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THE COHOMOLOGY OF MOTIVIC $A(2)$

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Abstract

Working over an algebraically closed field of characteristic zero, we compute the cohomology of the subalgebra $A(2)$ of the motivic Steenrod algebra that is generated by Sq^1, Sq^2, and Sq^4. The method of calculation is a motivic version of the May spectral sequence.

Speculatively assuming that there is a "motivic modular forms" spectrum with certain properties, we use an Adams-Novikov spectral sequence to compute the homotopy of such a spectrum at the prime 2.

1. Introduction

The purpose of this article is to present some algebraic calculations that are relevant to motivic homotopy theory. Recent work [**HKO**] [**DI**] has shown that the Adams and Adams-Novikov spectral sequences are useful for computing in the motivic stable homotopy category over an algebraically closed field of characteristic zero, after completion with respect to the motivic $\mathbb{F}_2$-Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$ that represents motivic $\mathbb{F}_2$-cohomology.

This program is built upon work of Voevodsky [**V1**] [**V2**] [**V3**]. Voevodsky has described the motivic $\mathbb{F}_2$-cohomology of an algebraically closed field of characteristic zero, which we write as $\mathbb{M}_2$. He also described the motivic $\mathbb{F}_2$-Steenrod algebra A. Much like the classical Steenrod algebra A_{cl}, A is generated by elements Sq^i, subject to motivic versions of the Adem relations.

In this article, we take Voevodsky's descriptions as given algebraic inputs, and we carry out further algebraic computations. The motivic Adams spectral sequence takes the cohomology of A, i.e., the ring $\mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$, as input. Just as in the classical situation, it is unlikely that we will ever have a complete description of this ring. However, much can be said in low dimensions [**DI**].

Classically, one way to approximate difficult calculations over A_{cl} is to consider instead the subalgebra $A(2)_{\mathrm{cl}}$ generated by Sq^1, Sq^2, and Sq^4. It is possible to give an explicit but lengthy description of the cohomology of $A(2)_{\mathrm{cl}}$ [**M2**] [**IS**]. The main

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purpose of this article is to carry out the motivic version of this computation. Namely, we completely describe the ring $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$.

The cohomology of $A(2)_{\mathrm{cl}}$ is the E_2-term of an Adams spectral sequence that converges to the homotopy of *tmf* at the prime 2. The original sources of this material are mostly unpublished work of Hopkins, Mahowald, and Miller [**H1**] [**H2**]. One might optimistically hope to construct a motivic spectrum *mmf*, defined over algebraically closed fields of characteristic zero, whose homotopy is similarly related to the cohomology of motivic $A(2)$. This is a potential homotopical application of our algebraic computations. In this article, we do not discuss any issues related to the existence of *mmf*.

Although it is theoretically possible to compute the homotopy of *tmf* at the prime 2 by starting with Ext groups over $A(2)_{\mathrm{cl}}$ and applying the Adams spectral sequence, it is difficult in practice. The motivic version of this computation is significantly more difficult. Even finding the E_3-term of the spectral sequence requires a prohibitive amount of bookkeeping, and there are a tremendous number of exotic extensions to resolve.

Classically, a more efficient way to compute the homotopy of *tmf* at the prime 2 is to start with the cohomology of the elliptic curves Hopf algebroid and apply the Adams-Novikov spectral sequence. This computation is entirely described in [**B**], based on work of Hopkins and Mahowald [**HM**].

At the end of this paper, we carry out the motivic version of this computation. We make several assumptions about motivic versions of the elliptic curves Hopf algebroid and the motivic Adams-Novikov spectral sequence. Based on these assumptions, we are able to describe the homotopy of the speculative motivic spectrum *mmf* at the prime 2.

1.1. Organization of the paper

Section 2 gives a brief review of the algebraic objects under consideration. In Section 3, we set up a motivic May spectral sequence that converges to the cohomology of $A(2)$. We also describe the E_∞-term of this spectral sequence. In Section 4, we compute Massey products and resolve all multiplicative extensions to give a complete description of the cohomology of $A(2)$ as a ring. Finally, Section 5 discusses an Adams-Novikov spectral sequence and describes the homotopy of the speculative motivic spectrum *mmf*.

The charts in Sections 3, 4, and 5 are the central contributions of the paper. We highly recommend that the reader obtain the color versions of these charts, as they are much easier to interpret.

Acknowledgements

We acknowledge the assistance of Robert Bruner. Section 3 relies very heavily on unpublished notes of Peter May on the cohomology of $A(2)_{\mathrm{cl}}$ [**M2**]. Section 5 relies very heavily on charts of Tilman Bauer [**B**]. We also thank Mark Behrens, Dan Dugger, and Mike Hill for useful conversations. Finally, we appreciate the careful reading by the referee, which yielded several technical corrections.

2. Background

In this section we review the basic algebraic facts about the objects under consideration. We are working in categories of bigraded objects. In a bidegree (p, q), we shall refer to p as the topological degree and q as the weight. The terminology is motivated by the relationship between motivic cohomology and classical homotopy theory.

Definition 2.1. The bigraded ring $\mathbb{M}_2$ is the polynomial ring $\mathbb{F}_2[\tau]$ on one generator τ of bidegree $(0, 1)$.

The relevance of $\mathbb{M}_2$ is that it is the motivic $\mathbb{F}_2$-cohomology of an algebraically closed field of characteristic zero [**V1**].

Definition 2.2. The motivic Steenrod algebra A is the $\mathbb{M}_2$-algebra generated by elements Sq^{2k} and Sq^{2k-1} for all $k \geqslant 1$, of bidegrees $(2k, k)$ and $(2k-1, k-1)$ respectively, where Sq^0 is the identity, and satisfying the following relations for $a < 2b$:

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_c \binom{b-1-c}{a-2c} \tau^? \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c .$$

The expression $\tau^?$ stands for either 1 or τ. The distinction is easily determined by consideration of bidegrees. For example, $\mathrm{Sq}^2 \mathrm{Sq}^2 = \tau \mathrm{Sq}^3 \mathrm{Sq}^1$.

The relevance of A is that it is the ring of motivic $\mathbb{F}_2$-cohomology operations over an algebraically closed field of characteristic zero [**V2**] [**V3**].

We consider $\mathbb{M}_2$ as an A-module, where Sq^i acts trivially on $\mathbb{M}_2$ for $i > 0$. Since $\mathbb{M}_2$ is concentrated in topological degree 0, this is the only possible action of A on $\mathbb{M}_2$.

Definition 2.3. The algebra $A(2)$ is the $\mathbb{M}_2$-subalgebra of A generated by Sq^1, Sq^2, and Sq^4.

Remark 2.4. The algebra A has a Milnor $\mathbb{M}_2$-basis consisting of elements of the form P^R, where $R = (r_1, r_2, \ldots)$ ranges over all finite sequences of non-negative integers. Just as in the classical case, $A(2)$ has an $\mathbb{M}_2$-basis consisting of elements of the form P^R, where $R = (r_1, r_2, r_3)$, $0 \leqslant r_1 \leqslant 7$, $0 \leqslant r_2 \leqslant 3$, and $0 \leqslant r_1 \leqslant 1$. See [**DI**] for more details on the motivic Milnor basis.

2.1. Ext **groups**

We will compute the tri-graded groups $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$.

We have

$$\mathrm{Ext}_{A(2)}^{0,(0,*)}(\mathbb{M}_2, \mathbb{M}_2) = \mathrm{Hom}_{A(2)}^{(0,*)}(\mathbb{M}_2, \mathbb{M}_2) = \mathbb{M}_2.$$

Here we abuse notation and write $\mathbb{M}_2$ where we really mean $\mathrm{Hom}_{\mathbb{M}_2}(\mathbb{M}_2, \mathbb{M}_2)$, the $\mathbb{M}_2$-dual of $\mathbb{M}_2$. The only important point is that now τ has bidegree $(0, -1)$. For fixed s and t, $\mathrm{Ext}_{A(2)}^{s,(t+s,*)}(\mathbb{M}_2, \mathbb{M}_2)$ is a module over $\mathrm{Ext}_{A(2)}^{0,(0,*)}(\mathbb{M}_2, \mathbb{M}_2)$. In particular, it is an $\mathbb{M}_2$-module and therefore decomposes as a sum of modules of the form $\mathbb{M}_2$ or $\mathbb{M}_2/\tau^k$. In Section 2.2, we will explain that the free part coincides with Ext over the classical version of $A(2)$.

2.2. Comparison with the classical Steenrod algebra

We write A_{cl} and $A(2)_{\mathrm{cl}}$ for the classical Steenrod algebra and its subalgebra generated by Sq^1, Sq^2, and Sq^4.

There is a ring map $A_{\mathrm{cl}} \to A$ that takes Sq^{2k} and Sq^{2k-1} to $\tau^k \mathrm{Sq}^{2k}$ and $\tau^k \mathrm{Sq}^{2k-1}$ respectively. After inverting τ, we have a map $A_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}] \to A[\tau^{-1}]$.

Lemma 2.5. *The map $A_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}] \to A[\tau^{-1}]$ is an isomorphism.*

In other words, the motivic Steenrod algebra and the classical Steenrod algebra are essentially the same after inverting τ. The proof of this lemma is a straightforward algebraic exercise. Similarly, there is a ring isomorphism $A(2)_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}] \to A(2)[\tau^{-1}]$ that takes Sq^1, Sq^2, and Sq^4 to Sq^1, $\tau \mathrm{Sq}^2$, and $\tau^2 \mathrm{Sq}^4$ respectively.

Proposition 2.6. *There are isomorphisms of rings*

$$\mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \cong \mathrm{Ext}_{A_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}].$$

and

$$\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2) \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \cong \mathrm{Ext}_{A(2)_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}].$$

The proof of Proposition 2.6 is given in [**DI**]. The idea is to use the isomorphism of Lemma 2.5 and the flatness of $\mathbb{M}_2 \to \mathbb{M}_2[\tau^{-1}]$.

Definition 2.7. The map

$$(-)_{\mathrm{cl}} \colon \mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \to \mathrm{Ext}_{A_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$$

is the localization map $\mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \to \mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}]$, followed by the isomorphism $\mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \to \mathrm{Ext}_{A_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}]$ of Proposition 2.6, followed by the map $\mathrm{Ext}_{A_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}] \to \mathrm{Ext}_{A_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ that sends τ to 1. The map

$$(-)_{\mathrm{cl}} \colon \mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2) \to \mathrm{Ext}_{A(2)_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$$

is defined similarly.

Remark 2.8. The point of Proposition 2.6 is that $(-)_{\mathrm{cl}}$ induces an isomorphism from the free part of $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ to $\mathrm{Ext}_{A(2)_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$, after tensoring over $\mathbb{M}_2$ with $\mathbb{F}_2$. On the other hand, $(-)_{\mathrm{cl}}$ applied to the non-free part of $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$, i.e., to copies of $\mathbb{M}_2/\tau^k$, is always zero.

Remark 2.9. In fact, $(-)_{\mathrm{cl}}$ is actually a map of E_2-terms of Adams spectral sequences, so differentials are preserved. We will not need this result. See [**DI**] for more details.

3. The motivic May spectral sequence

In this section we begin our computation of $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ by setting up a May spectral sequence [**M1**] and finding the E_∞-term of this spectral sequence. Elements of $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ will be graded in the form (s, a, w), where s is the stem, i.e., the topological degree minus the homological degree, a is Adams filtration, i.e., the homological degree, and w is the weight.

Let I be the two-sided $\mathbb{M}_2$-ideal of $A(2)$ generated by Sq^1, Sq^2, and Sq^4. Let $\mathrm{Gr}\, A(2)$ denote the associated graded algebra $A(2)/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$. Elements of $\mathrm{Gr}\, A(2)$ will be graded in the form (m, t, w), where m is the May filtration, i.e., the I-adic valuation, t is the topological degree, and w is the weight.

We will first need to compute $\mathrm{Ext}_{\mathrm{Gr}\, A(2)}(\mathbb{M}_2, \mathbb{M}_2)$. Elements of this ring will be graded in the form (m, s, a, w), where m is the May filtration, s is the stem, i.e., the topological degree minus the homological degree, a is the Adams filtration, i.e., the homological degree, and w is the weight.

Proposition 3.1. *There is a spectral sequence*

$$E_2 = \mathrm{Ext}^{(m,s,a,w)}_{\mathrm{Gr}\, A(2)}(\mathbb{M}_2, M_2) \Rightarrow \mathrm{Ext}^{(s,a,w)}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2).$$

We shall refer to this as the motivic May spectral sequence. As usual, it can be obtained by filtering the cobar complex by powers of I.

Let I_{cl} be the ideal of $A(2)_{\mathrm{cl}}$ generated by Sq^1, Sq^2, and Sq^4, and let $\mathrm{Gr}\, A(2)_{\mathrm{cl}}$ be the associated graded algebra.

Proposition 3.2.

(a) The tri-graded algebras $\mathrm{Gr}\, A(2)$ *and* $\mathrm{Gr}\, A(2)_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau]$ *are isomorphic.*

(b) The quadruply-graded rings $\mathrm{Ext}_{\mathrm{Gr}\, A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ *and* $\mathrm{Ext}_{\mathrm{Gr}\, A(2)_{cl}}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{M}_2$ *are isomorphic.*

The point is that the E_2-terms of the motivic and classical May spectral sequences are very similar. Moreover, since the comparison map $(-)_{\mathrm{cl}}$ of Definition 2.7 preserves filtrations, it induces a map from the motivic May spectral sequence to the classical May spectral sequence. This means that we can deduce information about differentials in the motivic situation from known information in the classical situation.

The proposition can be proved just as in [**DI**]. The main point is that the τ coefficients in the relations in A appear only on terms of higher filtration and thus do not affect the associated graded algebra.

The classical ring $\mathrm{Ext}_{\mathrm{Gr}\, A(2)_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ is computed in [**M2**] using the May spectral sequence. It is also computed in [**IS**] using a different method.

By Proposition 3.2 and the results in [**M1**], the ring $\mathrm{Ext}_{\mathrm{Gr}\, A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ is the cohomology of the differential graded $\mathbb{M}_2$-algebra $\mathbb{M}_2[h_{10}, h_{11}, h_{12}, h_{20}, h_{21}, h_{30}]$, where the differential is described in Table 1. See [**DI**] for an explanation of the degrees.

It is relatively straightforward to compute $\mathrm{Ext}_{\mathrm{Gr}\, A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ from the differentials given in Table 1. Table 2 lists the generators.

Proposition 3.3. *The ring* $\mathrm{Ext}_{\mathrm{Gr}\, A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ *is generated over* $\mathbb{M}_2$ *by the elements listed in Table 2, subject to the relations*

1. $h_0 h_1 = 0$.
2. $h_1 h_2 = 0$.
3. $h_2 b_{20} = h_0 h_0(1)$.
4. $h_2 h_0(1) = h_0 b_{21}$.
5. $h_0(1)^2 = b_{20} b_{21} + h_1^2 b_{30}$.

The proof of Proposition 3.3 is a straightforward lift of the analogous classical computation because of Proposition 3.2(b).

Table 1: Differentials for computing $\mathrm{Ext}_{\mathrm{Gr}\,A(2)}(\mathbb{F}_2, \mathbb{F}_2)$

x	degree	$d(x)$
h_{10}	$(1,0,1,0)$	0
h_{11}	$(1,1,1,1)$	0
h_{12}	$(1,3,1,2)$	0
h_{20}	$(2,2,1,1)$	$h_{10}h_{11}$
h_{21}	$(2,5,1,3)$	$h_{11}h_{12}$
h_{30}	$(3,6,1,3)$	$h_{10}h_{21} + h_{20}h_{12}$

Table 2: Generators for $\mathrm{Ext}_{\mathrm{Gr}\,A(2)}(\mathbb{M}_2, \mathbb{M}_2)$

generator	degree	description in terms of h_{ij}
h_0	$(1,0,1,0)$	h_{10}
h_1	$(1,1,1,1)$	h_{11}
h_2	$(1,3,1,2)$	h_{12}
b_{20}	$(4,4,2,2)$	h_{20}^2
b_{21}	$(4,10,2,6)$	h_{21}^2
b_{30}	$(6,12,2,6)$	h_{30}^2
$h_0(1)$	$(4,7,2,4)$	$h_{20}h_{21} + h_{11}h_{30}$

3.1. E_4-term of the motivic May spectral sequence

Having described the E_2-term $\mathrm{Ext}_{\mathrm{Gr}\,A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ of the motivic May spectral sequence, we are now ready to compute the higher terms. For dimension reasons, as in the classical case, the odd differentials must vanish. In particular, $E_3 = E_4$.

The d_2 differentials on the E_2-term are easy to analyze. They are entirely determined by the facts that d_2 commutes with the comparison map $(-)_{\mathrm{cl}}$ and that d_2 preserves the weight. Table 3 lists the d_2 differentials on all of our generators. From the data in this table, one can use the Leibniz rule to compute the d_2 differential on any element.

A straightforward computation now gives the E_4-term of the motivic May spectral sequence. The generators are listed in Table 4, and the relations are listed in the next theorem.

Table 3: d_2 differentials in the motivic May spectral sequence

x	$d_2(x)$	x	$d_2(x)$
h_0	0	b_{20}	$\tau h_1^3 + h_0^2 h_2$
h_1	0	b_{21}	h_2^3
h_2	0	b_{30}	$\tau h_1 b_{21}$
		$h_0(1)$	$h_0 h_2^2$

Table 4: Generators of the E_4-term of the motivic May spectral sequence

generator	degree	description in E_2
h_0	$(1,0,1,0)$	h_0
h_1	$(1,1,1,1)$	h_1
h_2	$(1,3,1,2)$	h_2
P	$(8,8,4,4)$	b_{20}^2
c	$(5,8,3,5)$	$h_1h_0(1)$
u	$(5,11,3,7)$	h_1b_{21}
α	$(7,12,3,6)$	h_0b_{30}
d	$(8,14,4,8)$	$h_1^2b_{30}+b_{20}b_{21}$
ν	$(7,15,3,8)$	h_2b_{30}
e	$(8,17,4,10)$	$h_0(1)b_{21}$
g	$(8,20,4,12)$	b_{21}^2
Δ	$(12,24,4,12)$	b_{30}^2

The notation is chosen to be compatible with the standard notation for elements in the cohomology of the classical Steenrod algebra. In particular, c_{cl}, d_{cl}, e_{cl}, and g_{cl} are the classical elements that are usually denoted by c_0, d_0, e_0, and g. The element P_{cl} is related to the classical Adams periodicity operator. The element Δ_{cl} is related to the element in the homotopy of *tmf* of the same name. The elements u, α, and ν have no classical analogues and are given arbitrary names.

Theorem 3.4. *The E_4-term of the motivic May spectral sequence is generated over $\mathbb{M}_2$ by the elements listed in Table 4, subject to the following relations:*

1. h_0h_1, h_1h_2, $h_0^2h_2+\tau h_1^3$, $h_0h_2^2$, h_2^3.
2. τu, τh_1^2c, τcd, τce, τcg.
3. $h_0^2\nu+\tau h_1d$, $h_0h_2\nu+\tau h_1e$, $h_2^2\nu+\tau h_1g$.
4. h_2d+h_0e, h_2e+h_0g, $h_2\alpha+h_0\nu$.
5. h_0c, h_2c, h_0u, h_2u, $h_1\alpha$, $h_1\nu$.
6. $c^2+h_1^2d$, $u^2+h_1^2g$, $cu+h_1^2e$, e^2+dg.
7. $ud+ce$, $ue+cg$, $\nu d+\alpha e$, $\nu e+\alpha g$.
8. $c\alpha$, $c\nu$, $u\alpha$, $u\nu$.
9. $\alpha^2+h_0^2\Delta$, $\alpha\nu+h_0h_2\Delta$, $\nu^2+h_2^2\Delta$.
10. $h_0^2d+Ph_2^2$, $h_0\alpha d+Ph_2\nu$, $d^2+h_1^4\Delta+Pg$.

Several observations can be made immediately. First, the E_4-term contains the polynomial ring $\mathbb{M}_2[P,\Delta]$, and the E_4-term is free as a module over $\mathbb{M}_2[P,\Delta]$. However, beware that the E_4-term is not of the form $\mathbb{M}_2[P,\Delta]\otimes_{\mathbb{M}_2} B$, because of the relations in part (10) of the theorem.

The E_4-term also contains the polynomial ring $\mathbb{M}_2[g]$, but the E_4-term is not free as a module over $\mathbb{M}_2[g]$. For example, $h_0^3g=0$, but h_0^3 is not zero. However, the ideal generated by g is free over $\mathbb{M}_2[g]$.

Table 5: Generators of the E_∞-term of the motivic May spectral sequence

generator	degree	generator	degree
h_0	$(1,0,1,0)$	ν	$(7,15,3,8)$
h_1	$(1,1,1,1)$	e	$(8,17,4,10)$
h_2	$(1,3,1,2)$	g	$(8,20,4,12)$
P	$(8,8,4,4)$	$\Delta h_1 = \langle \tau^2 g, h_2, h_1 \rangle$	$(13,25,5,13)$
c	$(5,8,3,5)$	$\Delta c = \langle h_2, \tau^2 g, c \rangle$	$(17,32,7,17)$
u	$(5,11,3,7)$	$\Delta u = \langle h_2, \tau^2 g, u \rangle$	$(17,35,7,19)$
α	$(7,12,3,6)$	Δ^2	$(24,48,8,24)$
d	$(8,14,4,8)$		

Some parts of the E_4-term are depicted in the chart on the next page. The horizontal axis is the stem, and the vertical axis is the Adams filtration. Solid circles indicate copies of $\mathbb{M}_2$, while open circles indicate copies of $\mathbb{M}_2/\tau$. Vertical lines indicate multiplication by h_0, lines of slope 1 indicate multiplication by h_1, and lines of slope $\frac{1}{3}$ indicate multiplication by h_2. Dashed lines indicate that the multiplication hits τ times a generator. For example, the relation $h_0^2 h_2 = \tau h_1^3$ occurs in the 3-stem.

Vertical arrows indicate infinite towers of copies of $\mathbb{M}_2$ connected by h_0 multiplications. Diagonal arrows indicate infinite towers of copies of $\mathbb{M}_2/\tau$ connected by h_1 multiplications.

For legibility, we have omitted most of the multiples of P. We have only shown a few elements (in red) to express multiplicative relations with elements that are not multiples of P. We have also omitted the strict multiples of Δ and Δ^2.

(For readers with the color version of the chart, the green parts of the figure consist of elements that are multiples of g, the blue parts consist of elements that are multiples of g^2, and the purple parts consist of elements that are multiples of g^3. Observe that the blue part is a shifted copy of the green part. Similarly, if the figure were larger, the purple part would be another shifted copy.)

3.2. E_∞-term of the motivic May spectral sequence

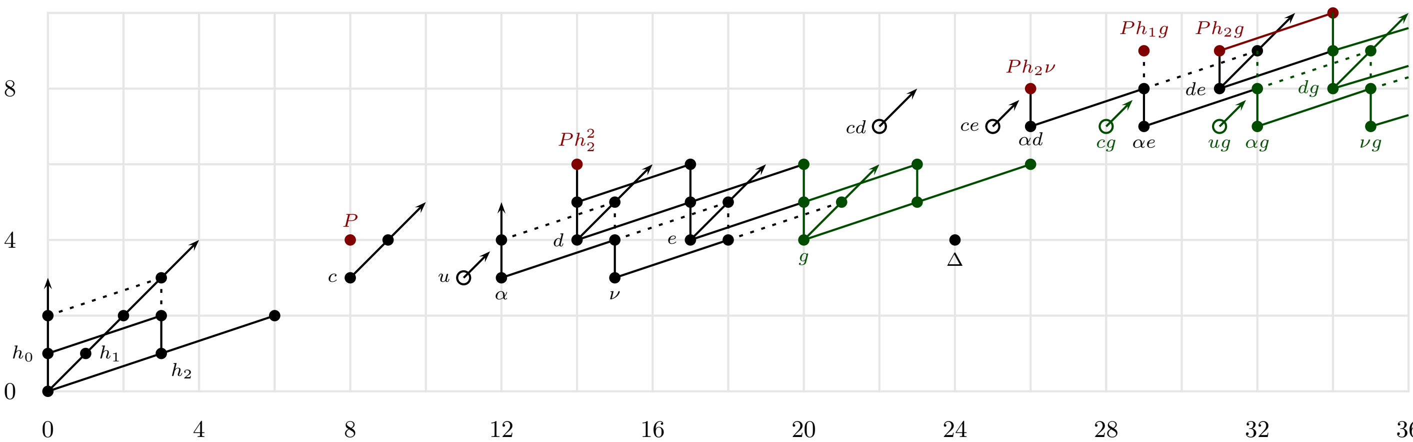
Having described the E_4-term of the motivic May spectral sequence, we are now ready to compute the E_∞-term.

As for the d_2 differentials, the d_4 differentials commute with the comparison map $(-)_{\mathrm{cl}}$ to the classical May spectral sequence. The only generator that supports a d_4 differential is Δ:

$$d_4(\Delta) = \tau^2 h_2 g.$$

A straightforward computation now gives the E_5-term of the motivic May spectral sequence. By inspection, there are no higher differentials, so $E_5 = E_\infty$. The generators of E_∞ are listed in Table 5, and the relations are listed in the next theorem.

Although Δ does not survive past E_4, we shall write Δ for the Massey product operators $\langle \tau^2 g, h_2, - \rangle$ and $\langle h_2, \tau^2 g, - \rangle$.



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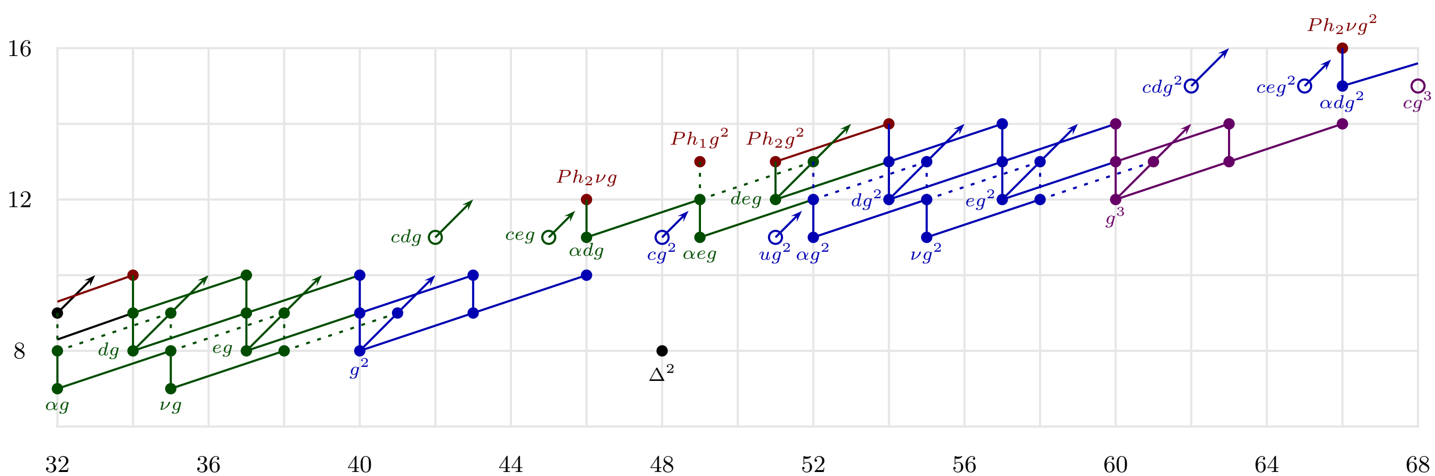

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Table 6: Multiplication table for E_∞

	c	u	α	d	ν	e	Δh_1	Δc	Δu
c	$h_1^2 d$	$h_1^2 e$	0	cd	0	ud	$h_1\Delta c$	$h_1 d\Delta h_1$	$h_1 e\Delta h_1$
u		$h_1^2 g$	0	ce	0	cg	$h_1\Delta u$	$h_1 e\Delta h_1$	$h_1 g\Delta h_1$
α			α^2	αd	$\alpha\nu$	νd	0	0	0
d				$h_1^3\Delta h_1 + Pg$	αe	de	$d\Delta h_1$	$d\Delta c$	$e\Delta c$
ν					ν^2	αg	0	0	0
e						dg	$e\Delta h_1$	$d\Delta u$	$g\Delta c$
Δh_1							$h_1^2\Delta^2$	$h_1 c\Delta^2$	$h_1 u\Delta^2$
Δc								$h_1^2 d\Delta^2$	$h_1^2 e\Delta^2$
Δu									$h_1^2 g\Delta^2$

Theorem 3.5. *The E_∞-term of the motivic May spectral sequence is generated over $\mathbb{M}_2$ by the elements listed in Table 5, subject to the relations expressed in the multiplication table Table 6, as well as the following relations:*

1. $h_0 h_1$, $h_1 h_2$, $h_0^2 h_2 + \tau h_1^3$, $h_0 h_2^2$, h_2^3.
2. τu, $\tau h_1^2 c$, $\tau h_1^2 \Delta c$, $\tau\Delta u$, τcd, τce, τcg, $\tau d\Delta c$, $\tau e\Delta c$, $\tau g\Delta c$, $\tau^2 h_2 g$.
3. $h_0^2\nu + \tau h_1 d$, $h_0 h_2\nu + \tau h_1 e$, $h_2^2\nu + \tau h_1 g$, $h_0\alpha\nu + \tau h_1^2\Delta h_1$.
4. $h_2 d + h_0 e$, $h_2 e + h_0 g$, $h_2\alpha + h_0\nu$.
5. $h_0 c$, $h_2 c$, $h_0 u$, $h_2 u$, $h_1\alpha$, $h_1\nu$, $h_0\Delta h_1$, $h_2\Delta h_1$, $h_0\Delta c$, $h_2\Delta c$, $h_0\Delta u$, $h_2\Delta u$, $h_0\nu^2$, $h_2\nu^2$.
6. $h_0^2 d + Ph_2^2$, $h_0\alpha d + Ph_2\nu$, $\alpha^2 d + P\nu^2$.
7. $\alpha^2\nu + \tau d\Delta h_1$, $\alpha\nu^2 + \tau e\Delta h_1$, $\nu^3 + \tau g\Delta h_1$, $\alpha^4 + h_0^4\Delta^2$.

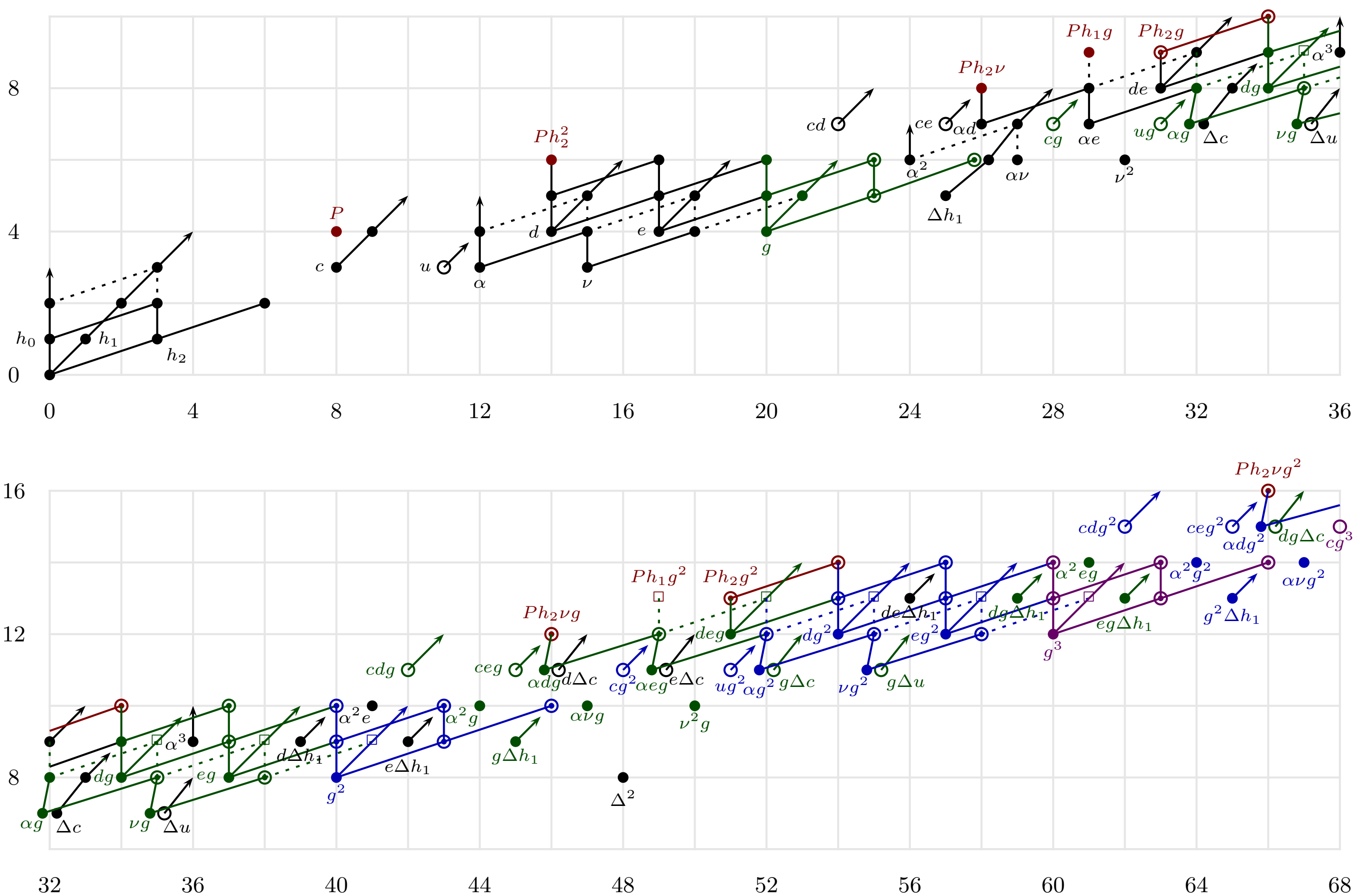
Note that the E_∞-term is free as a module over $\mathbb{M}_2[P, \Delta^2]$. It is not free as a module over $\mathbb{M}_2[g]$, but the ideal generated by g^2 is free over $\mathbb{M}_2[g]$.

Some parts of the E_∞-term are depicted in the chart on the next page. The horizontal axis is the stem, and the vertical axis is the Adams filtration. Solid circles indicate copies of $\mathbb{M}_2$, open circles indicate copies of $\mathbb{M}_2/\tau$, open circles with dots indicate copies of $\mathbb{M}_2/\tau^2$, and open boxes indicate copies of $\mathbb{M}_2/\tau^3$. Vertical lines indicate multiplication by h_0, lines of slope 1 indicate multiplication by h_1, and lines of slope $\frac{1}{3}$ indicate multiplication by h_2. Dashed lines indicate that the multiplication hits τ times a generator.

Most of the multiples of P are not shown. A few multiples of P are shown (in red) to express multiplicative relations with elements that are not multiples of P. Also, multiples of Δ^2 are not shown.

(For readers with the color version of the chart, multiples of g are shown in green, multiples of g^2 are shown in blue, and multiples of g^3 are shown in purple. If the diagram were larger, the purple portion of the diagram would be a shifted copy of the blue portion.)

Note the class $de\Delta h_1$ in the 56-stem. Classically $de\Delta h_1 = \alpha^3 g$, but motivically we oly have $\alpha^3 g = \tau de\Delta h_1$. Thus $\tau de\Delta h_1$ is a multiple of g, but $de\Delta h_1$ is not a multiple of g.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$g\Delta u$
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$dg\Delta h_1$
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$eg\Delta h_1$
cdg^2
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$g^2\Delta h_1$
ceg^2
αdg^2
$Ph_2\nu g^2$
$dg\Delta c$
cg^3
$\alpha\nu g^2$

4. Computation of $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$

With the E_∞-term of the motivic May spectral sequence in hand from the previous section, we have nearly computed $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$. It only remains to resolve some hidden extensions in the multiplicative structure.

4.1. Massey products

In this section we compute some Massey products in $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ that will be needed to resolve the extension problems. All Massey products that we consider have zero indeterminacy.

Lemma 4.1. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$,

1. $\tau h_1^2 = \langle h_0, h_1, h_0 \rangle$.
2. $h_0 h_2 = \langle h_1, h_0, h_1 \rangle$.
3. $h_2^2 = \langle h_1, h_2, h_1 \rangle$.

There are several ways of understanding these formulas.

The simplest is to note that the comparison map $(-)_{\mathrm{cl}}$ commutes with Massey products, and these formulas are already known in $\mathrm{Ext}_{A(2)_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$. From this, we can deduce the motivic formulas also.

Another approach is to compute a minimal resolution for $A(2)$ in low dimensions and to compute Yoneda products and Massey products explicitly in terms of chain maps.

Lemma 4.2. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$,

1. $\tau d = \langle h_0, h_1, \alpha \rangle$.
2. $\tau e = \langle h_0, h_1, \nu \rangle$.
3. $\tau e = \langle h_2, h_1, \alpha \rangle$.
4. $\tau g = \langle h_2, h_1, \nu \rangle$.

Proof. For the first formula, we have

$$h_1 \langle h_0, h_1, \alpha \rangle = \langle h_1, h_0, h_1 \rangle \alpha = h_0 h_2 \alpha = \tau h_1 d.$$

In the relevant dimension, multiplication by h_1 is injective, so $\tau d = \langle h_0, h_1, \alpha \rangle$.

The proofs of the other formulas are similar. □

Lemma 4.3. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$,

1. $c = \langle h_1, h_0, h_2^2 \rangle$.
2. $u = \langle h_1, h_2, h_2^2 \rangle$.
3. $\alpha = \langle h_0, h_1, h_2, \tau h_2^2 \rangle$.
4. $h_0 d = \langle \tau h_1 c, h_1, h_2 \rangle$.
5. $\alpha e = \langle \tau g, c, h_0 \rangle$.
6. $\alpha g = \langle \tau g, u, h_0 \rangle$.

Proof. The formulas are already true in the E_4-term of the motivic May spectral sequence. They can be verified by computations with explicit cocycles. There are no extension problems to resolve in passing from E_4 to $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$. □

We remind the reader that Δh_1, Δc, and Δu are defined as follows.

Definition 4.4. In $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$,

1. $\Delta h_1 = \langle \tau^2 g, h_2, h_1 \rangle$.
2. $\Delta c = \langle h_2, \tau^2 g, c \rangle$.
3. $\Delta u = \langle h_2, \tau^2 g, u \rangle$.

4.2. Multiplicative Extensions

Lemma 4.5. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$*, the elements* $\tau \Delta u$*,* $\tau d \Delta c$*,* $\tau e \Delta c$*, and* $\tau g \Delta c$ *are all zero.*

Proof. For the first element, note that $\tau \langle u, \tau^2 g, h_2 \rangle \subseteq \langle 0, \tau^2 g, h_2 \rangle = 0$.

For the second element,

$$\langle h_2, \tau^2 g, c \rangle \tau d = h_2 \langle \tau^2 g, c, \tau d \rangle,$$

which equals zero since $\langle \tau^2 g, c, \tau d \rangle$ is zero for dimension reasons. The argument for the last two elements is similar. $\square$

Lemma 4.6. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$*, we have*

1. $d \Delta u = e \Delta c$.
2. $e \Delta u = g \Delta c$.

Proof. We know from E_∞ that $e \Delta u$ equals either $g \Delta c$ or $g \Delta c + \tau \alpha g^2$. We have already shown that $\tau \cdot g \Delta c$ and $\tau \cdot e \Delta u$ are both zero, but $\tau^3 \alpha g^2$ is non-zero. Therefore, $e \Delta u = g \Delta c$.

The proof of the second formula is similar. $\square$

Lemma 4.7. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$*, we have the following formulas:*

1. $h_0 \Delta c = \tau h_0 \alpha g$.
2. $h_0 \Delta u = \tau h_0 \nu g$.
3. $h_2 \Delta c = \tau h_0 \nu g$.
4. $h_2 \Delta u = \tau h_2 \nu g$.

Proof. For the first formula,

$$h_0 \langle c, \tau^2 g, h_2 \rangle = \langle h_0, c, \tau g \rangle \tau h_2 = \tau h_2 \alpha e = \tau h_0 \alpha g.$$

The proof of the second formula is similar.

The third and fourth formulas follow easily by multiplying the first and second formulas by h_2, using that multiplication by h_0 is injective in the relevant dimensions. $\square$

Lemma 4.8. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$*, we have the following formulas:*

1. $\alpha c = \tau h_0^2 g$.
2. $\nu c = \tau h_0 h_2 g$.
3. $\alpha u = \tau h_0 h_2 g$.
4. $\nu u = \tau h_2^2 g$.

5. $\alpha \Delta h_1 = \tau^3 eg$.
6. $\nu \Delta h_1 = \tau^3 g^2$.

Proof. For the first formula,

$$\alpha \langle h_1, h_0, h_2^2 \rangle = \langle \alpha, h_1, h_0 \rangle h_2^2 = \tau d h_2^2 = \tau h_0^2 g.$$

The other calculations are similar. □

Lemma 4.9. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$*, the products* $\alpha \Delta c$*,* $\alpha \Delta u$*,* $\nu \Delta c$*, and* $\nu \Delta u$ *are all zero.*

Proof. For the first product,

$$\alpha \Delta c = \langle h_0, h_1, h_2, \tau h_2^2 \rangle \Delta c \subseteq \langle h_0, h_1, h_2, \tau^3 h_1 eg \rangle = \langle h_0, h_1, h_2, 0 \rangle = 0.$$

The second product vanishes for similar reasons.

For the third product,

$$\tau \nu \Delta c = \langle h_2, \tau^2 g, c \rangle \tau \nu = h_2 \langle \tau^2 g, c, \tau \nu \rangle,$$

which is zero because $\langle \tau^2 g, c, \tau \nu \rangle$ is zero for dimension reasons. Therefore, $\nu \Delta c$ does not equal $\tau \alpha \nu g$ since $\tau^2 \alpha \nu g$ is not zero. The fourth product vanishes for similar reasons. □

Lemma 4.10. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$*, we have the following formulas:*

1. $(\Delta h_1)^2 = h_1^2 \Delta^2 + \tau^2 \nu^2 g$.
2. $\Delta h_1 \cdot \Delta c = h_1 c \Delta^2$.
3. $\Delta h_1 \cdot \Delta u = h_1 u \Delta^2$.
4. $(\Delta c)^2 = h_1^2 d \Delta^2$.
5. $\Delta c \cdot \Delta u = h_1^2 e \Delta^2$.
6. $(\Delta u)^2 = h_1^2 g \Delta^2$.

Proof. For the first formula, we know from E_∞ that $(\Delta h_1)^2$ equals either $h_1^2 \Delta^2$ or $h_1^2 \Delta^2 + \tau^2 \nu^2 g$. We have already shown that $\nu (\Delta h_1)^2 = \tau^3 g^2 \Delta h_1$ is non-zero, but $\nu \cdot h_1^2 \Delta^2$ is zero. It follows that $(\Delta h_1)^2 = h_1^2 \Delta^2 + \nu^2 g$.

For the second formula, we know from E_∞ that $\Delta h_1 \cdot \Delta c$ equals either $h_1 c \Delta^2$ or $h_1 c \Delta^2 + \tau^4 e g^2$. We have already shown that $\tau g \cdot \Delta h_1 \Delta c$ and $\tau g \cdot h_1 c \Delta^2$ are both zero, but $\tau g \cdot \tau^4 e g^3$ is non-zero.

For the third formula, we know from E_∞ that $\Delta h_1 \cdot \Delta u$ equals $h_1 u \Delta^2$ or $h_1 u \Delta^2 + \tau^4 g^3$. We have already shown that $\tau \cdot \Delta h_1 \Delta u$ and $\tau \cdot h_1 u \Delta^2$ are zero, but $\tau \cdot \tau^4 g^3$ is non-zero.

For the fourth formula, we know from E_∞ that $(\Delta c)^2$ equals $h_1^2 d \Delta^2$ or $h_1^2 d \Delta^2 + \tau^2 \alpha^2 g^2$. Observe that

$$\tau \cdot (\Delta c)^2 = \langle h_2, \tau^2 g, c \rangle \tau \Delta c = h_2 \langle \tau^2 g, c, \tau \Delta c \rangle,$$

which must be zero because $\langle \tau^2 g, c, \tau \Delta c \rangle$ is zero for dimension reasons. Also, $\tau \cdot h_1^2 d \Delta^2$ is zero, but $\tau \cdot \tau^2 \alpha^2 g^2$ is non-zero.

For the fifth formula, we know from E_∞ that $\Delta c \cdot \Delta u$ equals $h_1^2 e \Delta^2$ or $h_1^2 e\Delta^2 + \tau^2 \alpha \nu g^2$. We already know that $\tau \cdot (\Delta c \cdot \Delta u)$ and $\tau \cdot h_1^2 e \Delta^2$ are zero, but $\tau \cdot \tau^2 \alpha \nu g$ is non-zero.

For the sixth formula, we know from E_∞ that $(\Delta u)^2$ equals $h_1^2 g \Delta^2$ or $h_1^2 g\Delta^2 + \tau^2 \nu^2 g^2$. We already know that $\tau \cdot (\Delta u)^2$ and $\tau \cdot h_1^2 g \Delta^2$ are zero, but $\tau \cdot \tau^2 \nu^2 g^2$ is non-zero. □

Lemma 4.11. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$*, we have* $\tau h_1^2 \Delta c = \tau^2 h_0 dg$ *but* $h_1^2 \Delta c$ *does not equal* $\tau h_0 dg$*.*

Proof. From E_∞, we know that $h_1 \cdot h_1^2 \Delta c$ is non-zero, but $h_1 \cdot \tau h_0 dg$ is zero. Therefore, $h_1^2 \Delta c$ and $\tau h_0 dg$ cannot be equal.

From E_∞, we also know that $h_1 \Delta c$ equals $c \Delta h_1$. Therefore,

$$\tau h_1^2 \Delta c = \tau h_1 c \Delta h_1 = \tau h_1 c \langle h_1, h_2, \tau^2 g \rangle = \langle \tau h_1 c, h_1, h_2 \rangle \tau^2 g.$$

It follows that $\tau h_1^2 \Delta c = \tau^2 h_0 dg$. □

Lemma 4.12. *In* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$*, we have* $\alpha^4 = h_0^4 \Delta^2 + \tau^4 P g^2$*.*

Proof. From E_∞, we know that α^4 equals either $h_0^4 \Delta^2$ or $h_0^4 \Delta^2 + \tau^4 P g^2$. We have already shown that $\Delta h_1 \cdot \alpha^4$ equals

$$\alpha^3 \cdot \tau^3 eg = \tau^3 \alpha^2 e \cdot \nu e = \tau^4 d \Delta h_1 \cdot dg = \tau^4 g \Delta h_1 (Pg + h_1^3 \Delta h_1) = \tau^4 P g^2 \Delta h_1.$$

However, $\Delta h_1 \cdot h_0^4 \Delta^2$ equals zero. It follows that α^4 must equal $h_0^4 \Delta^2 + \tau^4 P g^2$. □

4.3. Ring structure of $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$

Finally, we assemble the calculations made in the previous sections to compute $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ as a ring. The generators are listed in Table 7. All possible multiplicative extensions have been resolved in the previous section.

Table 7: Generators of $\mathrm{Ext}_{A(2)}$

generator	degree	generator	degree
h_0	$(0,1,0)$	ν	$(15,3,8)$
h_1	$(1,1,1)$	e	$(17,4,10)$
h_2	$(3,1,2)$	g	$(20,4,12)$
P	$(8,4,4)$	Δh_1	$(25,5,13)$
c	$(8,3,5)$	Δc	$(32,7,17)$
u	$(11,3,7)$	Δu	$(35,7,19)$
α	$(12,3,6)$	Δ^2	$(48,8,24)$
d	$(14,4,8)$		

Theorem 4.13. *The ring* $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ *is generated over* $\mathbb{M}_2$ *by the elements listed in Table 7, subject to the relations expressed in the multiplication table below (Table 8), as well as the following relations:*

1. h_0h_1, h_1h_2, $h_0^2h_2 + \tau h_1^3$, $h_0h_2^2$, h_2^3.
2. τu, $\tau h_1^2 c$, $\tau h_1^2 \Delta c + \tau^2 h_0 dg$, $\tau \Delta u$, τcd, τce, τcg, $\tau d\Delta c$, $\tau e\Delta c$, $\tau g\Delta c$, $\tau^2 h_2 g$.
3. $h_0^2\nu + \tau h_1 d$, $h_0h_2\nu + \tau h_1 e$, $h_2^2\nu + \tau h_1 g$, $h_0\alpha\nu + \tau h_1^2 \Delta h_1$.
4. $h_2 d + h_0 e$, $h_2 e + h_0 g$, $h_2\alpha + h_0\nu$.
5. $h_0 c$, $h_2 c$, $h_0 u$, $h_2 u$, $h_1\alpha$, $h_1\nu$, $h_0\Delta h_1$, $h_2\Delta h_1$, $h_0\Delta c + \tau h_0\alpha g$, $h_2\Delta c + \tau h_0\nu g$, $h_0\Delta u + \tau h_0\nu g$, $h_2\Delta u + \tau h_2\nu g$, $h_0\nu^2$, $h_2\nu^2$.
6. $h_0^2 d + Ph_2^2$, $h_0\alpha d + Ph_2\nu$, $\alpha^2 d + P\nu^2$.
7. $\alpha^2\nu + \tau d\Delta h_1$, $\alpha\nu^2 + \tau e\Delta h_1$, $\nu^3 + \tau g\Delta h_1$, $\alpha^4 + h_0^4\Delta^2 + \tau^4 Pg^2$.

Table 8: Multiplication table for $\mathrm{Ext}_{A(2)}$

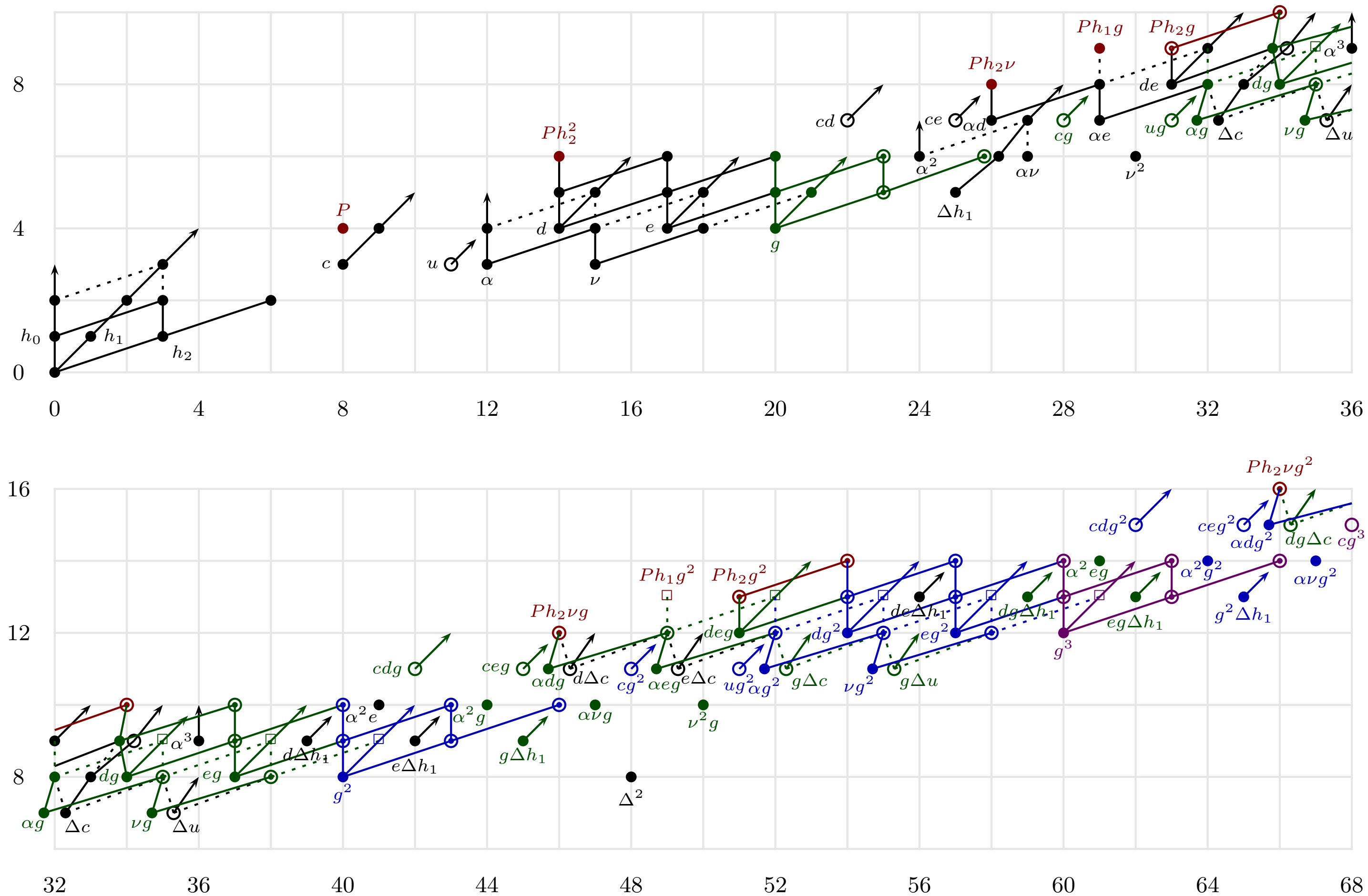
	c	u	α	d	ν
c	$h_1^2 d$	$h_1^2 e$	$\tau h_0^2 g$	cd	$\tau h_0 h_2 g$
u		$h_1^2 g$	$\tau h_0 h_2 g$	ce	$\tau h_2^2 g$
α			α^2	αd	$\alpha\nu$
d				$h_1^3\Delta h_1 + Pg$	αe
ν					ν^2

	e	Δh_1	Δc	Δu
c	ud	$h_1\Delta c$	$h_1 d\Delta h_1$	$h_1 e\Delta h_1$
u	cg	$h_1\Delta u$	$h_1 e\Delta h_1$	$h_1 g\Delta h_1$
α	νd	$\tau^3 eg$	0	0
d	de	$d\Delta h_1$	$d\Delta c$	$e\Delta c$
ν	αg	$\tau^3 g^2$	0	0
e	dg	$e\Delta h_1$	$d\Delta u$	$g\Delta c$
Δh_1		$h_1^2\Delta^2 + \tau^2\nu^2 g$	$h_1 c\Delta^2$	$h_1 u\Delta^2$
Δc			$h_1^2 d\Delta^2$	$h_1^2 e\Delta^2$
Δu				$h_1^2 g\Delta^2$

Some parts of $\mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ are depicted in the chart on the next page. The notation is the same as in the chart for the E_∞-term of the motivic May spectral sequence.

The most interesting difference occurs with Δc. Note that Δc supports exotic multiplications by h_0 and h_2. Note also that $\tau h_1^2\Delta c$ is no longer zero; this is an exotic τ-extension. In fact, $h_1^2\Delta c$ is the sum $(h_1^2\Delta c + \tau h_0 dg) + \tau h_0 dg$, where the first term is killed by τ and the second is killed by no power of τ.

Similarly to Δc, the classes Δu, $d\Delta c$, and $e\Delta c$ all support exotic multiplications by h_0 and h_2.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Δh_1
$\alpha\nu$
$Ph_2\nu$
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$\alpha^2 eg$
$eg\Delta h_1$
cdg^2
$\alpha^2 g^2$
ceg^2
αdg^2
$g^2\Delta h_1$
$Ph_2\nu g^2$
$dg\Delta c$
cg^3
$\alpha\nu g^2$

4.4. An Adams spectral sequence?

At this point, it is natural to wonder whether the cohomology of $A(2)$ is the E_2-term of an Adams spectral sequence that converges to the homotopy of some motivic spectrum that is analogous to *tmf*.

Assuming that such a spectral sequence exists, it is possible to entirely determine the d_2-differentials using two tools. The first is a map

$$\mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \to \mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$$

induced by the inclusion $A(2) \to A$, which ought to be a map of E_2-terms of spectral sequences. The second is the map $(-)_{\mathrm{cl}}\colon \mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2) \to \mathrm{Ext}_{A(2)_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$, which again ought to be a map of E_2-terms of spectral sequences.

For example, comparison to the full motivic Steenrod algebra implies that $d_2(e) = h_1^2 d$ [**DI**]. Then the relation $ud = ce$ implies that $d_2(u) = h_1^2 c$. Similar kinds of arguments allow one to compute the entire d_2 differential.

It is therefore possible to find explicitly the E_3-term of this hypothetical spectral sequence. Unfortunately, the E_3-term contains a large non-free part, none of which is seen in the classical situation. We will not here describe this messy calculation any further. In Section 5, we present a simpler way of making the same speculative calculation.

5. Adams-Novikov spectral sequence for "motivic modular forms"

In this section, we make a leap of faith and assume that there exists a motivic spectrum *mmf*, called "motivic modular forms", defined over $\mathrm{Spec}\,\mathbb{C}$. We assume that the topological realization of this motivic spectrum is the classical spectrum *tmf*. We also assume that the homotopy of *mmf* can be computed by an Adams-Novikov spectral sequence whose E_2-term is the cohomology of a version of the elliptic curves Hopf algebroid.

Recall that the elliptic curves Hopf algebroid localized at the prime 2 has the form $(\tilde{A}_{\mathrm{cl}}, \tilde{\Gamma}_{\mathrm{cl}})$, where $\tilde{A}_{\mathrm{cl}}$ is the ring $\mathbb{Z}_{(2)}[a_1, a_3, a_4, a_6]$ and $\tilde{\Gamma}_{\mathrm{cl}} = \tilde{A}_{\mathrm{cl}}[s, t]$ [**B**].

We write $\mathbb{M}_{(2)}$ for the ring $\mathbb{Z}_{(2)}[\tau]$, i.e., the 2-local motivic cohomology of a point.

Definition 5.1. The motivic elliptic curves Hopf algebroid localized at 2 is $(\tilde{A}, \tilde{\Gamma})$, where $\tilde{A} = \mathbb{M}_{(2)}[a_1, a_3, a_4, a_6]$ and $\tilde{\Gamma} = \tilde{A}[s, t]$. The bidegree of a_i is $(2i, i)$, while the bidegrees of s and t are $(2, 1)$ and $(6, 3)$ respectively.

The structure maps of $(\tilde{A}, \tilde{\Gamma})$ are defined to be compatible with the classical structure maps.

One may compute the cohomology of $(\tilde{A}, \tilde{\Gamma})$ as in [**B**, Section 7]. It turns out that the weights introduce no new complications. The answer is essentially the same as the classical answer, as shown in the chart on page 30 of [**B**]. In fact, the motivic answer is equal to the classical answer tensored over $\mathbb{Z}_{(2)}$ with $\mathbb{M}_{(2)}$. Also, the generators are assigned weights, but these are easy to determine. At location (s, t) of the chart, the weight of the generator is $\frac{s+t}{2}$.

We now have the E_2-term of a (speculative) spectral sequence for computing the homotopy groups of *mmf* at the prime 2. Next we consider differentials and compute the E_∞-term.

We assume that the topological realization of *mmf* is *tmf*. This implies that the motivic differentials are algebraically compatible with the classical differentials, i.e., they are equal after inverting τ. Using this fact, it turns out that the motivic differentials are entirely determined by the classical differentials. However, one must be careful with the weights. We describe this issue next.

The first classical differential is a d_3 that hits h_1^4. The input to this differential has weight 3 (since it lies in the 5-stem and has filtration 1), but h_1^4 has weight 4. It follows that d_3 hits τh_1^4 in the motivic calculation, not h_1^4 as in the classical calculation. As a result, instead of finding that $h_1^4 = 0$, we have that h_1^k is non-zero for all $k \geqslant 0$ but $\tau h_1^k = 0$ for $k \geqslant 4$.

The same situation occurs for the h_1-multiples of many of the classes in filtration 0. Classically, d_3-differentials tell us that these classes are killed by h_1^3. Motivically, these classes are not killed by h_1^k for any k, but they are killed by τh_1^3.

In order to simplify our discussion and make our diagrams legible, from here on we shall ignore these classes in filtration 0 and their h_1^k-multiples without further mention.

Classically, the next differential is $d_5(\Delta) = h_2 g$. Since Δ has weight 12 while $h_2 g$ has weight 14, the motivic differential is $d_5(\Delta) = \tau^2 h_2 g$. We observe that $h_2 g$ survives the motivic calculation, but $\tau^2 h_2 g = 0$.

The entire calculation proceeds similarly. Elementary bookkeeping with the weights shows that a non-zero d_{2k+1}-differential hits τ^k times a generator. In other words, non-zero d_{2k+1}-differentials produce classes that are killed by τ^k.

The results of the full analysis are shown in the three charts at the end of the paper. These are the E_∞-term of a (speculative) spectral sequence converging to the 2-complete homotopy groups of the (speculative) motivic spectrum *mmf*.

Boxes represent copies of $\mathbb{Z}_2[\tau]$. Solid dots represent copies of $\mathbb{Z}/2[\tau]$. A number k represents $\mathbb{Z}/2[\tau]/\tau^k$. A horizontal row of symbols at a single location represents extensions by 2. For example, at location $(3,1)$ there is a copy of $\mathbb{Z}/4[\tau]$, and at location $(23,5)$ there is a copy of $\mathbb{Z}/4[\tau]/\tau^2$. More interestingly, at location $(20,4)$ there is a copy of $\mathbb{Z}/8[\tau]$; at location $(40,8)$ there is a copy of $\mathbb{Z}/8[\tau]/4\tau^2$; and at location $(120,24)$ there is a copy of $\mathbb{Z}/8[\tau]/(\tau^{11}, 2\tau^6, 4\tau^2)$.

Observe that if we ignore the numbers and just consider the boxes and solid dots, we obtain the classical picture. This expresses the principle that the classical calculation is recovered by inverting τ.

Lines of slope 1 and 1/3 (in blue) show extensions by η and ν that are detected in the E_∞-term. They take generators to generators in the predictable way. The arrows of slope 1 (in blue) indicate infinite sequences of elements that are connected by h_1-multiplications and are killed by τ. We have not shown them explicitly in order to make the diagrams legible.

The other lines (in red) show exotic extensions by 2, η, and ν that are not detected in E_∞. Beware that these exotic extensions do not take generators to generators. The first example occurs in the 3-stem, where $4\nu = \tau\eta^3$. The next example is $\nu e[25,1] = \tau^2 \epsilon\overline{\kappa}$. As always, the required power of τ is easily determined by the weights. Here we write $e[s,t]$ for the homotopy element that is represented by the generator in stem s and filtration t. Note that ϵ and $\overline{\kappa}$ are names for $e[8,2]$ and $e[20,4]$.

The motivic calculation shows many more exotic extensions than the classical calculation. However, all of the motivic extensions are easily implied by the classical ones.

Remark 5.2. The classes $e[124,6]$, $e[144,10]$, $e[164,14]$, and $e[184,18]$ support non-exotic η-extensions, but they support exotic η-extensions after multiplication by τ^2.

The charts in [**B**] inadvertently failed to indicate exotic ν-extensions on $e[122,2]$, $e[142,6]$, $e[162,10]$, and $e[182,14]$. For completeness, we provide a proof.

Lemma 5.3. $\nu e[122,2] = \tau^9 e[125,21]$.

Proof. We need the following brackets:

$$e[25,1] = \langle \tau^2\overline{\kappa}, \nu, \eta\rangle$$
$$e[122,2] = \langle \eta, \tau^5\overline{\kappa}^3, \tau^6\overline{\kappa}^3\rangle.$$

These follow from Massey product calculations in the differential graded algebras (E_5, d_5) and (E_{23}, d_{23}) respectively.

By Toda shuffling, we compute

$$\begin{aligned}
\nu \cdot e[122,2] &= \nu\langle \eta, \tau^5\overline{\kappa}^3, \tau^6\overline{\kappa}^3\rangle \\
&= \langle \nu, \eta, \tau^5\overline{\kappa}^3\rangle \tau^6\overline{\kappa}^3 \\
&= \tau^6\overline{\kappa}\langle \nu, \eta, \tau^5\overline{\kappa}^3\rangle \overline{\kappa}^2 \\
&= \tau^4\langle \tau^2\overline{\kappa}, \nu, \eta\rangle \tau^5\overline{\kappa}^5 \\
&= \tau^9 e[25,1]\overline{\kappa}^5 \\
&= \tau^9 e[125,21].
\end{aligned}$$

□

Remark 5.4. One might be surprised to see that the classes $e[162,10]$ and $e[182,14]$ are hit by exotic η-extensions, yet support exotic ν-extensions. More precisely, we have

$$\nu\eta e[161,3] = \nu(\tau^3 e[162,10]) = \tau^{12} e[165,29].$$

But $\tau^{11} e[165,29] = 0$, so this is consistent with the fact that $\eta\nu = 0$.

Example 5.5. Note that $\pi_{0,*}mmf$ is $\mathbb{Z}_2[\tau]$. Hence for any k, $\pi_{k,*}mmf$ is a $\mathbb{Z}_2[\tau]$-module.

Consider $\pi_{120,*}mmf$. This module has 6 free generators of weight 60. These arise in filtration 0; only one is shown on the chart.

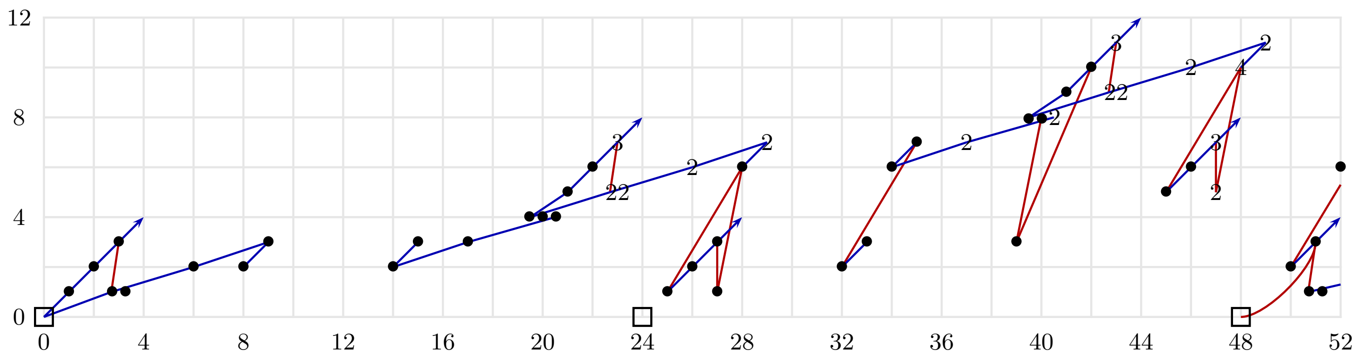
In addition, $\pi_{120,*}mmf$ contains many copies of $\mathbb{Z}/2[\tau]/\tau$; these arise as h_1-multiples and are not shown explicitly on the chart.

Finally, $\pi_{120,*}mmf$ contains a copy of $\mathbb{Z}/8[\tau]/(\tau^{11}, 2\tau^6, 4\tau^2)$, generated by $\overline{\kappa}^6$ with weight 72.

Example 5.6. The $\mathbb{Z}_2[\tau]$-module $\pi_{170,*}mmf$ contains a copy of

$$\mathbb{Z}/8[x,y,z]/(\tau^2 x, \tau^6 y, \tau^{11} z, 2x = \tau^4 y, 2y = \tau^6 z),$$

where the generators x, y, and z have weights 88, 92, and 98 respectively.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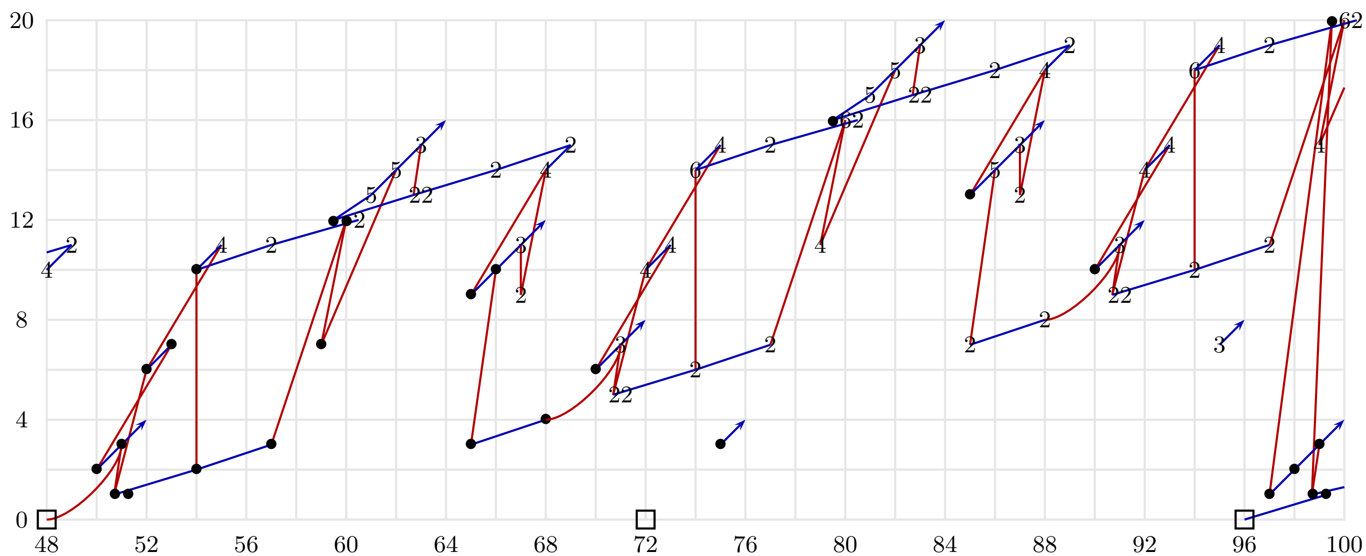
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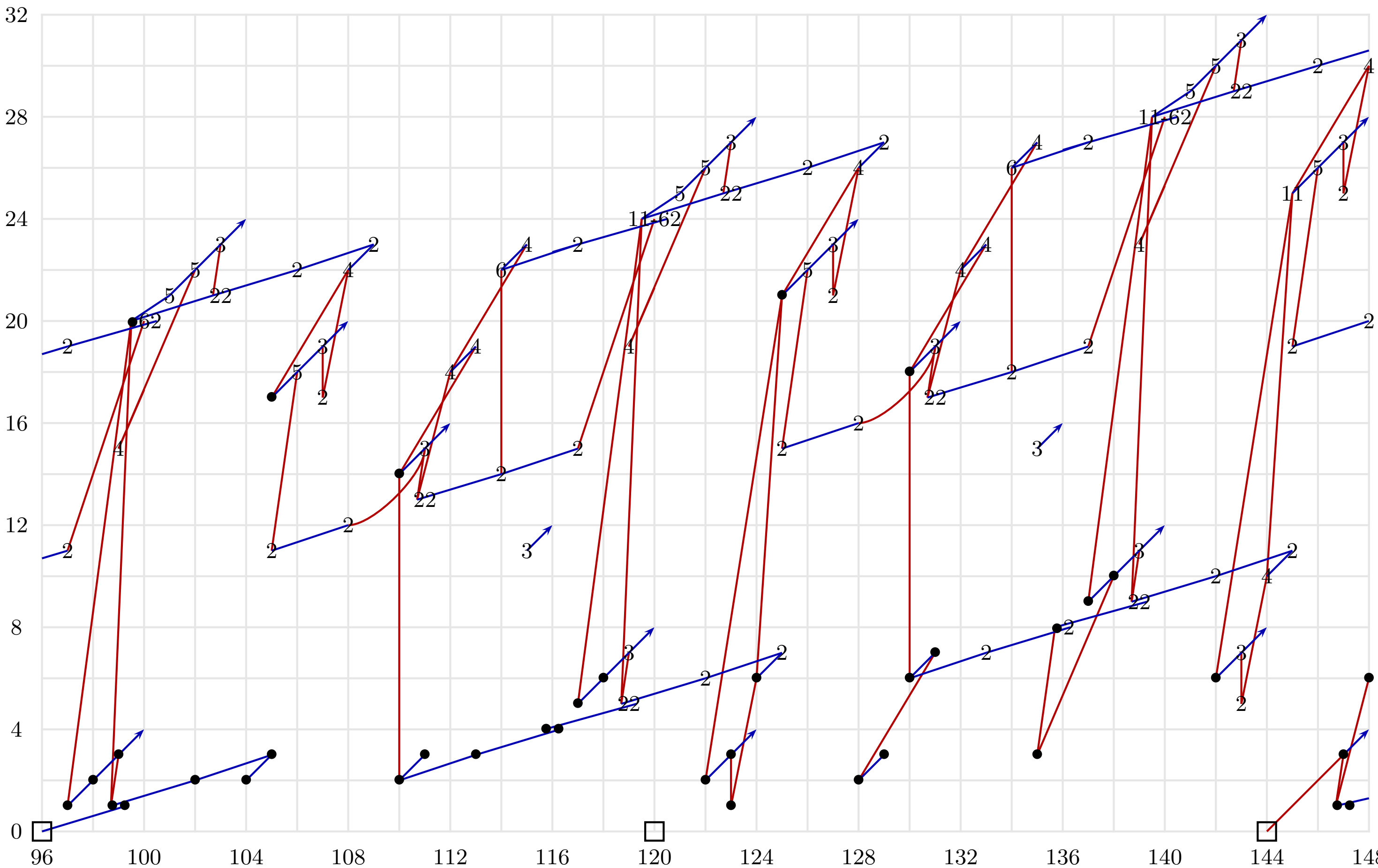
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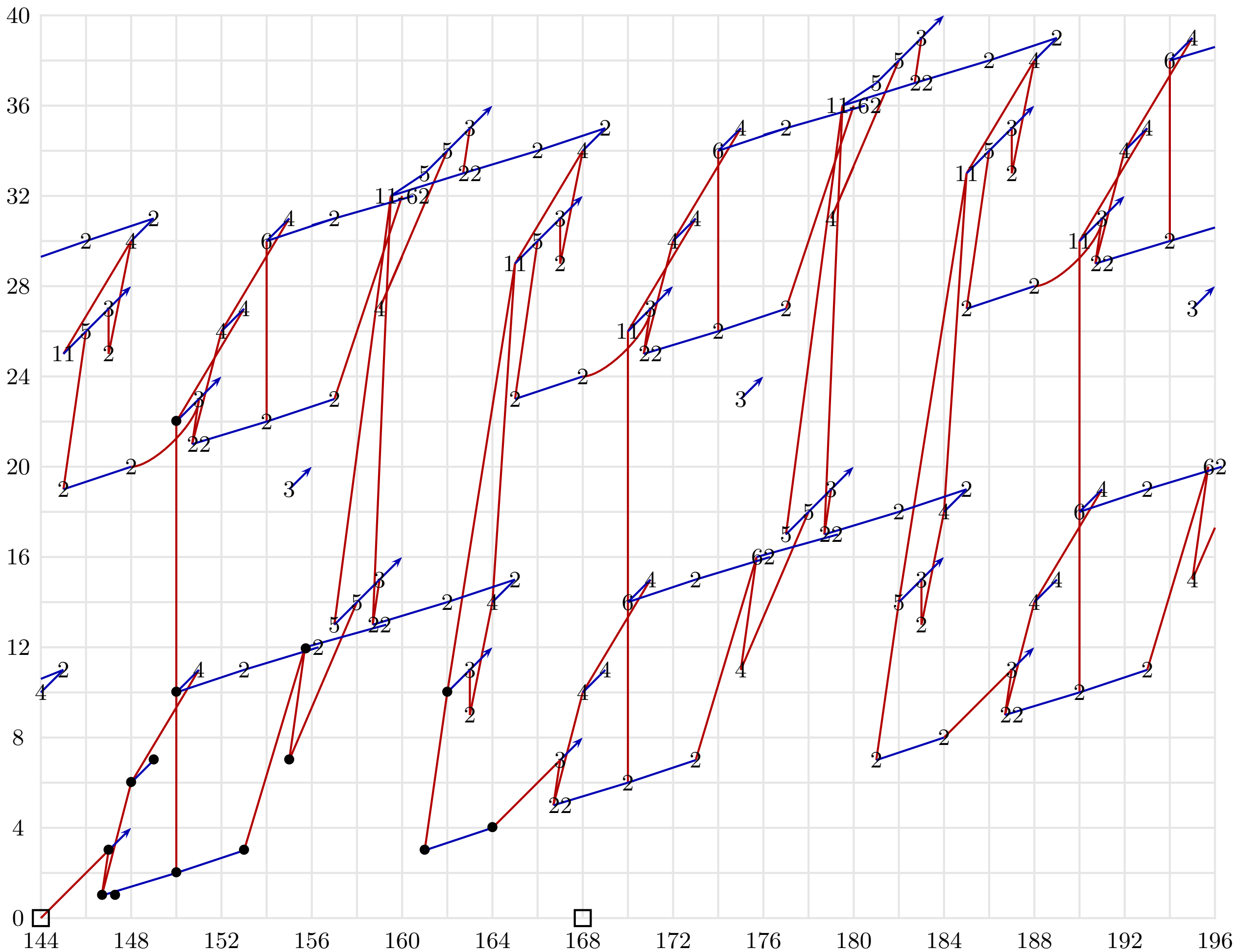
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