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DIVIDED POWER (CO)HOMOLOGY. PRESENTATIONS OF SIMPLE FINITE DIMENSIONAL MODULAR LIE SUPERALGEBRAS WITH CARTAN MATRIX

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(*communicated by Charles A. Weibel*)

Abstract

For modular Lie superalgebras, new notions are introduced: Divided power homology and divided power cohomology. For illustration, we explicitly give presentations (in terms of analogs of Chevalley generators) of finite dimensional Lie (super)algebras with indecomposable Cartan matrix in characteristic 2 (and — in the arXiv version of the paper — in other characteristics for completeness of the picture). In the modular and super cases, we define notions of Chevalley generators and Cartan matrix, and an auxiliary notion of the Dynkin diagram. The relations of simple Lie algebras of the A, D, E types are not only Serre ones. These non-Serre relations are same for Lie superalgebras with the same Cartan matrix and any distribution of parities of the generators. Presentations of simple orthogonal Lie algebras having no Cartan matrix (indigenous for characteristic 2) are also given.

To D.B. Fuchs on the occasion of his 70th birthday

1. Introduction

In what follows $\mathbb{K}$ is a field of characteristic $p > 0$, algebraically closed unless otherwise stated. The Lie (super)algebras considered are of finite dimension.

We recall, for clarity, several not well-known facts related to our new results on classification of non-degenerate bilinear forms and Lie (super)algebras preserving them: Lecturing on these results during the past several years we have encountered incredulity of the listeners based on several false premises intermixed with correct statements: "The question sounds classical and so had been solved by classics without doubt (the solution just has to be dug out from paper diluvium)", "this is known

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for quadratic forms (don't you know about Arf invariant?!)", "There are two non-isomorphic types of simple finite orthogonal groups acting on $2n$-dimensional space, so what's new?", and so on.

The problem of describing preserved bilinear forms has two levels: we can consider *linear transformations* (Linear Algebra) and *arbitrary coordinate changes* (Differential Geometry). In the literature, both levels are completely investigated, except for the case where $p = 2$. More precisely, the fact that the non-split and split forms of the Lie algebras that preserve the symmetric bilinear forms are not always isomorphic was never mentioned. Although known for the Chevalley groups preserving these forms, cf. [**St**], these facts do not follow from each other since there is no analog of Lie theorem on the correspondence between Lie groups and Lie algebras. Here we consider the Linear Algebra aspect, for the Differential Geometry related to the objects studied here, see [**Le2**].

1.1. Divided power (co)homology

Over $\mathbb{K}$, the notion of Lie **super**algebra (co)homology obtains one more dimension — the shearing parameter $\underline{N}$. Indeed, since the (co)chains, with trivial coefficients and differential forgotten, form a supercommutative superalgebra — an analog of the polynomial superalgebra (with values in a module for non-trivial coefficients), and the polynomial algebra has divided power analogs in the modular case, so does Lie **super**algebra (co)homology. For Lie algebras, this phenomenon does not exists since the supercommutative superalgebra of polynomials is generated by purely odd elements only.

This being the main idea, the only thing to do is to define the differentials. The appropriate definitions are given in the text and even implemented in the package `SuperLie`, see [**Gr**].

For an illustration, we consider defining relations (here), deformations of (finite dimensional) Lie superalgebras with indecomposable Cartan matrix and of queer series (in [**BGL2**]); in the sequels (in preparation) we consider deformations of their representations. In **these** problems, the effect of divided power (co)homology is only visible for $p = 2$. For completeness, however, the arXiv version of the paper (`arXiv:0911.0243`) contains presentations of all new (previously not covered in the literature) cases of presentations of finite dimensional Lie superalgebras with indecomposable Cartan matrix for $p > 2$.

1.2. Presentations of simple Lie (super)algebras: Overview

• **Over** $\mathbb{C}$, the most studied type of simple Lie algebras are finite dimensional ones and the $\mathbb{Z}$-graded of polynomial growth. The latter type splits into (twisted) loop algebras, vectorial Lie algebras (with polynomial coefficients) and Witt algebra (the vectorial Lie algebra with Laurent polynomials as coefficients).

The finite dimensional and (twisted) loop algebras can be defined by means of their Cartan matrix and Chevalley generators (we recall these notions in what follows). The explicit presentation was first published by Serre and certain relations, whose sufficiency was most difficult to prove, are referred to as *Serre relations*.

For simple vectorial Lie algebras, it was not even clear (until implicitly by V. Ufnarovsky in late 1970s for some cases) if they were finitely presented; for the explicit presentations eventually obtained, and references, see [**GLP**], where the Lie superalgebras are also considered. The relations are passable for a computer, but rather ugly for humans; the only message we can deduce from their description at the moment is that, in addition to the relations in the linear part of the vectorial Lie algebra, there are only finitely many relations (less than 10 for any type of the algebras).

In the super case, in addition to the Lie superalgebras of vectorial type and those with Cartan matrix, there are also the queer series whose presentation is clear in principle, but whose explicit form is even less appealing than that of vectorial Lie (super)algebras, see [**LSe**].

• **Over** $\mathbb{K}$, the vectorial Lie (super)algebras acquire one more parameter (a shearing vector $\underline{N}$, see (13)) and even the description of generators becomes too complicated, to say nothing of relations. For the *restricted case* and *sufficiently large characteristic* and dimension of the space on which the vectorial Lie algebra is realized, the answer is identical to that obtained in [**GLP**].

The Lie (super)algebra with more roots of one sign (say, positive than negative) is said to be *skew-symmetric* and *symmetric* otherwise.[1] For vectorial Lie (super)algebras, as well as for queer Lie superalgebras, the general picture of their presentations is clear and as long as the explicit answer is not really needed (as in [**LSg**], where somewhat awful relations found in [**GL1**] are used), we see no point in deriving it. In every particular case, it is easy by means of `SuperLie` [**Gr**] to anybody capable to use *Mathematica*.

For $p = 2$, several more types of simple Lie (super)algebras appear: Symmetric but without Cartan matrix (such as $\mathfrak{o}_I(n)$ and $\mathfrak{q}(\mathfrak{o}_I(n))$, see [**LeD**]), various deforms of the above-listed types. So, we arrive at the last cases left:

1. Lie (super)algebras with Cartan matrix;
2. Lie (super)algebras without Cartan matrix but not of vectorial type.

In this paper, we consider case 1 (and a series of examples of case 2: The Lie algebras[2] $\mathfrak{o}_I^{(1)}(2n)$). The first thing to do is to define the basic notions sufficiently clear: Unlike humans, computers can not work otherwise whereas we can not write this text without computer's assistance.

1.3. Lie superalgebras with Cartan matrix

The classification of *finite dimensional modular Lie algebras with indecomposable Cartan matrix* over $\mathbb{K}$ was obtained in [**WK**] with a gap corrected in [**Br3, Sk1**] (not even mentioned in [**KWK**]). Although in [**WK**] some notions used in the description of the classification were left undefined, the strategy was impeccable. In [**BGL**], we clarified the notions left somewhat vague during the time elapsed since publication of [**WK**] (Cartan matrix, restrictedness, Dynkin diagram) and superized them, as well as the key notion — that of Lie superalgebra — for the case where $p = 2$. Following ideas of Weisfeiler and Kac [**WK**], and with the help of `SuperLie` package [**Gr**], we

[1] For $p = 2$, there are simple skew-symmetric Lie (super)algebras distinct from vectorial Lie (super)algebras.

[2] The derived of $\mathfrak{g}(A)$ (or any other algebra with a commentary in parentheses like (A) after a "family name" $\mathfrak{g}$) should be denoted $\mathfrak{g}(A)^{(i)}$ but it is usually more convenient to denote it $\mathfrak{g}^{(i)}(A)$ (and similarly treat other commentaries).

classified *finite dimensional modular Lie* **super***algebras with indecomposable Cartan matrix*, see [**BGL**].

If a given indecomposable Cartan matrix A is invertible, the Lie (super)algebra $\mathfrak{g}(A)$ is simple, otherwise $\mathfrak{g}^{(i)}(A)/\mathfrak{c}$ — the quotient of its first (for $i=1$) or second (for $i=2$) derived algebra modulo the center $\mathfrak{c}$ — is simple if size$(A)>1$ (we say that the *size* of an $n\times n$ matrix is equal to n).

The simple Lie algebra $\mathfrak{g}^{(i)}(A)/\mathfrak{c}$ — in what follows in such situation i is equal to 1 or 2 (meaning that the derived series of algebras stabilizes) — does not possess any Cartan matrix although the conventional sloppy practice is to refer to the simple Lie (super)algebra $\mathfrak{g}^{(i)}(A)/\mathfrak{c}$ as "possessing a Cartan matrix".

Elduque interpreted about a dozen of exceptional (when the fact that they are exceptional was only conjectured; now it is proved) simple Lie superalgebras in characteristic 3 [**CE2**] in terms of super analogs of division algebras and collected them into a Supermagic Square (an analog of Freudenthal's Magic Square); the rest of the exceptional examples for $p=3$ and $p=5$, not entering the Elduque[3] Supermagic Square (the ones described for the first time in `arXiv:math/0611391, math/0611392` and [**BGL**]) are, nevertheless, somehow affiliated to the Elduque Supermagic Square [**El3**].

Very interesting, we think, is the situation in characteristic 2. *A posteriori* we see that the list of Lie **super**algebras in characteristic 2 of the form $\mathfrak{g}(A)$ with an indecomposable matrix A is as follows:

*In characteristic 2, take any finite dimensional simple Lie algebra of the form $\mathfrak{g}(A)$ with indecomposable Cartan matrix A ([**WK**]) and declare some of the Chevalley generators of $\mathfrak{g}(A)$ odd (the corresponding diagonal elements of A should be changed accordingly $\bar{0}$ to 0 and $\bar{1}$ to 1, see subsect. 4.5). Do this for each of the inequivalent Cartan matrices of $\mathfrak{g}(A)$ and for any distribution of parities I of the Chevalley generators. Construct the Lie* **super***algebra $s(\mathfrak{g})(A,I)$ from these generators by the rules (25) explicitly described in this paper. For the Lie superalgebra $s(\mathfrak{g})(A,I)$, list all its inequivalent Cartan matrices.*

Such superization may turn a given orthogonal Lie algebra into ortho-orthogonal or periplectic Lie superalgebra; the three exceptional Lie algebras of $\mathfrak{e}$ type turn into seven non-isomorphic Lie superalgebras of $\mathfrak{e}$ type, whereas the $\mathfrak{wk}$ type Lie algebras turn into $\mathfrak{bgl}$ type Lie superalgebras.

The Lie superalgebra $s(\mathfrak{g})(A,I)$ is simple if A is invertible, otherwise pass to $s(\mathfrak{g})(A,I)^{(i)}/\mathfrak{c}$, where i can be equal to 1 or 2. We normalize the Cartan matrix so as to make the parameter I redundant and do not mention it in what follows.

In [**BGL**], we also listed *all* inequivalent Cartan matrices A for each given Lie (super)algebra $\mathfrak{g}(A)$. Although the number of inequivalent Cartan matrices grows with the size of A, it is easy to describe all possibilities for serial Lie (super)algebras. Certain exceptional Lie superalgebras have dozens of inequivalent Cartan matrices; nevertheless, there are several reasons to list all of them: To classify all $\mathbb{Z}$-gradings of a given $\mathfrak{g}(A)$ (in particular, inequivalent Cartan matrices) is a very natural problem. Besides, sometimes the knowledge of the best, for the occasion, $\mathbb{Z}$-grading is

[3]Although the first, as far as we know, superization of Freudenthal's Square was performed by Martinez [**Mz**] (for $p=0$), Elduque went much further. It is instructive, however, to compare the two squares.

important, cf. [**RU**] (all simple roots non-isotropic), [**LSS**] (all simple roots odd); for computations "by hand" the cases where only one simple root is odd are useful. In particular, the defining relations between the natural (Chevalley) generators of $\mathfrak{g}(A)$ are of completely different form for inequivalent $\mathbb{Z}$-gradings and this is used in [**RU**].

1.4. More on motivations

We illustrate our definition of divided power (co)homology with a final result of independent interest — presentations of simple modular Lie superalgebras with Cartan matrix as well as presentations of simple modular Lie algebras with Cartan matrix in the cases neglected so far: for $p = 5$, 3, and 2.

Recently we observe a rise of interest in presentations (by means of generators and defining relations) of simple (and close to simple) Lie (super)algebras occasioned by various applications of this technical result, see [**GL1, Sa, Di, iPR**] and references therein, where presentations in terms of various other types of generators (*Jacobson, Silvester-t'Hooft, extremal*, etc.) are given. Sometimes these other types of generators can be used as an alternative to Chevalley generators; it is desirable, however, to know the situations in which some of them are better (use less time to construct the basis of the algebra they generate) than the others or unavoidable as seems to be the case for Lie algebras of "matrices of complex size" ([**GL1**]). Kornyak compared time needed to present a given simple finite dimensional Lie algebra (over $\mathbb{C}$) in terms of Chevalley generators and Serre relations with same in terms of Jacobson generators and Grozman-Leites relations, see [**GL1**]; the usefulness (in the above sense) of extremal generators [**Di, iPR**] is not yet compared with other presentations, which is a pity: presentation in terms of them is rather cumbersome.

For $p = 2$, non-Serre relations appear even between the Chevalley generators of simple Lie algebras. This is a new result.

Representations of quantum groups — the deforms $U_q(\mathfrak{g})$ of the enveloping algebras — at q equal to a root of unity resemble, even over $\mathbb{C}$, representations of Lie algebras in positive characteristic and this is one more application that brings the modular Lie (super)algebras and an **explicit** form of their presentations to the limelight.

1.5. Disclaimer

Although presentation — description in terms of generators and relations — is one of the accepted ways to represent a given algebra, it seems that an *explicit* form of the presentation is worth the trouble to obtain only if this presentation is often in need, or (which is usually the same) is sufficiently neat. The Chevalley generators of simple finite dimensional Lie algebras over $\mathbb{C}$ satisfy simple and neat relations ("Serre relations") and are often needed for various calculations and theoretical discussions. Relations between their analogs in the super case, although not so neat (certain "non-Serre relations" appear), are still tolerable, at least, for most Cartan matrices.

The defining relations expressed in terms of other generators, different from Chevalley ones, are a bit too complicated to be used by humans and were of academic interest until recently Grozman's package `SuperLie` ([**Gr**]) made the task of finding the explicit expression of the defining relations for many types of Lie algebras and superalgebras a routine exercise for anybody capable to use *Mathematica*.

What we usually need to know about defining relations is that there are finitely many of them; hence the fact that some simple loop superalgebras with Cartan matrix are **not** finitely presentable in terms of Chevalley generators was unexpected (although obvious as an afterthought). The explicit form of defining relations for the dozens or hundreds of systems of simple roots for the Lie superalgebras of e type (for $p = 2$) can be easily obtained using `SuperLie`, whereas for the exceptional simple Lie superalgebras for $p > 2$, it seems natural to list the relations explicitly.

1.6. Main results

The definitions of new and clarification of classical[4] notions, especially, the definition of divided power (co)homology.

We also define *Chevalley generators* and describe presentations of finite dimensional modular Lie (super)algebras of the form $\mathfrak{g}(A)$ and $\mathfrak{g}(A)^{(i)}/\mathfrak{c}$ with indecomposable Cartan matrix A in terms of these generators.

If $p = 2$, the non-Serre defining relations for each Lie superalgebra with indecomposable Cartan matrix are the same as for the Lie algebra with the same (assuming $0 = \bar{0}$ and $1 = \bar{1}$ on the main diagonal) Cartan matrix. (This is proved for the exceptional cases and $\mathfrak{sl}$ series; for the other series this is a conjecture backed up by numerous examples.)

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2. What a Lie superalgebra in characteristic 2 is

Let us give a naive definition of a Lie superalgebra for $p = 2$. (For a scientific one, as a Lie algebra in the category of supervarieties, needed, for example, for a rigorous study and interpretation of odd parameters of deformations, see [**LSh**].) We define a Lie superalgebra as a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that the even part $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, the odd part $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module (made into the two-sided one by symmetry; more exactly, by *anti*-symmetry, but if $p = 2$, it is the same) and on $\mathfrak{g}_{\bar{1}}$ a *squaring* (roughly speaking, the halved bracket) is defined as a map

$$\begin{gathered} x \mapsto x^2 \quad \text{such that } (ax)^2 = a^2x^2 \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } a \in \mathbb{K}, \text{ and} \\ (x+y)^2 - x^2 - y^2 \text{ is a bilinear form on } \mathfrak{g}_{\bar{1}} \text{ with values in } \mathfrak{g}_{\bar{0}}. \end{gathered} \tag{1}$$

(We use a minus sign, so the definition also works for $p \neq 2$.) The origin of this operation is as follows: If $\operatorname{char} \mathbb{K} \neq 2$, then for any Lie superalgebra $\mathfrak{g}$ and any odd

[4] The reader might be interested in related problems, especially those posed by Deligne, see [**LL**].

element $x \in \mathfrak{g}_{\bar{1}}$, the Lie superalgebra $\mathfrak{g}$ contains the element x^2 which is equal to the even element $\frac{1}{2}[x, x] \in \mathfrak{g}_{\bar{0}}$. It is desirable to keep this operation for the case of $p = 2$, but, since it can not be defined in the same way, we define it separately, and then define the bracket of odd elements to be (this equation is valid for $p \neq 2$ as well):

$$[x, y] := (x + y)^2 - x^2 - y^2. \tag{2}$$

We also assume, as usual, that

- if $x, y \in \mathfrak{g}_{\bar{0}}$, then $[x, y]$ is the bracket on the Lie algebra;
- if $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}_{\bar{1}}$, then $[x, y] := l_x(y) = -[y, x] = -r_x(y)$, where l and r are the left and right $\mathfrak{g}_{\bar{0}}$-actions on $\mathfrak{g}_{\bar{1}}$, respectively.

The Jacobi identity involving odd elements now takes the following form:

$$[x^2, y] = [x, [x, y]] \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}, y \in \mathfrak{g}. \tag{3}$$

If $\mathbb{K} \neq \mathbb{Z}/2\mathbb{Z}$, we can replace the condition (3) on two odd elements by a simpler one:

$$[x, x^2] = 0 \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \tag{4}$$

Because of the squaring, the definition of derived algebras should be modified. For any Lie superalgebra $\mathfrak{g}$, set $\mathfrak{g}^{(0)} := \mathfrak{g}$ and

$$\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}] + \text{Span}\{g^2 \mid g \in \mathfrak{g}_{\bar{1}}\}, \quad \mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \text{Span}\{g^2 \mid g \in \mathfrak{g}^{(i)}_{\bar{1}}\}. \tag{5}$$

An even linear map $r\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is said to be a *representation of the Lie superalgebra* $\mathfrak{g}$ (and the superspace V is said to be a $\mathfrak{g}$-*module*) if

$$\begin{array}{l} r([x, y]) = [r(x), r(y)] \quad \text{for any } x, y \in \mathfrak{g}; \\ r(x^2) = (r(x))^2 \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \end{array} \tag{6}$$

2.1. Examples: Lie superalgebras preserving non-degenerate (anti-)symmetric forms

We say that two bilinear forms B and B' on a superspace V are *equivalent* if there is an even invertible linear map $M\colon V \longrightarrow V$ such that

$$B'(x, y) = B(Mx, My) \quad \text{for any } x, y \in V. \tag{7}$$

We fix some basis in V and identify a given bilinear form with its Gram matrix in this basis; we also identify any linear operator on V with its supermatrix in a fixed basis.

Then two bilinear forms (rather supermatrices) are equivalent if and only if there is an even invertible matrix M such that

$$B' = MBM^T, \quad \text{where } T \text{ is for transposition.} \tag{8}$$

A bilinear form B on V is said to be *symmetric* if $B(v, w) = B(w, v)$ for any $v, w \in V$; a bilinear form is said to be *anti-symmetric* if $B(v, v) = 0$ for any $v \in V$.

A homogeneous[5] linear map F is said to preserve a bilinear form B, if[6]

$$B(Fx,y)+(-1)^{p(x)p(F)}B(x,Fy)=0 \quad \text{for any } x,y\in V.$$

All linear maps preserving a given bilinear form constitute a Lie sub(super)algebra $\mathfrak{aut}_B(V)$ of $\mathfrak{gl}(V)$ denoted $\mathfrak{aut}_B(n)\subset\mathfrak{gl}(n)$ in matrix realization and consisting of the supermatrices X such that

$$BX+(-1)^{p(X)}X^{st}B=0,$$

where the *supertransposition* st acts as follows (in the standard format):

$$st\colon \begin{pmatrix}A & B\\ C & D\end{pmatrix}\longrightarrow\begin{pmatrix}A^t & -C^t\\ B^t & D^t\end{pmatrix}.$$

Consider the case of purely even space V of dimension n over a field of characteristic $p\neq 2$. Every non-zero form B can be uniquely represented as the sum of a symmetric and an anti-symmetric form and it is possible to consider automorphisms and equivalence classes of each summand separately.

If the ground field $\mathbb{K}$ of characteristic $p>2$ satisfies[7] $\mathbb{K}^2=\mathbb{K}$, then there is just one equivalence class of non-degenerate symmetric even forms, and the corresponding Lie algebra $\mathfrak{aut}_B(V)$ is called *orthogonal* and denoted $\mathfrak{o}_B(n)$ (or just $\mathfrak{o}(n)$). Non-degenerate anti-symmetric forms over V exist only if n is even; in this case, there is also just one equivalence class of non-degenerate antisymmetric even forms; the corresponding Lie algebra $\mathfrak{aut}_B(n)$ is called *symplectic* and denoted $\mathfrak{sp}_B(2k)$ (or just $\mathfrak{sp}(2k)$). Both algebras $\mathfrak{o}(n)$ and $\mathfrak{sp}(2k)$ are simple.

If $p=2$, the space of anti-symmetric bilinear forms is a subspace of symmetric bilinear forms. Also, instead of a unique representation of a given form as a sum of an anti-symmetric and symmetric form, we have a subspace of symmetric forms and the quotient space of non-symmetric forms; it is not immediately clear what to take for a representative of a given non-symmetric form. For an answer and classification, see Lebedev's thesis [**LeD**] and [**Le1**]. There are no new simple Lie superalgebras associated with non-symmetric forms, so we confine ourselves to symmetric ones.

Instead of orthogonal and symplectic Lie algebras we have two different types of orthogonal Lie algebras (see Theorem 2.2). Either the derived algebras of these algebras or their quotient modulo center are simple if n is large enough, so the canonical expressions of the forms B are needed as a step towards classification of simple Lie algebras in characteristic 2 which is an open problem, and as a step towards a version of this problem for Lie superalgebras, even less investigated.

In [**Le1**], Lebedev showed that, with respect to the above natural equivalence of forms (8), every **even** symmetric non-degenerate form on a superspace of dimension

[5]Hereafter, as always in Linear Algebra in superspaces, all formulas of linear algebra defined on homogeneous elements only are supposed to be extended to arbitrary ones by linearity.

[6]Hereafter, p denotes both parity defining a superstructure and the characteristic of the ground field; the context is, however, always clear.

[7]Aside: We thought that one should require perfectness of $\mathbb{K}$, i.e., $\mathbb{K}^p=\mathbb{K}$ but the referee suggested a simple counterexample for $\mathbb{K}=\mathbb{Z}/3$ with 2 non-equivalent types of non-degenerate symmetric forms. In this paper $\mathbb{K}$ is algebraically closed; over fields algebraically non-closed, there are more types of symmetric forms.

$n_{\bar 0}|n_{\bar 1}$ over a perfect (i.e., such that every element of $\mathbb{K}$ has a square root[8]) field of characteristic 2 is equivalent to a form of the shape (here: $i=\bar 0$ or $\bar 1$ and each n_i may equal to 0),

$$B=\begin{pmatrix}B_{\bar 0}&0\\0&B_{\bar 1}\end{pmatrix},\quad \text{where } B_i=\begin{cases}1_{n_i}&\text{if } n_i \text{ is odd,}\\ \text{either } 1_{n_i} \text{ or } \Pi_{n_i}&\text{if } n_i \text{ is even,}\end{cases}$$

and where

$$\Pi_n=\begin{cases}\begin{pmatrix}0&1_k\\1_k&0\end{pmatrix}&\text{if } n=2k,\\ \begin{pmatrix}0&0&1_k\\0&1&0\\1_k&0&0\end{pmatrix}&\text{if } n=2k+1.\end{cases}$$

(In other words, the bilinear forms with matrices 1_n and Π_n are equivalent if n is odd and non-equivalent if n is even.) The Lie superalgebra preserving B — by analogy with the orthosymplectic Lie superalgebras $\mathfrak{osp}$ in characteristic 0 we call it *ortho-orthogonal* and denote $\mathfrak{oo}_B(n_{\bar 0}|n_{\bar 1})$ — is spanned by the supermatrices which in the standard format are of the form

$$\begin{pmatrix}A_{\bar 0}&B_{\bar 0}C^TB_{\bar 1}^{-1}\\C&A_{\bar 1}\end{pmatrix},\quad \begin{matrix}\text{where } A_{\bar 0}\in\mathfrak{o}_{B_{\bar 0}}(n_{\bar 0}),\ A_{\bar 1}\in\mathfrak{o}_{B_{\bar 1}}(n_{\bar 1}), \text{ and}\\ C \text{ is arbitrary } n_{\bar 1}\times n_{\bar 0} \text{ matrix.}\end{matrix}$$

Since, as is easy to see,

$$\mathfrak{oo}_{\Pi I}(n_{\bar 0}|n_{\bar 1})\simeq\mathfrak{oo}_{I\Pi}(n_{\bar 1}|n_{\bar 0}),$$

we do not have to consider the Lie superalgebra $\mathfrak{oo}_{\Pi I}(n_{\bar 0}|n_{\bar 1})$ separately unless we study Cartan prolongations where the difference between these two incarnations of one algebra is vital: For the one, the prolong is finite dimensional (the automorphism algebra of the $p=2$ analog of the Riemann geometry), for the other one it is infinite dimensional (an analog of the Lie superalgebra of Hamiltonian vector fields).

For an **odd** symmetric form B on a superspace of dimension $(n_{\bar 0}|n_{\bar 1})$ over a field of characteristic 2 to be non-degenerate, we need $n_{\bar 0}=n_{\bar 1}$, and every such form B is equivalent to $\Pi_{k|k}$, where $k=n_{\bar 0}=n_{\bar 1}$, and which is same as Π_{2k} if the superstructure is forgotten. This form is preserved by linear transformations with supermatrices in the standard format of the shape

$$\begin{pmatrix}A&C\\D&A^T\end{pmatrix},\quad \text{where } A\in\mathfrak{gl}(k),\ C \text{ and } D \text{ are symmetric } k\times k \text{ matrices.}\qquad (9)$$

As over $\mathbb{C}$ or $\mathbb{R}$, the Lie superalgebra of linear maps preserving B will be referred to as *periplectic*, as A. Weil suggested, and denoted $\mathfrak{pe}_B(k)$ or just $\mathfrak{pe}(k)$. Note that even the superdimensions of the characteristic 2 versions of the Lie (super)algebras $\mathfrak{aut}_B(k)$ differ from their analogs in other characteristics for both even and odd forms B.

[8]Since $a^2-b^2=(a-b)^2$ if $p=2$, it follows that no element can have two distinct square roots.

Now observe that

The fact that two bilinear forms are inequivalent does not, generally, imply that the Lie (super)algebras that preserve them are not isomorphic. (10)

In [**Le1**], Lebedev proved that for the *non-degenerate symmetric* forms, the implication spoken about in (10) is, however, true (bar a few exceptions), and therefore we have several types of non-isomorphic Lie (super) algebras (except for occasional isomorphisms intermixing the types, e.g., $\mathfrak{oo}_{I\Pi} \simeq \mathfrak{oo}_{\Pi I}$ and $\mathfrak{oo}_{\Pi\Pi}^{(1)}(6|2) \simeq \mathfrak{pe}^{(1)}(4)$).

2.1.1. Known facts: The case $p = 2$

1) With any symmetric bilinear form B the quadratic form $Q(x) := B(x, x)$ is associated. Arf has discovered *the Arf invariant* — an important invariant of non-degenerate quadratic forms in characteristic 2; for an exposition, see [**Dye**]. Two such forms are equivalent if and only if their Arf invariants are equal.

The other way round, given a quadratic form Q, we define a symmetric bilinear form, called *the polar form* of Q, by setting

$$B_Q(x, y) = Q(x + y) - Q(x) - Q(y).$$

The Arf invariant can not, however, be used for classification of symmetric bilinear forms because one symmetric bilinear form can serve as the polar form for two non-equivalent (and having different Arf invariants) quadratic forms. Moreover,

- *not every symmetric bilinear form can be represented as a polar form.*
- *If $p = 2$, the correspondence $Q \longleftrightarrow B_Q$ is not one-to-one.*

2) Recall that the space of anti-symmetric forms (their matrices are zero-diagonal ones) is a subspace in the space of symmetric forms. Albert [**A**] classified symmetric bilinear forms over a field of characteristic 2 and proved that (we have in mind symmetric forms only)

(1) two anti-symmetric forms of equal ranks are equivalent;

(2) every non-anti-symmetric form has a matrix which is equivalent to a diagonal matrix;

(3) if $\mathbb{K}$ is perfect, then every two non-anti-symmetric forms of equal ranks are equivalent.

Remarks 2.1. 1) Over a field $\mathbb{K}$ of characteristic 2, Albert also obtained certain results on the classification of quadratic forms (considered as elements of the quotient space of all bilinear forms modulo the space of anti-symmetric forms). In particular, he showed that if $\mathbb{K}$ is algebraically closed, then every quadratic form is equivalent to exactly one of the forms

$$x_1x_{r+1} + \dots + x_rx_{2r} \quad \text{or} \quad x_1x_{r+1} + \dots + x_rx_{2r} + x_{2r+1}^2, \tag{11}$$

where $2r$ is the rank of the form. Lebedev [**Le1**] used this result in the study of Lie algebras preserving the contact structure.

2) Lebedev [**Le1**] also suggested canonical forms (or rather of their classes modulo the subspace of symmetric forms) of non-symmetric bilinear forms and classified them. This result is also related to a result of Albert and — rather unexpectedly — with contact structures on superspaces.

Theorem 2.2 ([**Le1**]). *Let $\mathbb{K}$ be a perfect field of characteristic 2. Let V be an n-dimensional space over $\mathbb{K}$.*

1) *For n odd, there is only one equivalence class of non-degenerate symmetric bilinear forms on V.*

2) *For n even, there are two equivalence classes of non-degenerate symmetric bilinear forms, one — with at least one non-zero element on the main diagonal — contains 1_n and the other one — all its Gram matrices are* zero-diagonal — *contains $S_n := \operatorname{antidiag}(1, \dots, 1)$ and Π_n.*

In view of (10) the statement of the next Lemma (proved in [**Le1, BGL**]) is non-trivial.

Lemma 2.3. 1) *The Lie algebras $\mathfrak{o}_I(2k)$ and $\mathfrak{o}_\Pi(2k)$ are not isomorphic (though are of the same dimension); the same applies to their derived algebras:*

2) $\mathfrak{o}_I^{(1)}(2k) \not\simeq \mathfrak{o}_\Pi^{(1)}(2k)$, *though* $\dim \mathfrak{o}_I^{(1)}(2k) = \dim \mathfrak{o}_\Pi^{(1)}(2k)$*;*

3) $\mathfrak{o}_I^{(2)}(2k) \not\simeq \mathfrak{o}_\Pi^{(2)}(2k)$ *unless $k = 1$.*

Based on these results, Lebedev described all the (four) possible analogs of the Poisson bracket, and (there exists just one) contact bracket. Similar results for the odd bilinear form yield a description of the anti-bracket (a.k.a. Buttin bracket), and the (peri)contact bracket, compare [**Le2**] with [**LSh**]. The quotients of the Poisson and Buttin Lie (super)algebras modulo center — analogs of Lie algebras of Hamiltonian vector fields, and their divergence-free subalgebras — are also described in [**Le2**].

3. Analogs of functions and vector fields for $p > 0$

3.1. Divided powers

Let us consider the supercommutative superalgebra $\mathbb{C}[x]$ of polynomials in a indeterminates $x = (x_1, \dots, x_a)$, for convenience ordered in a "standard format", i.e., so that the first m indeterminates are even and the rest n ones are odd ($m + n = a$). Among the integer bases of $\mathbb{C}[x]$ (i.e., the bases, in which the structure constants are integers), there are two canonical ones, — the monomial one (it is more conventional) and the basis of *divided powers*, which is constructed in the following way.

For any multi-index $\underline{r} = (r_1, \dots, r_a)$, where $r_1, \dots, r_m$ are non-negative integers, and $r_{m+1}, \dots, r_n$ are 0 or 1, we set

$$u_i^{(r_i)} := \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{(\underline{r})} := \prod_{i=1}^{a} u_i^{(r_i)}.$$

These $u^{(\underline{r})}$ form an integer basis of $\mathbb{C}[x]$. Clearly, their multiplication relations are

$$\begin{gathered} u^{(\underline{r})} \cdot u^{(\underline{s})} = \prod_{i=m+1}^{n} \min(1, 2 - r_i - s_i) \cdot (-1)^{\sum\limits_{m<i<j\leqslant a} r_j s_i} \cdot \binom{\underline{r}+\underline{s}}{\underline{r}} u^{(\underline{r}+\underline{s})}, \\ \text{where} \quad \binom{\underline{r}+\underline{s}}{\underline{r}} := \prod_{i=1}^{m} \binom{r_i+s_i}{r_i}. \end{gathered} \tag{12}$$

In what follows, for clarity, we will write exponents of divided powers in parentheses, as above, especially if the usual exponents might be encountered as well.

Now, for an arbitrary field $\mathbb{K}$ of characteristic $p > 0$, we may consider the supercommutative superalgebra $\mathbb{K}[u]$ spanned by elements $u^{(\underline{r})}$ with multiplication relations (12). For any m-tuple $\underline{N} = (N_1, \dots, N_m)$, where N_i are either positive integers or infinity, denote (we assume that $p^\infty = \infty$)

$$\mathcal{O}(m; \underline{N}) := \mathbb{K}[u; \underline{N}] := \operatorname{Span}_{\mathbb{K}} \left(u^{(\underline{r})} \mid r_i \begin{cases} < p^{N_i} & \text{for } i \leqslant m \\ = 0 \text{ or } 1 & \text{for } i > m \end{cases} \right). \tag{13}$$

As is clear from (12), $\mathbb{K}[u; \underline{N}]$ is a subalgebra of $\mathbb{K}[u]$. The algebra $\mathbb{K}[u]$ and its subalgebras $\mathbb{K}[u; \underline{N}]$ are called the *algebras of divided powers*; they can be considered as analogs of the polynomial algebra.

Only one of these numerous algebras of divided powers $\mathcal{O}(n; \underline{N})$ are indeed generated by the indeterminates declared: If $N_i = 1$ for all i. Otherwise, in addition to the u_i, we have to add $u_i^{(p^{k_i})}$ for all $i \leqslant m$ and all k_i such that $1 < k_i < N_i$ to the list of generators. Since any derivation D of a given algebra is determined by the values of D on the generators, we see that $\mathfrak{der}(\mathcal{O}[m; \underline{N}])$ has more than m functional parameters (coefficients of the analogs of partial derivatives) if $N_i \neq 1$ for at least one i. Define *distinguished partial derivatives* by setting

$$\partial_i(u_j^{(k)}) = \delta_{ij} u_j^{(k-1)} \quad \text{for any } k < p^{N_j}.$$

The simple vectorial Lie algebras over $\mathbb{C}$ have only one parameter: the number of indeterminates. If $\operatorname{char} \mathbb{K} = p > 0$, the vectorial Lie algebras acquire one more parameter: $\underline{N}$. For Lie superalgebras, $\underline{N}$ only concerns the even indeterminates.

The Lie (super)algebra of all derivations $\mathfrak{der}(\mathcal{O}[m; \underline{N}])$ turns out to be not so interesting as its *Lie subsuperalgebra of distinguished derivations*: Let

$$\begin{gathered} \mathfrak{vect}(m; \underline{N}|n) \ \text{ a.k.a } W(m; \underline{N}|n) \ \text{ a.k.a} \\ \mathfrak{der}_{dist} \mathbb{K}[u; \underline{N}] = \operatorname{Span}_{\mathbb{K}} \left(u^{(\underline{r})} \partial_k \mid r_i \begin{cases} < p^{N_i} & \text{for } i \leqslant m, \\ = 0 \text{ or } 1 & \text{for } i > m; \end{cases} \quad 1 \leqslant k \leqslant n \right) \end{gathered} \tag{14}$$

be the *general vectorial Lie algebra of distinguished derivations*. The next notions are analogs of the polynomial algebra of the dual space.

3.2. Symmetric differential forms and exterior differential forms

In what follows, as is customary in modern geometry, we use the antisymmetric $\wedge$ product for the analogs of the *exterior differential* forms, and the symmetric $\circ$ product for the *symmetric differential* forms, e.g., analogs of the metrics. We can also consider the divided power versions of the exterior and symmetric forms because both types of forms generate (in the divided sense) supercommutative superalgebras depending not only on the u_i, as above, but also on du_i, such that $p(du_i) = p(u_i)$ in the symmetric case and $p(du_i) = p(u_i) + \bar{1}$ in the exterior case. Usually we suppress the $\wedge$ or $\circ$ signs, since all is clear from the context, unless both multiplications are needed simultaneously. We have, however, to distinguish the non-divided $\wedge$ or $\circ$ from their divided counterparts $\overset{d}{\wedge}$ and $\overset{d}{\circ}$. This is important since both non-divided and divided products are often needed simultaneously. Fortunately, in this paper, we only need divided products, so for simplicity of notations, *having in mind more appropriate notation $\overset{d}{\wedge}$, we use $\wedge$.*

Considering *exterior* differential forms, we use divided powers $dx_i^{(\wedge k)}$ with multiplication relations (12), where the indeterminates are now the dx_i of parity $p(x_i)+\bar{1}$, and the Lie derivative along the vector field X is given by the formula

$$L_X(dx_i^{(\wedge k)}) = (L_X dx_i)\wedge dx_i^{(\wedge k-1)}.$$

Note that if we consider divided power differential forms in characteristic 2, then, for x_i odd, we have $dx_i\wedge dx_i = 2(dx_i^{(\wedge 2)}) = 0$. (If x_i is even, then $dx_i\wedge dx_i=0$, anyway.)

Considering divided powers of chains and cochains of Lie superalgebras affects the formula for the (co)chain differentials. For cochains of a given Lie superalgebra $\mathfrak{g}$, this only means that a divided power of an odd element must be differentiated as a whole:

$$d(\varphi^{(\wedge k)}) = d\varphi\wedge\varphi^{(\wedge(k-1))} \text{ for any } \varphi\in(\mathfrak{g}^*)_{\bar{1}}. \tag{15}$$

For chains, the modification is a little more involved: Let $g_1,\dots,g_n$ be a basis of $\mathfrak{g}$. Then for chains of $\mathfrak{g}$ with coefficients in a right module M, and $m\in M$, we have

$$\begin{aligned} d\left(m\otimes\bigwedge_{i=1}^{n} g_i^{(\wedge r_i)}\right) = & \sum_{p(g_k)=\bar{1},\ r_k\geqslant 2} m\otimes\bigwedge_{i<k} g_i^{(\wedge r_i)}\wedge g_k^2\wedge g_k^{(\wedge(r_k-2))}\wedge\bigwedge_{i>k} g_i^{(\wedge r_i)}+\\ & \sum_{1\leqslant k<l\leqslant n,\quad r_k,r_l\geqslant 1} (-1)^{\sum\limits_{k<j<l} r_j p(g_j)} m\otimes\bigwedge_{i<k} g_i^{(\wedge r_i)}\wedge\\ & [g_k,g_l]\wedge g_k^{(\wedge(r_k-1))}\wedge\bigwedge_{k<i<l} g_i^{(\wedge r_i)}\wedge g_l^{(\wedge(r_l-1))}\wedge\bigwedge_{i>l} g_i^{(\wedge r_i)}+\\ & \sum_{r_k\geqslant 1}(-1)^{p(g_k)\sum\limits_{j<k} r_j p(g_j)}(mg_k)\otimes\bigwedge_{i<k} g_i^{(\wedge r_i)}\wedge g_k^{(\wedge(r_k-1))}\wedge\bigwedge_{i>k} g_i^{(\wedge r_i)}. \end{aligned} \tag{16}$$

Denote the divided power cohomology by $DPH^{i,\underline{N}}(\mathfrak{g};M)$ and divided power homology by $DPH_{i,\underline{N}}(\mathfrak{g};M)$. Note that if $\mathfrak{g}$ is a Lie **super**algebra and $p=2$, we can not interpret its generating relations in terms of the 2nd homology $H_2(\mathfrak{g})$, as we do for $p\neq 2$: Instead, we **must** use divided powers homology $DPH_{2,\underline{N}}(\mathfrak{g}) := DPH_{2,\underline{N}}(\mathfrak{g};\mathbb{K})$ (with $\underline{N}$ such that $N_i\geqslant 2$ for all i) since otherwise we won't be able to take into account the relations of the form $x^2=0$ for x odd.

Problem 3.1. *To define the divided power (co)homology as the derived functor, we have to completely modify the representation theory and, in particular, the notion of the universal enveloping algebra. We do not know a precise definition of the "divided power universal enveloping algebra" but conjecture that is can be found along the way hinted at in [**LL**].*

3.2.1. A useful Lemma

We computed cohomology using Grozman's `Mathematica`-based package `SuperLie`. The formula of the following lemma was helpful in the computations. For any finite dimensional Lie (super)algebra $\mathfrak{g}$, all cochains with non-trivial coefficients in a $\mathfrak{g}$-module M can be expressed as sums of tensor products of the form $m\otimes\omega$, where $m\in M$ and $\omega\in\bigwedge^{\bullet}(\mathfrak{g}^*)$. We are working with a fixed basis of M and the dual basis of $\mathfrak{g}^*$. For simplicity, the following Lemma is formulated for Lie algebras, its superization is routine, by means of the Sign Rule.

Lemma 3.2. *For any $c = m \otimes \omega$, where $m \in M$ and $\omega \in \bigwedge^r(\mathfrak{g}^*)$, let dc denote the coboundary of c in the complex with coefficients in M, while $d\omega$ denotes the coboundary in the complex with trivial coefficients and dm denotes the coboundary of $m \in M$ considered as a 0-cochain in the complex with coefficients in M. If $c = m \otimes \omega$, then $dc = m \otimes d\omega + dm \wedge \omega$.*

Proof. For any $x_1, \dots, x_{r+1} \in \mathfrak{g}$, we have:

$$\begin{aligned} dc(x_1, \dots, x_{r+1}) = & \sum_{1 \leqslant i < j \leqslant r+1} (-1)^{i+j-1} m \otimes \omega([x_i, x_j], x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_{r+1}) + \\ & + \sum_{1 \leqslant i \leqslant r+1} (-1)^i x_i(m) \otimes \omega(x_1, \dots, \hat{x_i}, \dots, x_{r+1}) = \\ & = (m \otimes d\omega)(x_1, \dots, x_{r+1}) + (dm \wedge \omega)(x_1, \dots, x_{r+1}). \end{aligned}$$

□

3.2.1.1. In characteristic 2 The following definition of Lie algebra cohomology in char $= 2$ is implemented in `SuperLie`. The wedge product of vector spaces is defined without a normalization factor:

$$a \wedge b = a \otimes b + b \otimes a.$$

For 1-cochains with trivial coefficients, the codifferential is defined as an operation dual to the Lie bracket:

$$d \colon \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \wedge \mathfrak{g}^*.$$

For q-cochains with trivial coefficients, d is defined via the Leibniz rule. For cochains with coefficients in a module M, we set

$$\begin{aligned} d(m) &:= \sum_{1 \leqslant i \leqslant \dim M} g_i(m) \otimes g_i^*, \\ d(m \otimes \omega) &:= d(m) \wedge \omega + m \otimes d(\omega) \end{aligned}$$

for any $m \in M$, any r-cochain ω, where $r > 0$, and any basis g_i of M, cf. Lemma 3.2.

4. What $\mathfrak{g}(A)$ is

4.1. Warning: $\mathfrak{psl}$ has no Cartan matrix. The relatives of $\mathfrak{sl}$ and $\mathfrak{psl}$ that have Cartan matrices

For the most reasonable definition of Lie algebra with Cartan matrix over $\mathbb{C}$, see [**K**]. The same definition applies, practically literally, to Lie superalgebras and to modular Lie algebras and to modular Lie superalgebras. However, the usual sloppy practice is to attribute Cartan matrices to (usually simple) Lie (super)algebras none of which, strictly speaking, has a Cartan matrix!

Although it may look strange for those with non-super experience over $\mathbb{C}$, neither the simple modular Lie algebra $\mathfrak{psl}(pk)$, nor the simple modular Lie superalgebra $\mathfrak{psl}(a|pk+a)$, nor — in characteristic 0 — the simple Lie superalgebra $\mathfrak{psl}(a|a)$ possesses a Cartan matrix. Their central extensions ($\mathfrak{sl}(pk)$, the modular Lie superalgebra $\mathfrak{sl}(a|pk+a)$, and — in characteristic 0 — the Lie superalgebra $\mathfrak{sl}(a|a)$) do not have Cartan matrix, either.

Their relatives possessing a Cartan matrix are, respectively, $\mathfrak{gl}(pk)$, $\mathfrak{gl}(a|pk+a)$, and $\mathfrak{gl}(a|a)$, and for the grading operator we take the matrix unit $E_{1,1}$.

Since all the Lie (super)algebras involved (the simple one, its central extension, the derivation algebras thereof) are often needed simultaneously (and only representatives of one of these types of Lie (super)algebras are of the form $\mathfrak{g}(A)$), it is important to have (preferably short and easy to remember) notation for each of them. For example, in addition to $\mathfrak{psl}$, $\mathfrak{sl}$, $\mathfrak{pgl}$ and $\mathfrak{gl}$, we have:

for $p=3$: $\mathfrak{e}(6)$ is of dimension 79, then $\dim\mathfrak{e}(6)^{(1)}=78$, whereas the "simple core" is $\mathfrak{e}(6)^{(1)}/\mathfrak{c}$ of dimension 77;

$\mathfrak{g}(2)$ is not simple, its "simple core" is isomorphic to $\mathfrak{psl}(3)$;

for $p=2$: $\mathfrak{e}(7)$ is of dimension 134, then $\dim\mathfrak{e}(7)^{(1)}=133$, whereas the "simple core" is $\mathfrak{e}(7)^{(1)}/\mathfrak{c}$ of dimension 132;

$\mathfrak{g}(2)$ is not simple, its "simple core" is isomorphic to $\mathfrak{psl}(4)$;

the orthogonal Lie algebras and their super analogs are considered in detail later.

In our main examples, $\operatorname{sdim}\mathfrak{g}(A)^{(1)}/\mathfrak{c}=d|B$ whereas the notation $D/d|B$ means that $\operatorname{sdim}\mathfrak{g}(A)=D|B$. The general formula is

$$d=D-2(\operatorname{size}(A)-\operatorname{rk}(A)) \tag{17}$$

4.2. Generalities

Let $A=(A_{ij})$ be an $n\times n$-matrix with elements in $\mathbb{K}$ with $\operatorname{rk}A=n-l$. Complete A to an $(n+l)\times n$-matrix $\begin{pmatrix}A\\B\end{pmatrix}$ of rank n. (Thus, B is an $l\times n$-matrix.)

Let the elements $e_i^\pm, h_i$, where $i=1,\dots,n$, and d_k, where $k=1,\dots,l$, generate a Lie superalgebra denoted $\tilde{\mathfrak{g}}(A,I)$, where $I=(p_1,\dots p_n)\in(\mathbb{Z}/2)^n$ is a collection of parities ($p(e_i^\pm)=p_i$, the parities of the d_k's being $\bar{0}$), free except for the relations

$$\begin{aligned}&[e_i^+,e_j^-]=\delta_{ij}h_i;\quad [h_i,e_j^\pm]=\pm A_{ij}e_j^\pm;\quad [d_k,e_j^\pm]=\pm B_{kj}e_j^\pm;\\&[h_i,h_j]=[h_i,d_k]=[d_k,d_m]=0\quad\text{for any } i,j,k,m.\end{aligned} \tag{18}$$

The Lie superalgebra $\tilde{\mathfrak{g}}(A,I)$ is $\mathbb{Z}^n$-graded with

$$\begin{aligned}&\deg e_i^\pm=(0,\dots,0,\pm1,0,\dots,0)\\&\deg h_i=\deg d_k=(0,\dots,0)\quad\text{for any } i,k.\end{aligned} \tag{19}$$

Let $\mathfrak{h}$ denote the linear span of the h_i's and d_k's. Let $\tilde{\mathfrak{g}}(A,I)^\pm$ denote the Lie subsuperalgebras in $\tilde{\mathfrak{g}}(A,I)$ generated by $e_1^\pm,\dots,e_n^\pm$. Then

$$\tilde{\mathfrak{g}}(A,I)=\tilde{\mathfrak{g}}(A,I)^-\oplus\mathfrak{h}\oplus\tilde{\mathfrak{g}}(A,I)^+,$$

where the homogeneous component of degree $(0,\dots,0)$ is just $\mathfrak{h}$.

The Lie subsuperalgebras $\tilde{\mathfrak{g}}(A,I)^\pm$ are homogeneous in this $\mathbb{Z}^n$-grading, and there is a

$$\text{maximal homogeneous (in this } \mathbb{Z}^n\text{-grading) ideal } \mathfrak{r} \text{ such that } \mathfrak{r}\cap\mathfrak{h}=0. \tag{20}$$

The ideal $\mathfrak{r}$ is just the sum of homogeneous ideals whose homogeneous components of degree $(0,\dots,0)$ are trivial.

As $\operatorname{rk} A = n - l$, there exists an $l \times n$-matrix $T = (T_{ij})$ of rank l such that

$$TA = 0. \tag{21}$$

Let

$$c_i = \sum_{j=1}^{n} T_{ij} h_j, \quad \text{where } i = 1, \dots, l. \tag{22}$$

Then, from the properties of the matrix T, we deduce that

a) the elements c_i are linearly independent; let $\mathfrak{c}$ be the space they span;
b) the elements c_i are central, because

$$[c_i, e_j^\pm] = \pm \left(\sum_{k=1}^{n} T_{ik} A_{kj} \right) e_j^\pm = \pm (TA)_{ij} e_j^\pm \overset{(21)}{=} 0. \tag{23}$$

Observe that $\mathfrak{c}$ does not depend on T.

The Lie (super)algebra $\mathfrak{g}(A, I)$ is defined as the quotient $\tilde{\mathfrak{g}}(A, I)/\mathfrak{r}$ and is called the *Lie (super)algebra with Cartan matrix A (and parities I)*. Note that this coincides with the definition in [**CE**] of the *contragredient* Lie superalgebras, although written in a slightly different way. Condition (20) modified as

$$\text{maximal homogeneous (in this } \mathbb{Z}^n\text{-grading) ideal } \mathfrak{s} \text{ such that } \mathfrak{s} \cap \mathfrak{h} = \mathfrak{c} \tag{24}$$

leads to what in [**CE**] is called the *centerless contragredient* Lie superalgebra, cf. [**Bi**].

By abuse of notation we denote by $e_i^\pm, h_i, d_k$ and $\mathfrak{c}$ their images in $\mathfrak{g}(A, I)$ and $\mathfrak{g}(A, I)^{(1)}$.

The Lie superalgebra $\mathfrak{g}(A, I)$ inherits, clearly, the $\mathbb{Z}^n$-grading of $\tilde{\mathfrak{g}}(A, I)$. The non-zero elements $\alpha \in \mathbb{Z}^n \subset \mathbb{R}^n$ such that the homogeneous component $\mathfrak{g}(A, I)_\alpha$ is non-zero are called *roots*. The set R of all roots is called *the root system* of $\mathfrak{g}$. Clearly, the subspaces $\mathfrak{g}_\alpha$ are purely even or purely odd, and the corresponding roots are said to be *even* or *odd*.

The additional to (18) relations that turn $\tilde{\mathfrak{g}}(A, I)^\pm$ into $\mathfrak{g}(A, I)^\pm$ are of the form $R_i = 0$ whose left sides are implicitly described as follows:

$$\text{the } R_i \text{ that generate the maximal ideal } \mathfrak{r}. \tag{25}$$

The explicit description of these additional relations forms the main bulk of this paper.

4.3. Roots and weights

In this subsection, $\mathfrak{g}$ denotes one of the algebras $\mathfrak{g}(A, I)$ or $\tilde{\mathfrak{g}}(A, I)$.

The elements of $\mathfrak{h}^*$ are called *weights*. For a given weight α, the *weight subspace* of a given $\mathfrak{g}$-module V is defined as

$$V_\alpha = \{x \in V \mid \text{an integer } N > 0 \text{ exists such that } (\alpha(h) - \operatorname{ad}_h)^N x = 0 \text{ for any } h \in \mathfrak{h}\}.$$

Any non-zero element $x \in V$ is said to be *of weight* α. For the roots, which are particular cases of weights if $p = 0$, the above definition is inconvenient: In the modular analog of the following useful statement summation should be over roots defined in the previous subsection.

Statement 4.1 ([**K**]). *Over $\mathbb{C}$, the space $\mathfrak{g}$ can be represented as a direct sum of subspaces*

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha.$$

Note that $\mathfrak{h} \subsetneq \mathfrak{g}_0$ over $\mathbb{K}$, e.g., all weights of the form $p\alpha$ over $\mathbb{C}$ become 0.

4.4. Systems of simple and positive roots

In this subsection, $\mathfrak{g} = \mathfrak{g}(A, I)$, and R is the root system of $\mathfrak{g}$.

For any subset $B = \{\sigma_1, \dots, \sigma_m\} \subset R$, we set (we denote by $\mathbb{Z}_+$ the set of non-negative integers):

$$R_B^{\pm} = \{\alpha \in R \mid \alpha = \pm \sum n_i\sigma_i, \;\; n_i \in \mathbb{Z}_+\}.$$

The set B is called a *system of simple roots* of R (or $\mathfrak{g}$) if $\sigma_1, \dots, \sigma_m$ are linearly independent and $R = R_B^+ \cup R_B^-$. Note that R contains basis coordinate vectors, and therefore spans $\mathbb{R}^n$; thus, any system of simple roots contains exactly n elements.

Let $(\cdot,\cdot)$ be the standard Euclidean inner product in $\mathbb{R}^n$. A subset $R^+ \subset R$ is called a *system of positive roots* of R (or $\mathfrak{g}$) if there exists $x \in \mathbb{R}^n$ such that

$$\begin{aligned} &(\alpha, x) \in \mathbb{R}\backslash\{0\} \text{for any } \alpha \in R,\\ &R^+ = \{\alpha \in R \mid (\alpha, x) > 0\}. \end{aligned} \tag{26}$$

Since R is a finite (or, at least, countable if $\dim \mathfrak{g}(A) = \infty$) set, so the set

$$\{y \in \mathbb{R}^n \mid \text{there exists } \alpha \in R \text{ such that } (\alpha, y) = 0\}$$

is a finite/countable union of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$, so it has zero measure. So for almost every x, condition (26) holds.

By construction, any system B of simple roots is contained in exactly one system of positive roots, which is precisely R_B^+.

Statement 4.2. *Any finite system R^+ of positive roots of $\mathfrak{g}$ contains exactly one system of simple roots. This system consists of all the positive roots (i.e., elements of R^+) that can not be represented as a sum of two positive roots.*

We can not give an *a priori* proof of the fact that each set of all positive roots each of which is not a sum of two other positive roots consists of linearly independent elements. This is, however, true for finite dimensional Lie algebras and superalgebras $\mathfrak{g}(A, I)$ if $p \neq 2$.

4.5. Normalization convention

Clearly,

$$\text{the rescaling } e_i^{\pm} \mapsto \sqrt{\lambda_i} e_i^{\pm}, \text{ sends } A \text{ to } A' := \operatorname{diag}(\lambda_1, \dots, \lambda_n) \cdot A. \tag{27}$$

Two pairs (A, I) and (A', I') are said to be *equivalent* (and we write $(A, I) \sim (A', I')$) if (A', I') is obtained from (A, I) by a composition of a permutation of parities and a rescaling $A' = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \cdot A$, where $\lambda_1 \dots \lambda_n \neq 0$. Clearly, equivalent pairs determine isomorphic Lie superalgebras.

The rescaling affects only the matrix A_B, not the set of parities I_B. The Cartan matrix A is said to be *normalized* if

$$A_{jj} = 0 \quad \text{or } 1, \text{ or } 2, \tag{28}$$

where we let $A_{jj} = 2$ only if $p_j = \bar{0}$; in order to distinguish between the cases where $p_j = \bar{0}$ and $p_j = \bar{1}$, we write $A_{jj} = \bar{0}$ or $\bar{1}$, instead of 0 or 1, if $p_j = \bar{0}$.

We will only consider normalized Cartan matrices; for them, we do not have to describe I.

The row with a 0 or $\bar{0}$ on the main diagonal can be multiplied by any nonzero factor; usually (not only in this paper) we multiply the rows so as to make A_B symmetric, if possible.

A posteriori, for each **finite dimensional** Lie (super)algebra of the form $\mathfrak{g}(A)$ with indecomposable Cartan matrix A, the matrix A is symmetrizable (i.e., it can be made symmetric by operation (27)) for any p. For affine and almost affine Lie (super)algebra of the form $\mathfrak{g}(A)$ this is not so, cf. [**CCLL**]

4.6. Equivalent systems of simple roots

Let $B = \{\alpha_1, \dots, \alpha_n\}$ be a system of simple roots. Choose non-zero elements $e_i^{\pm}$ in the 1-dimensional (by definition) superspaces $\mathfrak{g}_{\pm\alpha_i}$; set $h_i = [e_i^+, e_i^-]$, let $A_B = (A_{ij})$, where the entries A_{ij} are recovered from relations (18), and let

$$I_B = \{p(e_1), \cdots, p(e_n)\}.$$

Lemma 6.3 claims that all the pairs (A_B, I_B) are equivalent to each other.

Two systems of simple roots B_1 and B_2 are said to be *equivalent* if the pairs $(A_{B_1}, I_{B_1}) \sim (A_{B_2}, I_{B_2})$.

For the role of the "best" (first among equals) order of indices we propose the one that minimizes the value

$$\max_{i,j\in\{1,\dots,n\} \text{ such that } (A_B)_{ij}\neq 0} |i-j| \tag{29}$$

(i.e., gather the non-zero entries of A as close to the main diagonal as possible).

4.6.1. Chevalley generators and Chevalley bases

We often denote the set of generators corresponding to a normalized matrix by $X_1^{\pm}, \dots, X_n^{\pm}$ instead of $e_1^{\pm}, \dots, e_n^{\pm}$; and call them, together with the elements $H_i := [X_i^+, X_i^-]$, and the derivatives d_j added for convenience for all i and j, the *Chevalley generators*.

For $p = 0$ and normalized Cartan matrices of simple finite dimensional Lie algebras, there exists only one (up to signs) basis containing $X_i^{\pm}$ and H_i in which $A_{ii} = 2$ for all i and all structure constants are integer, cf. [**St**]. Such a basis is called the *Chevalley* basis.

Observe that, having normalized the Cartan matrix of $\mathfrak{o}(2n+1)$ so that $A_{ii} = 2$ for all $i \neq n$ but $A_{nn} = 1$, we get **another** basis with integer structure constants. We think that this basis also qualifies to be called *Chevalley basis*; for the Lie superalgebras, the basis normalized as in (28) is even more appropriate.

Conjecture 4.3. *If $p > 2$, then for finite dimensional Lie (super)algebras with indecomposable Cartan matrices normalized as in (28), there also exists only one (up to signs) analog of the Chevalley basis.*

We had no idea how to describe analogs of Chevalley bases for $p = 2$ until appearance of the recent paper [**CR**]; clearly, its methods should solve the problem.

5. Ortho-orthogonal and periplectic Lie superalgebras

In this section, $p = 2$ and $\mathbb{K}$ is perfect. We also assume that $n_{\bar 0}, n_{\bar 1} > 0$.

5.1. Non-degenerate even supersymmetric bilinear forms and ortho–orthogonal Lie superalgebras

For $p = 2$, there are, in general, four equivalence classes of inequivalent non-degenerate even supersymmetric bilinear forms on a given superspace. Any such form B on a superspace V of superdimension $n_{\bar 0}|n_{\bar 1}$ can be decomposed as follows:

$$B = B_{\bar 0} \oplus B_{\bar 1},$$

where $B_{\bar 0}$, $B_{\bar 1}$ are symmetric non-degenerate forms on $V_{\bar 0}$ and $V_{\bar 1}$, respectively. For $i = \bar 0, \bar 1$, the form B_i is equivalent to 1_{n_i} if n_i is odd, and equivalent to 1_{n_i} or Π_{n_i} if n_i is even. So every non-degenerate even symmetric bilinear form is equivalent to one of the following forms (some of them are defined not for all dimensions):

$B_{II} = 1_{n_{\bar 0}} \oplus 1_{n_{\bar 1}}$; $\quad B_{I\Pi} = 1_{n_{\bar 0}} \oplus \Pi_{n_{\bar 1}}$ if $n_{\bar 1}$ is even;
$B_{\Pi I} = \Pi_{n_{\bar 0}} \oplus 1_{n_{\bar 1}}$ if $n_{\bar 0}$ is even; $\quad B_{\Pi\Pi} = \Pi_{n_{\bar 0}} \oplus \Pi_{n_{\bar 1}}$ if $n_{\bar 0}, n_{\bar 1}$ are even.

We denote the Lie superalgebras that preserve the respective forms by $\mathfrak{oo}_{II}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}_{\Pi I}(n_{\bar 0}|n_{\bar 1})$, $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$, respectively. Now let us describe these algebras.

5.1.1. $\mathfrak{oo}_{II}(n_{\bar 0}|n_{\bar 1})$

If $n \geqslant 3$, then the Lie superalgebra $\mathfrak{oo}^{(1)}_{II}(n_{\bar 0}|n_{\bar 1})$ is simple. This Lie superalgebra **has no Cartan matrix**.

5.1.2. $\mathfrak{oo}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$ $\quad (n_{\bar 1} = 2k_{\bar 1})$

The Lie superalgebra $\mathfrak{oo}^{(1)}_{I\Pi}(n_{\bar 0}|n_{\bar 1})$ is simple, it has Cartan matrix if and only if $n_{\bar 0}$ is odd; this matrix has the following form (up to a format; all possible formats — corresponding to $* = 0$ or $* = \bar 0$ — are described in the table on page 266 below):

$$\begin{pmatrix} \ddots & \ddots & \ddots & \vdots \\ \ddots & * & 1 & 0 \\ \ddots & 1 & * & 1 \\ \cdots & 0 & 1 & 1 \end{pmatrix} \qquad (30)$$

5.1.3. $\mathfrak{oo}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})$ $(n_{\bar 0}=2k_{\bar 0}, n_{\bar 1}=2k_{\bar 1})$
If $n=n_{\bar 0}+n_{\bar 1}\geqslant 6$, then

$$\begin{aligned}&\text{if } k_{\bar 0}+k_{\bar 1} \text{ is odd, then the Lie superalgebra } \mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1}) \text{ is simple;}\\ &\text{if } k_{\bar 0}+k_{\bar 1} \text{ is even, then the Lie superalgebra } \mathfrak{oo}^{(2)}_{\Pi\Pi}(n_{\bar 0}|n_{\bar 1})/\mathbb{K}1_{n_{\bar 0}|n_{\bar 1}} \text{ is simple.}\end{aligned} \tag{31}$$

Each of these simple Lie superalgebras is also close to a Lie superalgebra with Cartan matrix. To describe this Cartan matrix Lie superalgebra in most simple terms, we will choose a slightly different realization of $\mathfrak{oo}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$: Let us consider it as the algebra of linear transformations that preserve the bilinear form $\Pi(2k_{\bar 0}+2k_{\bar 1})$ in the format $k_{\bar 0}|k_{\bar 1}|k_{\bar 0}|k_{\bar 1}$. Then the algebra $\mathfrak{oo}^{(i)}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$ is spanned by supermatrices of format $k_{\bar 0}|k_{\bar 1}|k_{\bar 0}|k_{\bar 1}$ and of the form

$$\begin{pmatrix}A & C\\ D & A^T\end{pmatrix} \text{ where } \begin{matrix} A\in\begin{cases}\mathfrak{gl}(k_{\bar 0}|k_{\bar 1}) & \text{if } i\leqslant 1,\\ \mathfrak{sl}(k_{\bar 0}|k_{\bar 1}) & \text{if } i\geqslant 2,\end{cases}\\ C,D \text{ are } \begin{cases}\text{symmetric matrices} & \text{if } i=0;\\ \text{symmetric zero-diagonal matrices} & \text{if } i\geqslant 1.\end{cases}\end{matrix} \tag{32}$$

If $i\geqslant 1$, these derived algebras have a non-trivial central extension given by the following cocycle:

$$F\left(\begin{pmatrix}A & C\\ D & A^T\end{pmatrix},\begin{pmatrix}A' & C'\\ D' & A'^T\end{pmatrix}\right)=\sum_{1\leqslant i<j\leqslant k_{\bar 0}+k_{\bar 1}}(C_{ij}D'_{ij}+C'_{ij}D_{ij}) \tag{33}$$

(note that this expression resembles $\frac{1}{2}\operatorname{tr}(CD'+C'D)$). We will denote this central extension of $\mathfrak{oo}^{(i)}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$ by $\mathfrak{ooc}(i,2k_{\bar 0}|2k_{\bar 1})$.

Let

$$I_0:=\operatorname{diag}(1_{k_{\bar 0}|k_{\bar 1}},0_{k_{\bar 0}|k_{\bar 1}}). \tag{34}$$

Then the corresponding Cartan matrix Lie superalgebra is[9]

$$\begin{aligned}&\mathfrak{ooc}(2,2k_{\bar 0}|2k_{\bar 1})\ltimes\mathbb{K}I_0 && \text{if } k_{\bar 0}+k_{\bar 1} \text{ is odd;}\\ &\mathfrak{ooc}(1,2k_{\bar 0}|2k_{\bar 1})\ltimes\mathbb{K}I_0 && \text{if } k_{\bar 0}+k_{\bar 1} \text{ is even.}\end{aligned} \tag{35}$$

The corresponding Cartan matrix has the following form (up to a format; all possible formats — corresponding to $*=0$ or $*=\bar 0$ — are described in the table on page 266 below):

$$\begin{pmatrix}\ddots & \ddots & \ddots & \vdots & \vdots\\ \ddots & * & 1 & 0 & 0\\ \ddots & 1 & * & 1 & 1\\ \cdots & 0 & 1 & \bar 0 & 0\\ \cdots & 0 & 1 & 0 & \bar 0\end{pmatrix} \tag{36}$$

[9]Hereafter the simbol $A\ltimes B$ denotes the semidirect sum of the two algebras, in which A is an ideal.

5.2. The non-degenerate odd supersymmetric bilinear forms. Periplectic Lie superalgebras

In this subsect., $m \geqslant 3$.

$$\begin{array}{l}\text{If } m \text{ is odd, then the Lie superalgebra } \mathfrak{pe}_B^{(2)}(m) \text{ is simple;}\\ \text{If } m \text{ is even, then the Lie superalgebra } \mathfrak{pe}_B^{(2)}(m)/\mathbb{K}1_{m|m} \text{ is simple.}\end{array} \tag{37}$$

If we choose the form B to be $\Pi_{m|m}$, then the algebras $\mathfrak{pe}_B^{(i)}(m)$ consist of matrices of the form (32); the only difference from $\mathfrak{oo}_{\Pi\Pi}^{(i)}$ is the format which in this case is $m|m$.

Each of these simple Lie superalgebras has a 2-structure. Note that if $p \neq 2$, then the Lie superalgebra $\mathfrak{pe}_B(m)$ and its derived algebras are not close to Cartan matrix Lie superalgebras (because, for example, their root system is not symmetric). If $p = 2$ and $m \geqslant 3$, then they **are** close to Cartan matrix Lie superalgebras; here we describe them.

The algebras $\mathfrak{pe}_B^{(i)}(m)$, where $i > 0$, have non-trivial central extensions with cocycles (33); we denote these central extensions by $\mathfrak{pec}(i, m)$. Let us introduce another matrix

$$I_0 := \operatorname{diag}(1_m, 0_m). \tag{38}$$

Then the Cartan matrix Lie superalgebras are

$$\begin{array}{l}\mathfrak{pec}(2, m) \ltimes \mathbb{K}I_0 \text{ if } m \text{ is odd;}\\ \mathfrak{pec}(1, m) \ltimes \mathbb{K}I_0 \text{ if } m \text{ is even.}\end{array} \tag{39}$$

The corresponding Cartan matrix has the form (36); the only condition on its format is that the last two simple roots must have distinct parities. The corresponding Dynkin diagram is shown in the table on page 266; all its nodes, except for the “horns”, may be both $\otimes$ or $\odot$, see (42).

5.3. Superdimensions

The following expressions (with a + sign) are the superdimensions of the relatives of the ortho-orthogonal and periplectic Lie superalgebras that possess Cartan matrices. To get the superdimensions of the simple relatives, one should replace +2 and +1 by −2 and −1, respectively, in the two first lines and the four last ones:

$$\begin{array}{lll}
\dim \mathfrak{oc}(1; 2k) \ltimes \mathbb{K}I_0 & = 2k^2 - k \pm 2 & \text{if } k \text{ is even;}\\
\dim \mathfrak{oc}(2; 2k) \ltimes \mathbb{K}I_0 & = 2k^2 - k \pm 1 & \text{if } k \text{ is odd;}\\
\dim \mathfrak{o}^{(1)}(2k+1) & = 2k^2 + k & \\
\operatorname{sdim} \mathfrak{oo}^{(1)}(2k_{\bar 0}+1|2k_{\bar 1}) & = 2k_{\bar 0}^2 + k_{\bar 0} + 2k_{\bar 1}^2 + k_{\bar 1} \mid 2k_{\bar 1}(2k_{\bar 0}+1) & \\
\operatorname{sdim} \mathfrak{ooc}(1; 2k_{\bar 0}|2k_{\bar 1}) \ltimes \mathbb{K}I_0 & = 2k_{\bar 0}^2 - k_{\bar 0} + 2k_{\bar 1}^2 - k_{\bar 1} \pm 2 \mid 4k_{\bar 0}k_{\bar 1} & \text{if } k_{\bar 0} + k_{\bar 1} \text{ is even;}\\
\operatorname{sdim} \mathfrak{ooc}(2; 2k_{\bar 0}|2k_{\bar 1}) \ltimes \mathbb{K}I_0 & = 2k_{\bar 0}^2 - k_{\bar 0} + 2k_{\bar 1}^2 - k_{\bar 1} \pm 1 \mid 4k_{\bar 0}k_{\bar 1} & \text{if } k_{\bar 0} + k_{\bar 1} \text{ is odd;}\\
\operatorname{sdim} \mathfrak{pec}(1; m) \ltimes \mathbb{K}I_0 & = m^2 \pm 2 \mid m^2 - m & \text{if } m \text{ is even;}\\
\operatorname{sdim} \mathfrak{pec}(2; m) \ltimes \mathbb{K}I_0 & = m^2 \pm 1 \mid m^2 - m & \text{if } m \text{ is odd}
\end{array} \tag{40}$$

5.3.1. Summary: The types of Lie superalgebras preserving non-degenerate symmetric forms

In addition to the isomorphisms $\mathfrak{oo}_{\Pi I}(a|b) \simeq \mathfrak{oo}_{I\Pi}(b|a)$, there is the only "occasional" isomorphism intermixing the types of Lie superalgebras preserving non-degenerate symmetric forms: $\mathfrak{oo}^{(1)}_{\Pi\Pi}(6|2) \simeq \mathfrak{pe}^{(1)}(4)$.

Let $\widehat{\mathfrak{g}} := \mathfrak{g} \ltimes \mathbb{K}I_0$. We have the following types of non-isomorphic Lie (super)algebras:

no relative has Cartan matrix	with Cartan matrix
$\mathfrak{oo}_{II}(2n+1\|2m+1)$, $\mathfrak{oo}_{II}(2n+1\|2m)$	$\widehat{\mathfrak{oc}(i;2n)}$, $\mathfrak{o}^{(1)}(2n+1)$; $\widehat{\mathfrak{pec}(i;k)}$
$\mathfrak{oo}_{II}(2n\|2m)$, $\mathfrak{oo}_{I\Pi}(2n\|2m)$; $\mathfrak{o}_I(2n)$;	$\mathfrak{ooc}(\widehat{i;2n\|}2m)$, $\mathfrak{oo}^{(1)}_{I\Pi}(2n+1\|2m)$

(41)

6. Dynkin diagrams

A usual way to represent simple Lie algebras over $\mathbb{C}$ with integer Cartan matrices is via graphs called, in the finite dimensional case, *Dynkin diagrams*. The Cartan matrices of certain interesting infinite dimensional simple Lie *super*algebras $\mathfrak{g}$ (even over $\mathbb{C}$) can be non-symmetrizable or (for any p in the super case and for $p>0$ in the non-super case) have entries belonging to the ground field $\mathbb{K}$. Still, it is always possible to assign an analog of the Dynkin diagram to each (modular) Lie (super)algebra with Cartan matrix, of course) provided the edges and nodes of the graph (Dynkin diagram) are rigged with an extra information. Although these analogs of the Dynkin graphs are not uniquely recovered from the Cartan matrix (and the other way round), they give a graphic presentation of the Cartan matrices and help to observe some hidden symmetries.

Namely, the *Dynkin diagram* of a normalized $n \times n$ Cartan matrix A is a set of n nodes connected by multiple edges, perhaps endowed with an arrow, according to the usual rules ([**K**]) or their modification, most naturally formulated by Serganova: compare [**Se, FLS**] with [**FSS**]. In what follows, we recall these rules, and further improve them to fit the modular case.

6.1. Nodes

To every simple root there corresponds

$$\begin{cases} \text{a node } \circ & \text{if } p(\alpha_i)=\bar 0 \text{ and } A_{ii}=2,\\ \text{a node } * & \text{if } p(\alpha_i)=\bar 0 \text{ and } A_{ii}=\bar 1;\\ \text{a node } \bullet & \text{if } p(\alpha_i)=\bar 1 \text{ and } A_{ii}=1;\\ \text{a node } \otimes & \text{if } p(\alpha_i)=\bar 1 \text{ and } A_{ii}=0,\\ \text{a node } \odot & \text{if } p(\alpha_i)=\bar 0 \text{ and } A_{ii}=\bar 0.\end{cases} \tag{42}$$

The Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{o}(3)^{(1)}$ with Cartan matrices (2) and $(\bar 1)$, respectively, and the Lie superalgebra $\mathfrak{osp}(1|2)$ with Cartan matrix (1) are simple.

The Lie algebra $\mathfrak{gl}(2)$ with Cartan matrix $(\bar 0)$ and the Lie superalgebra $\mathfrak{gl}(2|2)$ with Cartan matrix (0) are solvable of dim 4 and sdim $2|2$, respectively. Their derived algebras are the *Heisenberg algebra* $\mathfrak{hei}(2) := \mathfrak{hei}(2|0) \simeq \mathfrak{sl}(2)$ and the *Heisenberg superalgebra* $\mathfrak{hei}(0|2) \simeq \mathfrak{sl}(1|1)$ of (super)dimension 3 and $1|2$, respectively.

Remark 6.1. A posteriori (from the classification of simple Lie superalgebras with Cartan matrix and of polynomial growth) we find out that **for** $p = 0$, the simple root $\odot$ can only occur if $\mathfrak{g}(A, I)$ grows faster than polynomially. Thanks to classification again, if $\dim \mathfrak{g} < \infty$, the simple root $\odot$ can not occur if $p > 3$; whereas for $p = 3$, the Brown Lie algebras are examples of $\mathfrak{g}(A)$ with a simple root of type $\odot$; for $p = 2$, such roots are routine.

6.2. Edges

If $p = 2$ and $\dim \mathfrak{g}(A) < \infty$, the Cartan matrices considered are symmetric. If $A_{ij} = a$, where $a \neq 0$ or 1, then we rig the edge connecting the ith and jth nodes by a label a.

If $p > 2$ and $\dim \mathfrak{g}(A) < \infty$, then A is symmetrizable, so let us symmetrize it, i.e., consider DA for an invertible diagonal matrix D. Then, if $(DA)_{ij} = a$, where $a \neq 0$ or -1, we rig the edge connecting the ith and jth nodes by a label a.

If all off-diagonal entries of A belong to $\mathbb{Z}/p$ and their representatives are selected to be non-positive integers, we can draw the Dynkin diagram as for $p = 0$, i.e., connect the ith node with the jth one by $\max(|A_{ij}|, |A_{ji}|)$ edges rigged with an arrow $>$ pointing from the ith node to the jth if $|A_{ij}| > |A_{ji}|$ or in the opposite direction if $|A_{ij}| < |A_{ji}|$.

6.3. Reflections

Let R^+ be a system of positive roots of Lie superalgebra $\mathfrak{g}$, and let $B = \{\sigma_1, \ldots, \sigma_n\}$ be the corresponding system of simple roots with some corresponding pair $(A = A_B, I = I_B)$. Then the set $(R^+ \backslash \{\sigma_k\}) \coprod \{-\sigma_k\}$ is a system of positive roots for any $k \in \{1, \ldots, n\}$. This operation is called *the reflection in* σ_k; it changes the system of simple roots by the formulas

$$r_{\sigma_k}(\sigma_j) = \begin{cases} -\sigma_j & \text{if } k = j, \\ \sigma_j + B_{kj}\sigma_k & \text{if } k \neq j, \end{cases} \tag{43}$$

where

$$B_{kj} = \begin{cases} -\dfrac{2A_{kj}}{A_{kk}} & \text{if } A_{kk} \neq 0 \text{ and } -\dfrac{2A_{kj}}{A_{kk}} \in \mathbb{Z}/p\mathbb{Z}, \\ p-1 & \text{if } A_{kk} \neq 0 \text{ and } -\dfrac{2A_{kj}}{A_{kk}} \notin \mathbb{Z}/p\mathbb{Z}, \\ 1 & \text{if } p_k = \bar{1}, \ \ A_{kk} = 0, \ \ A_{kj} \neq 0, \\ 0 & \text{if } p_k = \bar{1}, \ \ A_{kk} = A_{kj} = 0, \\ p-1 & \text{if } p_k = \bar{0}, \ \ A_{kk} = \bar{0}, \ \ A_{kj} \neq 0, \\ 0 & \text{if } p_k = \bar{0}, \ \ A_{kk} = \bar{0}, \ \ A_{kj} = 0, \end{cases} \tag{44}$$

where we consider $\mathbb{Z}/p\mathbb{Z}$ as a subfield of $\mathbb{K}$.

Remark 6.2. The description of the numbers B_{ik} is empirical and based on classification [**BGL**]: For infinite-dimensional Lie (super)algebras these numbers might be different. In principle, in the second, fourth and penultimate cases, the matrix (44) can be equal to $kp - 1$ for any $k \in \mathbb{N}$, and in the last case any element of $\mathbb{K}$ may occur. For $\dim \mathfrak{g} < \infty$, this does do not happen (and it is of interest to investigate at least the simplest infinite dimensional case — the modular analog of [**CCLL**]).

The values $-\dfrac{2A_{kj}}{A_{kk}}$ and $-\dfrac{A_{kj}}{A_{kk}}$ are elements of $\mathbb{K}$, while the roots are elements of a vector space over $\mathbb{R}$. Therefore

These expressions in the first and third cases in (44) *should be understood as "the minimal non-negative integer congruent to* $-\dfrac{2A_{kj}}{A_{kk}}$ *or* $-\dfrac{A_{kj}}{A_{kk}}$*, respectively".* (If $\dim \mathfrak{g} < \infty$, these expressions are always congruent to integers. There is known just one exception: If $p = 2$ and $A_{kk} = A_{jk}$, then $-\dfrac{2A_{jk}}{A_{kk}}$ should be understood as 2, not 0.)

The name "reflection" is used because in the case of (semi)simple finite-dimensional Lie algebras this action extended on the whole R by linearity is a map from R to R, and it does not depend on R^+, only on σ_k. This map is usually denoted by r_{σ_k} or just r_k. The map r_{σ_i} extended to the $\mathbb{R}$-span of R is reflection in the hyperplane orthogonal to σ_i relative the bilinear form dual to the Killing form.

The reflections in the even (odd) roots are said to be *even* (*odd*). A simple root is called *isotropic*, if the corresponding row of the Cartan matrix has zero on the diagonal, and *non-isotropic* otherwise. The reflections that correspond to isotropic or non-isotropic roots will be referred to accordingly.

If there are isotropic simple roots, the reflections r_α do not, as a rule, generate a version of the *Weyl group* because the product of two reflections in nodes not connected by one (perhaps, multiple) edge is not defined. These reflections just connect pair of "neighboring" systems of simple roots and there is no reason to expect that we can multiply two distinct such reflections. In the general case (of Lie superalgebras and $p > 0$), the action of a given isotropic reflections (43) can not, generally, be extended to a linear map $R \longrightarrow R$. For Lie superalgebras over $\mathbb{C}$, one can extend the action of reflections by linearity to the root lattice but this extension preserves the root system only for $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(2m+1|2n)$, cf. [**Se1**].

If σ_i is an odd isotropic root, then the corresponding reflection sends one set of Chevalley generators into a new one:

$$\tilde{X}_i^{\pm} = X_i^{\mp}; \quad \tilde{X}_j^{\pm} = \begin{cases} [X_i^{\pm}, X_j^{\pm}] & \text{if } A_{ij} \neq 0, \bar{0}, \\ X_j^{\pm} & \text{otherwise.} \end{cases} \tag{45}$$

6.3.1. On neighboring root systems

Serganova [**Se**] proved (for $p = 0$) that there is always a chain of reflections connecting B_1 with some system of simple roots B_2' equivalent to B_2 in the sense of definition 4.6. Here is the modular version of Serganova's Lemma. Observe that Serganova's statement is not weaker: Serganova used only odd reflections.

Lemma 6.3 ([**LCh**])**.** *For any two systems of simple roots B_1 and B_2 of any simple finite dimensional Lie superalgebra with Cartan matrix, there is always a chain of reflections connecting B_1 with B_2.*

7. Presentations of $\mathfrak{g}(A)$

7.1. Serre relations, see [GL2]

Let A be an $n\times n$ matrix. We find the defining relations by induction on n with the help of the Hochschild–Serre spectral sequence (for its description for Lie superalgebras, which has certain subtleties, see [**Po**]). For the basis of induction consider the following cases of Dynkin diagrams with one vertex and no edges:

$$\begin{array}{lll} \circ \text{ or } \bullet & \text{no relations, i.e., } \mathfrak{g}^{\pm} \text{ are free Lie superalgebras} & \text{if } p\neq 3;\\ \bullet & \operatorname{ad}^2_{X^{\pm}}(X^{\pm})=0 & \text{if } p=3;\\ \otimes & [X^{\pm},X^{\pm}]=0. & \end{array} \tag{46}$$

Set $\deg X_i^{\pm}=0$ for $1\leqslant i\leqslant n-1$ and $\deg X_n^{\pm}=\pm 1$. Let $\mathfrak{g}^{\pm}=\oplus\mathfrak{g}_i^{\pm}$ and $\mathfrak{g}=\oplus\mathfrak{g}_i$ be the corresponding $\mathbb{Z}$-gradings. Set $\mathfrak{g}_{\pm}=\mathfrak{g}^{\pm}/\mathfrak{g}_0^{\pm}$. From the Hochschild–Serre spectral sequence for the pair $\mathfrak{g}_0^{\pm}\subset\mathfrak{g}^{\pm}$ we get (for more detail, see [**LCh**]):

$$H_2(\mathfrak{g}_{\pm})\subset H_2(\mathfrak{g}_0^{\pm})\oplus H_1(\mathfrak{g}_0^{\pm};H_1(\mathfrak{g}_{\pm}))\oplus H_0(\mathfrak{g}_0^{\pm};H_2(\mathfrak{g}_{\pm})). \tag{47}$$

It is clear that

$$H_1(\mathfrak{g}_{\pm})=\mathfrak{g}_1^{\pm},\quad H_2(\mathfrak{g}_{\pm})=\wedge^2(\mathfrak{g}_1^{\pm})/\mathfrak{g}_2^{\pm}. \tag{48}$$

So, the second summand in (47) provides us with relations of the form:

$$\begin{array}{ll} (\operatorname{ad}_{X_n^{\pm}})^{k_{ni}}(X_i^{\pm})=0 & \text{if the } n\text{-th simple root is not } \otimes\\ [X_n,X_n]=0 & \text{if the } n\text{-th simple root is } \otimes\,. \end{array} \tag{49}$$

while the third summand in (47) is spanned by the $\mathfrak{g}_0^{\pm}$-lowest vectors in

$$\wedge^2(\mathfrak{g}_1^{\pm})/(\mathfrak{g}_2^{\pm}+\mathfrak{g}^{\pm}\wedge^2(\mathfrak{g}_1^{\pm})). \tag{50}$$

Let the matrix $B=(B_{ij})$ be as in formula (44). The following proposition, whose proof is straightforward, illustrates the usefulness of our normalization of Cartan matrices as compared with other options:

Proposition 7.1. *The numbers k_{in} and k_{ni} in (49) are expressed in terms of (B_{ij}) as follows:*

$$\begin{array}{ll} (\operatorname{ad}_{X_i^{\pm}})^{1+B_{ij}}(X_j^{\pm})=0 & \textit{for } i\neq j\\ [X_i^{\pm},X_i^{\pm}]=0 & \textit{if } A_{ii}=0 \textit{ and } p\neq 2\\ \left(X_i^{\pm}\right)^2=0 & \textit{if } A_{ii}=0 \textit{ and } p=2. \end{array} \tag{51}$$

Usually, only the relations of the first line in (51) are said to be ***Serre relations*** *for the Lie superalgebra* $\mathfrak{g}(A)$, but it is convenient to incorporate (and we propose to do so) the relations (18) and (51) to the set of *Serre relations.*

If $p=3$, then the relation

$$[X_i^{\pm},[X_i^{\pm},X_i^{\pm}]]=0\quad\text{for } X_i^{\pm} \text{ odd and } A_{ii}=1 \tag{52}$$

is not a consequence of the Jacobi identity; for simplicity, however, we will include it in the set of Serre relations.

7.2. Non-Serre relations

These are relations that correspond to the third summand in (47). Let us consider the simplest case: $\mathfrak{sl}(m|n)$ in the realization with the system of simple roots

$$\circ - \dots - \circ - \otimes - \circ - \dots - \circ \tag{53}$$

Then $H_2(\mathfrak{g}_\pm)$ from the third summand in (47) is just $\wedge^2(\mathfrak{g}_\pm)$. For simplicity, we confine ourselves to the positive roots. Let $X_1, \dots, X_{m-1}$ and $Y_1, \dots, Y_{n-1}$ be the root vectors corresponding to even roots separated by the root vector Z corresponding to the root $\otimes$.

If $n = 1$ or $m = 1$, then $\wedge^2(\mathfrak{g})$ is an irreducible $\mathfrak{g}_{\bar 0}$-module and there are no non-Serre relations. If $n \neq 1$ and $m \neq 1$, then $\wedge^2(\mathfrak{g})$ splits into 2 irreducible $\mathfrak{g}_{\bar 0}$-modules. The lowest component of one of them corresponds to the relation $[Z, Z] = 0$, the other one corresponds to the non-Serre-type relation

$$[[X_{m-1}, Z], [Y_1, Z]] = 0. \tag{54}$$

If, instead of $\mathfrak{sl}(m|n)$, we would have considered the Lie algebra $\mathfrak{sl}(m + n)$, the same argument would have led us to the two relations, both of Serre type:

$$\mathrm{ad}_Z^2(X_{m-1}) = 0, \qquad \mathrm{ad}_Z^2(Y_1) = 0.$$

In what follows we give an explicit description of the defining relations in terms of the Chevalley generators of the Lie (super)algebras of the form $\mathfrak{g}(A)$ or their simple subquotients $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$.

8. The Lie (super)algebras of the form $\mathfrak{g}(A)$ or their simple subquotients $\mathfrak{g}^{(1)}(A)/\mathfrak{c}$

8.1. Over $\mathbb{C}$

Kaplansky was the first (see his newsletters in [**Kapp**]) to discover the exceptional algebras $\mathfrak{ag}(2)$ and $\mathfrak{ab}(3)$ (he dubbed them Γ_2 and Γ_3, respectively) and a parametric family $\mathfrak{osp}(4|2;\alpha)$ (he dubbed it $\Gamma(A, B, C)$)); our notation reflect the fact that $\mathfrak{ag}(2)_{\bar 0} = \mathfrak{sl}(2) \oplus \mathfrak{g}(2)$ and $\mathfrak{ab}(3)_{\bar 0} = \mathfrak{sl}(2) \oplus \mathfrak{o}(7)$ ($\mathfrak{o}(7)$ is B_3 in Cartan's nomenclature). Kaplansky's description (irrelevant to us at the moment except for the fact that A, B and C are on equal footing and $A + B + C = 0$) of what we now identify as $\mathfrak{osp}(4|2;\alpha)$, a parametric family of deforms of $\mathfrak{osp}(4|2)$, made an S_3-symmetry of the parameter manifest (to A. A. Kirillov, and he informed us, in 1976). Indeed, since $A + B + C = 0$, and $\alpha \in \mathbb{C} \cup \infty$ is the ratio of the two remaining parameters, we get an S_3-action on the plane $A + B + C = 0$ which in terms of α is generated by the transformations:

$$\alpha \longmapsto -1 - \alpha, \qquad \alpha \longmapsto \frac{1}{\alpha}. \tag{55}$$

This symmetry should have immediately sprang to mind since $\mathfrak{osp}(4|2;\alpha)$ is strikingly similar to $\mathfrak{wk}(3;a)$ found 5 years earlier, cf. (58), and since $S_3 \simeq \mathrm{SL}(2;\mathbb{Z}/2)$.

Figure 8.1 depicts the fundamental domains of the S_3-action. The other transformations generated by (55) are

$$\alpha \longmapsto -\frac{1+\alpha}{\alpha}, \quad \alpha \longmapsto -\frac{1}{\alpha+1}, \quad \alpha \longmapsto -\frac{\alpha}{\alpha+1}.$$

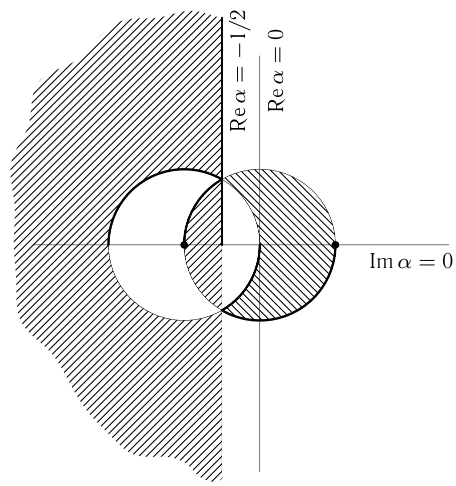


Figure 1:

8.1.1. Notation: On matrices with a "–" sign and other notation in the lists of inequivalent Cartan matrices

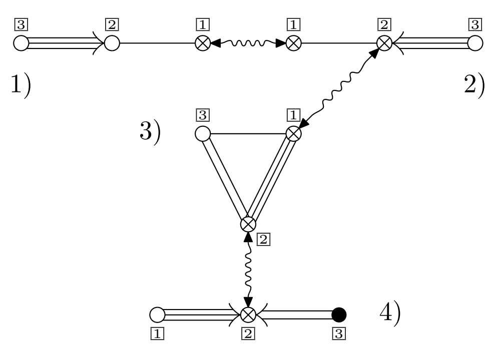
The rectangular matrix at the beginning of each list of inequivalent Cartan matrices for each Lie superalgebra shows the result of odd reflections (the number of the row is the number of the Cartan matrix in the list below, the number of the column is the number of the root (given by small boxed number) in which the reflection is made; the cells contain the results of reflections (the number of the Cartan matrix obtained) or a "–" if the reflection is not appropriate because $A_{ii} \neq 0$. Some of the Cartan matrices thus obtained are equivalent, as indicated.

The number of the matrix A such that $\mathfrak{g}(A)$ has only one odd simple root is *boxed*, that with all simple roots odd is **underlined**. The nodes are numbered by small boxed numbers; the curly lines with arrows depict odd reflections.

8.1.2. Cartan matrices

Recall that $\mathfrak{ag}(2)$ of sdim $= 17|14$ has the following Cartan matrices

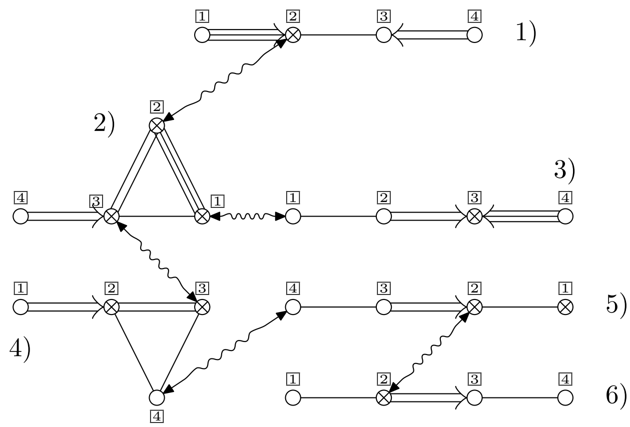
$$\begin{pmatrix} 2 & - & - \\ 1 & 3 & - \\ - & 2 & 4 \\ - & - & 3 \end{pmatrix}$$



$$\boxed{1)}\ \begin{pmatrix}0&-1&0\\-1&2&-3\\0&-1&2\end{pmatrix}\quad 2)\ \begin{pmatrix}0&-1&0\\-1&0&3\\0&-1&2\end{pmatrix}\quad 3)\ \begin{pmatrix}0&-3&1\\-3&0&2\\-1&-2&2\end{pmatrix}\quad 4)\ \begin{pmatrix}2&-1&0\\-3&0&2\\0&-1&1\end{pmatrix} \tag{56}$$

Recall that $\mathfrak{ab}(3)$ of sdim $= 24|16$ has the following Cartan matrices

$$\begin{pmatrix}-&2&-&-\\3&1&4&-\\2&-&-&-\\-&-&2&5\\-&6&-&4\\-&5&-&-\end{pmatrix}$$



$$\boxed{1)}\ \begin{pmatrix}2&-1&0&0\\-3&0&1&0\\0&-1&2&-2\\0&0&-1&2\end{pmatrix}\quad 2)\ \begin{pmatrix}0&-3&1&0\\-3&0&2&0\\1&2&0&-2\\0&0&-1&2\end{pmatrix}\quad \boxed{3)}\ \begin{pmatrix}2&-1&0&0\\-1&2&-1&0\\0&-2&0&3\\0&0&-1&2\end{pmatrix}$$
$$4)\ \begin{pmatrix}2&-1&0&0\\-2&0&2&-1\\0&2&0&-1\\0&-1&-1&2\end{pmatrix}\quad 5)\ \begin{pmatrix}0&1&0&0\\-1&0&2&0\\0&-1&2&-1\\0&0&-1&2\end{pmatrix}\quad \boxed{6)}\ \begin{pmatrix}2&-1&0&0\\-1&2&-1&0\\0&-2&2&-1\\0&0&-1&0\end{pmatrix} \tag{57}$$

8.2. Modular Lie algebras and Lie superalgebras

8.2.1. $p = 2$, Lie algebras

Weisfeiler and Kac [**WK**] discovered two new parametric families that we denote $\mathfrak{wk}(3;a)$ and $\mathfrak{wk}(4;a)$ (*Weisfeiler and Kac algebras*).

$\mathfrak{wk}(3;a)$, where $a \neq 0, -1$, of dim 18 is a non-super version of $\mathfrak{osp}(4|2;a)$ (although no $\mathfrak{osp}$ exists for $p = 2$); the dimension of its simple subquotient $\mathfrak{wk}(3;a)^{(1)}/\mathfrak{c}$ is equal to 16; the inequivalent Cartan matrices are:

$$1)\ \begin{pmatrix}\bar{0}&a&0\\a&\bar{0}&1\\0&1&\bar{0}\end{pmatrix},\ 2)\ \begin{pmatrix}\bar{0}&1+a&a\\1+a&\bar{0}&1\\a&1&\bar{0}\end{pmatrix}$$

$\mathfrak{wk}(4;a)$, where $a \neq 0, -1$, of dim $= 34$; the inequivalent Cartan matrices are:

$$1)\ \begin{pmatrix}\bar{0}&a&0&0\\a&\bar{0}&1&0\\0&1&\bar{0}&1\\0&0&1&\bar{0}\end{pmatrix},\ 2)\ \begin{pmatrix}\bar{0}&1&1+a&0\\1&\bar{0}&a&0\\a+1&a&\bar{0}&a\\0&0&a&\bar{0}\end{pmatrix},\ 3)\ \begin{pmatrix}\bar{0}&a&0&0\\a&\bar{0}&a+1&0\\0&a+1&\bar{0}&1\\0&0&1&\bar{0}\end{pmatrix}$$

Weisfeiler and Kac investigated also which of these algebras are isomorphic and the answer is as follows:

$$\begin{aligned} &\mathfrak{wk}(3;a) \simeq \mathfrak{wk}(3;a') \Longleftrightarrow a' = \frac{\alpha a+\beta}{\gamma a+\delta}, \text{where } \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}/2) \\ &\mathfrak{wk}(4;a) \simeq \mathfrak{wk}(4;a') \Longleftrightarrow a' = \frac{1}{a}. \end{aligned} \tag{58}$$

8.2.2. $p=2$**, Lie superalgebras**

The same Cartan matrices as for $\mathfrak{wk}$ algebras but with arbitrary distribution of 0's on the main diagonal correspond to Lie superalgebras $\mathfrak{bgl}(3;a)$ and $\mathfrak{bgl}(4;a)$ discovered in [**BGL**]. The conditions when they are isomorphic are the same as in (58), they have the same inequivalent Cartan matrices, and are considered also only if $a\neq 0,1$ (since otherwise they are not simple). We have $\operatorname{sdim}\mathfrak{bgl}(3;a) = 10/8|8$ and $\operatorname{sdim}\mathfrak{bgl}(4;a) = 18|16$.

8.3. Systems of simple roots of the $\mathfrak{e}$-type Lie superalgebras

Observe that if $p=2$ and the Cartan matrix has no parameters, the reflections do not change the shape of the Dynkin diagram. Therefore, for the $\mathfrak{e}$-superalgebras, it suffices to list distributions of parities of the nodes in order to describe the Dynkin diagrams. Since there are tens and even hundreds of diagrams in these cases, this possibility saves a lot of space, see the lists of all inequivalent Cartan matrices of the $\mathfrak{e}$-type Lie superalgebras. For the lists of the inequivalent systems of simple roots of the $\mathfrak{e}$-type Lie superalgebras, see the arXiv version of the paper (`arXiv:0911.0243`).

9. Defining relations in characteristic 2

To save space, in what follows we omit indicating the Serre relations; their fulfillment is assumed. Additionally there appear relations of a new type (non-Serre relations). Here we describe them. We have proved them analytically only for Lie (super)-algebras of $\mathfrak{sl}$ type and their relatives. Relations for the rest of the (super)algebras are results of computations with `SuperLie`. For serial Lie (super)algebras (like $\mathfrak{o}$, $\mathfrak{oo}$, $\mathfrak{osp}$, $\mathfrak{spe}$), the relations are conjectural.

9.1. Results

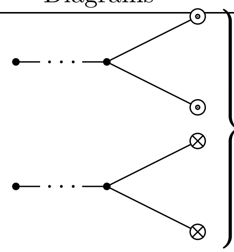
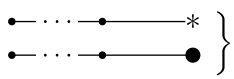
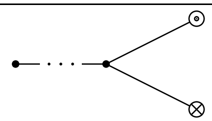
Here we consider the classical Lie algebras and superalgebras as preserving the volume element or a non-degenerate bilinear form. We usually interpret the exceptional Lie (super)algebras as preserving a non-integrable distribution, cf. [**Shch**] but here we just construct them from their Cartan matrices.

For subalgebras of $\mathfrak{gl}$, we set $x_i = E_{i,i+1}, y_i = E_{i+1,i}, h_i = E_{i,i} - E_{i+1,i+1}$; the Lie sub(super)algebra $\mathfrak{n}$ consists of upper-triangular (super)matrices.

Theorem 9.1. *For* $\mathfrak{g} = \mathfrak{sl}(n+1)$ *or* $\mathfrak{sl}(a|b)$*, where* $a+b=n+1$*: In characteristic* >2*, the Serre relations (51) define* $\mathfrak{n}$*; in characteristic* 2*, the following additional relations are required:*

$$[[x_{i-1},x_i],[x_i,x_{i+1}]] = 0 \quad \text{for } 1<i<n. \tag{59}$$

$p=2$**, Lie superalgebras. Dynkin diagrams for** $p=2$

Diagrams	$\mathfrak{g}$	v	ev	od	png	$ng \leqslant \min(*\,,*)$
1)  2)	$\mathfrak{ooc}(2;2k_{\bar 0}\vert 2k_{\bar 1})\ltimes\mathbb{K}I_0$ if $k_{\bar 0}+k_{\bar 1}$ is odd; $\mathfrak{ooc}(1;2k_{\bar 0}\vert 2k_{\bar 1})\ltimes\mathbb{K}I_0$ if $k_{\bar 0}+k_{\bar 1}$ is even.	$k_{\bar 0}+k_{\bar 1}$	$k_{\bar 0}-2$	$k_{\bar 1}$	$\bar 0$	$2k_{\bar 0}-4, 2k_{\bar 1}$
			$k_{\bar 1}$	$k_{\bar 0}-2$	$\bar 1$	$2k_{\bar 0}-3, 2k_{\bar 1}-1$
			$k_{\bar 1}-2$	$k_{\bar 0}$	$\bar 0$	$2k_{\bar 0}, 2k_{\bar 1}-4$
			$k_{\bar 0}$	$k_{\bar 1}-2$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-3$
			$k_{\bar 0}-1$	$k_{\bar 1}-1$		$2k_{\bar 0}-2, 2k_{\bar 1}-1$
			$k_{\bar 1}-1$	$k_{\bar 0}-1$		$2k_{\bar 0}-1, 2k_{\bar 1}-2$
3) 4) 	$\mathfrak{oo}_{I\Pi}^{(1)}(2k_{\bar 0}+1\vert 2k_{\bar 1})$	$k_{\bar 0}+k_{\bar 1}$	$k_{\bar 0}-1$	$k_{\bar 1}$	$\bar 0$	$2k_{\bar 0}-2, 2k_{\bar 1}$
			$k_{\bar 1}$	$k_{\bar 0}-1$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-1$
			$k_{\bar 1}-1$	$k_{\bar 0}$	$\bar 0$	$2k_{\bar 0}, 2k_{\bar 1}-2$
			$k_{\bar 0}$	$k_{\bar 1}-1$	$\bar 1$	$2k_{\bar 0}-1, 2k_{\bar 1}-1$
5) 	$\mathfrak{pec}(2;m)\ltimes\mathbb{K}I_0$ for m odd; $\mathfrak{pec}(1;m)\ltimes\mathbb{K}I_0$ for m even.	m				

Notation

The Dynkin diagrams in the table correspond to Cartan matrix Lie superalgebras close to ortho-orthogonal and periplectic Lie superalgebras. Each thin black dot may be $\otimes$ or $\odot$; the last five columns show conditions on the diagrams; in the last four columns, it suffices to satisfy conditions in any one row. Horizontal lines in the last four columns separate the cases corresponding to different Dynkin diagrams. The notation are: v is the total number of nodes in the diagram; ng is the number of "grey" nodes $\otimes$'s among the thin black dots; png is the parity of this number; ev and od are the number of thin black dots such that the number of $\otimes$'s to the left from them is even and odd, respectively.

Remark 9.2. In characteristic $p > 0$, the Lie algebra $\mathfrak{sl}(pk)$ is not simple, since it contains the center $\mathfrak{c} = \{\lambda \cdot 1_{pk} \mid \lambda \in \mathbb{K}\}$. The corresponding simple Lie algebra $\mathfrak{sl}(pk)/\mathfrak{c}$ is denoted by $\mathfrak{psl}(pk)$. Since the reduction from $\mathfrak{sl}(pk)$ to $\mathfrak{psl}(pk)$ does not affect the structure of $\mathfrak{n}$, its presentation is the same for $\mathfrak{gl}(pk)$, $\mathfrak{sl}(pk)$, and $\mathfrak{psl}(pk)$. Same applies to any other Lie (super)algebra with non-invertible Cartan matrix.

Theorem 9.3. *Let the nodes of the Dynkin diagram of $\mathfrak{e}(8)$ be numbered as usual:*
1 2 3 4 5 6 7
8

For $\mathfrak{g} = \mathfrak{e}(n)$ or $\mathfrak{g} = \mathfrak{e}(n; i)$: In characteristic 2, in the case of $\mathfrak{g} = \mathfrak{e}(8)$, the following list of relations must be added to the Serre relations:

$$\begin{array}{l}
[[x_1, x_2], [x_2, x_3]] = 0;\\
[[x_2, x_3], [x_3, x_4]] = 0;\\
[[x_3, x_4], [x_4, x_5]] = 0;\\
[[x_4, x_5], [x_5, x_6]] = 0;\\
[[x_5, x_6], [x_6, x_7]] = 0;\\
[[x_4, x_5], [x_5, x_8]] = 0;\\
[[x_5, x_6], [x_5, x_8]] = 0;\\
[[x_4, [x_5, x_6]], [x_4, [x_5, x_8]]] = 0;\\
[[x_4, [x_5, x_6]], [x_8, [x_5, x_6]]] = 0;\\
[[x_4, [x_5, x_8]], [x_8, [x_5, x_6]]] = 0;\\
[[x_3, [x_4, [x_5, x_6]]], [x_3, [x_4, [x_5, x_8]]]] = 0;\\
[[x_4, [x_5, [x_6, x_7]]], [x_8, [x_5, [x_6, x_7]]]] = 0;\\
[[x_2, [x_3, [x_4, [x_5, x_6]]]], [x_2, [x_3, [x_4, [x_5, x_8]]]]] = 0;\\
[[x_1, [x_2, [x_3, [x_4, [x_5, x_6]]]]], [x_1, [x_2, [x_3, [x_4, [x_5, x_8]]]]]] = 0.
\end{array} \tag{60}$$

To obtain the corresponding lists of relations for $\mathfrak{e}(6)$ or $\mathfrak{e}(7)$, one should delete the relations containing the "extra" x_i and renumber the rest of the x_i, i.e:
1) delete the relations containing x_1 for $\mathfrak{e}(7)$, x_1 and x_2 for $\mathfrak{e}(6)$;
2) decrease all indices of the x_i by 1 for $\mathfrak{e}(7)$, by 2 for $\mathfrak{e}(6)$.

Proof: Direct computer calculations.

Remark 9.4. Here is a shorter way to describe these relations. Let *a chain of nodes* for a Dynkin diagram with n nodes be a sequence $i_1, \dots, i_k$, where $k \geqslant 2$ and
1) $i_j \in \overline{1, n}$ for all $j = 1, \dots, k$;
2) $i_j \neq i_{j'}$ for $j \neq j'$;
3) nodes with numbers i_j and i_{j+1} are connected for all $j = 1, \dots, k-1$.

The above non-Serre relations (both for $\mathfrak{sl}(n+1)$ and $\mathfrak{e}(n)$ can be represented in the form

$$[\ [x_{i_1}, [\dots, [x_{i_{k-1}}, x_{i_k}]\dots]],\ [x_{i_1}, [\dots, [x_{i_{k-1}}, x_{i'_k}]\dots]]\ ] = 0, \tag{61}$$
where $i_1, \dots, i_{k-1}, i_k$ and $i_1, \dots, i_{k-1}, i'_k$ are two chains of nodes that differ only in the last element.

All the relations that can be represented in the form (61) are necessary.

In what follows we only consider the Lie algebras $\mathfrak{g}(A)$; *the non-Serre relations of Lie superalgebras $s(\mathfrak{g}(A))$ from which $\mathfrak{g}(A)$ can be obtained by means of forgetful functor are the same as those of $\mathfrak{g}(A)$.*

Theoretically, there could be redundant ones among them, we can only conjecture (by analogy with $\mathfrak{sl}$ and $\mathfrak{e}$ types) that no redundancies occur.

9.1.1. $\mathfrak{g}=\mathfrak{o}_B(2n)$

The orthogonal algebra is, by definition, the Lie algebra of linear transformations preserving a given non-degenerate symmetric bilinear form B. The bilinear form is usually taken with the Gram matrix 1_{2n} or Π_{2n}. In characteristic >2, these two forms are equivalent over any perfect field. The corresponding Lie algebra has the same defining relations as in characteristic 0, so in this subsection we only consider $p=2$.

It turns out ([**Le1**]) that these two forms are not equivalent over any ground field $\mathbb{K}$ of characteristic 2. If $\mathbb{K}$ is perfect, then any non-degenerate symmetric bilinear form is equivalent to one of these two forms: It is equivalent to Π_n, if it is zero-diagonal; otherwise, it is equivalent to 1_n.

The orthogonal Lie algebras corresponding to these two forms (we denote them $\mathfrak{o}_I(n)$ and $\mathfrak{o}_\Pi(n)$, respectively) are not isomorphic and have different properties.

In particular, *only* $\mathfrak{o}_\Pi(2n)$ *for* $n\geqslant 3$ *is close to an algebra with a Cartan matrix* (same as in characteristic 0). The corresponding algebra $\mathfrak{g}^{(1)}(A)$ is $\mathfrak{oc}(2;2n)$ (i.e., the central extension of $\mathfrak{o}_\Pi^{(2)}(2n)$, given by the formula (33)).

9.1.2. $\mathfrak{oc}(2;2n)$

The algebra $\mathfrak{o}_\Pi^{(2)}(2n)$ (whose central extension is $\mathfrak{oc}(2;2n)$) consists of matrices of the following form (where $ZD(n)$ denotes the space (Lie algebra if $p=2$) of symmetric zero-diagonal $n\times n$-matrices):

$$\begin{pmatrix} A & B\\ C & A^T\end{pmatrix},\qquad \begin{array}{l}\text{where } A\in\mathfrak{sl}(n);\\ B,C\in ZD(n).\end{array}$$

The Chevalley generators of $\mathfrak{oc}(2;2n)$ are:

$$\begin{array}{l}
x_i=E_{i,i+1}+E_{n+i+1,n+i}\quad\text{for } 1\leqslant i\leqslant n-1;\\
x_n=E_{n-1,2n}+E_{n,2n-1};\\
y_i=x_i^T\quad\text{for } 1\leqslant i\leqslant n;\\
h_i=E_{i,i}+E_{i+1,i+1}+E_{n+i,n+i}+E_{n+i+1,n+i+1}\quad\text{for } 1\leqslant i\leqslant n-1;\\
h_n=h_{n-1}+z,
\end{array}$$

where z is central element.

Theorem 9.5. *In characteristic* 2*, for* $\mathfrak{oc}(2;2n)$*, where* $n\geqslant 4$*, the defining relations for* $\mathfrak{n}$ *are Serre relations plus the following ones:*

$$[[x_{i-1},x_i],[x_i,x_{i+1}]]=0\quad\textit{for } 2\leqslant i\leqslant n-2;$$

$$\begin{array}{l}
[[x_{n-3},x_{n-2}],[x_{n-2},x_n]]=0;\\
[[x_{n-2},x_{n-1}],[x_{n-2},x_n]]=0;\\
[[x_{n-3},[x_{n-2},x_{n-1}]],[x_n,[x_{n-1},x_{n-2}]]=0;\\
[[x_{n-3},[x_{n-2},x_n]],[x_n,[x_{n-1},x_{n-2}]]=0;
\end{array}$$

and, for $1\leqslant i\leqslant n-3$*,*

$$[[x_{n-1},[x_i,[x_{i+1},\dots,[x_{n-3},x_{n-2}]\dots]]],[x_n,[x_i,[x_{i+1},\dots,[x_{n-3},x_{n-2}]\dots]]]]=0.$$

(We don't consider the case of $n = 3$ in the theorem because $\mathfrak{oc}(2;6)$ is isomorphic to $\mathfrak{sl}(4)$.)

9.1.3. $\mathfrak{g} = \mathfrak{o}_I^{(1)}(2n)$

As shown above, if $n \geqslant 2$, then $\mathfrak{o}_I(2n) \not\simeq \mathfrak{o}_I^{(1)}(2n) \simeq \mathfrak{o}_I^{(2)}(2n)$ (and if $n = 1$, then the algebra $\mathfrak{o}_I(2n)$ is nilpotent). So any set of generators of $\mathfrak{o}_I(2n)$ contains "extra" (as compared with generators of $\mathfrak{o}_I^{(1)}(2n)$) generators $a_1, \dots, a_{2n}$. The relations containing these generators say nothing new about the structure of the simple (and, thus, more interesting) algebra $\mathfrak{o}_I^{(1)}(2n)$. Because of this and because we want to make the set of generators we use as small as possible, we consider the algebra $\mathfrak{o}_I^{(1)}(2n)$. It consists of symmetric zero-diagonal $2n \times 2n$-matrices. We can choose the following generators (for the whole algebra since in this case there is no apparent analog of the maximal nilpotent subalgebra $\mathfrak{n}$):

$$X_i = E_{i,i+1} + E_{i+1,i} \quad \text{for } 1 \leqslant i \leqslant 2n-1.$$

Theorem 9.6. *The following are the defining relations for $\mathfrak{o}_I^{(1)}(2n)$, $n \geqslant 2$:*

$$\begin{array}{ll}
[X_i, X_j] = 0 & \text{for } 1 \leqslant i,j \leqslant 2n-1, |i-j| \geqslant 2;\\
\left.\begin{array}{r}[X_i,[X_i,X_{i+1}]] = x_{i+1}\\ [X_{i+1},[X_i,X_{i+1}]] = x_i\end{array}\right\} & \text{for } 1 \leqslant i \leqslant 2n-2;\\
[[X_{i-1},X_i],[X_i,X_{i+1}]] = 0 & \text{for } 2 \leqslant i \leqslant 2n-2.
\end{array}$$

Proof. (Sketch of.) The algebra $\mathfrak{o}_I^{(1)}(2n)$ is filtered:

$$0 = L_0 \subset \dots \subset L_{2n-1},$$

where L_k consists of all symmetric zero-diagonal matrices M such that $M_{ij} = 0$ for all i, j such that $|i-j| > k$. The associated graded algebra is isomorphic to the algebra of upper-triangular matrices, i.e., a maximal nilpotent subalgebra of $\mathfrak{sl}(2n)$. So we can use Theorem 9.1. □

Remark 9.7. Presentations of the Lie algebra $\mathfrak{o}_I^{(1)}(2n+1)$, where $n \geqslant 1$, are similar in shape.

9.1.4. $\mathfrak{g} = \mathfrak{o}_B(2n+1)$

For this algebra, again, the case of characteristic > 2 does not differ from the case of characteristic 0, so we only consider the case of characteristic 2. Then, if the ground field is perfect, all the non-degenerate symmetric bilinear form over a linear space of dimension $2n+1$ are equivalent. We choose the form Π_{2n+1}.

9.1.4.1. $\mathfrak{g} = \mathfrak{o}_\Pi(2n+1)$ It is easy to see that

$$\mathfrak{o}_\Pi(2n+1) \not\simeq \mathfrak{o}_\Pi^{(1)}(2n+1) \text{ and } \mathfrak{o}_\Pi^{(1)}(2n+1) \simeq \mathfrak{o}_\Pi^{(2)}(2n+1) \text{ for } n \geqslant 1.$$

So, as for $\mathfrak{o}_I(2n)$, we consider the first derived algebra $\mathfrak{o}_\Pi^{(1)}(2n+1)$. The algebra $\mathfrak{o}_\Pi^{(1)}(2n+1)$ consists of matrices of the following form:

$$\begin{pmatrix} A & X & B \\ Y^T & 0 & X^T \\ C & Y & A^T \end{pmatrix}, \quad \begin{array}{l} \text{where } A \in \mathfrak{gl}(n); B, C \in ZD(n); \\ X, Y \text{ are } n\text{-vectors.} \end{array}$$

This algebra has a Cartan matrix. The Chevalley generators are:

$$\begin{array}{l} x_i = E_{i,i+1} + E_{n+i+2,n+i+1} \quad \text{for } 1 \leqslant i \leqslant n-1; \\ x_n = E_{n,n+2} + E_{n+1,2n+1}; \\ y_i = x_i^T \quad \text{for } 1 \leqslant i \leqslant n; \\ h_i = E_{i,i} + E_{i+1,i+1} + E_{n+i+1,n+i+1} + E_{n+i+2,n+i+2} \quad \text{for } 1 \leqslant i \leqslant n-1; \\ h_n = E_{n,n} + E_{2n+1,2n+1}. \end{array}$$

Theorem 9.8. *In characteristic* 2*, for* $\mathfrak{g} = \mathfrak{o}_\Pi^{(1)}(2n+1)$*, the defining relations for* $\mathfrak{n}$ *are the Serre relations plus the following ones:*

$$[[x_{i-1}, x_i], [x_i, x_{i+1}]] = 0 \quad \text{for } 2 \leqslant i \leqslant n-2.$$

9.1.5. $\mathfrak{g}(2)$

The Cartan matrix of $\mathfrak{g}(2)$ reduced modulo 2 coincides with Cartan matrix of $\mathfrak{sl}(3)$. There is, however, another approach: Select the Chevalley basis in the Lie algebra $\mathfrak{g}(2)$ as explicitly described in [**FH**], p. 346. Reducing the integer structure constants reduced modulo 2 we get a **simple** Lie algebra $\mathfrak{g}(2)_\mathbb{K}$ (its basis is that of $\mathfrak{g}(2)$). This Lie algebra is isomorphic to $\mathfrak{psl}(4)$.

9.1.6. $\mathfrak{f}(4)$

There is no $\mathbb{Z}$-form of $\mathfrak{f}(4)$ such that the algebra $\mathfrak{f}(4)_\mathbb{K}$ is still simple.

9.1.7. $\mathfrak{wk}(3; a)$ and $\mathfrak{bgl}(3; a)$

The non-Serre relations are:

For the first Cartan matrix:	For the second Cartan matrix:
$[[x_1, x_2], [x_3, [x_1, x_2]]] = 0$	$[x_2, [x_1, x_3]] = a[x_3, [x_1, x_2]]$
$[[x_2, x_3], [x_3, [x_1, x_2]]] = 0$	

9.1.8. $\mathfrak{wk}(4;a)$ and $\mathfrak{bgl}(4;a)$

The non-Serre relations are:

For the first Cartan matrix:
$[[x_2,x_3],[x_3,x_4]]=0$
$[[x_1,x_2],[x_3,[x_1,x_2]]]=0$
$[[x_2,x_3],[x_3,[x_1,x_2]]]=0$
$[[x_4,[x_2,x_3]],[[x_1,x_2],[x_3,x_4]]]=0$
$[[[x_1,x_2],[x_2,x_3]],[[x_1,x_2],[x_4,[x_2,x_3]]]]=0$

For the second Cartan matrix:
$[x_2,[x_1,x_3]]=(1+a)[x_3,[x_1,x_2]]$
$[[x_2,x_3],[x_3,x_4]]=0$
$[[x_1,x_3],[x_4,[x_1,x_3]]]=0$
$[[x_3,x_4],[x_4,[x_1,x_3]]]=0$

For the third Cartan matrix:
$[[x_1,x_2],[x_3,[x_1,x_2]]]=0$
$[[x_2,x_3],[x_3,[x_1,x_2]]]=0$
$[[x_2,x_3],[x_4,[x_2,x_3]]]=0$
$[[x_3,x_4],[x_4,[x_2,x_3]]]=0$
$[[x_3,[x_1,x_2]],[x_4,[x_2,x_3]]]=(1+a)[[x_3,x_4],[[x_1,x_2],[x_2,x_3]]]$

9.1.9. $\mathfrak{g}=\mathfrak{oo}(n|m)$, $p=2$

Here we consider Cartan matrix Lie superalgebras close to some of the ortho-orthogonal algebras. There are two kinds of such Cartan matrices:

1) The Cartan matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & \vdots \\ \ddots & * & 1 & 0 \\ \ddots & 1 & 0 & 1 \\ \dots & 0 & 1 & 1 \end{pmatrix}$$

generates $\mathfrak{oo}^{(1)}_{I\Pi}(2k_{\bar 0}|2k_{\bar 1}+1)$, $k_{\bar 0}+k_{\bar 1}=n$ (parities of the rows of the matrix may be different; the connection between these parities and $k_{\bar 0},k_{\bar 1}$ is described in the table on page 266). The corresponding non-Serre relations are as in Theorem 9.8.

2) The Cartan matrix

$$\begin{pmatrix} \ddots & \ddots & \ddots & \vdots & \vdots \\ \ddots & * & 1 & 0 & 0 \\ \ddots & 1 & * & 1 & 1 \\ \cdots & 0 & 1 & \bar 0 & 0 \\ \cdots & 0 & 1 & 0 & 0 \end{pmatrix}$$

generates an algebra close to $\mathfrak{oo}_{\Pi\Pi}(2k_{\bar 0}|2k_{\bar 1})$, $k_{\bar 0}+k_{\bar 1}=n$ (parities of the rows of the matrix may be different; the connection between these parities and $k_{\bar 0},k_{\bar 1}$ is described in the table on page 266; the exact description of the Cartan matrix Lie superalgebra is in subsection 5.1.3). The corresponding non-Serre relations are as in Theorem 9.5.

9.1.10. $\mathfrak{g}=\mathfrak{ag}(2)$ and $\mathfrak{g}=\mathfrak{ab}(3)$ in case $p=2$

The Cartan matrices (56) and (57) being reduced modulo 2 do not produce anything "resembling" $\mathfrak{ag}(2)$ or $\mathfrak{ab}(3)$. (In particular, the Lie superalgebra that corresponds to the Cartan matrices 1) and 2) in (56) is isomorphic to $\mathfrak{sl}(1|3)$.)

We do not know an integer basis of $\mathfrak{ag}(2)$ or $\mathfrak{ab}(3)$ in which the corresponding Lie superalgebra in characteristics $p=2$ is simple. Cunha and Elduque suggested, nevertheless, their $p=3$ analogs, see [**CE**].

10. Proofs for $p=2$: Lie algebras

10.1. $\mathfrak{g}=\mathfrak{sl}(n+1)$, $p=2$

The elements E_{ij}, where $1\leqslant i<j\leqslant n+1$, form a basis of the algebra $\mathfrak{n}$. In particular, $x_i=E_{i,i+1}$. Clearly, we have

$$[E_{ij},E_{kl}]=\delta_{jk}E_{il}+\delta_{il}E_{kj}.$$

Let $\overline{\mathfrak{h}}$ be the algebra of diagonal matrices. The elements E_{ii}, where $1\leqslant i\leqslant n+1$, form a basis of $\overline{\mathfrak{h}}$. Let the ω_i be the dual basis elements.

We consider the weights of $\mathfrak{n}$ with respect to $\overline{\mathfrak{h}}$. The weight of E_{ij} is equal to $\omega_i+\omega_j$.

Recall several facts about homology.

Lemma 10.1. *Set*

$$M_c=\{E_{i_1j_1},\dots,E_{i_mj_m}\}\ \textit{for a basic chain}\ c=E_{i_1j_1}\wedge\dots\wedge E_{i_mj_m}.$$

If for any $E_{ij}\in M_c$ and any k such that $i<k<j$, at least one of the elements E_{ik} and E_{kj} lies in M_c, then c can not appear with non-zero coefficient in decomposition of a boundary with respect to basic chains.

Proof. Clearly, it suffices to show that c can not appear with non-zero coefficient in the decomposition of the differential of a basic chain with respect to basic chains. It follows from the formula for the differential d that any basic chains that appears with non-zero coefficient in decomposition of the differential of a basic chain F with respect to basic chains, can be obtained from F by replacing E_{ik} and E_{kj} by E_{ij} for some i,j,k. If c satisfies the hypothesis of the Lemma, then c can not be obtained in such a way from any F. □

The elements of $C_2(\mathfrak{n};\mathbb{K})$ have weights of two types: $\omega_i+\omega_j$ and $\omega_i+\omega_j+\omega_k+\omega_l$. Consider them:

<u>I. A weight $\alpha=\omega_i+\omega_j$, where $1\leqslant i<j\leqslant n+1$.</u> The following chains form a basis of $C_2(\mathfrak{n};\mathbb{K})_\alpha$:

$$\begin{array}{lll} E_{ik}\wedge E_{kj}, & i<k<j; & d(E_{ik}\wedge E_{kj})=E_{ij};\\ E_{ki}\wedge E_{kj}, & 1\leqslant k<i; & d(E_{ki}\wedge E_{kj})=0;\\ E_{ik}\wedge E_{jk}, & j<k\leqslant n+1; & d(E_{ik}\wedge E_{jk})=0.\end{array}$$

Thus, the following cycles form a basis of $C_2(\mathfrak{n};\mathbb{K})_\alpha$:

$$\begin{array}{ll} E_{ik}\wedge E_{kj}+E_{i,k+1}\wedge E_{k+1,j}, & i<k<j-1;\\ E_{ik}\wedge E_{jk}, & j<k\leqslant n+1.\end{array}$$

We consider them:

1) $E_{ik} \wedge E_{kj} + E_{i,k+1} \wedge E_{k+1,j} = d(E_{ik} \wedge E_{k,k+1} \wedge E_{k+1,j})$, so this is a boundary.

2) $E_{ki} \wedge E_{kj}$, where $1 \leqslant k < i$; in this case, we consider three subcases:

a) $j - i > 1$: In this case, $E_{ki} \wedge E_{kj} = d(E_{ki} \wedge E_{k,j-1} \wedge E_{j-1,j})$.

b) $i - k > 1$: In this case, $E_{ki} \wedge E_{kj} = d(E_{k,i-1} \wedge E_{i-1,i} \wedge E_{kj})$.

c) $i - k = j - i = 1$, i.e., $i = k + 1$; $j = k + 2$. In this case, according to Lemma 10.1, the basic chain $E_{ki} \wedge E_{kj}$ can not appear with non-zero coefficient in decomposition of a boundary with respect to basic chains; so this is a non-trivial cycle. It gives us the relation

$$[E_{k,k+1}, E_{k,k+2}] = 0, \qquad \text{i.e., } [x_k, [x_k, x_{k+1}]] = 0.$$

Here $k \in \overline{1, n-1}$.

3) This case is completely analogous to the previous one; it gives us the relation

$$[x_k, [x_{k-1}, x_k]] = 0,$$

where $k \in \overline{2, n}$.

II. A weight $\alpha = \omega_i + \omega_j + \omega_i + \omega_j$, where $1 \leqslant i < j < k < l \leqslant n + 1$. Clearly, the space $C_2(\mathfrak{n}; \mathbb{K})_\alpha$ has the following basis:

$$c_{\alpha,1} = E_{ij} \wedge E_{kl}, \qquad c_{\alpha,2} = E_{ik} \wedge E_{jl}, \qquad c_{\alpha,3} = E_{il} \wedge E_{jk}.$$

All this three chains are cycles, i.e., $Z_2(\mathfrak{n}; \mathbb{K})_\alpha = C_2(\mathfrak{n}; \mathbb{K})_\alpha$. Here we have three subcases:

1) $j - i > 1$. Then

$$c_{\alpha,1} = d(E_{i,i+1} \wedge E_{i+1,j} \wedge E_{kl});$$
$$c_{\alpha,2} = d(E_{i,i+1} \wedge E_{i+1,k} \wedge E_{jl});$$
$$c_{\alpha,3} = d(E_{i,i+1} \wedge E_{i+1,l} \wedge E_{jk}).$$

2) $l - k > 1$. Then, similarly to the previous case,

$$c_{\alpha,1} = d(E_{ij} \wedge E_{k,l-1} \wedge E_{l-1,l});$$
$$c_{\alpha,2} = d(E_{ik} \wedge E_{j,l-1} \wedge E_{l-1,l});$$
$$c_{\alpha,3} = d(E_{jk} \wedge E_{i,l-1} \wedge E_{l-1,l}).$$

3) $j - i = l - k = 1$, i.e., $j = i + 1; l = k + 1$. Then, from Lemma 10.1, $c_{\alpha,1}$ is a non-trivial cycle. It gives the relation

$$[E_{i,i+1}, E_{k,k+1}] = 0, \qquad \text{i.e., } [x_i, x_k] = 0.$$

Here $i, k \in \overline{1, n}$, and $k - i \geqslant 2$.

For the other cycles, we need to consider the two subcases:

a) $k - j > 1$. Then

$$c_{\alpha,2} = d(E_{i,k-1} \wedge E_{k-1,k} \wedge E_{jl}); \quad c_{\alpha,3} = d(E_{il} \wedge E_{j,k-1} \wedge E_{k-1,k}).$$

b) $k - j = 1$, i.e., $i = j - 1$; $k = j + 1$; $l = i + 2$. It is easy to see (like in the proof of Lemma 10.1) that the only two chains such that $c_{\alpha,2}$ or $c_{\alpha,3}$ appear with non-zero coefficients in the decomposition of their differentials with respect to basic chains are

$$E_{j-1,j} \wedge E_{j,j+1} \wedge E_{j,j+2} \quad \text{and} \quad E_{j-1,j+1} \wedge E_{j,j+1} \wedge E_{j+1,j+2}.$$

The differentials of both these chains are equal to $c_{\alpha,2} + c_{\alpha,3}$. So we can consider one of the chains $c_{\alpha,2}$ or $c_{\alpha,3}$ as a non-trivial cycle. The cycle $c_{\alpha,2}$ gives the relation

$$[E_{j-1,j+1}, E_{j,j+2}] = 0, \qquad \text{i.e., } [[x_{j-1}, x_j], [x_j, x_{j+1}]] = 0,$$

and $c_{\alpha,3}$ gives an equivalent (taking other relations into account) relation

$$[E_{j-1,j+2}, E_{j,j+1}] = 0 \qquad \text{i.e., } [[x_{j-1}, [x_j, x_{j+1}]], x_j] = 0.$$

Here $j \in \overline{2, n-1}$.

10.2. Proofs: Lie superalgebras

In the exceptional cases, the relations are obtained by means of `SuperLie`. For the $\mathfrak{sl}$ series, the arguments of the non-super case are applicable. For the other series, the answers are conjectural but we tested them by means of `SuperLie` for small values of superdimensions, and hence are sure.

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