Homology, Homotopy and Applications, vol. 12(2), 2010, pp.371–400

THE RATIONAL HOMOTOPY TYPE OF THE SPACE OF SELF-EQUIVALENCES OF A FIBRATION

YVES FÉLIX, GREGORY LUPTON AND SAMUEL B. SMITH

(*communicated by Charles A. Weibel*)

Abstract

Let $\mathrm{Aut}(p)$ denote the space of all self-fibre-homotopy equivalences of a fibration $p\colon E \to B$. When E and B are simply connected CW complexes with E finite, we identify the rational Samelson Lie algebra of this monoid by means of an isomorphism:

$$\pi_*(\mathrm{Aut}(p)) \otimes \mathbb{Q} \cong H_*(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W)).$$

Here $\wedge V \to \wedge V \otimes \wedge W$ is the Koszul-Sullivan model of the fibration and $\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W)$ is the DG Lie algebra of derivations vanishing on $\wedge V$. We obtain related identifications of the rationalized homotopy groups of fibrewise mapping spaces and of the rationalization of the nilpotent group $\pi_0(\mathrm{Aut}_\sharp(p))$, where $\mathrm{Aut}_\sharp(p)$ is a fibrewise adaptation of the submonoid of maps inducing the identity on homotopy groups.

1. Introduction

Given a fibration $p\colon E \to B$ of connected CW complexes, let $\mathrm{Aut}(p)$ denote the space of unpointed fibre-homotopy self-equivalences $f\colon E \to E$ topologized as a subspace of $\mathrm{Map}(E, E)$. By a theorem of Dold [**8**, Th. 6.3], $\mathrm{Aut}(p)$ corresponds to the space of ordinary homotopy self-equivalences $f\colon E \to E$ satisfying $p \circ f = p$, The space $\mathrm{Aut}(p)$ is a monoid with multiplication given by composition of maps. In general, $\mathrm{Aut}(p)$ is a disconnected space with possibly infinitely many components. The group of components $\pi_0(\mathrm{Aut}(p))$ is the group of fibre-homotopy equivalence classes of self-fibre-homotopy equivalences of p. We denote this group by $\mathcal{E}(p)$. We make a general study of $\mathrm{Aut}(p)$, especially in rational homotopy theory.

The monoid $\mathrm{Aut}(p)$ appears as an object of interest in many different situations. When B is a point, $\mathrm{Aut}(p) \simeq \mathrm{Aut}(E)$ is the monoid of (free) self-homotopy equivalences of the space E, and $\mathcal{E}(p) = \mathcal{E}(E)$, the group of self-equivalences of E. Taking $p\colon PB \to B$ to be the path-space fibration, we have $\mathrm{Aut}(p) \simeq \Omega B$ and $\mathcal{E}(p) = \pi_1(B)$

Received February 21, 2009, revised October 11, 2010; published on November 19, 2010.
2000 Mathematics Subject Classification: 55P62, 55Q15.
Key words and phrases: fibre-homotopy equivalence, Samelson Lie algebra, function space, Sullivan minimal model, derivation.
This article is available at `http://intlpress.com/HHA/v12/n2/a13`

(see Example 2.4, below). When p is a covering map, $\mathrm{Aut}(p)$ contains the group of deck transformations of the covering.

With regard to the homotopy type of $\mathrm{Aut}(p)$, under reasonable hypotheses on p, the monoid $\mathrm{Aut}(p)$ is a grouplike space of CW type (see Proposition 2.2). The path components of a well-pointed grouplike space are all of the same homotopy type. Thus we focus on the path component of the identity map which we denote $\mathrm{Aut}(p)_\circ$. Our first main result gives a complete description of the rational H-homotopy type of this connected grouplike space when E is finite.

Before describing our main results, we review some background and notation. Recall the homotopy groups $\pi_*(G)$ of a connected, grouplike space G admit a natural bilinear pairing $[\, ,\,]$ called the *Samelson product* (see [**29**, Ch. III]). If G has multiplication μ, then the pair (G, μ) admits a rationalization as in [**18**] yielding an H-space $(G_\mathbb{Q}, \mu_\mathbb{Q})$ which is unique up to H-equivalence. We refer to the H-homotopy type of the pair $(G_\mathbb{Q}, \mu_\mathbb{Q})$ as the *rational* H-*type* of (G, μ). If (G, μ) is grouplike, then $(G_\mathbb{Q}, \mu_\mathbb{Q})$ is also. In this case, the rational H-type of G is completely determined by the rational Samelson algebra $\pi_*(G) \otimes \mathbb{Q}, [\, ,\,]$. Specifically, two grouplike spaces are rationally H-equivalent if and only if they have isomorphic Samelson Lie algebras [**27**, Cor. 1]. We identify the rational Samelson Lie algebra of $\mathrm{Aut}(p)_\circ$ in the context of Sullivan's rational homotopy theory. Our reference for rational homotopy theory is [**11**].

In [**28**], Sullivan defined a functor $A_{PL}(-)$ from topological spaces to commutative differential graded algebras over $\mathbb{Q}$ (DG algebras for short). The functor $A_{PL}(-)$ is connected to the cochain algebra functor $C^*(-;\mathbb{Q})$ by a sequence of natural quasi-isomorphisms. Let $\wedge V$ denote the free commutative graded algebra on the graded rational vector space V. A DG algebra (A, d) is a *Sullivan algebra* if $A \cong \wedge V$ and if V admits a basis (v_i) indexed by a well-ordered set such that $d(v_i) \in \wedge(v_j, j < i)$. If the differential d has image in the decomposables of $\wedge V$, then we say (A, d) is *minimal*. Filtering by product length, a minimal DG algebra is seen to be a Sullivan algebra. A DG algebra (A, d) is a *Sullivan model* for X if (A, d) is a Sullivan algebra and there is a quasi-isomorphism $(A, d) \to A_{PL}(X)$. If (A, d) is minimal, then it is the *Sullivan minimal model* of X.

A fibration $p\colon E \to B$ of simply connected CW complexes admits a *relative minimal model* (see [**11**, Prop. 15.6]). This is an injection of DG algebras $I\colon (\wedge V, d) \to (\wedge V \otimes W, D)$ equipped with quasi-isomorphisms η_B and η_E, which make the following diagram commutative:

$$\begin{tikzcd}
(\wedge V, d) \arrow[r, "I"] \arrow[d, "\eta_B"', "\simeq"] & (\wedge V \otimes \wedge W, D) \arrow[d, "\eta_E"', "\simeq"] \\
A_{PL}(B) \arrow[r, "A_{PL}(p)"] & A_{PL}(E).
\end{tikzcd}$$

Here $(\wedge V, d)$ is the Sullivan minimal model of B, while $(\wedge V \otimes \wedge W, D)$ is a Sullivan (but generally non-minimal) model of E. The differential D satisfies

$$D(W) \subset (\wedge^+ V \otimes \wedge W) \oplus (\wedge V \otimes \wedge^{\geqslant 2}(W))$$

and further, W admits a basis w_i indexed by a well-ordered set such that $D(w_i) \in \wedge V \otimes \wedge(w_j, j < i)$.

A derivation θ of degree n of a DG algebra (A, d) will mean a linear map *lowering* degrees by n and satisfying $\theta(ab) = \theta(a)b - (-1)^{n|a|}a\theta(b)$ for $a, b \in A$. We write $\mathrm{Der}^n(A)$ for the vector space of all degree n derivations. The graded vector space $\mathrm{Der}^*(A)$ has the structure of a DG Lie algebra with the commutator bracket $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2 \circ \theta_1$ and differential $\mathcal{D}(\theta) = [d, \theta]$. Given a DG subalgebra $B \subseteq A$, we write $\mathrm{Der}^*_B(A)$ for the space of derivations of A that vanish on B. The bracket and differential evidently restrict to give a DG Lie algebra $(\mathrm{Der}^*_B(A), \mathcal{D})$. More generally, given a DG algebra map $\phi\colon A \to A'$, we write $\mathrm{Der}^n(A, A'; \phi)$ for the space of degree n linear maps satisfying $\theta(ab) = \theta(a)\phi(b) + (-1)^{n|a|}\phi(a)\theta(b)$ with differential $\mathcal{D}(\theta) = d_{A'} \circ \theta - (-1)^{|\theta|}\theta \circ d_A$. The pair $(\mathrm{Der}^*(A, A'; \phi), \mathcal{D})$ is then a DG vector space. Given a DG subalgebra $B \subseteq A$ we have the DG vector subspace $(\mathrm{Der}^*_B(A, A'; \phi), \mathcal{D})$ of derivations that vanish on B.

We now describe our main results. Given a fibration $p\colon E \to B$ of simply connected CW complexes, choose and fix a relative minimal model $I\colon (\wedge V, d) \to (\wedge V \otimes \wedge W, D)$ as above. We have:

Theorem 1.1. *Let $p\colon E \to B$ be a fibration of simply connected* CW *complexes with E finite. There is an isomorphism of graded Lie algebras*

$$\pi_*(\mathrm{Aut}(p)_\circ) \otimes \mathbb{Q} \cong H_*(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W)).$$

By the remarks above, Theorem 1.1 completely determines the rational H-type of $\mathrm{Aut}(p)$. Taking $B = *$, we recover Sullivan's Lie algebra isomorphism

$$\pi_*(\mathrm{Aut}(E)) \otimes \mathbb{Q} \cong H_*(\mathrm{Der}(\wedge W))$$

described in [**28**, §11]. Here $(\wedge W, d)$ is the minimal model for E.

We obtain Theorem 1.1 as a sharpened special case of a general calculation of the rational homotopy groups of a fibrewise mapping space. Let $p'\colon E' \to B'$ be a second fibration and $f\colon E' \to E$ be a fibrewise map covering a map $g\colon B' \to B$. Let

$$\mathrm{Map}_g(E', E) = \{h\colon E' \to E \mid p \circ h = g \circ p'\}$$

denote the function space of maps over g topologized as a subspace of $\mathrm{Map}(E', E)$. Write $\mathrm{Map}_g(E', E; f)$ for the path component of f. We compute the rational homotopy groups of this space as an extension of Sullivan's original approach. Fix a relative minimal model $I'\colon (\wedge V', d) \to (\wedge V' \otimes \wedge W', D)$ for $p'\colon E' \to B'$ as above. The map $f\colon E' \to E$ has a model $\mathcal{A}_f\colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W'$. A homotopy class $\alpha\colon S^n \to \mathrm{Map}_g(E', E; f)$ has adjoint $F\colon E' \times S^n \to E$ satisfying $p \circ F(x, s) = g \circ p'(x)$ and $F(x, *) = f(x)$ for all $(x, s) \in E' \times S^n$. We show F has a DG model

$$\mathcal{A}_F\colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes \left(\wedge(u)/\langle u^2\rangle\right)$$

satisfying

$$\mathcal{A}_F(\chi) = \mathcal{A}_f(\chi) + u\theta(\chi) \quad \text{for } \chi \in \wedge V \otimes \wedge W.$$

Here $|u| = n$ so that $(\wedge(u)/\langle u^2\rangle, 0)$ is a Sullivan model for S^n. The degree n map θ is then an $\mathcal{A}_f$-derivation as defined above, and we show θ vanishes on $\wedge V$; that is, θ is an element of $\mathrm{Der}^n_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f)$. We prove the assignment $\alpha \mapsto \theta$ gives rise to a rational isomorphism:

Theorem 1.2. *Let $p'\colon E' \to B'$ and $p\colon E \to B$ be fibrations of simply connected* CW *complexes with E' finite. Let $f\colon E' \to E$ be a fibrewise map with model $\mathcal{A}_f$ as above. There is an isomorphism of vector spaces*

$$\pi_n(\mathrm{Map}_g(E', E; f)) \otimes \mathbb{Q} \cong H_n(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f))$$

for $n \geqslant 2$. Further, the group $\pi_1(\mathrm{Map}_g(E', E; f))$ is nilpotent and satisfies

$$\mathrm{rank}(\pi_1(\mathrm{Map}_g(E', E; f))) = \dim_{\mathbb{Q}}\left(H_1(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f))\right).$$

Again, taking $B' = B = *$, we recover known results; in this case the vector space isomorphism

$$\pi_n(\mathrm{Map}(X, Y; f)) \otimes \mathbb{Q} \cong H_n(\mathrm{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$$

for $n \geqslant 2$ appears in [**5, 7, 21**], where $\mathcal{M}_f\colon \mathcal{M}_Y \to \mathcal{M}_X$ is the Sullivan minimal model of $f\colon X \to Y$. The result on $\mathrm{rank}(\pi_1(\mathrm{Map}(X, Y; f)))$ is [**22**, Th. 1]. Our proof of Theorem 1.2 is an adaptation of the proofs of [**21**, Th. 2.1] and [**22**, Th. 1] with adjustments made for the fibrewise setting.

Finally, with regard to the group $\mathcal{E}(p)$ of path components of $\mathrm{Aut}(p)$, we remark that this group is not, generally, a nilpotent group and so not directly amenable to rational homotopy theory. In Section 6, we consider the subgroup $\mathcal{E}_\sharp(p)$ of $\mathcal{E}(p)$ consisting of homotopy equivalence classes of maps $f\colon E \to E$ over B inducing the identity on the image and the cokernel of the connecting homomorphism in the long exact sequence on homotopy groups of the fibration. The group $\mathcal{E}_\sharp(p)$ is nilpotent and localizes well for E finite by the fibrewise extension of results of Maruyama [**23**]. Let $D_0\colon W \to V$ be the linear part of D in the Sullivan model for p. Write $W = W_0 \oplus W_1$, where $W_0 = \ker D_0$ and W_1 is a vector space complement. Denote by $\mathrm{Der}^0_\#(\wedge V \otimes \wedge W)$ the vector space of derivations θ of degree 0 of $\wedge V \otimes \wedge W$ that satisfy $\theta(V) = 0$, $\mathcal{D}(\theta) = 0$, $\theta(W_0) \subset V \oplus \wedge^{\geqslant 2}(V \oplus W)$, and $\theta(W_1) \subset V \oplus W_0 \oplus \wedge^{\geqslant 2}(V \oplus W)$. Define

$$H_0(\mathrm{Der}_\sharp(\wedge V \otimes \wedge W)) = \mathrm{coker}\{\mathcal{D}\colon \mathrm{Der}^1_{\wedge V}(\wedge V \otimes \wedge W) \to \mathrm{Der}^0_\#(\wedge V \otimes \wedge W)\}.$$

The case $B = *$ in the following result is originally due to Sullivan [**28**, §11]. See also [**26**, Th. 12].

Theorem 1.3. *Let $p\colon E \to B$ be a fibration of simply connected* CW *complexes with E finite. There is a group isomorphism*

$$\mathcal{E}_\#(p)_{\mathbb{Q}} \cong H_0(\mathrm{Der}_\sharp(\wedge V \otimes \wedge W)).$$

The paper is organized as follows. In Section 2, we prove several general results concerning $\mathrm{Aut}(p)$. We prove Theorem 1.2 in Section 3 as a consequence of some results in fibrewise DG homotopy theory. We prove Theorem 1.1 in Section 4 and deduce some consequences in Section 5. Finally, in Section 6 we turn to the group of components $\mathcal{E}(p)$ of $\mathrm{Aut}(p)$ and prove Theorem 1.3. We conclude with an example to show that $\mathcal{E}_\#(p)$ is not generally a subgroup of $\mathcal{E}_\#(E)$.

2. Basic homotopy theory of $\mathrm{Aut}(p)$

Function spaces are not generally suitable for and well-behaved under localization. In this section we show that, under reasonable hypotheses, the fibrewise function spaces are nilpotent CW complexes admitting natural localizations. This result is a direct consequence of the corresponding, standard results on function spaces. We then give a variety of examples concerning the monoid $\mathrm{Aut}(p)$.

Start with a commutative diagram of connected CW complexes

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd} \tag{1}$$

with vertical maps fibrations. Composition with p gives a fibration of ordinary function spaces

$$p_*\colon \mathrm{Map}(E', E; f) \to \mathrm{Map}(E', B; p \circ f).$$

The fibre of p_* over the basepoint $p \circ f$ is the space

$$\mathcal{F} = \{h\colon E' \to E \mid h \simeq f \text{ and } p \circ h = p \circ f\}.$$

The path component of f in $\mathcal{F}$ is just the fibrewise function space $\mathrm{Map}_g(E', E; f)$ introduced above.

Following Hilton-Mislin-Roitberg [**18**], we say that a space X is a *nilpotent* space if X is a connected CW complex with $\pi_1(X)$ nilpotent and acting nilpotently on the higher homotopy groups of X. As shown in [**18**, Ch. II], the class of nilpotent spaces admit $\mathbb{P}$-localizations for $\mathbb{P}$ any fixed collection of primes. That is, there is a space $X_{\mathbb{P}}$ and a map $\ell_X\colon X \to X_{\mathbb{P}}$ such that $X_{\mathbb{P}}$ is a $\mathbb{P}$-local space and ℓ_X is a $\mathbb{P}$-local-equivalence. Given a map $f\colon X \to Y$ with Y nilpotent, we write $f_{\mathbb{P}} = \ell_Y \circ f\colon X \to Y_{\mathbb{P}}$. We will mostly be interested in the case $\mathbb{P}$ is empty in which case we write $f_{\mathbb{Q}}\colon X \to Y_{\mathbb{Q}}$.

The assignment $X \mapsto X_{\mathbb{P}}$ is functorial at the level of homotopy classes of maps [**18**, Prop. II.3.4]. We observe a refinement of this fact for fibrations. Suppose $p\colon E \to B$ is a fibration of nilpotent spaces. We may construct $\mathbb{P}$-localizations for E and B creating a commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\ell_E"] \arrow[d, "p"'] & E_{\mathbb{P}} \arrow[d, "q"] \\ B \arrow[r, "\ell_B"] & B_{\mathbb{P}}, \end{tikzcd}$$

where q is a fibration. To see this, first construct $\mathbb{P}$-localizations $\ell'_E\colon E \to E'_{\mathbb{P}}$ and $\ell_B\colon B \to B_{\mathbb{P}}$ as relative CW complexes. Then $p \circ \ell_B\colon E \to B_{\mathbb{P}}$ extends to a map $q'\colon E'_{\mathbb{P}} \to B_{\mathbb{P}}$. Transform q' into a fibration by injecting q' into the associated homotopy fibration yielding $q\colon E_{\mathbb{P}} \to B_{\mathbb{P}}$. Let $\ell_E\colon E \to E_{\mathbb{P}}$ denote the composition of the equivalence $E'_{\mathbb{P}} \simeq E_{\mathbb{P}}$ with ℓ'_E.

Now suppose given a commutative diagram (1) with all spaces nilpotent. We then have a commutative square:

$$\begin{tikzcd} E' \arrow[r, "f_{\mathbb{P}}"] \arrow[d, "p'"'] & E_{\mathbb{P}} \arrow[d, "p"] \\ B' \arrow[r, "g_{\mathbb{P}}"] & B_{\mathbb{P}} \end{tikzcd}$$

and a map

$$(\ell_E)_*\colon \mathrm{Map}_g(E', E; f) \to \mathrm{Map}_{g_{\mathbb{P}}}(E', E_{\mathbb{P}}; f_{\mathbb{P}})$$

induced by composition with ℓ_E. We prove

Proposition 2.1. *In the commutative square (1), suppose E' is a finite* CW *complex and E and B are nilpotent spaces. Then $\mathrm{Map}_g(E', E; f)$ is a nilpotent space and composition with ℓ_E induces a $\mathbb{P}$-localization map*

$$(\ell_E)_*\colon \mathrm{Map}_g(E', E; f) \to \mathrm{Map}_{g_{\mathbb{P}}}(E', E_{\mathbb{P}}; f_{\mathbb{P}}).$$

Proof. By [**25**], the function spaces $\mathrm{Map}(E', E; f)$ and $\mathrm{Map}(E', B; p \circ f)$ are of CW type. Thus the fibre of $p_*\colon \mathrm{Map}(E', E; f) \to \mathrm{Map}(E', B; p \circ f)$ is CW as well by [**19**, Lem. 2.4]. Further, by [**18**, Cor. II.2.6 and Th. II.3.11], $\mathrm{Map}(E', E; f)$ and $\mathrm{Map}(E', B; p \circ f)$ are nilpotent spaces with rationalizations induced by composition with ℓ_E and ℓ_B. By [**18**, Th. II.2.2], each component of the fibre of p_* is nilpotent. (As remarked at the end of the proof [**18**, p. 63], the result holds for non-connected fibres.) Thus $\mathrm{Map}_g(E', E; f)$ is a nilpotent space. Finally, we see ℓ_E induces a $\mathbb{P}$-localization by the Five Lemma. □

With regard to the monoid $\mathrm{Aut}(p)_\circ$, we may sharpen the first part of this result:

Proposition 2.2. *Let $p\colon E \to B$ be a fibration of connected* CW *complexes with connected fibre F. If either E or B is finite, then $\mathrm{Aut}(p)$ has the homotopy type of a* CW *complex and the H-homotopy type of a loop-space.*

Proof. For the CW structure in the case B is finite, we use the identity $\mathrm{Aut}(p) \simeq \Omega\mathrm{Map}(B, BF; h)$, where BF is the base of the universal fibre with fibre $F = p^{-1}(b_0)$ and $h\colon B \to BF$ is the classifying map (see [**17**, Th. 1] and [**6**, Th. 3.3]). Since BF is CW in this case, applying [**25**] again gives the result. In either case, $\mathrm{Aut}(p)$ is a strictly associative CW monoid and so admits a Dold-Lashof classifying space $\mathrm{BAut}(p)$. Thus $\mathrm{Aut}(p) \simeq \Omega\mathrm{BAut}(p)$ [**15**, Satz.7.3] □

In our main results, we consider $\mathrm{Aut}(p)$ for E finite. By Proposition 2.2 this restriction is not necessary for nilpotence since a connected CW monoid is automatically nilpotent. However, we will make use of the second statement in Proposition 2.1 in the proof of Theorems 1.1 and 1.2.

We next recall an interesting invariant of a connected grouplike space G, the *homotopical nilpotency* of G as studied by Berstein and Ganea [**4**]. It is defined as follows: Using the homotopy inverse, we have commutator maps $\varphi_n\colon G^n \to G$: Here φ_1 is the identity, $\varphi_2(g, h) = ghg^{-1}h^{-1}$ is the usual commutator, and $\varphi_n = \varphi_2 \circ (\varphi_{n-1} \times \varphi_1)$. The homotopical nilpotency $\mathrm{Hnil}(G)$ is then the least integer n such that φ_{n+1} is null-homotopic. The *rational homotopical nilpotency* $\mathrm{Hnil}_{\mathbb{Q}}(G)$ of $G = (G, \mu)$ is defined to be the homotopical nilpotency of $G_{\mathbb{Q}} = (G_{\mathbb{Q}}, \mu_{\mathbb{Q}})$. The inequality $\mathrm{Hnil}_{\mathbb{Q}}(G) \leqslant \mathrm{Hnil}(G)$ is direct from definitions. When $\mathrm{Hnil}(G) = 1$ (respectively, $\mathrm{Hnil}_{\mathbb{Q}}(G) = 1$) we say G is *homotopy abelian* (respectively, *rational homotopy abelian*).

When $G = \Omega X$ is a loop-space, $\mathrm{Hnil}(G)$ is directly related to the nilpotency $\mathrm{Nil}(\pi_*(G))$ of the Samelson bracket $[\,,\,]$ on $\pi_*(G)$ and to the length $\mathrm{WL}(X)$ of the longest Whitehead bracket in $\pi_*(X)$. If X is a nilpotent space, write $\mathrm{WL}_{\mathbb{Q}}(X)$ for the Whitehead length of the rationalization of $X_{\mathbb{Q}}$ of X.

Proposition 2.3. *Let G be a connected* CW *loop space, $G \simeq \Omega X$ for some simply connected space X. Then*

$$\mathrm{Hnil}_{\mathbb{Q}}(G) = \mathrm{Nil}(\pi_*(G) \otimes \mathbb{Q}) = \mathrm{WL}_{\mathbb{Q}}(X) \leqslant \mathrm{WL}(X) = \mathrm{Nil}(\pi_*(G)) \leqslant \mathrm{Hnil}(G).$$

Proof. The equality $\mathrm{WL}(X) = \mathrm{Nil}(\pi_*(G))$ (and its rationalization) is a consequence of the identification of the Whitehead product with the Samelson product via the isomorphism $\pi_*(\Omega X) \cong \pi_{*+1}(X)$ [**29**, Th. X.7.10]. The inequality $\mathrm{Nil}(\pi_*(G)) \leqslant \mathrm{Hnil}(G)$ is [**4**, Th. 4.6]. The equality $\mathrm{Nil}(\pi_*(G) \otimes \mathbb{Q}) = \mathrm{WL}_{\mathbb{Q}}(X)$ is [**1**, Lem. 4.2] (see also [**26**, Th. 3]). Finally, the inequality $\mathrm{WL}(X_{\mathbb{Q}}) \leqslant \mathrm{WL}(X)$ is immediate from the definition. □

We give some examples and direct calculations regarding $\mathrm{Aut}(p)$.

Example 2.4. Let PB be the space of Moore paths at $b_0 \in B$:

$$PB = \{(\omega, r) \mid r \geqslant 0, \omega\colon (R_{\geqslant 0}, 0) \to (B, b_0), \text{such that } \omega(s) = \omega(r) \text{ for } s \geqslant r\}.$$

Let $p\colon PB \to B$ given by $p(\omega, r) = \omega(r)$ be the path-space fibration. Then there is an H-equivalence $\mathrm{Aut}(p) \simeq \Omega B$, where $\Omega B = \{(\sigma, s) \in PB \mid \sigma(s) = b_0\}$ is the space of Moore loops at b_0. To see this, define $\theta\colon \Omega B \to \mathrm{Aut}(p)$ given by $\theta((\sigma, s))(\omega, r) = (\sigma * \omega, s + r)$ and $\psi\colon \mathrm{Aut}(p) \to \Omega B$. Given $f \in \mathrm{Aut}(p)$, f restricts to an equivalence $f\colon \Omega B \to \Omega B$. Define $\psi(f) = f(c_{b_0}, 0)$, where c_{b_0} is the constant loop. Clearly, $\psi \circ \theta = \mathrm{id}$. On the other hand, a homotopy $H\colon \mathrm{Aut}(p) \times [0, 1] \to \mathrm{Aut}(p)$, between $\theta \circ \psi$ and the identity is defined by

$$H(f, t)(\omega, r) = f(\omega_{tr}, tr) * (\omega_t^b, r - tr).$$

Here $\omega_s^b(t) = \omega(t + s)$, and

$$\omega_s(t) = \begin{cases} \omega(t) & t \leqslant s \\ \omega(s) & t \geqslant s. \end{cases}$$

Using Proposition 2.3, we conclude

$$\mathrm{Hnil}(\mathrm{Aut}(p)_\circ) \geqslant \mathrm{WL}(\widetilde{B}) \quad \text{and} \quad \mathrm{Hnil}_{\mathbb{Q}}(\mathrm{Aut}(p)_\circ) = \mathrm{WL}_{\mathbb{Q}}(\widetilde{B}),$$

where $\widetilde{B}$ denotes the universal cover of B.

Example 2.5. Observe that when $\pi\colon F \times B \to B$ is the (trivial) product fibration then by adjointness we have

$$\mathrm{Aut}(\pi) \cong \mathrm{Map}(B, \mathrm{Aut}(F))$$

with pointwise multiplication in the latter space. More generally, by the fibre-homotopy invariance of $\mathrm{Aut}(p)$ we have this identification for any fibre-homotopy trivial fibration $p\colon E \to B$ with fibre $F = p^{-1}(b_0)$. By [**20**, Th. 4.10], we have

$$\mathrm{Hnil}(\mathrm{Aut}(p)_\circ) = \mathrm{Hnil}(\mathrm{Aut}(F)_\circ)$$

in this case.

Example 2.6. Let F be a simply connected complex with $H^*(F; \mathbb{Q})$ and $\pi_*(F) \otimes \mathbb{Q}$ both finite-dimensional and $H^{\mathrm{odd}}(F; \mathbb{Q}) = 0$. A fundamental open problem in rational homotopy theory asks whether the rational Serre spectral sequence collapses at the E_2-term for every fibration $p\colon E \to B$ of simply connected CW complexes with fibre F (see [**11**, p. 516, Prob.1]). A positive answer to this question was conjectured by Halperin [**14**]. Halperin's conjecture has been affirmed in many cases, including (finite products of) even-dimensional spheres, complex projective spaces and homogeneous spaces G/H of equal rank compact pairs.

By [**24**, Th. A], F satisfies Halperin's conjecture if and only if $\mathrm{Aut}(F)_\circ$ has vanishing even degree rational homotopy groups. Thus $\mathrm{Nil}(\pi_*(\mathrm{Aut}(F)_\circ) \otimes \mathbb{Q}) = 1$ in this case for degree reasons and so $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{Aut}(F)_\circ) = 1$. By Example 2.5, we conclude that $\mathrm{Aut}(p)_\circ$ is rationally homotopy abelian for any trivial fibration $p\colon E \to B$ of simply connected spaces with fibre F satisfying Halperin's conjecture.

This result extends to non-trivial fibrations via the identity

$$\mathrm{Aut}(p)_\circ \simeq \Omega_\circ \mathrm{Map}(B, \mathrm{BAut}_1(F); h).$$

Here we write $\mathrm{BAut}_1(F) = \mathrm{BAut}(F)_\circ$ for the classifying space of the connected monoid $\mathrm{Aut}(F)_\circ$. By [**24**, Th. A] again, $\mathrm{BAut}_1(F)$ has evenly graded rational homotopy groups and so $\mathrm{BAut}_1(F)$ is a rational grouplike space. We thus have an equivalence $\mathrm{Map}(B, \mathrm{BAut}_1(F)_{\mathbb{Q}}; h_{\mathbb{Q}}) \simeq \mathrm{Map}(B, \mathrm{BAut}_1(F)_{\mathbb{Q}}; 0)$, which loops to an H-equivalence

$$\Omega_\circ \mathrm{Map}(B, \mathrm{BAut}_1(F)_{\mathbb{Q}}; h_{\mathbb{Q}}) \simeq \Omega_\circ \mathrm{Map}(B, \mathrm{BAut}_1(F)_{\mathbb{Q}}; 0).$$

Combining these two equivalences, we see

$$\begin{aligned}(\mathrm{Aut}(p)_\circ)_{\mathbb{Q}} &\simeq \Omega_\circ \mathrm{Map}(B, \mathrm{BAut}_1(F)_{\mathbb{Q}}; 0)\\ &\simeq \mathrm{Map}(B, \Omega_\circ \mathrm{BAut}_1(F)_{\mathbb{Q}}; 0)\\ &\simeq \mathrm{Map}(B, (\mathrm{Aut}(F)_\circ)_{\mathbb{Q}}; 0).\end{aligned}$$

If F satisfies Halperin's conjecture, then $\mathrm{Aut}(F)_\circ$ is rationally homotopy abelian and so $\mathrm{Aut}(p)_\circ$ is as well.

Example 2.7. Let $p\colon E \to B$ be a principal G-bundle. Multiplication by an element of G induces a morphism of H-spaces $G \to \mathrm{Aut}(p)$, while evaluation at the identity gives a left inverse $\mathrm{Aut}(p) \to G$. Thus G is a retract of $\mathrm{Aut}(p)$. It follows easily that

$$\mathrm{Hnil}(\mathrm{Aut}(p)_\circ) \geqslant \mathrm{Hnil}(G_\circ).$$

3. Rational homotopy groups of fibrewise mapping spaces

We now consider the diagram of fibrations (1) above defining the fibrewise mapping space $\mathrm{Map}_g(E', E; f)$. We assume all the spaces E', E, B', B are simply connected CW complexes. Our calculation of $\pi_n(\mathrm{Map}_g(E', E; f)) \otimes \mathbb{Q}$ will follow the line of proof of [**21**, Th. 2.1]. Namely, we will define a homomorphism Φ' from the ordinary homotopy group of the function space to the DG vector space of derivations and prove Φ' induces an isomorphism Φ after rationalization.

As in [**21**], the construction of Φ' depends on some DG algebra homotopy theory. Let $\alpha \in \pi_n(\mathrm{Map}_g(E', E; f))$ be represented by $a\colon S^n \to \mathrm{Map}_g(E', E; f)$ with adjoint $F\colon E' \times S^n \to E$. Write $i\colon E' \to E' \times S^n$ for the based inclusion and $q\colon E' \times S^n \to B'$ for p' composed with the projection, so that $q \circ i = p'$. The map F is then a map "under f" and "over g". That is, we have $F \circ i = f$ and $p \circ F = g \circ q$. To define $\Psi'(\alpha)$, we show the Sullivan model of such a map F can be taken as a DGA map "under" a model for g and "over" one for f. We prove, in fact, that this assignment sets up a bijection between homotopy classes of maps when $p\colon E \to B$ is replaced by its rationalization. Precise definitions and statements follow.

Let Z be any nilpotent space and consider the diagram

$$\begin{tikzcd} E' \arrow[d, "i"'] \arrow[dr, "f"] & \\ E' \times Z \arrow[r, dotted, "F"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B. \end{tikzcd} \tag{2}$$

If F makes the diagram strictly commute, then we say that F is a map *over* g and *under* f. We say two such maps F_0 and F_1 are *homotopic over* g *and under* f, if there is a homotopy H from F_0 to F_1 through maps over g and under f, i.e., if H makes the diagram

$$\begin{tikzcd} E' \times I \arrow[d, "i\times 1"'] \arrow[dr, "T_f"] & \\ E' \times Z \times I \arrow[r, dotted, "H"'] \arrow[d, "q\times 1"'] & E \arrow[d, "p"] \\ B' \times I \arrow[r, "T_g"] & B \end{tikzcd} \tag{3}$$

commute, where T_f and T_g are stationary homotopies at f and g, respectively. Write $[E' \times Z, E]_{\mathrm{o/u}}$ for the set of homotopy classes of maps over g and under f.

On the DG algebra side, let $\eta_Z\colon (C, d) \to (A(Z), \delta_Z)$ be a Sullivan model for Z. We then have a corresponding diagram:

$$\begin{tikzcd} \wedge V \arrow[r, "\mathcal{M}_g"] \arrow[d, "I"'] & \wedge V' \arrow[d, "I''"] \\ \wedge V \otimes \wedge W \arrow[r, dotted, "\Gamma"] \arrow[dr, "\mathcal{A}_f"'] & \wedge V' \otimes \wedge W' \otimes C \arrow[d, "P"] \\ & \wedge V' \otimes \wedge W'. \end{tikzcd} \tag{4}$$

Here $I\colon \wedge V \to \wedge V \otimes \wedge W$ is the inclusion $I(\chi) = \chi \otimes 1$, for $\chi \in \wedge V$, I'' is the inclusion $I''(\chi') = \chi' \otimes 1 \otimes 1$, for $\chi' \in \wedge V'$, and $P = (1 \otimes 1) \cdot \varepsilon$ is the obvious projection so that the composition $P \circ I''$ gives the inclusion $I'\colon \wedge V' \to \wedge V' \otimes \wedge W'$, with $I'(\chi') = \chi' \otimes 1$, for $\chi' \in \lambda V'$. The inclusion $J\colon \wedge V' \otimes \wedge W' \to \wedge V' \otimes \wedge W' \otimes C$ splits P. A DG algebra map Γ making the diagram (4) strictly commute will be called a map *under* $\mathcal{M}_g$ and *over* $\mathcal{A}_f$. Recall that a DG homotopy between Γ_0 and Γ_1 may be taken as a map

$$\mathcal{H}\colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes C \otimes \wedge(t, dt),$$

with $\pi_i \circ \mathcal{H} = \Gamma_i$ for $i = 0, 1$. Here $\wedge(t, dt)$ is the contractible DG algebra with $|t| = 0$; the maps $\pi_i\colon \wedge V' \otimes \wedge W' \otimes C \otimes \wedge(t, dt) \to \wedge V' \otimes \wedge W' \otimes C$ correspond to sending $t \mapsto i$ and $dt \mapsto 0$ [**11**, §12.b]. We say Γ_0 and Γ_1 are *homotopic under* $\mathcal{M}_g$ *and over* $\mathcal{A}_f$ if $\mathcal{H}$ is a DG homotopy through maps under $\mathcal{M}_g$ and over $\mathcal{A}_f$, i.e., if $\mathcal{H}$ makes the diagram

$$\begin{tikzcd}
\wedge V \arrow[rr, "\mathcal{T}_g"] \arrow[d, "I"'] & & \wedge V' \otimes \wedge(t, dt) \arrow[d, "I'' \otimes 1"] \\
\wedge V \otimes \wedge W \arrow[rr, dotted, "\mathcal{H}"] \arrow[drr, "\mathcal{T}_f"'] & & \wedge V' \otimes \wedge W' \otimes C \otimes \wedge(t, dt) \arrow[d, "P \otimes 1"] \\
& & \wedge V' \otimes \wedge W' \otimes \wedge(t, dt)
\end{tikzcd}$$

commute, where $\mathcal{T}_g$ and $\mathcal{T}_f$ are stationary homotopies at $\mathcal{M}_g$ and $\mathcal{A}_f$, respectively. We write $[\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W' \otimes C]_{\mathrm{u/o}}$ for the set of homotopy classes of maps under $\mathcal{M}_g$ and over $\mathcal{A}_f$. We will define an assignment $F \mapsto \Gamma$ and show that it leads to a bijection of homotopy sets when $p\colon E \to B$ is rationalized.

We begin by choosing and fixing models for g and f. First, we claim a model $\mathcal{A}_f$ for f may be constructed as a map under $\mathcal{M}_g$. That is, we have $\mathcal{A}_f \circ I = I' \circ \mathcal{M}_g$. Writing $A_{PL}(X) = (A(X), \delta_X)$ for the DG algebra of Sullivan polynomial forms, the relative model of the map $A(p)\colon (A(B), \delta_B) \to (A(E), \delta_E)$ is of the form $(A(B) \otimes \wedge W, D)$ [**11**, §14]. Choose a minimal model $\eta_B\colon (\wedge V, d) \to (A(B), \delta_B)$ (the Sullivan minimal model of B) and obtain a quasi-isomorphism

$$\eta_E\colon (\wedge V \otimes \wedge W, D) \to (A(E), \delta_E).$$

Now recall the *surjective trick*: Given a graded algebra U, we define $(S(U), \delta)$ to be the contractible DG algebra on a basis $U \oplus \delta(U)$. Given a DG algebra map $\eta\colon (B, d) \to (A, d)$, this manoeuvre results in a diagram

$$\begin{tikzcd}
B \arrow[r, shift left, "\alpha"] \arrow[d, "\eta"'] & B \otimes S(A) \arrow[l, shift left, "\beta"] \arrow[dl, two heads, "\gamma"] \\
A &
\end{tikzcd}$$

in which γ is a surjection, and both α and β are quasi-isomorphisms. Here $\eta = \gamma \circ \alpha$.

This trick is used in the standard construction of the model $\mathcal{M}_g$ for g. Write $\eta_B\colon (\wedge V, d) \to (A(B), \delta_B)$ and $\eta_{B'}\colon (\wedge V', d') \to (A(B'), \delta_{B'})$ for Sullivan minimal models for B and B'. Convert $\eta_{B'}$ into a surjection $\gamma_{B'}\colon \wedge V' \otimes S(A(B')) \to A(B')$ as above. We then lift the composite $A(g) \circ \eta_B$ through the surjective quasi-isomorphism $\gamma_{B'}$, using the standard lifting lemma [**11**, Lem. 12.4]. We thus obtain $\phi_g\colon \wedge V \to \wedge V' \otimes S(A(B'))$. Now set $\mathcal{M}_g = \beta_{B'} \circ \phi_g$. All this is summarized in the following diagram:

$$\begin{tikzcd}
 & & \wedge V' \otimes S(A(B')) \arrow[dl, shift left, "\beta_{B'}", "\simeq"'] \arrow[ddl, two heads, bend left, "\gamma_{B'}", "\simeq"'] \\
\wedge V \arrow[urr, bend left, "\phi_g"] \arrow[r, dotted, "\mathcal{M}_g"'] \arrow[d, "\eta_B"', "\simeq"] & \wedge V' \arrow[ur, shift left, "\alpha_{B'}", "\simeq"'] \arrow[d, "\eta_{B'}"', "\simeq"] & \\
A(B) \arrow[r, "A(g)"'] & A(B'). &
\end{tikzcd}$$

Here the symbol $\simeq$ indicates that a map is a quasi-isomorphism. By construction, we have $\gamma_{B'} \circ \phi_g = A(g) \circ \eta_B$. We will use the letters ϕ, α, β, γ, and η, with suitable subscripts, in a consistent way for diagrams of the above form. Notice that α is the obvious inclusion, and β is the obvious projection.

We apply the same construction to obtain a model for $\mathcal{A}_f$. However, in this case we use a relative version of the lifting lemma. Converting the vertical quasi-isomorphisms in the commutative diagram

$$\begin{tikzcd}
\wedge V' \arrow[r, "I'"] \arrow[d, "\eta_{B'}"', "\simeq"] & \wedge V' \otimes \wedge W' \arrow[d, "\simeq"', "\eta_{E'}"] \\
A(B') \arrow[r, "A(p')"'] & A(E')
\end{tikzcd}$$

to surjections results in a commutative diagram

$$\begin{tikzcd}
\wedge V' \otimes S(A(B')) \arrow[r, "I' \otimes S(A(p'))"] \arrow[d, two heads, "\gamma_{B'}"', "\simeq"] & \wedge V' \otimes \wedge W' \otimes S(A(E')) \arrow[d, two heads, "\simeq"', "\gamma_{E'}"] \\
A(B') \arrow[r, "A(p')"'] & A(E'),
\end{tikzcd}$$

in which $S(A(p'))\colon S(A(B')) \to S(A(E'))$ is the map induced by $A(p')\colon A(B') \to A(E')$. We incorporate this into the following *relative lifting problem*:

$$\begin{tikzcd}
\wedge V \arrow[r, "I' \otimes S(A(p')) \circ \phi_g"] \arrow[d, "I"'] & \wedge V' \otimes \wedge W' \otimes S(A(E')) \arrow[d, two heads, "\simeq"', "\gamma_{E'}"] \\
\wedge V \otimes \wedge W \arrow[r, "A(f) \circ \eta_E"'] \arrow[ur, dotted, "\phi_f"] & A(E').
\end{tikzcd} \tag{5}$$

The relative lifting lemma as in [**11**, Lem. 14.4] provides the lift ϕ_f that makes both upper and lower triangles commute. But we now break-off from our development of ideas to give here an extension of that result, and also its development into [**11**, Lem. 14.6], both of which which we need for the sequel and neither of which we can find in the literature.

Proposition 3.1 (Under and Over Lifting Lemma)**.** *Suppose given a diagram of* DG *algebra maps*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "i"'] & & B \arrow[r, "r_B"'] \arrow[d, "\gamma", "\simeq"'] & B' \arrow[l, bend right, "i_B"'] \arrow[d, "\gamma'"] \\
A \otimes \wedge V \arrow[rr, "\phi"'] \arrow[urr, dotted, "\Psi"] \arrow[urrr, "\psi"'] & & C \arrow[r, "r_C"] & C' \arrow[l, bend left, "i_C"]
\end{tikzcd}$$

that satisfies

- *commutativity:* $\gamma \circ f = \phi \circ i$ *and* $\gamma' \circ r_B = r_C \circ \gamma$*;*
- i_B *and* i_C *are splittings that are "natural," in that we have* $r_B \circ i_B = 1_{B'}$*,* $r_C \circ i_C = 1_{C'}$*, and also* $\gamma \circ i_B = i_C \circ \gamma'$*;*
- ψ *is a lift of* $r_C \circ \phi$ *through* γ' *relative to* $r_B \circ f$*, in that* $r_B \circ f = \psi \circ i$ *and* $\gamma' \circ \psi = r_C \circ \phi$*;*
- γ *is a quasi-isomorphism that is onto* $\ker(r_C)$ *(not necessarily surjective).*

Then there exists a lift Ψ *of* ϕ *through* γ *that is under* f *and over* ψ*:* $\gamma \circ \Psi = \phi$*,* $\Psi \circ i = f$*, and* $r_B \circ \Psi = \psi$*.*

Proof. We assume that V admits a decomposition $V = \oplus_{i \geqslant 1} V(i)$, with respect to which $A \otimes \wedge V$ satisfies the nilpotency condition $d\big(V(i)\big) \subseteq A \otimes \wedge V(< i)$ and proceed by induction on i. Induction starts with $i = 0$, where we already have the lift f. Now suppose that Ψ has been constructed on $A \otimes \wedge V(< i)$ and that $v \in V(i)$ for some $i \geqslant 1$. Then $dv \in A \otimes \wedge V(< i)$, and so $\Psi(dv)$ is defined. We have $d(\Psi(dv)) = \Psi(d^2(v)) = 0$, so $\Psi(dv) \in \mathcal{Z}(B)$. Furthermore, we have $\gamma_*([\Psi(dv)]) = 0$, since $\gamma \circ \Psi(dv) = \phi(dv) = d\phi(v)$. Since γ is a quasi-isomorphism, $\exists b \in B$ with $\Psi(dv) = db$. Now consider $\phi(v) - \gamma(b) \in C$: we have $d\big(\phi(v) - \gamma(b)\big) = d\phi(v) - \gamma(db) = \phi(dv) - \gamma \circ \Psi(dv) = 0$, and so $\phi(v) - \gamma(b) \in \mathcal{Z}(C)$. Now "polarize" this cycle, using the splitting i_C, by writing

$$\phi(v) - \gamma(b) = \big[\big(\phi(v) - \gamma(b)\big) - i_C \circ r_C\big(\phi(v) - \gamma(b)\big)\big] + i_C \circ r_C\big(\phi(v) - \gamma(b)\big). \quad (6)$$

Observe that we have

$$i_C \circ r_C\big(\phi(v) - \gamma(b)\big) = i_C \circ \gamma' \circ \psi(v) - \gamma \circ i_B \circ r_B(b) = \gamma\big(i_B \circ \psi(v) - i_B \circ r_B(b)\big).$$

Also, writing $\chi = \big(\phi(v) - \gamma(b)\big) - i_C \circ r_C\big(\phi(v) - \gamma(b)\big)$, we have that $\chi \in \ker(r_C) \cap \mathcal{Z}(C)$. Since γ is a quasi-isomorphism, $\exists \beta \in \mathcal{Z}(B)$ with $\gamma_*([\beta]) = [\chi]$, and so $\gamma(\beta) = \chi + d\xi$ for some $\xi \in C$. Then we have

$$\begin{aligned}
\gamma\big(\beta - i_B \circ r_B(\beta)\big) &= \gamma(\beta) - \gamma \circ i_B \circ r_B(\beta) = \gamma(\beta) - i_C \circ r_C \circ \gamma(\beta) \\
&= \chi + d\xi - i_C \circ r_C(\chi + d\xi) = \chi + d\xi - i_C \circ r_C d\xi \\
&= \chi + d\big(\xi - i_C \circ r_C(\xi)\big).
\end{aligned}$$

Since $\xi - i_C \circ r_C(\xi) \in \ker(r_C)$, and γ is onto $\ker(r_C)$, there exists $b' \in B$ with $\gamma(b') = \xi - i_C \circ r_C(\xi)$. Without loss of generality, we may choose $b' \in \ker(r_B)$, since we have

$\gamma(b') = \gamma(b') - i_C \circ r_C \circ \gamma(b') = \gamma\big(b' - i_B \circ r_B(b')\big)$. So we have

$$\gamma\big(\beta - i_B \circ r_B(\beta)\big) = \chi + d\gamma(b'), \quad \text{or} \quad \chi = \gamma\big(\beta - i_B \circ r_B(\beta) - db'\big).$$

Substituting this last identity and the one obtained earlier into (6) now gives

$$\phi(v) - \gamma(b) = \gamma\big(\beta - i_B \circ r_B(\beta) - db'\big) + \gamma\big(i_B \circ \psi(v) - i_B \circ r_B(b)\big),$$

so we have

$$\phi(v) = \gamma\big(b - i_B \circ r_B(b) + \beta - i_B \circ r_B(\beta) - db' + i_B \circ \psi(v)\big).$$

Now define

$$\Psi(v) = \big(b - i_B \circ r_B(b)\big) + \big(\beta - i_B \circ r_B(\beta)\big) - db' + i_B \circ \psi(v).$$

Observe that $\Psi(v) - i_B \circ \psi(v) \in \ker(r_B)$. Evidently, we have $\gamma \circ \Psi(v) = \phi(v)$, and $r_B \circ \Psi(v) = r_B \circ i_B \circ \psi(v) = \psi(v)$ as desired. Induction is complete, and the result follows. □

We extend this to a result on lifting homotopy classes in the following (cf. [**3**, Prop. II.2.11], [**11**, Props. 12.9 and 14.6], and [**21**, Prop. A.4]).

Proposition 3.2 (Under and Over Homotopy Lifting Lemma)**.** *Suppose given a commutative diagram of* DG *algebra maps*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"'] & B \arrow[r, two heads, "\gamma", "\simeq"'] \arrow[d, "r_B"'] & C \arrow[d, "r_C"'] \\
A \otimes \wedge V \arrow[r, "f"'] \arrow[ur, dotted] & B' \arrow[r, "\gamma'"] \arrow[u, bend right, "i_B"'] & C' \arrow[u, bend right, "i_C"']
\end{tikzcd}$$

with $\gamma\colon B \to C$ a surjective quasi-isomorphism and i_B, i_C natural splittings as in Proposition 3.1. That is, we have $r_B \circ i_B = 1_{B'}$, $r_C \circ i_C = 1_{C'}$ and also $\gamma \circ i_B = i_C \circ, \gamma'$. Then γ induces a bijection of homotopy sets

$$\lambda_*\colon [A \otimes \wedge V, B]_{\mathrm{u/o}} \to [A \otimes \wedge V, C]_{\mathrm{u/o}}.$$

Proof. Homotopy classes in $[A \otimes \wedge V, B]_{\mathrm{u/o}}$ are represented by maps F that make the left-hand diagram below commute, and homotopies H between two such maps make the right-hand diagram below commute:

$$\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "g"] & \\
A \otimes \wedge V \arrow[r, "F"'] \arrow[dr, "f"'] & B \arrow[d, "r_B"] \\
& B'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "T_g"] & \\
A \otimes \wedge V \arrow[r, "H"'] \arrow[dr, "T_f"'] & B \otimes \wedge(t, dt) \arrow[d, "r_B \otimes 1"] \\
& B' \otimes \wedge(t, dt).
\end{tikzcd}$$

Homotopy classes in $[A \otimes \wedge V, C]_{\mathrm{u/o}}$ are represented by maps $\mathcal{F}$ that make the left-hand diagram below commute, and homotopies $\mathcal{H}$ between two such maps make the right-hand diagram below commute:

$$\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "\gamma\circ g"] & \\
A\otimes\wedge V \arrow[r, "\mathcal{F}"'] \arrow[dr, "\gamma'\circ f"'] & C \arrow[d, "r_C"] \\
& C'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "i"'] \arrow[dr, "T_{\gamma\circ g}"] & \\
A\otimes\wedge V \arrow[r, "\mathcal{H}"'] \arrow[dr, "T_{\gamma'\circ f}"'] & C\otimes\wedge(t,dt) \arrow[d, "r_C\otimes 1"] \\
& C'\otimes\wedge(t,dt).
\end{tikzcd}$$

In these diagrams, T_g denotes the stationary homotopy at g, for example. Evidently, we have $(\gamma'\otimes 1)\circ T_f = T_{\gamma'\circ f}$ and $(\gamma\otimes 1)\circ T_g = T_{\gamma\circ g}$. So γ_* gives a well-defined map of under-and-over homotopy classes. Now suppose given some $[\phi]\in[A\otimes\wedge V, C]_{\mathrm{u/o}}$. We have the following diagram,

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"'] & B \arrow[r, "r_B"] \arrow[d, "\gamma", "\simeq"'] & B' \arrow[l, bend right, "i_B"'] \arrow[d, "\gamma'"] \\
A\otimes\wedge V \arrow[r, "\phi"'] \arrow[ur, dotted] \arrow[urr, "f"' near end] & C \arrow[r, "r_C"] & C', \arrow[l, bend left, "i_C"]
\end{tikzcd}$$

and it follows from Proposition 3.1 that γ_* is surjective.

Finally, we show that γ_* is injective. Suppose we have two maps $F_0, F_1\colon A\otimes\wedge V\to B$ under g and over f, and $\gamma\circ F_0$ and $\gamma\circ F_1$ are homotopic via a homotopy $\mathcal{H}\colon A\otimes\wedge V\to C\otimes\wedge(t,dt)$ under $\gamma\circ f$ and over $\gamma'\circ g$. Form the commutative cube

$$\begin{tikzcd}
& Q \arrow[rr] \arrow[dd] & & B'\times B' \arrow[dd, "\gamma'\times\gamma'"] \\
P \arrow[ur, "r"] \arrow[rr] \arrow[dd] & & B\times B \arrow[ur, "r_B\times r_B"] \arrow[dd, "\gamma\times\gamma" near start] & \\
& C'\otimes\wedge(t,dt) \arrow[rr, "(1\cdot\varepsilon_0,1\cdot\varepsilon_1)"' near start] & & C'\times C' \\
C\otimes\wedge(t,dt) \arrow[ur, "r_C\otimes 1"] \arrow[rr, "(1\cdot\varepsilon_0,1\cdot\varepsilon_1)"'] & & C\times C \arrow[ur, "r_C\times r_C"'] &
\end{tikzcd}$$

in which the front and back faces are pullbacks, so that

$$P = \big(C\otimes\wedge(t,dt)\big)\times_{C\times C}\big(B\times B\big) \quad\text{and}\quad Q=\big(C'\otimes\wedge(t,dt)\big)\times_{C'\times C'}\big(B'\times B'\big).$$

The map $r\colon P\to Q$ is the one induced on the pullbacks so as to make the cube commute. Since the forwards maps are composed of r_C and r_B, they admit natural splittings and a natural splitting $i\colon Q \to P$ of r is induced. Denote by Γ the map induced from the pullback diagram as follows:

$$\begin{tikzcd}
B \otimes \wedge(t,dt) \arrow[dr, "\Gamma"] \arrow[drr, bend left, "(1\cdot\varepsilon_0, 1\cdot\varepsilon_1)"] \arrow[ddr, bend right, "\gamma\otimes 1"', "\simeq"] & & \\
& P \arrow[r] \arrow[d, "\simeq"] & B \times B \arrow[d, "\simeq"', "\gamma\times\gamma"] \\
& C \otimes \wedge(t,dt) \arrow[r, two heads, "(1\cdot\varepsilon_0, 1\cdot\varepsilon_1)"'] & C \times C.
\end{tikzcd}$$

It follows from properties of the pullback that Γ is a surjective quasi-isomorphism. We now include Γ, and the corresponding map $\Gamma'\colon B' \otimes \wedge(t,dt) \to Q$ obtained from the back face of the above pullback cube, into the right-hand part of the following diagram:

$$\begin{tikzcd}
A \arrow[r, "T_g"] \arrow[d, "i"'] & B \otimes \wedge(t,dt) \arrow[r, "r_B\otimes 1"'] \arrow[d, "\simeq"', "\Gamma"] & B' \otimes \wedge(t,dt) \arrow[l, bend right, "i_B\otimes 1"'] \arrow[d, "\Gamma'"] \\
A \otimes \wedge V \arrow[r, "{\left(\mathcal{H},(F_0,F_1)\right)}"'] \arrow[ur, dotted] \arrow[urr, "T_f"] & C \arrow[r, "r"] & C'. \arrow[l, bend left, "i"]
\end{tikzcd}$$

The lift obtained from Proposition 3.1 gives the desired homotopy from F_0 to F_1 under g and over f. $\square$

Now we return to the development of ideas that preceded Proposition 3.1. In diagram (5), Proposition 3.1, applied to the case in which $B' = C' = \mathbb{Q}$, and r_B, r_C, and ψ are the augmentations, provides the relative lift. That is, we obtain a lifting ϕ_f in diagram (5) and set $\mathcal{A}_f = \beta_{E'} \circ \phi_f$. It follows from the definitions that we have $I' \circ \mathcal{M}_g = \mathcal{A}_f \circ I$.

Having chosen and fixed models for f and g, we now extend their construction to Sullivan models for maps F over g and under f as in (2).

Proposition 3.3. *A Sullivan model $\mathcal{A}_F$ for a map F that makes (2) commute may be chosen so that $\Gamma = \mathcal{A}_F$ makes (4) commute. Further, if F_0 and F_1 are homotopic over g and under f, then $\mathcal{A}_{F_0}$ and $\mathcal{A}_{F_1}$ are* DG *homotopic under $\mathcal{M}_g$ and over $\mathcal{A}_f$.*

Proof. The existence of a model $\mathcal{A}_F$ of the desired form follows directly from Proposition 3.1. Apply the Sullivan functor $A(-)$ to diagram (2), and incorporate the result, along with the models just constructed, into the following diagram:

$$
\begin{tikzcd}
\wedge V \arrow[r, "(I''\otimes S(A(q)))\circ\phi_g"] \arrow[d, "I"'] & \wedge V'\otimes\wedge W'\otimes C\otimes S(A(E'\times Z)) \arrow[r, "P\otimes S(A(i))"'] \arrow[d, "\simeq"', "\gamma_{E'\times Z}"] & \wedge V'\otimes\wedge W'\otimes S(A(E')) \arrow[l, bend right, "J\otimes S(A(p_1))"'] \arrow[d, "\gamma_{E'}"] \\
\wedge V\otimes\wedge W \arrow[r, "A(F)\circ\eta_E"'] \arrow[ur, dotted, "\phi_F"] \arrow[urr, "\phi_f"] & A(E'\times Z) \arrow[r, "A(i)"] & A(E') \arrow[l, bend left, "A(p_1)"]
\end{tikzcd}
$$

Proposition 3.1 gives an under-over lift ϕ_F, and we set $\mathcal{A}_F = \beta_{E'\times Z}\circ\phi_F$.

We show the relation of homotopy over-and-under is preserved through passing to models in two steps. First, we establish that there is a well-defined map of homotopy classes

$$[E'\times Z, E]_{\mathrm{o/u}} \to [\wedge V\otimes\wedge W, A(E'\times Z)]_{\mathrm{u/o}}.$$

Suppose $H\colon E'\times Z\times I\to E$ is a homotopy from F_0 to F_1 which is over g and under f, as in diagram (3). Apply $A(-)$ to that diagram, and adapt the argument of [**11**, Prop. 12.6] as follows (some of the notation in what follows is adopted from there.): From the diagram

$$
\begin{tikzcd}
A(E'\times Z)\otimes\wedge(t,dt) \arrow[rrr, "A(i)\otimes 1"'] \arrow[dr, "(\mathrm{id}\cdot\varepsilon_0,\mathrm{id}\cdot\varepsilon_1)"] \arrow[dd, "\simeq", "A(p^{E'\times Z})\cdot A(p^I)"'] & & & A(E')\otimes\wedge(t,dt) \arrow[lll, bend right, "A(p_1)\otimes 1"'] \arrow[dl, "(\mathrm{id}\cdot\varepsilon_0,\mathrm{id}\cdot\varepsilon_1)"'] \arrow[dd, "A(p^{E'})\cdot A(p^I)"] \\
& A(E'\times Z)\times A(E'\times Z) & A(E')\times A(E') \arrow[l] & \\
A(E'\times Z\times I) \arrow[rrr, "A(i\times 1)"] \arrow[ur, "(A(j_0),A(j_1))"'] & & & A(E'\times I) \arrow[lll, bend left, "A(p_1\times 1)"] \arrow[ul, "(A(j_0),A(j_1))"]
\end{tikzcd}
$$

we adjust the left-hand vertical map into a map γ that is onto the kernel of $A(i\times 1)$, using $U = \ker(A(j_0), A(j_1))\cap\ker(A(i\times 1))$. Now apply Proposition 3.1 to the diagram

$$
\begin{tikzcd}
\wedge V \arrow[r, "(A(g\circ q)\otimes 1)\circ\eta_B"] \arrow[d, "I"'] & A(E'\times Z)\otimes\wedge(t,dt)\otimes S(U) \arrow[r, "A(i)\otimes 1"'] \arrow[d, "\simeq"', "\gamma"] & A(E')\otimes\wedge(t,dt) \arrow[l, bend right, "A(p_1)\otimes 1"'] \arrow[d, "A(p^{E'})\cdot A(p^I)"] \\
\wedge V\otimes\wedge W \arrow[r, "A(H)\circ\eta_E"'] \arrow[ur, dotted, "\phi_H"] \arrow[urr, "A(f)\circ\eta_E"] & A(E'\times Z\times I) \arrow[r, "A(i\times 1)"] & A(E'\times I) \arrow[l, bend left, "A(p_1\times 1)"]
\end{tikzcd}
$$

to obtain a DG homotopy $\mathcal{G} = \beta \circ \phi_H$ from $\eta_E \circ A(F_0)$ to $\eta_E \circ A(F_1)$ that is a DG homotopy under $A(g) \circ \eta_B$ and over $A(f) \circ \eta_E$. Thus far, we have established that there is a well-defined map of homotopy classes

$$\mathcal{S}\colon [E' \times Z, E]_{\mathrm{o/u}} \to [\wedge V \otimes \wedge W, A(E' \times Z)]_{\mathrm{u/o}}.$$

We now want to lift this correspondence up to minimal models. Converting the quasi-isomorphism

$$\eta_{E'\times Z}\colon \wedge V' \otimes \wedge W' \otimes C \to A(E' \times Z)$$

to a surjection gives, amongst other data, a surjective quasi-isomorphism

$$\gamma_{E'\times Z}\colon \wedge V' \otimes \wedge W' \otimes C \otimes S\big(A(E' \times Z)\big) \to A(E' \times Z)$$

and the retraction map $\beta\colon \wedge V' \otimes \wedge W' \otimes C \otimes S\big(A(E' \times Z)\big) \to \wedge V' \otimes \wedge W' \otimes C$, which we observe is also a surjective quasi-isomorphism.

The diagrams

$$\begin{tikzcd}
\wedge V \arrow[r, "g"] \arrow[d, "i"'] & \wedge V' \otimes \wedge W' \otimes C \otimes S\big(A(E' \times Z)\big) \arrow[r, two heads, "\gamma_{E'\times Z}", "\simeq"'] \arrow[d, "P\otimes S(A(i))"'] & A(E' \times Z) \arrow[d, "A(i)"'] \\
\wedge V \otimes \wedge W \arrow[r, "f"'] \arrow[ur, dotted] & \wedge V' \otimes \wedge W' \otimes S\big(A(E')\big) \arrow[r, "\gamma_{E'}"] \arrow[u, bend right, "J\otimes S(A(p_1))"'] & A(E') \arrow[u, bend right, "A(p_1)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\wedge V \arrow[r, "g"] \arrow[d, "i"'] & \wedge V' \otimes \wedge W' \otimes C \otimes S\big(A(E' \times Z)\big) \arrow[r, two heads, "\beta_{E'\times Z}", "\simeq"'] \arrow[d, "P\otimes S(A(i))"'] & \wedge V' \otimes \wedge W' \otimes C \arrow[d, "P"'] \\
\wedge V \otimes \wedge W \arrow[r, "f"'] \arrow[ur, dotted] & \wedge V' \otimes \wedge W' \otimes S\big(A(E')\big) \arrow[r, "\beta_{E'}"] \arrow[u, bend right, "J\otimes S(A(p_1))"'] & \wedge V' \otimes \wedge W' \arrow[u, bend right, "J"']
\end{tikzcd}$$

yield, respectively, the bijections

$$\begin{aligned}(\gamma_{E'\times Z})_*\colon [\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W' \otimes C \otimes S\big(A(E' \times Z)\big)]_{\mathrm{u/o}} \\ \to [\wedge V \otimes \wedge W, A(E' \times Z)]_{\mathrm{u/o}}\end{aligned}$$

and

$$\begin{aligned}(\beta_{E'\times Z})_*\colon [\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W' \otimes C \otimes S\big(A(E' \times Z)\big)]_{\mathrm{u/o}} \\ \to [\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W' \otimes C]_{\mathrm{u/o}}.\end{aligned}$$

Combined with our previous work, we now have a well-defined map of homotopy classes

$$(\beta_{E'\times Z})_* \circ ((\gamma_{E'\times Z})_*)^{-1} \circ \mathcal{S}\colon [E' \times Z, E]_{\mathrm{o/u}} \to [\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W' \otimes C]_{\mathrm{u/o}}$$

induced by the assignment $F \mapsto \mathcal{A}_F$. $\square$

Now we turn to rationalization of the above correspondence. Consider the diagram

$$\begin{array}{ccc} E' & \xrightarrow{f_{\mathbb{Q}}} & E_{\mathbb{Q}} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p_{\mathbb{Q}}} \\ B' & \xrightarrow{g_{\mathbb{Q}}} & B_{\mathbb{Q}} \end{array} \tag{7}$$

obtained from (1) by replacing p, f and g by their rationalizations. (Recall we are assuming the spaces E, E and B, B' are all simply connected.) Let

$$\Psi\colon [E' \times Z, E_{\mathbb{Q}}]_{\mathrm{o/u}} \to [\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W' \otimes C]_{\mathrm{u/o}}$$

denote the corresponding map. The following result extends the standard correspondence between homotopy classes of maps into a rational space and DG algebra homotopy classes of maps between Sullivan models [**11**, Props. 12.7 and 17.13].

Proposition 3.4. *Suppose the spaces E' and Z are finite CW complexes. Then Ψ is a bijection of sets.*

Proof. First assume $p_{\mathbb{Q}}\colon E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ is a principal $K(W_n, n)$-fibration for W_n a rational space concentrated in degree n. Let $F\colon E' \times Z \to E_{\mathbb{Q}}$ be a map over $g_{\mathbb{Q}}$ and under $f_{\mathbb{Q}}$. Define a class $\gamma_F \in H^n(Z; W_m) \cong \mathrm{Hom}(W_m, H^*(Z; \mathbb{Q}))$ as follows: Let $j\colon Z \to E' \times Z$ denote the inclusion and observe $p_{\mathbb{Q}} \circ F \circ j \simeq *$. Thus $F \circ j$ is homotopic to a map $G_F\colon Z \to K(W_n, n)$, or equivalently, a class $\gamma_F \in H^n(Z; W_m)$.

Conversely, given a class $\gamma \in H^n(Z; W_m)$ we construct a map F_γ over g and under f as follows: Let $\mathcal{P}\colon E_{\mathbb{Q}} \times K(W_n, n) \to E_{\mathbb{Q}}$ denote the fibrewise action. Here $\mathcal{P}$ is a map over $1_{B_{\mathbb{Q}}}$ and under the inclusion $E_{\mathbb{Q}} \to E_{\mathbb{Q}} \times K(W_n, n)$. Let $G\colon Z \to K(W_n, n)$ be the map induced by γ. Define F_γ to be the composite

$$E' \times Z \xrightarrow{f_{\mathbb{Q}} \times G} E_{\mathbb{Q}} \times K(W_n, n) \xrightarrow{\mathcal{P}} E_{\mathbb{Q}}.$$

It is direct to check that F_γ is a map over $g_{\mathbb{Q}}$ and under $f_{\mathbb{Q}}$ and that the assignments $F \mapsto \gamma_F$ and $\gamma \mapsto F_\gamma$ set up a bijection

$$[E' \times Z, E_{\mathbb{Q}}]_{\mathrm{o/u}} \equiv H^n(Z; W_n).$$

Now suppose Γ is a DG map making (4) commute. Then we may write $\Gamma(\chi) = \mathcal{A}_f(\chi) + \eta(\chi)$, where $\eta(\chi) = 0$ for $\chi \in \wedge V$ and $P(\eta(\chi)) = 0$. Given $w \in W_n$, since $D(w) \in \wedge V$, it follows that $P'(\eta(w))$ is a cycle of C. Let $\gamma_\Gamma \in H^n(Z; W_n)$ denote the class corresponding to $P' \circ \eta$ restricted to W_n, where $P'\colon \wedge V' \otimes \wedge W' \otimes C \to C$ is the projection. The assignment $\Gamma \mapsto \gamma_\Gamma$ then gives a well-defined surjection

$$\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W' \otimes C]_{\mathrm{u/o}} \twoheadrightarrow H^n(Z; W_n).$$

Finally, observe that the spatial realization F of γ_Γ constructed in the preceding paragraph has Sullivan model Γ. This is direct from the fact that $\mathcal{P}$ is a map over $1_{B_{\mathbb{Q}}}$ and under $E_{\mathbb{Q}} \to E_{\mathbb{Q}} \times K(W_n, n)$. Suppose Γ_0 and Γ_1 satisfy $\gamma_{\Gamma_0} = \gamma_{\Gamma_1} \in H^n(Z; W_n)$. Writing γ for this class, we obtain a map F_γ over $g_{\mathbb{Q}}$ and under $f_{\mathbb{Q}}$ with two Sullivan models Γ_0 and Γ_1. By Proposition 3.3, Γ_0 and Γ_1 are homotopic under $\mathcal{M}_g$ and over $\mathcal{A}_f$, and the result is proved in this case.

Now proceed by induction over a Moore-Postnikov factorization of the fibration $p_{\mathbb{Q}}\colon E_{\mathbb{Q}} \to B_{\mathbb{Q}}$. Let $(p_{\mathbb{Q}})_n\colon (E_{\mathbb{Q}})_n \to (E_{\mathbb{Q}})_{n-1}$ be the nth fibration, a principal fibration with fibre $K(W_n, n)$. The Sullivan model for $(p_{\mathbb{Q}})_n$ is of the form $\wedge V \to \wedge V \otimes \wedge W_{(n)}$, where $W_{(n)} = \bigoplus_{k\leqslant n} W_k$. Since $E' \times Z$ is finite, for n large, composition with the canonical map $h_n\colon E_{\mathbb{Q}} \to (E_n)_{\mathbb{Q}}$ yields a bijection

$$[E' \times Z, E_{\mathbb{Q}}]_{\mathrm{o/u}} \equiv [E' \times Z, (E_n)_{\mathbb{Q}}]_{\mathrm{o/u}}.$$

Similarly, the inclusion $W_{(n)} \to W$ for such n induces a bijection

$$[\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W' \otimes C]_{\mathrm{u/o}} \equiv [\wedge V \otimes \wedge W_{(n)}, \wedge V' \otimes \wedge W' \otimes C]_{\mathrm{u/o}}. \qquad \square$$

We apply the foregoing to define $\Phi(\alpha)$ for an element $\alpha \in \pi_n(\mathrm{Map}_g(E', E; f))$. Let $F\colon E' \times S^n \to E$ be the adjoint of α. By Proposition 3.3, F has a Sullivan model of the form

$$\mathcal{A}_F\colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes (\wedge(u)/\langle u^2\rangle)$$

with $\mathcal{A}_F(\chi) = \mathcal{A}_f(\chi) + u\theta(\chi)$ for $\chi \in \wedge V \otimes \wedge W$ and where $\mathcal{A}_f(\chi) = \mathcal{M}_g(\chi)$ and $\theta(\chi) = 0$ for $\chi \in \wedge V$. Here $(\wedge(u)/\langle u^2\rangle, 0)$ is a Sullivan model for S^n with $|u| = n$. The map θ is thus linear of degree n vanishing on $\wedge V$. The following facts are standard and direct to check:

1. $\mathcal{A}_F(\chi_1\chi_2) = \mathcal{A}_F(\chi_1)\mathcal{A}_F(\chi_2) \Longrightarrow \theta$ is an $\mathcal{A}_f$-derivation.
2. $\mathcal{A}_F \circ D = D \circ \mathcal{A}_F \Longrightarrow \theta$ is an $\mathcal{A}_f$-derivation cycle.

Lemma 3.5. *The homology class $\langle\theta\rangle \in H_n(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f))$ is independent of the choice of representative of $\alpha \in \pi_n(\mathrm{Map}_g(E', E; f))$.*

Proof. Suppose $a, b\colon S^n \to \mathrm{Map}_g(E', E; f)$ both represent $\alpha \in \pi_n(\mathrm{Map}_g(E', E; f))$. Observe that a homotopy from a to b has adjoint $H\colon E' \times S^n \times I \to E$ giving rise to a commutative diagram as in (3), with $Z = S^n$. This translates into a Sullivan model $\mathcal{H}$ for H that fits into the following commutative diagram:

$$\begin{tikzcd}
\wedge V \arrow[rr, "\mathcal{T}_g"] \arrow[d, "I"'] & & \wedge V' \otimes \wedge(t, dt) \arrow[d, "I''\otimes 1"] \\
\wedge V \otimes \wedge W \arrow[rr, dotted, "\mathcal{H}"] \arrow[drr, "\mathcal{T}_f"'] & & \wedge V' \otimes \wedge W' \otimes (\wedge(u)/\langle u^2\rangle) \otimes \wedge(t, dt) \arrow[d, "P\otimes 1"] \\
& & \wedge V' \otimes \wedge W' \otimes \wedge(t, dt).
\end{tikzcd}$$

For $\chi \in \wedge V \otimes \wedge W$, we may write (being particular with the order of terms in each sum)

$$\mathcal{H}(\chi) = \mathcal{A}_f(\chi) + \sum_{i\geqslant 0} u\, t^i\, \psi_i(\chi) + \sum_{i\geqslant 0} u\, t^i\, dt\, \phi_i(\chi),$$

for linear maps ψ_i and ϕ_i. Because the homotopy $\mathcal{H}$ is under $\mathcal{M}_g$, we have $\psi_i(V) = 0$ and $\phi_i(V) = 0$ for each i. From our definition above, and since the original homotopy was from a to b, it follows that ψ_0, respectively $\sum_{i\geqslant 0}\psi_i$, is the derivation θ_a, respectively θ_b, that corresponds under Φ' to $[a]$, respectively $[b]$. We must show that

$\theta_b - \theta_a = \sum_{i \geqslant 1} \psi_i$ is a boundary. In fact, the identity $\mathcal{H}(\chi_1\chi_2) = \mathcal{H}(\chi_1)\mathcal{H}(\chi_2)$ easily yields that each ψ_i and each ϕ_i is an $\mathcal{A}_f$-derivation. Then the identity $\mathcal{H} \circ D = D \circ \mathcal{H}$ yields that

$$\mathcal{D}(\phi_i) = [d, \phi_i] = (i+1)\psi_{i+1},$$

for each $i \geqslant 0$. Hence we have

$$\theta_b - \theta_a = \sum_{i \geqslant 0} \frac{1}{i+1} \mathcal{D}(\phi_i),$$

and the cohomology class is well-defined. □

Define

$$\Phi' \colon \pi_n(\mathrm{Map}_g(E', E; f)) \to H_n(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f))$$

by $\Phi'(\alpha) = \langle \theta \rangle$. The following result contains the first assertion of Theorem 1.2.

Theorem 3.6. *Suppose given a commutative diagram*

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

with vertical maps fibrations and all spaces simply connected CW *complexes. The map*

$$\Phi' \colon \pi_n(\mathrm{Map}_g(E', E; f)) \to H_n(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f))$$

defined above is a homomorphism for $n \geqslant 2$. If E' is finite, then the rationalization

$$\Phi \colon \pi_n(\mathrm{Map}_g(E', E; f)) \otimes \mathbb{Q} \longrightarrow H_n(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f))$$

of Φ' is an isomorphism for $n \geqslant 2$.

Proof. To show that Φ' is a homomorphism for $n \geqslant 2$, let $\alpha, \beta \in \pi_n(\mathrm{Map}_g(E', E; f))$ with adjoints $F, G \colon E' \times S^n \to E$. Let $(F \mid G) \colon E' \times (S^n \vee S^n) \to E$ be the map induced by F and G. We then have a commutative diagram (2) with $Z = S^n \vee S^n$. A Sullivan model for $S^n \vee S^n$ is $(\wedge(u, v)/\langle u^2, uv, v^2\rangle, 0)$ with $|u| = |v| = n$. Applying Proposition 3.3, we see $(F \mid G)$ has the Sullivan model

$$\chi \mapsto \chi + u\theta_a(\chi) + v\theta_b(\chi) \colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes (\wedge(u, v)/\langle u^2, uv, v^2\rangle)$$

for $\chi \in \wedge V \otimes \wedge W$. Using that $(F \mid G) \circ (1_{E'} \times i_1) = F$ and $(F \mid G) \circ (1_{E'} \times i_2) = G$, where $i_j \colon S^n \to S^n \vee S^n$ are the inclusions, yields that θ_a and θ_b are cycle representatives for $\Phi'(\alpha)$ and $\Phi'(\beta)$, respectively. The map $(F \mid G) \circ (1_{E'} \times \sigma)$ is adjoint to the sum $\alpha + \beta$ where $\sigma \colon S^n \to S^n \vee S^n$ is the pinch map. The result now follows from the fact that a Sullivan model for σ is given by

$$u, v \mapsto w \colon \wedge (u, v)/\langle u^2, uv, v^2\rangle \to \wedge(w)/\langle w^2\rangle.$$

Now assume E' is finite. By Proposition 2.1, composition with a rationalization $\ell_E \colon E \to E_{\mathbb{Q}}$ gives a rationalization $\ell_E \colon \mathrm{Map}_g(E', E; f) \to \mathrm{Map}_{g_{\mathbb{Q}}}(E', E_{\mathbb{Q}}; f_{\mathbb{Q}})$.

We take

$$\Phi\colon \pi_n(\mathrm{Map}_{g_{\mathbb{Q}}}(E', E_{\mathbb{Q}}; f_{\mathbb{Q}})) \to H_n(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f))$$

to be the map Φ' corresponding to the diagram (7).

Suppose $\Phi(\alpha) = 0$ for $\alpha \in \pi_n(\mathrm{Map}_{g_{\mathbb{Q}}}(E', E_{\mathbb{Q}}; f_{\mathbb{Q}}))$. Then by Proposition 3.3, a Sullivan model $\mathcal{A}_F$ for the adjoint $F\colon E' \times S^n \to E_{\mathbb{Q}}$ is given by

$$\chi \mapsto \mathcal{A}_f(\chi) + u\theta(\chi)\colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes (\wedge(u)/\langle u^2\rangle),$$

and $\theta = \mathcal{D}(\overline{\theta})$ for some $\overline{\theta} \in \mathrm{Der}^{n+1}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f)$. Define a DG algebra homotopy $\mathcal{H}$ from the map

$$\chi \mapsto \mathcal{A}_f(\chi)\colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes (\wedge(u)/\langle u^2\rangle)$$

to the map $\mathcal{A}_F$ by the rule

$$\chi \mapsto \mathcal{A}_f(\chi) + tu\theta(\chi) + (-1)^n dtu\overline{\theta}(\chi)\colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes (\wedge(u)/\langle u^2\rangle) \otimes \wedge(t, dt).$$

Proposition 3.4 with $Z = S^n$ gives a homotopy $H\colon E' \times S^n \times I \to E_{\mathbb{Q}}$, over $g_{\mathbb{Q}}$ and under $f_{\mathbb{Q}}$, between the adjoint of the trivial class in $\pi_n(\mathrm{Map}_{g_{\mathbb{Q}}}(E', E_{\mathbb{Q}}; f_{\mathbb{Q}}))$ and F. It follows that Φ is injective.

Finally, to prove Φ is onto for $n \geqslant 2$, let $\theta \in \mathrm{Der}^n_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f)$ be a cycle. Define a DG algebra map Γ by

$$\chi \mapsto \mathcal{A}_f(\chi) + u\theta(\chi)\colon \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes C.$$

Using Proposition 3.4 with $Z = S^n$, we obtain a map $F\colon E' \times S^n \to E_{\mathbb{Q}}$ that is under $f_{\mathbb{Q}}$ and over $g_{\mathbb{Q}}$. The adjoint of F is a class $\alpha \in \pi_n(\mathrm{Map}_{g_{\mathbb{Q}}}(E', E_{\mathbb{Q}}; f_{\mathbb{Q}}))$ with $\Phi(\alpha) = \langle\theta\rangle$. □

We note that $\pi_1(\mathrm{Map}_g(E', E; f))$ is not, in general, abelian and so Φ cannot, in general, be an isomorphism (cf. [**22**, Ex. 1.1]). The proof that Φ' is a homomorphism breaks down if $n = 1$ because $Z = S^1 \vee S^1$ is a non-nilpotent space. In fact, Φ is generally not a homomorphism in degree 1.

By Proposition 2.1, $\pi_1(\mathrm{Map}_g(E', E; f)$ is a nilpotent group, and thus, it has a well-defined rank. We have:

Theorem 3.7. *With hypotheses as in Theorem 3.6, if E' is finite, then*

$$\mathrm{rank}(\pi_1(\mathrm{Map}_g(E', E; f))) = \dim(H_1(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W, \wedge V' \otimes \wedge W'; \mathcal{A}_f)).$$

Proof. The proof is an adaptation of [**22**, Th. 1] similar to the preceding result. Here we use a Moore-Postnikov factorization of $p_{\mathbb{Q}}\colon E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ in place of the absolute Postnikov tower of Y for $\mathrm{Map}(X, Y; f)$ used there. Also, we use Theorem 3.6 in place of [**21**, Th. 2.1]. We omit the details. □

4. The rational Samelson Lie Algebra of $\mathrm{Aut}(p)_\circ$

In this section, we sharpen Theorem 3.6 in the case f and g are the respective identity maps to prove Theorem 1.1. Fix a fibration $p\colon E \to B$ of simply connected

CW complexes with E finite. Observe $\mathrm{Aut}(p)_\circ = \mathrm{Map}_{1_B}(E, E; 1_E)$. We prove the map $\Phi'\colon \pi_1(\mathrm{Aut}(p)_\circ) \to H_1(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W))$ induces an isomorphism after rationalization. We then show Φ' induces an isomorphism

$$\Phi\colon \pi_*(\mathrm{Aut}(p)_\circ) \otimes \mathbb{Q} \to H_*(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W))$$

of graded Lie algebras.

Let $\alpha \in \pi_p(\mathrm{Aut}(p)_\circ)$ and $\beta \in \pi_q(\mathrm{Aut}(p)_\circ)$ be homotopy classes with adjoints $F\colon E \times S^p \to E$ and $G\colon E \times S^q \to E$. Let

$$\theta_a \in \mathrm{Der}^p_{\wedge V}(\wedge V \otimes \wedge W) \quad \text{and} \quad \theta_b \in \mathrm{Der}^q_{\wedge V}(\wedge V \otimes \wedge W)$$

be cycle representatives for $\Phi'(\alpha)$ and $\Phi'(\beta)$, respectively. Define a homotopy class $\alpha * \beta \in [S^p \times S^q, \mathrm{Aut}(p)_\circ]$ to be the composite

$$S^p \times S^q \xrightarrow{\alpha\times\beta} \mathrm{Aut}(p)_\circ \times \mathrm{Aut}(p)_\circ \xrightarrow{\mu} \mathrm{Aut}(p)_\circ,$$

where μ is the multiplication (composition of maps) in $\mathrm{Aut}(p)_\circ$. Write

$$F * G\colon E \times S^p \times S^q \to E$$

for the adjoint map. Let $(\wedge(u, v)/\langle u^2, v^2\rangle, 0)$ denote the Sullivan model for $S^p \times S^q$, where $|u| = p$ and $|v| = q$.

Lemma 4.1. *A Sullivan model for the map $F * G$ is the DG algebra map*

$$\mathcal{A}_{F*G}\colon \wedge V \otimes \wedge W \to \wedge V \otimes \wedge W \otimes \wedge(u, v)/\langle u^2, v^2\rangle$$

given by

$$\mathcal{A}_{F*G}(\chi) = \chi + u\theta_a(\chi) + v\theta_b(\chi) + uv\theta_a \circ \theta_b(\chi)$$

for $\chi \in \wedge V \otimes \wedge W$.

Proof. The map $F * G$ is the composite

$$E \times S^p \times S^q \xrightarrow{F\times 1_{S^q}} E \times S^q \xrightarrow{E} .$$

By the Künneth Theorem, a Sullivan model $\mathcal{A}_{F\times 1_{S^q}}$ for the product $F \times 1_{S^q}$ is the product of the Sullivan models

$$\mathcal{A}_F\colon \wedge V \otimes \wedge W \to \wedge V \otimes \wedge W \otimes (\wedge(u)/\langle u^2\rangle) \quad \text{and} \quad 1\colon \wedge(v)/\langle v^2\rangle \to \wedge(v)/\langle v^2\rangle.$$

Thus, given $\chi \in \wedge V \otimes \wedge W$, we see:

$$\begin{aligned}\mathcal{A}_{F*G}(\chi) &= \mathcal{A}_{F\times 1_{S^q}}(\mathcal{A}_G(\chi)) \\ &= \mathcal{A}_{F\times 1_{S^q}}(\chi + v\theta_b(\chi)) \\ &= \mathcal{A}_F(\chi) + \mathcal{A}_F(v\theta_b(\chi)) \\ &= \chi + u\theta_a(\chi) + v\theta_b(\chi) + uv\theta_a(\theta_b(\chi)).\end{aligned}$$ □

We can now extend Theorem 3.6 to the fundamental group for the monoid $\mathrm{Aut}(p)_\circ$.

Theorem 4.2. *Let $p\colon E \to B$ be a fibration of simply connected* CW *complexes with E finite. Then the map*

$$\Phi'\colon \pi_1(\mathrm{Aut}(p)_\circ) \to H_1(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W))$$

is a homomorphism inducing an isomorphism

$$\Phi\colon \pi_1(\mathrm{Aut}(p)_\circ) \otimes \mathbb{Q} \xrightarrow{\cong} H_1(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W)).$$

Proof. To prove Φ' is a homomorphism, let $\alpha, \beta \in \pi_1(\mathrm{Aut}(p)_\circ))$ and recall the basic identity

$$\alpha \cdot \beta = (\alpha * \beta) \circ \Delta\colon S^1 \to \mathrm{Aut}(p)_\circ,$$

where Δ is the diagonal map and the left-hand product is the usual multiplication in the fundamental group. Thus the adjoint to $\alpha \cdot \beta$ is the composition

$$S^1 \times E \xrightarrow{\Delta \times 1_E} S^1 \times S^1 \times E \xrightarrow{F * G} E.$$

The result now follows directly from Lemma 4.1 and the fact that a Sullivan model for Δ is the map $\wedge(u, v) \to \wedge(w)$ given by $u, v \mapsto w$. The proof that Φ is a bijection is now the same as in the proof in Theorem 3.6. $\square$

We next prove that the map $\Phi'\colon \pi_*(\mathrm{Aut}(p)_\circ) \to H_*(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W))$ preserves brackets, which completes the proof of Theorem 1.1.

Theorem 4.3. *Let $p\colon E \to B$ be a fibration of simply connected* CW *complexes with E finite. Let $\alpha \in \pi_p(\mathrm{Aut}(p)_\circ)$ and $\beta \in \pi_q(\mathrm{Aut}(p)_\circ)$. Then*

$$\Phi'([\alpha, \beta]) = [\Phi'(\alpha), \Phi'(\beta)]\,,$$

where the left-hand bracket is the Samelson product in $\pi_*(\mathrm{Aut}(p)_\circ)$ and the right-hand bracket is that induced by the commutator in $\mathrm{Der}^*_{\wedge V}(\wedge V \otimes \wedge W)$.

Proof. By Proposition 2.2, $\mathrm{Aut}(p)_\circ$ is a grouplike space under composition. Let $\nu\colon \mathrm{Aut}(p)_\circ \to \mathrm{Aut}(p)_\circ$ be a homotopy inverse. Define $\overline{F}$ to be the composite

$$\overline{F}\colon E \times S^p \xrightarrow{1_E \times \alpha} E \times \mathrm{Aut}(p)_\circ \xrightarrow{1_E \times \nu} E \times \mathrm{Aut}(p)_\circ \xrightarrow{\omega} E,$$

where ω is the evaluation map. Then $\overline{F}$ is adjoint to $-\alpha = \nu_\sharp(\alpha) \in \pi_p(\mathrm{Aut}(p)_\circ)$ and so has the Sullivan model

$$\mathcal{A}_{\overline{F}}(\chi) = \chi - u\theta_a(\chi)\colon\ \wedge V \otimes \wedge W \to \wedge V \otimes \wedge W \otimes (\wedge(u)/\langle u^2\rangle)$$

for $\chi \in \wedge V \otimes \wedge W$, where $\langle \theta_a \rangle = \Phi'(\alpha)$.

Now given two classes $\alpha \in \pi_p(\mathrm{Aut}(p)_\circ)$ and $\beta \in \pi_q(\mathrm{Aut}(p)_\circ)$, the Samelson product is defined by means of the map $\gamma\colon S^p \times S^q \to \mathrm{Aut}(p)_\circ$ defined by

$$\gamma(x, y) = \alpha(x) \circ \beta(y) \circ \overline{\alpha(x)} \circ \overline{\beta(y)},$$

where $\overline{\alpha} = \nu_\sharp(\alpha)$. The map γ has adjoint Γ given by the composite

$$\begin{tikzcd}
E \times S^p \times S^q \arrow[r, "1_E \times \Delta \times \Delta"] \arrow[ddr, "\Gamma"'] & E \times S^p \times S^p \times S^q \times S^q \arrow[d, "1_E \times 1_{S^p} \times T \times 1_{S^q}"] \\
& E \times S^p \times S^q \times S^p \times S^q \arrow[d, "{[F,G]}"] \\
& E,
\end{tikzcd}$$

where T is transposition and

$$[F,G] = F \circ (G \times 1_{S^n}) \circ (\overline{F} \times 1_{S^p \times S^q}) \circ (\overline{G} \times 1_{S^p \times S^q \times S^p})\,.$$

We see the map $[F,G]$ has the Sullivan model

$$\mathcal{A}_{[F,G]} \colon \wedge V \otimes \wedge W \to \wedge V \otimes \wedge W \otimes (\wedge(u, v, \overline{u}, \overline{v})/\langle u^2, v^2, \overline{u}^2, \overline{v}^2\rangle),$$

given by

$$\begin{aligned}
\mathcal{A}_{[F,G]}(\chi) = \chi &+ u\theta_a(\chi) + v\theta_b(\chi) - \overline{u}\theta_a(\chi) - \overline{v}\theta_b(\chi) \\
&+ uv\theta_a \circ \theta_b(\chi) + \overline{u}\overline{v}\theta_a \circ \theta_b(\chi) - u\overline{v}\theta_a \circ \theta_b(\chi) - v\overline{u}\theta_b \circ \theta_a(\chi) \\
&+ \text{terms involving } u\overline{u} \text{ or } v\overline{v}.
\end{aligned}$$

Thus Γ has the Sullivan model

$$\mathcal{A}_\Gamma(\chi) = \chi + uv[\theta_a, \theta_b](\chi) \colon \wedge V \otimes \wedge W \to \wedge V \otimes \wedge W \otimes \wedge(u, v)/\langle u^2, v^2\rangle,$$

for $\chi \in \wedge V \otimes \wedge W$. The result now follows from the definition of the Samelson product: The restriction of Γ to $E \times (S^p \vee S^q)$ is null and so Γ induces $\Gamma' \colon E \times S^{p+q} \to E$ satisfying $\Gamma \simeq \Gamma' \circ (1_E \times q)$, where $q \colon S^p \times S^q \to S^{p+q}$ is the projection onto the smash product. Finally, using the fact that q induces the map

$$w \mapsto uv \colon \wedge(w)/\langle w^2\rangle \to \wedge(u, v)/\langle u^2, v^2\rangle,$$

we see $\Phi'(\gamma)$ is represented by $[\theta_a, \theta_b]$. $\square$

Remark 4.4. The techniques above can be applied to describe a related space of fibrewise self-homotopy equivalences. Let $F = p^{-1}(x)$ be the fibre over the basepoint of $p \colon E \to B$ as above. The restriction $\mathrm{res} \colon \mathrm{Aut}(p) \to \mathrm{Aut}(F)$ is a multiplicative continuous map. Denote the kernel by $\mathrm{Aut}^F(p)$. This monoid is of interest in the study of gauge groups. Denote by $\overline{\mathrm{Der}}_{\wedge V}(\wedge V \otimes \wedge W)$ the subcomplex of $\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge W)$ consisting of derivations θ such that $\theta(V) = 0$ and $\theta(W) \subset \wedge^+ V \otimes \wedge W$. Then we have a Lie algebra isomorphism

$$\pi_*(\mathrm{Aut}^F(p)_\circ) \otimes \mathbb{Q} \cong H_*(\overline{\mathrm{Der}}_{\wedge V}(\wedge V \otimes \wedge W)).$$

The details of the proof are extensive but entirely similar to the above.

5. Applications and examples

We apply Theorem 1.1 to expand on some of the examples mentioned in Section 2. We begin with the path-space fibration $p \colon PB \to B$ as discussed in Example 2.4. Let

B be a simply connected CW complex. As in [**11**, §16b], p has a relative minimal model of the form

$$I\colon (\wedge V, d) \to (\wedge V \otimes \wedge \overline{V}, D)$$

where, as usual, $(\wedge V, d)$ is the Sullivan minimal model for B. Here $\overline{V}$ is the desuspension of V, $(\overline{V})^n \cong V^{n+1}$. We recover part of the main result of [**10**].

Theorem 5.1 (Félix-Halperin-Thomas). *Let B be a simply connected* CW *complex and $p\colon PB \to B$ the path-space fibration. Then there is an isomorphism of graded Lie algebras*

$$\pi_*(\Omega B) \otimes \mathbb{Q} \cong H_*(\mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge \overline{V})).$$

Proof. The result is a direct consequence of Example 2.4 and Theorem 1.1. □

We next give a general bound for $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{Aut}(p)_\circ)$ in terms of the rational homotopy groups of the fibre.

Theorem 5.2. *Let $p\colon E \to B$ be a fibration of simply connected* CW *complexes with E finite. Let $F = p^{-1}(*)$ be the fibre. Then*

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{Aut}(p)_\circ) \leqslant \mathrm{card}\{n \mid \pi_n(F) \otimes \mathbb{Q} \neq 0\}.$$

Proof. A non-trivial commutator $[\theta_1, \ldots, [\theta_{k-1}, \theta_k]]$ in $\mathrm{Der}^*_{\wedge V}(\wedge V \otimes \wedge W)$ gives rise to a sequence $w_1, \ldots, w_k$ of vectors in W with degrees $n_1 < \cdots < n_k$. To see this, choose w_k so that $[\theta_1, \ldots, [\theta_{k-1}, \theta_k]](w_k) \neq 0$. Choose a non-vanishing summand $\theta_{i_1} \circ \cdots \circ \theta_{i_k}(w_k) \neq 0$. Choose w_{k-1} so that w_{k-1} appears in $\theta_{i_k}(w_k)$ and $\theta_{i_1} \circ \cdots \circ \theta_{i_{k-1}}(w_{k-1}) \neq 0$. Proceed by induction. The result then follows from Theorem 1.1. □

Finally, we specialize to a case where we can make a complete calculation of the rational H-type of $\mathrm{Aut}(p)_\circ$.

Theorem 5.3. *Let $p\colon E \to B$ be a fibration with fibre $F = S^{2n+1}$ and E and B simply connected* CW *complexes with E finite. Suppose $i_\sharp\colon \pi_{2n+1}(S^{2n+1}) \to \pi_{2n+1}(E)$ is injective. Then* $\mathrm{Aut}(p)_\circ$ *is rationally homotopy abelian and, for each $q \geqslant 1$, we have isomorphisms*

$$\pi_q(\mathrm{Aut}(p)_\circ) \otimes \mathbb{Q} \cong H_{2n+1-q}(B; \mathbb{Q}).$$

Proof. Let $(\wedge V, d) \to (\wedge V \otimes \wedge(u), D)$ be a relative minimal model for p. A derivation $\theta \in \mathrm{Der}_{\wedge V}(\wedge V \otimes \wedge(u))$ is determined by the element $\theta(u) \in (\wedge V)^{2n+1-q}$. The derivation θ is a cocycle (resp. a coboundary) if $\theta(u)$ is a cocycle (resp. a coboundary). Further, the commutator bracket of any two such derivations is directly seen to be trivial. The result thus follows from Theorem 1.1. □

6. On the group of components of $\mathrm{Aut}(p)$

As mentioned in the introduction, the group $\mathcal{E}(p) = \pi_0(\mathrm{Aut}(p))$ of path components of $\mathrm{Aut}(p)$ does not generally localize well. First, $\mathcal{E}(p)$ is often non-nilpotent. Even when $\mathcal{E}(p)$ is nilpotent, it may not satisfy $\mathcal{E}(p)_{\mathbb{Q}} = \mathcal{E}(p_{\mathbb{Q}})$ — take $p\colon S^n \to *$, for an easy example. However, by work of Dror-Zabrodsky [**9**] and Maruyama (e.g., [**23**]), various natural subgroups of $\mathcal{E}(p)$ are nilpotent and do localize well. We consider one such example which we denote $\mathcal{E}_\sharp(p)$. We note that different versions of groups of fibrewise equivalences have also been considered [**12, 13, 16**]. Our definition of $\mathcal{E}_\sharp(p)$ is chosen to allow for an identification of $(\mathcal{E}_\sharp(p))_\mathbb{Q}$ in the spirit of Theorem 1.1.

Suppose, as usual, that $p\colon E \to B$ is a fibration of simply connected CW complexes with E finite. Then any fibrewise self-map of E is fibrewise homotopic to a based self-map. Further, any two based fibrewise self-maps of E that are freely fibrewise homotopic are also based fibrewise homotopic. Thus, in what follows, we work with based fibre-self-equivalences of $p\colon E \to B$.

Consider the long exact sequence of the fibration

$$\cdots \to \pi_{i+1}(B) \xrightarrow{\partial} \pi_i(F) \xrightarrow{j_\#} \pi_i(E) \xrightarrow{p_\#} \pi_i(B) \to \cdots .$$

Each $f \in \mathrm{Aut}(p)$ induces an automorphism $f_\#$ of this sequence, which is the identity on $\pi_*(B)$ and on $\mathrm{im}\,\partial \subseteq \pi_*(F)$. Thus $f_\#$ induces an automorphism of cokernels

$$\overline{f_\#}\colon \frac{\pi_*(F)}{\mathrm{im}\,\partial} \to \frac{\pi_*(F)}{\mathrm{im}\,\partial}.$$

Define $\mathcal{E}_\#(p)$ to be the group of based homotopy classes of based equivalences in $\mathrm{Aut}(p)$ that induce the identity on the cokernels above, through the dimension of E. Notice that this reduces to the subgroup of classes that induce the identity on $\pi_*(F)$ through degree equal to the dimension of E if the fibration is "Whitehead trivial," i.e., if the connecting homomorphism ∂ is trivial (through the dimension of E) as considered in [**12**]. When p is trivial, it reduces to the subgroup $\mathcal{E}_\sharp(E)$ of classes that induce the identity on $\pi_*(E)$ through degree equal to the dimension of E as in [**23**].

Theorem 6.1. *Let $p\colon E \to B$ be a fibration of simply connected* CW *complexes with E finite. Then $\mathcal{E}_\sharp(p)$ is a nilpotent group. Given any set of primes $\mathbb{P}$ we have*

$$\mathcal{E}_\#(p)_\mathbb{P} \cong \mathcal{E}_\#(p_\mathbb{P}).$$

Proof. Observe $\mathcal{E}_\sharp(p)$ acts nilpotently on the normal chain

$$0 \lhd \mathrm{im}\,\partial \lhd \pi_*(F)$$

through the dimension of E. That $\mathcal{E}_\sharp(p)$ is a nilpotent group follows from [**9**]. The result on localization is now obtained by adjusting Maruyama's argument [**23**] to the fibrewise setting by replacing the use of a Postnikov decomposition of E with Moore-Postnikov decomposition of $p\colon E \to B$. Compare [**16**, Th. 1.5]. $\square$

By virtue of Theorem 6.1, we may identify $\mathcal{E}_\#(p)_\mathbb{Q}$ in terms of certain automorphisms of the Sullivan model of E. As usual, let $(\wedge V, d) \to (\wedge V \otimes \wedge W, D)$ denote the minimal model of p. Let $D_0\colon W \to V$ be the linear part of D, and decompose W as $W = W_0 \oplus W_1$ with $W_0 = \ker D_0$ and W_1 a complement. Topologically, W_1 may be identified with the dual of $\mathrm{im}\,\partial = \ker j_\sharp$ and W_0 with the dual of $\mathrm{coker}\,\partial = \mathrm{im}\, j_\sharp$ after rationalization. Write $\mathrm{Aut}_\sharp(\wedge V \otimes \wedge W)$ for the group of automorphisms φ of $\wedge V \otimes W$, which are the identity on $\wedge V$ and have linear part $\varphi_0\colon W \to V \oplus W$ satisfying $(\varphi_0 - 1)(W_0) \subseteq V$ and $(\varphi_0 - 1)(W_1) \subseteq V \oplus W_0$. As in [**2**], we may identify $\mathcal{E}_\#(p_\mathbb{Q})$ with the group of homotopy classes of $\mathrm{Aut}_\sharp(\wedge V \otimes \wedge W)$.

Recall from the introduction that $\mathrm{Der}^0_{\#}(\wedge V \otimes \wedge W)$ denotes the vector space of derivations θ of degree 0 of $\wedge V \otimes \wedge W$ that satisfy $\theta(V) = 0$, $\mathcal{D}(\theta) = 0$, $\theta(W_0) \subset V \oplus \wedge^{\geqslant 2}(V \oplus W)$, and $\theta(W_1) \subset V \oplus W_0 \oplus \wedge^{\geqslant 2}(V \oplus W)$. The derivation differential defines a linear map

$$\mathcal{D}\colon \mathrm{Der}^1_{\wedge V}(\wedge V \otimes \wedge W) \to \mathrm{Der}^0_{\#}(\wedge V \otimes \wedge W), \qquad \theta \mapsto \mathcal{D}(\theta) = D\theta + \theta D.$$

We are writing

$$H_0(\mathrm{Der}_{\sharp}(\wedge V \otimes \wedge W)) = \mathrm{coker}, \mathcal{D}.$$

As in [**26**, Prop. 12] and [**28**, §11], the correspondence $\theta \mapsto e^\theta$ gives a bijection (with inverse $\varphi \mapsto \log(\varphi)$) from $\mathrm{Der}^0_{\#}(\wedge V \otimes \wedge W)$ to $\mathrm{Aut}_{\sharp}(\wedge V \otimes \wedge W)$. Under this bijection, compositions of automorphisms $e^\theta \circ e^\phi$ correspond to "Baker-Campbell-Hausdorff" products of derivations; i.e.,

$$\log(e^\theta \circ e^\phi) = \theta + \phi + \frac{1}{2}[\theta, \phi] + \frac{1}{12}[\theta, [\theta, \phi]] + \cdots .$$

Theorem 6.2. *Let $p\colon E \to B$ be a fibration of simply connected* CW *complexes with E finite. The assignment $\theta \mapsto e^\theta$ induces an isomorphism of groups*

$$\Psi\colon H_0(\mathrm{Der}_{\sharp}(\wedge V \otimes \wedge W)) \to \mathcal{E}_{\#}(p_{\mathbb{Q}}),$$

where Baker-Campbell-Hausdorff composition of derivations is understood in the left-hand term.

Proof. To see that Ψ is well-defined, we observe that the assignment $\theta \mapsto e^\theta$ restricts to a correspondence between boundaries in $\mathrm{Der}^0_{\sharp}(\wedge V \otimes \wedge W)$ and automorphisms in $\mathrm{Aut}_{\sharp}(\wedge V \otimes \wedge W)$ homotopic to the identity there. Suppose $\theta = \mathcal{D}(\theta_1)$ with $\theta_1 \in$, $\mathrm{Der}^1_{\wedge V}(\wedge V \otimes \wedge W)$. Then we define a derivation θ_2 of $\wedge V \otimes \wedge W \otimes \wedge(t, dt)$ by $\theta_2(V) = \theta_2(t) = \theta_2(dt) = 0$ and $\theta_2(w) = t\theta_1(w)$ for $w \in W$. The map

$$\mathcal{H} = e^{\mathcal{D}(\theta_2)}\colon \wedge V \otimes \wedge W \to \wedge V \otimes \wedge W \otimes \wedge(t, dt)$$

satisfies $p_1 \circ \mathcal{H} = e^\theta$ and $p_0 \circ \mathcal{H}$ is the identity. Conversely, a given homotopy $\mathcal{H}\colon \wedge V \otimes \wedge W \to \wedge V \otimes \wedge W \otimes \wedge(t, dt)$ between the identity and φ in $\mathrm{Aut}_{\sharp}(\wedge V \otimes \wedge W)$ may be chosen to be fibrewise in the sense of Proposition 3.4. Taking the dt component of $\log(\mathcal{H})$, we obtain a derivation $\theta \in \mathrm{Der}^1_{\wedge V}(\wedge V \otimes \wedge W)$ with $\mathcal{D}(\theta) = \log(\varphi)$. The proof that the induced map Ψ is a bijection now follows the same line as the proof of Theorem 1.2. □

We remark that this result may be used to analyze the nilpotency of $\mathcal{E}_{\#}(p_{\mathbb{Q}})$, and ultimately, since we have $\mathcal{E}_{\#}(p_{\mathbb{Q}}) \cong \mathcal{E}_{\#}(p)_{\mathbb{Q}}$, that of $\mathcal{E}_{\#}(p)$, in terms of bracket lengths in $H_0(\mathrm{Der}_{\#}(\wedge V \otimes \wedge W))$.

Finally, observe that the results of this section may also be applied to give cardinality results for the order of $\mathcal{E}(p)$. As a simple example, let $\pi\colon B \times S^n \to B$ be a trivial fibration. If $H^n(B; \mathbb{Q}) = 0$, then $\mathcal{E}_{\sharp}(\pi)$ is a finite group. On the other hand, denote by V^n the dual of the image of the rational Hurewicz map in $H^n(B; \mathbb{Q})$. Then, clearly $\mathcal{E}_{\#}(\pi_{\mathbb{Q}}) \cong H_0(\mathrm{Der}_{\sharp}(\wedge V \otimes \wedge W)) \cong V^n$. Thus if V^n is non-trivial, then $\mathcal{E}(\pi)$ is infinite in this case. We conclude with one example involving a non-trivial fibration.

Example 6.3. Let $p\colon S^7 \times S^3 \to S^4$ be the composition $p = \eta \circ p_1$, where η denotes the Hopf map and p_1 projection onto the first factor. Then the fibre is $S^3 \times S^3$. A relative minimal model for p is given by

$$\wedge(v_4, v_7) \to \wedge(v_4, v_7) \otimes \wedge(w_3, w_3') \to \wedge(w_3, w_3'),$$

with (non-minimal) differential given by $D(v_7) = v_4^2$, $D(w_3') = v_4$ and $D = 0$ on other generators. With reference to our notation above, we have $V = \langle v_4, v_7\rangle$, $W_0 = \langle w_3\rangle$, and $W_1 = \langle w_3'\rangle$. Define a differential $\theta \in \mathrm{Der}^0_{\#}(\wedge V \otimes \wedge W)$ by setting $\theta(w_3') = w_3$, and $\theta = 0$ on other generators. A direct check shows that θ represents a non-zero class in $H_0(\mathrm{Der}_\sharp(\wedge V \otimes \wedge W))$. From Theorem 6.3 and the discussion of this section, it follows that $\mathcal{E}_{\#}(p)$ has infinite order.

Under the isomorphism of Theorem 6.3, the derivation θ evidently corresponds to the automorphism φ of $\wedge V \otimes \wedge W$ given by $\varphi(w_3') = w_3' + w_3$ and $\varphi = 1$ on other generators. Notice that this automorphism does not correspond to an element of $\mathcal{E}_{\#}(F)$—in the "non-fibrewise" notation of [**9**]. Also, since in this case the total space has the rational homotopy type of $S^3 \times S^7$, we see that φ does not correspond to an element of $\mathcal{E}_{\#}(E)$, which is rationally trivial. We note the automorphism ϕ is induced by the map

$$(x, y) \mapsto (x, xy)\colon S^3 \times S^7 \to S^3 \times S^7 \quad \text{for } x \in S^3, y \in S^7.$$

The example thus demonstrates that, in general, it is necessary to consider the relative minimal model of a fibration and not just the minimal models of the spaces involved to analyze $\mathcal{E}_{\#}(p)$ rationally.

References

[**1**] M. Arkowitz and C.R. Curjel, Homotopy commutators of finite order. II, *Quart. J. Math. Oxford Ser. (2)* **15** (1964), no. 1, 316–326

[**2**] M. Arkowitz and G. Lupton, On the nilpotency of subgroups of self-homotopy equivalences, in Algebraic topology: new rends in localization and periodicity (Sant Feliu de Guíols, 1994), Progr. Math. 1–22, **136**, Birkhäuser, Basel, 1996.

[**3**] H.J. Baues, *Algebraic homotopy*, Cambridge Studies in Advanced Mathematics **15**, Cambridge University Press, Cambridge, 1989.

[**4**] I. Berstein and T. Ganeaa, Homotopical nilpotency, *Illinois J. Math.* **5** (1961), no. 1,99–130.

[**5**] J. Block and A. Lazarev, André-Quillen cohomology and rational homotopy of function spaces, *Adv. Math.* **193** (2005), no. 1, 18–39.

[**6**] P. Booth, P. Heath, C. Morgan and R. Piccinini, *H*-spaces of self-equivalences of fibrations and bundles, *Proc. London Math. Soc. (3)* **49** (1984), no. 1, 111–127.

[**7**] U. Buijs and A. Murillo, The rational homotopy Lie algebra of function spaces, *Comment. Math. Helv.* **83** (2008), no. 4, 723–739.

[**8**] A. Dold, Partitions of unity in the theory of fibrations, *Ann. of Math. (2)* **78** (1963), no. 2, 223–255.

[9] E. Dror and A. Zabrodsky, Unipotency and nilpotency in homotopy equivalences, *Topology* **18** (1979), no. 3, 187–197.

[10] Y. Félix, S. Halperin and J.-C. Thomas, Sur certaines algébres de Lie de dérivations, *Annales Institut Fourier* **32** (1982), no. 4, 143–150.

[11] Y. Félix, S. Halperin and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Math. **205**, Springer-Verlag, New York, 2001.

[12] Y. Félix and A. Murillo, A note on the nilpotency of subgroups of self-homotopy equivalences, *Bull. London Math. Soc.* **29** (1997), no. 4, 486–488.

[13] Y. Félix and J.-C. Thomas, Nilpotent subgroups of the group of fibre homotopy equivalences, *Publ. Mat.* **39** (1995), no. 1, 95–106.

[14] S. Halperin, Finiteness in the minimal models of Sullivan, *Trans. Amer. Math. Soc.* **230** (1977), 173–199.

[15] M. Fuchs, Verallgemeinerte Homotopie-Homomorphismen und klassifizierende Räume, *Math. Ann.* **161** (1965), 197–230.

[16] A. Garvín, A. Murillo, P. Pavešić and A. Viruel, Nilpotency and localization of groups of fibre homotopy equivalences, *Contemp. Math.* **274** (2001), 145–157.

[17] D.H. Gottlieb, On fibre spaces and the evaluation map, *Ann. of Math. (2)* **87** (1968), no. 1, 42–55.

[18] P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Publishing Co., Amsterdam, 1975.

[19] P.J. Kahn, Some function spaces of CW type, *Proc. Amer. Math. Soc.* **90** (1984), no. 4, 599–607.

[20] G. Lupton, N.C. Phillips, C.L. Schochet and S.B. Smith, Banach algebras and rational homotopy theory, *Trans. Amer. Math. Soc.* **361** (2009), 267–295.

[21] G. Lupton and S.B. Smith, Rationalized evaluation subgroups of a map I: Sullivan models, derivations and G-sequences, *J. Pure and Applied Algebra* **209** (2007), no. 1, 159–171.

[22] G. Lupton and S.B. Smith, Rank of the fundamental group of any component of a function space, *Proc. Amer. Math. Soc.* **135** (2007), no. 8, 2649–2659.

[23] K. Maruyama, Localization of a certain subgroup of self-homotopy equivalences, *Pacific J. Math.* **136** (1989), 293–301.

[24] W. Meier, Rational universal fibrations and flag manifolds, *Math. Ann.* **258** (1982), no. 3, 329–340.

[25] J. Milnor, On spaces having the homotopy type of CW-complex, *Trans. Amer. Math. Soc.* **90** (1959), 272–280.

[26] P. Salvatore, Rational homotopy nilpotency of self-equivalences, *Topology and its Appl.* **77** (1997), no. 1, 37–50.

[27] H. Scheerer, On rationalized H- and co-H-spaces. With an appendix on decomposable H- and co-H-spaces, *Manuscripta Math.* **51** (1985), no. 1–3, 63–87.

[28] D. Sullivan, Infinitesimal computations in topology, *Publ. I.H.E.S.* **47** (1977), 269–331.

[29] G. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math. **61**, Springer-Verlag, New York, 1978.

Yves Félix felix@math.ucl.ac.be

Institut Mathématique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgique

Gregory Lupton G.Lupton@csuohio.edu

Department of Mathematics, Cleveland State University, Cleveland, OH 44115, USA

Samuel B. Smith smith@sju.edu

Department of Mathematics, Saint Joseph's University, Philadelphia, PA 19131, USA