F}}$ is exact and*
3. *there is an isomorphism $S^{-1}R \otimes_R M \cong M_{\mathcal{F}}$ for every module M.*

Proof. By Theorem 1.1, for every R-module M the complex $\mathrm{Null}_{\mathcal{T}} M$ has no homology in positive degrees. By [**3**, Proposition 3.1], $\mathrm{Null}_{\mathcal{T}} R$ has no homology in negative degrees. Moreover, a result of Miller [**8**] (see also [**3**, Proposition 2.10]) shows that for every R-module M, $\mathrm{Null}_{\mathcal{T}} M \simeq \mathrm{Null}_{\mathcal{T}} R \otimes^{\mathbf{L}}_R M$. This implies that $\mathrm{Null}_{\mathcal{T}} M$ has no homology in negative degrees.

We conclude that for every R-module M, $\mathrm{Null}_{\mathcal{T}} M$ has homology only in degree 0, and therefore, by Lemma 3.5, $\mathrm{Null}_{\mathcal{T}} M$ is quasi-isomorphic to $M_{\mathcal{F}}$. Since the functor $\mathrm{Null}_{\mathcal{T}}$ is exact, so is $(-)_{\mathcal{F}}$. Since $\mathrm{Null}_{\mathcal{T}} R \simeq \mathrm{Null}_{\mathcal{C}_S} R$, Dwyer's result [**3**, 3.2] shows that $R_{\mathcal{F}} \cong S^{-1}R$. Finally, the quasi-isomorphism $\mathrm{Null}_{\mathcal{T}} M \simeq \mathrm{Null}_{\mathcal{T}} R \otimes^{\mathbf{L}}_R M$ implies $S^{-1}R \otimes_R M \cong M_{\mathcal{F}}$. □

Example 5.5. This example is of a topological nature. Let M be a discrete monoid and let $k = \mathbb{Z}/p\mathbb{Z}$ for some prime p. The ring R we consider is the monoid ring $R = k[M]$; it has a natural augmentation $R \to k$ with augmentation ideal I. We also make the following assumptions:

1. The classifying space $\mathrm{B}M$ of M has a finite fundamental group.
2. The augmentation ideal I is finitely generated as a left R-module.
3. There is a projective resolution $P = \cdots P_2 \to P_1 \to P_0$ of k over R such that every P_n is finitely generated as an R-module.

Let $\mathcal{T}$ be the hereditary torsion class of I-torsion R-modules; then $\langle \mathcal{T} \rangle = \langle k \rangle$ and hence $\mathrm{Cell}_{\mathcal{T}} \simeq \mathrm{Cell}_k$. Denote by $R^\vee$ the left R-module $\mathrm{Hom}_k(R, k)$. From the results of Dwyer, Greenlees and Iyengar [**5**, 6.15 and 7.5], it is easy to conclude that $\mathrm{Cell}_k R^\vee$ is quasi-isomorphic to the cochain complex (with coefficients in k) of a certain space, which we describe next. Let $(\mathrm{B}M)^\wedge_p$ be the Bousfield-Kan p-completion of the classifying space of M. The space $\Omega(\mathrm{B}M)^\wedge_p$ is the loop-space of $(\mathrm{B}M)^\wedge_p$. So, $\mathrm{Cell}_k R^\vee$ is quasi-isomorphic to $\mathrm{C}^*(\Omega(\mathrm{B}M)^\wedge_p; k)$ — the singular cochain complex of $\Omega(\mathrm{B}M)^\wedge_p$ with coefficients in k. By Theorem 1.1, there exists an injective R-module E such that

$$H^n(\Omega(\mathrm{B}M)^\wedge_p; k) \cong \mathrm{Ext}^{n-1}_{\mathrm{End}_R(E)}(\mathrm{Hom}_R(R^\vee, E), E) \text{ for } n \geqslant 2.$$

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