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REMARKS ON FINITE SUBSET SPACES

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(*communicated by J. Daniel Christensen*)

Abstract

This paper expands on and refines some known and less well-known results about the finite subset spaces of a simplicial complex X including their connectivity and manifold structure. It also discusses the inclusion of the singletons into the three-fold subset space and shows that this subspace is weakly contractible but generally non-contractible unless X is a cogroup. Some homological calculations are provided.

1. Statement of results

Let X be a topological space (always assumed to be path-connected), and k a positive integer. It has become increasingly useful in recent years to study the space

$$\mathrm{Sub}_n X := \{\{x_1, \ldots, x_\ell\} \subset X \mid \ell \leqslant n\}$$

of all finite subsets of X of cardinality at most n [**1, 3, 9, 15, 19, 23**]. This space is topologized as the identification space obtained from X^n by identifying two n-tuples if and only if the sets of their coordinates coincide [**4**]. The functors $\mathrm{Sub}_n(-)$ are homotopy functors in the sense that if $X \simeq Y$, then $\mathrm{Sub}_n(X) \simeq \mathrm{Sub}_n(Y)$. If $k \leqslant n$, then $\mathrm{Sub}_k X$ naturally embeds in $\mathrm{Sub}_n X$. We write $j_n\colon X \hookrightarrow \mathrm{Sub}_n X$ for the inclusion given by $j_n(x) = \{x\}$.

This paper takes advantage of the close relationship between finite subset spaces and symmetric products to deduce a number of useful results about them.

As a starting point, we discuss cell structures on finite subset spaces. We observe in Section 3 that if X is a finite d-dimensional simplicial complex, then $\mathrm{Sub}_n X$ is an nd-dimensional CW-complex and of which $\mathrm{Sub}_k X$ for $k \leqslant n$ is a subcomplex (Proposition 3.1). Furthermore, $\mathrm{Sub}\, X := \coprod_{n \geqslant 1} \mathrm{Sub}_n X$ has the structure of an abelian CW-monoid (without unit) whenever X is a simplicial complex.

In Section 4 we address a connectivity conjecture stated in [**25**]. We recall that a space X is r-connected if $\pi_i(X) = 0$ for $i \leqslant r$. A contractible space is r-connected for all positive r. In [**25**] Tuffley proves that $\mathrm{Sub}_n X$ is $n-2$-connected and conjectures that it is $n+r-2$-connected if X is r-connected. We are able to confirm his conjecture for the three-fold subset spaces. In fact we show

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Theorem 1.1. *If X is r-connected, $r \geqslant 1$ and $n \geqslant 3$, then $\mathrm{Sub}_n X$ is $r+1$-connected.*

In Section 5 we address a somewhat surprising fact about the embeddings

$$\mathrm{Sub}_k X \hookrightarrow \mathrm{Sub}_n X, \quad k \leqslant n.$$

A theorem of Handel [**9**] asserts that the inclusion $j\colon \mathrm{Sub}_k(X) \hookrightarrow \mathrm{Sub}_{2k+1}(X)$ for any $k \geqslant 1$ is trivial on homotopy groups (i.e. "weakly trivial"). This is, of course, not enough to conclude that j is the trivial map, and in fact it need not be. Let $\mathrm{Sub}_k(X, x_0)$ be the subspace of $\mathrm{Sub}_k X$ of all finite subsets containing the basepoint $x_0 \in X$. Handel's result is deduced from the more basic fact that the inclusion $j_{x_0}\colon \mathrm{Sub}_k(X, x_0) \hookrightarrow \mathrm{Sub}_{2k-1}(X, x_0)$ is weakly trivial. The following theorem implies that these maps are often not null-homotopic.

Theorem 1.2. *The embeddings*

$$j_{x_0}\colon X \hookrightarrow \mathrm{Sub}_3(X, x_0), \quad x \mapsto \{x, x_0\}$$

and

$$j\colon X \hookrightarrow \mathrm{Sub}_3(X), \qquad x \mapsto \{x\},$$

are both null-homotopic if X is a cogroup. If $X = S^1 \times S^1$ is the torus, then both j and j_{x_0} are non-trivial in homology and are hence essential.

For a definition of a cogroup, see Section 5. In particular, suspensions are cogroups. The second half of Theorem 1.2 follows from a general calculation given in Section 5 which exhibits a model for $\mathrm{Sub}_3(X, x_0)$ and uses it to show that its homology is an explicit quotient of the homology of the symmetric square $\mathrm{SP}^2 X$ by a submodule determined by the coproduct on $H_*(X)$. One deduces, in particular, a homotopy equivalence between $\mathrm{Sub}_3(\Sigma X, x_0)$ and the *reduced* symmetric square $\overline{\mathrm{SP}}^2(\Sigma X)$ (cf. Section 2.1 and Proposition 5.6). The methods in Section 5 are taken up again in [**12**] where an explicit spectral sequence is devised to compute $H_*(\mathrm{Sub}_n X)$ for any finite simplicial complex X and any $n \geqslant 1$.

The final two sections of this paper deal with manifold structures on $\mathrm{Sub}_n X$ and top homology groups. It is known that $\mathrm{Sub}_2 X = \mathrm{SP}^2 X$ is a closed manifold if and only if X is closed of dimension 2. This is a consequence of the fact that $\mathrm{SP}^2(\mathbb{R}^d)$ is not a manifold if $d > 2$, while $\mathrm{SP}^2(\mathbb{R}^2) \cong \mathbb{R}^4$ [**20**]. The following complete description is due to Wagner [**26**]:

Theorem 1.3. *Let X be a closed manifold of dimension $d \geqslant 1$. Then $\mathrm{Sub}_n X$ is a closed manifold if and only if either*

(i) $d = 1$ *and* $n = 3$, *or*

(ii) $d = 2$ *and* $n = 2$.

This result is established in Section 7 where we use, in the case $d \geqslant 2$, the connectivity result of Theorem 1.1, one observation from [**17**] and some homological calculations from [**13**]. In the case $d = 1$, we reproduce Wagner's cute argument. Furthermore in that section, we refine a result of Handel's [**9**] on the top homology groups of $\mathrm{Sub}_n X$ when X is a manifold. We point out that if X is a closed orientable manifold of dimension $d \geqslant 2$, then the top homology group $H_{nd}(\mathrm{Sub}_n X)$ is trivial if d is odd and is $\mathbb{Z}$ if d is even. This group is always trivial if X is not orientable (see Section 6).

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2. Basic constructions

All spaces X in this paper are path-connected, paracompact, and have a chosen basepoint x_0.

The way we will think of $\mathrm{Sub}_n X$ is as a quotient of the n-th symmetric product $\mathrm{SP}^n X$. This symmetric product is the quotient of X^n by the permutation action of the symmetric group $\mathfrak{S}_n$. The quotient map $\pi\colon X^n \longrightarrow \mathrm{SP}^n X$ sends $(x_1, \dots, x_n)$ to the equivalence class $[x_1, \dots, x_n]$. It will be useful sometimes to write such an equivalence class as an an abelian product $x_1 \cdots x_n$, $x_i \in X$. There are topological embeddings

$$j_n\colon X \hookrightarrow \mathrm{SP}^n X, \qquad x \mapsto x x_0^{n-1}. \tag{1}$$

The finite subset space $\mathrm{Sub}_n X$ is obtained from $\mathrm{SP}^n X$ through the identifications

$$[x_1, \dots, x_n] \sim [y_1, \dots, y_n] \Longleftrightarrow \{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}.$$

In multiplicative notation, elements of $\mathrm{Sub}_n X$ are products $x_1 x_2 \cdots x_k$ with $k \leqslant n$, and subject to the identifications $x_1^2 x_2 \cdots x_k \sim x_1 x_2 \cdots x_k$.

The topology of $\mathrm{Sub}_n X$ is the quotient topology inherited from $\mathrm{SP}^n X$ or X^n [**9**]. When X is Hausdorff, this topology is equivalent to the so-called *Vietoris finite* topology whose basis of open sets are sets of the form

$$[U_1, \dots, U_k] := \{S \in \mathrm{Sub}_n X \mid S \subset \bigcup_{i=1}^k U_i \text{ and } S \cap U_i \neq \emptyset \text{ for each } i\},$$

where U_i is open in X [**26**]. When X is a metric space, $\mathrm{Sub}_k X$ is again a metric space under the Hausdorff metric, and hence it inherits a third and equivalent topology [**26**]. In all cases, for any topology we use, continuous maps between spaces induce continuous maps between their finite subset spaces.

Example 2.1. Of course $\mathrm{Sub}_1 X = X$ and $\mathrm{Sub}_2 X = \mathrm{SP}^2 X$. Generally, if $\boldsymbol{\Delta}^{n+1} X \subset \mathrm{SP}^{n+1} X$ denotes the image of the fat diagonal in X^{n+1}, that is

$$\boldsymbol{\Delta}^{n+1} X := \{x_1^{i_1} \dots x_r^{i_r} \in \mathrm{SP}^{n+1} X \mid r \leqslant n, \sum i_j = n+1 \text{ and } \ i_j > 0\},$$

then there is a map

$$q\colon \boldsymbol{\Delta}^{n+1} X \longrightarrow \mathrm{Sub}_n X, \quad x_1^{i_1} \dots x_r^{i_r} \longrightarrow \{x_1, \dots, x_r\},$$

and a pushout diagram

$$\begin{array}{ccc} \mathbf{\Delta}^{n+1}X & \xrightarrow{i} & \mathrm{SP}^{n+1}X \\ \downarrow{\scriptstyle q} & & \downarrow \\ \mathrm{Sub}_n X & \longrightarrow & \mathrm{Sub}_{n+1} X. \end{array} \tag{2}$$

This is quite clear since we obtain $\mathrm{Sub}_{n+1} X$ by identifying points in the fat diagonal to points in $\mathrm{Sub}_n X$. In particular, when $n = 2$, we have the pushout

$$\begin{array}{ccc} X \times X & \xrightarrow{i} & \mathrm{SP}^3 X \\ \downarrow{\scriptstyle q} & & \downarrow \\ \mathrm{SP}^2 X & \longrightarrow & \mathrm{Sub}_3 X, \end{array} \tag{3}$$

where $q(x, y) = xy$ and $i(x, y) = x^2y$. The homology of $\mathrm{Sub}_3(X)$ can then be obtained from a Mayer-Vietoris sequence. Some calculations for the three-fold subset spaces are in Section 5.

There are two immediate and non-trivial consequences of the above pushouts. Albrecht Dold shows in [**7**] that the homology of the symmetric products of a CW-complex X only depends on the homology of X. The pushout diagram in (2) shows that, in the case of the finite subset spaces, this homology also depends on the *cohomology structure of* X. This general fact for the three- and four-fold subset spaces is further discussed in [**22**].

The second consequence of (2) is that it yields an important corollary.

Corollary 2.2. $\mathrm{Sub}_n X$ *is simply connected for* $n \geqslant 3$.

Proof. We use the following known facts about symmetric products: $\pi_1(\mathrm{SP}^n X) \cong H_1(X; \mathbb{Z})$ whenever $n \geqslant 2$, and the inclusion $j_n\colon X \hookrightarrow \mathrm{SP}^n X$ induces the abelianization map at the level of fundamental groups. (P.A. Smith [**21**] proves this for $n = 2$, but his argument applies for $n > 2$ [**22**].) For $n \geqslant 3$, consider the composite

$$X \xrightarrow{\alpha} \mathbf{\Delta}^n X \xrightarrow{i} \mathrm{SP}^n X$$

with $\alpha(x) = [x, x_0, \ldots, x_0]$. The induced map $j_{n*} = i_* \circ \alpha_*$ on π_1 is surjective, as we pointed out, and hence so is i_*. Assume we know that $\pi_1(\mathrm{Sub}_3(X)) = 0$. Then the fact that i_* is surjective implies immediately, by the Van-Kampen theorem and the pushout diagram in (2), that $\pi_1(\mathrm{Sub}_4 X) = 0$. By induction, we see that $\pi_1(\mathrm{Sub}_n X) = 0$ for larger n. Therefore, we need only establish the claim for $n = 3$. For that we apply Van Kampen to diagram (3). Consider the maps

$$\tau\colon x_0 \times X \hookrightarrow X \times X \xrightarrow{i} \mathrm{SP}^3 X$$

and

$$\beta\colon X \times x_0 \to X \times X \xrightarrow{q} \mathrm{SP}^2 X.$$

Now $i(x, y) = x^2y$ so that $\tau(x_0, x) = x_0^2x = j_3(x)$ and $\beta(x, x_0) = xx_0 = j_2(x)$. Since the j_k's are surjective on π_1 it follows that τ and β are surjective on π_1. Therefore, for any classes $u \in \pi_1(\mathrm{SP}^3 X)$ and $v \in \pi_1(\mathrm{SP}^2 X)$, $\exists$ a class $w \in \pi_1(X \times X)$ such that $i_*(w) = u$ and $q_*(w) = v$. This shows that $\pi_1(\mathrm{Sub}_3 X) = 0$. $\square$

This corollary also follows from [**5, 25**], where it is shown that $\mathrm{Sub}_n X$ is $(n-2)$-connected for $n \geqslant 3$. However, the proof above is completely elementary.

2.1. Reduced constructions

For the spaces under consideration, the natural inclusion $\mathrm{Sub}_{n-1} X \subset \mathrm{Sub}_n X$ is a cofibration [**9**]. We write $\overline{\mathrm{Sub}}_n X := \mathrm{Sub}_n X / \mathrm{Sub}_{n-1} X$ for the cofiber. Similarly, $\mathrm{SP}^{n-1} X$ embeds in $\mathrm{SP}^n X$ as the closed subset of all configurations $[x_1, \ldots, x_n]$ with x_i at the basepoint for some i. We set $\overline{\mathrm{SP}}^n X := \mathrm{SP}^n X / \mathrm{SP}^{n-1} X$, the symmetric smash product.

Note that even though $\mathrm{SP}^2 X$ and $\mathrm{Sub}_2 X$ are the same, there is an essential difference between their reduced analogs. The difference here comes from the fact that the inclusion $X \hookrightarrow \mathrm{Sub}_2 X$ is the composite $X \xrightarrow{\Delta} X \times X \longrightarrow \mathrm{SP}^2 X \cong \mathrm{Sub}_2 X$, where Δ is the diagonal, while $j_2 \colon : X \hookrightarrow \mathrm{SP}^2 X$ is the basepoint inclusion.

Example 2.3. When $X = S^1$, $\mathrm{SP}^2(S^1)$ is the closed Möbius band. If we view this band as a square with two sides identified along opposite orientations, then $S^1 = \mathrm{SP}^1(S^1) \hookrightarrow \mathrm{SP}^2(S^1)$ embeds into this band as an edge (see figures on p. 1124 of [**23**]). Hence this embedding is homotopic to the embedding of an equator, and so $\overline{\mathrm{SP}}^2(S^1)$ is contractible. On the other hand, $S^1 = \mathrm{Sub}_1(S^1)$ embeds into $\mathrm{Sub}_2(S^1) = \mathrm{SP}^2(S^1)$ as the diagonal $x \mapsto \{x, x\} = [x, x]$, which is the boundary of the Möbius band, and so $\overline{\mathrm{Sub}}_2(S^1) = \mathbb{R}P^2$.

Example 2.4. When $X = S^2$, $\mathrm{SP}^2(S^2)$ is the complex projective plane $\mathbb{P}^2$, $\mathrm{SP}^1(S^2) = \mathbb{P}^1$ is a hyperplane, and $\overline{\mathrm{SP}}^2(S^2) = S^4$. On the other hand, $\overline{\mathrm{Sub}}_2(S^2)$ has the following description: Write $\mathbb{P}^1$ for $\mathbb{C} \cup \{\infty\}$. Then $\overline{\mathrm{Sub}}_2(S^2)$ is the quotient of $\mathbb{P}^2$ by the image of the Veronese embedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$, $z \mapsto [z^2 : -2z : 1]$, $\infty \mapsto [1 : 0 : 0]$. To see this, identify $\mathrm{SP}^n(\mathbb{C})$ with $\mathbb{C}^n$ by sending $(z_1, \ldots, z_n)$ to the coefficients of the polynomial $(x - z_1) \cdots (x - z_n)$. This extends to the compactifications to give an identification of $\mathrm{SP}^n(S^2)$ with $\mathbb{P}^n$ ([**10**, Chapter 4]). When $n = 1$, (z, z) is mapped to the coefficients of $(x - z)(x - z)$, that is to $(z^2, -2z)$. Note that the diagonal $S^2 \longrightarrow \mathrm{SP}^2(S^2) = \mathbb{P}^2$ is multiplication by 2 on the level of H_2 so that, in particular, $H_4(\overline{\mathrm{Sub}}_2(S^2)) = \mathbb{Z}$, $H_2(\overline{\mathrm{Sub}}_2(S^2)) = \mathbb{Z}_2$, and all other reduced homology groups are zero.

3. Cell decomposition

If X is a simplicial complex, then there is a standard way to pick a $\mathfrak{S}_n$-equivariant simplicial decomposition for the product X^n so that the quotient map $X^n \longrightarrow \mathrm{SP}^n X$ induces a cellular structure on $\mathrm{SP}^n X$. We argue that this same cellular structure descends to a cell structure on $\mathrm{Sub}_n X$. The construction of this cell structure for the symmetric products is fairly classical [**14, 18**]. The following is a review and slight expansion:

Proposition 3.1. *Let X be a simplicial complex. For $n \geqslant 1$, there exist cellular decompositions for X^n, $\mathrm{SP}^n X$ and $\mathrm{Sub}_n X$ so that all of the quotient maps*

$$X^n \to \mathrm{SP}^n X \to \mathrm{Sub}_n X$$

and the concatenation pairings $+$ are cellular

$$\begin{array}{ccc} \mathrm{SP}^r X \times \mathrm{SP}^s X & \xrightarrow{+} & \mathrm{SP}^{r+s} X \\ \downarrow & & \downarrow \\ \mathrm{Sub}_r X \times \mathrm{Sub}_s X & \xrightarrow{+} & \mathrm{Sub}_{r+s} X. \end{array} \tag{4}$$

Furthermore, the subspaces $\boldsymbol{\Delta}^n, \mathrm{SP}^{n-1} X \subset \mathrm{SP}^n X$ and $\mathrm{Sub}_{n-1} X \subset \mathrm{Sub}_n X$ are subcomplexes.

Proof. Both $\mathrm{SP}^n X$ and $\mathrm{Sub}_n X$ are obtained from X^n via identifications. If for some simplicial (hence cellular) structure on X^n, derived from that on X, these identifications become simplicial (i.e. they identify simplices to simplices), then the quotients will have a cellular structure and the corresponding quotient maps will be cellular with respect to these structures.

As we know, one obtains a nice and natural $\mathfrak{S}_n$-equivariant simplicial structure on the product if one works with *ordered* simplicial complexes [**8, 14, 18**]. We write $X_\bullet$ for the abstract simplicial (i.e. triangulated) complex of which X is the realization. So we assume $X_\bullet$ to be endowed with a partial ordering on its vertices which restricts to a total ordering on each simplex. Let $\prec$ be that ordering. A point $w = (v_1, \ldots, v_n)$ is a vertex in $X_\bullet^n$ if and only if v_i is a vertex of $X_\bullet$. Different vertices

$$w_0 = (v_{01}, v_{02}, \ldots, v_{0n}), \ldots, w_k = (v_{k1}, v_{k2}, \ldots, v_{kn}) \tag{5}$$

span a k-simplex in $X_\bullet^n$ if, and only if, for each i, the $k+1$ vertices $v_{0i}, v_{1i}, \ldots, v_{ki}$ are contained in a simplex of X and $v_{0i} \prec v_{1i} \prec \cdots \prec v_{ki}$. We write $\varpi := [w_0, \ldots, w_k]$ for such a simplex.

The permutation action of $\tau \in \mathfrak{S}_n$ on $\varpi = [w_0, \ldots, w_k]$ is given by

$$\tau\varpi = [\tau w_0, \ldots, \tau w_n].$$

This is a well-defined simplex since the factors of each vertex

$$w_j = (v_{j_1 1}, v_{j_2 2}, \ldots, v_{j_n n})$$

are permuted simultaneously according to τ, and hence the order $\prec$ is preserved. The permutation action is then simplicial and $\mathrm{SP}^n X$ inherits a CW-structure by passing to the quotient.

Fact 1. If a point $p := (x_1, x_2, \ldots, x_n) \in X^n$ is such that $x_{i_1} = x_{i_2} = \cdots = x_{i_r}$, then p lies in some k-simplex ϖ whose vertices $[w_0, \ldots, w_k]$ are such that $v_{ji_1} = v_{ji_2} = \cdots = v_{ji_r}$ for $j = 0, \ldots, k$. This implies that the fat diagonal is a simplicial subcomplex. It also implies that any permutation that fixes such a point p must fix the vertices of the simplex it lies in and hences fixes it pointwise. In other words, if a permutation leaves a simplex invariant then it must fix it pointwise.

Fact 2. If $p = (x_1, x_2, \ldots, x_n) \in \varpi$ is a simplex with vertices $w_0, \ldots, w_k$ as in (5), and if $\pi\colon X^n \longrightarrow X^i$ is any projection, then $\pi(p)$ lies in the simplex with vertices

$\pi(w_0), \dots, \pi(w_k)$ (which may or may not be equal). For instance, $\pi(p) := (x_1, \dots, x_i)$ lies in the simplex with vertices $(v_{01}, v_{02}, \dots, v_{0i}), \dots, (v_{k1}, v_{k2}, \dots, v_{ki})$.

We are now in a position to see that $\mathrm{Sub}_n X$ is a CW-complex. Recall that $\mathrm{Sub}_n X = X^n/_\sim$, where

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) \Longleftrightarrow \{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}.$$

Clearly, if $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$, then $\tau(x_1, \dots, x_n) \sim \tau(y_1, \dots, y_n)$ for $\tau \in \mathfrak{S}_n$. We wish to show that these identifications are simplicial. Let us argue through an example (the general case being identical). We have the identifications in $\mathrm{Sub}_6 X$:

$$p := (x, x, x, y, y, z) \sim (x, x, y, y, y, z) =: q. \tag{6}$$

By using Fact 2 applied to the projection, skipping the third coordinate and then Fact 1, we can see that p and q lie in simplices with vertices of the form

$$(v_1, v_1, ?, v_2, v_2, v_3).$$

By using Fact 1 again, p lies in a simplex σ_p with vertices of the form

$$(v_1, v_1, v_1, v_2, v_2, v_3),$$

while q lies in a simplex σ_q with vertices of the form $(v_1, v_1, v_2, v_2, v_2, v_3)$. It follows that the identification (6) identifies vertices of σ_p with vertices of σ_q, and hence identifies σ_p with σ_q as desired.

In conclusion, the quotient $\mathrm{Sub}_n X$ inherits a cellular structure and the composite

$$X^n \xrightarrow{\pi} \mathrm{SP}^n X \xrightarrow{q} \mathrm{Sub}_n X$$

is cellular. Since the pairing (4) is covered by $X^r \times X^s \longrightarrow X^{r+s}$, which is simplicial (by construction), and since the projections are cellular, the pairing (4) must be cellular. □

Remark 3.2. We could have worked with simplicial sets instead [**5**]. Similarly, Mostovoy (private communication) indicates how to construct a simplicial set $\mathrm{Sub}_n X$ out of a simplicial set X such that $|\mathrm{Sub}_n X| = \mathrm{Sub}_n |X|$. This approach will be further discussed in [**12**].

The following corollary is also obtained in [**5**].

Corollary 3.3. *For X a simplicial complex,* $\mathrm{Sub}_k X$ *has a* CW*-decomposition with top cells in $k \dim X$, so that $H_*(\mathrm{Sub}_k X) = 0$ for $* > k \dim X$.*

We collect a couple more corollaries

Corollary 3.4. *If X is a d-dimensional complex with $d \geqslant 2$, then the quotient map $\mathrm{SP}^n X \to \mathrm{Sub}_n X$ induces a homology isomorphism in top dimension nd.*

Proof. When X is as in the hypothesis, $\mathrm{Sub}_{n-1} X$ is a codimension d subcomplex of $\mathrm{Sub}_n X$ and since $d \geqslant 2$, $H_{nd}(\mathrm{Sub}_n X) = H_{nd}(\mathrm{Sub}_n X, \mathrm{Sub}_{n-1} X)$. On the other hand, Proposition 3.1 implies that $\mathbf{\Delta}^n X$ is a codimension d subcomplex of $\mathrm{SP}^n X$ so that $H_{nd}(\mathrm{SP}^n X) \cong H_{nd}(\mathrm{SP}^n X, \mathbf{\Delta}^n X)$ as well. But according to diagram (2), we have the homeomorphism

$$\mathrm{SP}^n X/\mathbf{\Delta}^n X \cong \mathrm{Sub}_n X/\mathrm{Sub}_{n-1} X.$$

Combining these facts yields the claim. □

Corollary 3.5. *Both* $\mathrm{SP}^k X$ *and the fat diagonal* $\mathbf{\Delta}^k \subset \mathrm{SP}^k X$ *have the same connectivity as* X*, and this is sharp.*

Proof. If X is an r-connected ordered simplicial complex, then X admits a simplicial structure so that the r-skeleton X_r is contractible in X to some point $x_0 \in X$. With such a simplicial decomposition we can consider Liao's induced decomposition $X^k_\bullet$ on X^k and its r-skeleton X^k_r. Note that

$$X^k_r \subset \bigcup_{i_1+\cdots+i_k \leqslant r} X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k} \subset (X_r)^k.$$

If $F\colon X_r \times I \longrightarrow X$ is a deformation of X_r to x_0, then F^k is a deformation of $(X_r)^k$; hence X^k_r, to $(x_0, \ldots, x_0)$ in X^k, and this deformation is $\mathfrak{S}_k$ equivariant. Since the r-skeleton of $\mathrm{SP}^k X$ is the $\mathfrak{S}_k$-quotient of X^k_r, it is then itself contractible in $\mathrm{SP}^k X$, and this proves the first claim. Similarly, the simplicial decomposition we have introduced on X^k includes the fat diagonal Λ^k as a subcomplex with r-skeleton $\Lambda^k_r := \Lambda^k \cap X^k_r$. The deformation F^k preserves the fat diagonal and so it restricts to Λ^k and to an equivariant deformation $F^k\colon \Lambda^k_r \times I \longrightarrow \Lambda^k$. This means that the r-skeleton of $q(\Lambda^k) =: \mathbf{\Delta}^k \subset \mathrm{SP}^k X$ is itself contractible in $\mathbf{\Delta}^k$, and the second claim follows. This bound is sharp for symmetric products since when $X = S^2$, $\mathrm{SP}^2(S^2) = \mathbb{P}^2$. It is sharp for the fat diagonal as well since $\mathbf{\Delta}^3 X \cong X \times X$ has exactly the same connectivity of X. □

4. Connectivity

As we have established in Corollary 2.2, finite subset spaces $\mathrm{Sub}_n X$, $n \geqslant 3$, are always simply connected. In this section we further relate the connectivity of $\mathrm{Sub}_k X$ to that of X. We first need the following useful result proved in [**11**]:

Theorem 4.1. *If* X *is* r*-connected with* $r \geqslant 1$*, then* $\overline{\mathrm{SP}}^n X$ *is* $2n + r - 2$*-connected.*

Example 5.7 shows that $\overline{\mathrm{SP}}^2(S^k)$ is $k+1$-connected as asserted. Note that

$$\overline{\mathrm{SP}}^2(S^2) = S^4$$

is 3-connected, so Theorem 4.1 is sharp.

Corollary 4.2 ([**18**, Corollary 4.7])**.** *If* X *is* r*-connected,* $r \geqslant 1$*, then*

$$H_*(X) \cong H_*(\mathrm{SP}^n X)$$

for $* \leqslant r + 2$*. This isomorphism is induced by the map* j_n *adjoining the basepoint.*

Proof. We give a short proof based on Theorem 4.1. By Steenrod's homological splitting [**18**]

$$H_*(\mathrm{SP}^n X) \cong \bigoplus_{k=1}^{n} H_*(\mathrm{SP}^k X, \mathrm{SP}^{k-1} X) = \bigoplus_{k=2}^{n} \tilde{H}_*(\overline{\mathrm{SP}}^k X) \oplus H_*(X) \tag{7}$$

with $\mathrm{SP}^0 X = \emptyset$, but $\tilde{H}_*(\overline{\mathrm{SP}}^k X) = 0$ for $* \leqslant 2k + r - 2$. The result follows. □

Remark 4.3. Note that Corollary 4.2 cannot be improved to $r = 0$ (i.e. X-connected). It fails already for the wedge $X = S^1 \vee S^1$ and $n = 2$ since $\mathrm{SP}^2(S^1 \vee S^1) \simeq S^1 \times S^1$ (see [**13**]) and hence $H_2(\mathrm{SP}^2(S^1 \vee S^1)) \ncong H_2(S^1 \vee S^1)$. Note also that (7) implies that $H_*(X)$ embeds into $H_*(\mathrm{SP}^n X)$ for all $n \geqslant 1$, a fact we will find useful below.

Proposition 4.4. *Suppose X is r-connected, $r \geqslant 1$. Then $\mathrm{Sub}_k X$ is $r+1$-connected whenever $k \geqslant 3$.*

Proof. Write $x_0 \in X$ for the basepoint and assume $k \geqslant 3$. Remember that the $\mathrm{Sub}_k X$ are simply connected for $k \geqslant 3$ (Corollary 2.2) so by the Hurewicz theorem if they have trivial homology up to degree $r+1$, then they are connected up to that level. We will now show by induction that $H_*(\mathrm{Sub}_k X) = 0$ for $* \leqslant r+1$. The first step is to show that $H_*(\mathrm{SP}^k X, \boldsymbol{\Delta}^k) = H_*(\mathrm{Sub}_k X, \mathrm{Sub}_{k-1} X) = 0$ for $* \leqslant r+1$. We write $i\colon \boldsymbol{\Delta}^k \hookrightarrow \mathrm{SP}^k X$ for the inclusion.

From the fact that $\boldsymbol{\Delta}^k$ and $\mathrm{SP}^k X$ have the same connectivity as X (Corollary 3.5), their homology vanishes up to degree r which implies similarly that the relative groups are trivial up to that degree. On the other hand, X embeds in $\boldsymbol{\Delta}^k$ via $x \mapsto [x, x_0, \ldots, x_0]$ (this is a well-defined map since $k \geqslant 3$) and, since the composite $j_k\colon X \longrightarrow \boldsymbol{\Delta}^{\mathbf{k}} \xrightarrow{i} \mathrm{SP}^k X$ is an isomorphism on H_{r+1} (Corollary 4.2), we see that the map $i_*\colon H_{r+1}(\boldsymbol{\Delta}^k) \longrightarrow H_{r+1}(\mathrm{SP}^k X)$ is surjective. Hence, $H_{r+1}(\mathrm{SP}^k X, \boldsymbol{\Delta}^k) = 0$.

Now since $0 = H_*(\mathrm{SP}^k X, \boldsymbol{\Delta}^k) = H_*(\mathrm{Sub}_k X, \mathrm{Sub}_{k-1} X)$ for $* \leqslant r+1$, it follows that

$$H_*(\mathrm{Sub}_{k-1} X) \cong H_*(\mathrm{Sub}_k X) \qquad \text{for } * \leqslant r$$

and that

$$H_{r+1}(\mathrm{Sub}_{k-1} X) \to H_{r+1}(\mathrm{Sub}_k X) \quad \text{is surjective.}$$

So if we prove that $H_*(\mathrm{Sub}_3 X) = 0$ for $* \leqslant r+1$, then by induction we will have proved our claim.

Consider the homology long exact sequences for

$$(\mathrm{Sub}_3 X, \mathrm{Sub}_2 X) \qquad \text{and} \qquad (\mathrm{SP}^3 X, \boldsymbol{\Delta}^3 X),$$

where again we identify $\boldsymbol{\Delta}^3 X$ with $X \times X$. We obtain commutative diagrams

$$\begin{array}{ccccccccc}
\longrightarrow & H_{r+2}(\mathrm{Sub}_3 X, \mathrm{Sub}_2 X) & \longrightarrow & H_{r+1}(\mathrm{Sub}_2 X) & \xrightarrow{i_*} & H_{r+1}(\mathrm{Sub}_3 X) & \longrightarrow & 0 \\
& \cong\uparrow & & q_*\uparrow & & \pi_*\uparrow & & \\
\longrightarrow & H_{r+2}(\mathrm{SP}^3 X, X^2) & \longrightarrow & H_{r+1}(X^2) & \xrightarrow{\alpha_*} & H_{r+1}(\mathrm{SP}^3 X) & \longrightarrow & 0,
\end{array}$$

where $\alpha(x,y) = x^2 y$ and $\pi\colon \mathrm{SP}^3 X \longrightarrow \mathrm{Sub}_3 X$ is the quotient map. We want to show that $i_* = 0$ so that by exactness $H_{r+1}(\mathrm{Sub}_3 X) = 0$. Now q_* is surjective since the composite

$$X \longrightarrow X \times \{x_0\} \hookrightarrow X \times X \longrightarrow \mathrm{SP}^2 X = \mathrm{Sub}_2 X$$

induces an isomorphism on H_{r+1} by Corollary 4.2. Showing that $i_* = 0$ comes down, therefore, to showing that $\pi_* \circ \alpha_* = 0$. But note that for $r \geqslant 1$, which is the connectivity of X, classes in $H_{r+1}(X \times X)$ are necessarily spherical and we have the following commutative diagram:

$$\begin{array}{ccccc}\pi_{r+1}X \times \pi_{r+1}(X) & \xrightarrow{\cong} & \pi_{r+1}(X\times X) & \longrightarrow & \pi_{r+1}(\mathrm{Sub}_3(X)) \\ & & \big\downarrow h & & \big\downarrow h \\ & & H_{r+1}(X\times X) & \xrightarrow{\pi_*\circ\alpha_*} & H_{r+1}(\mathrm{Sub}_3(X)),\end{array}$$

where h is the Hurewicz homomorphism. The top map is trivial since when restricted to each factor $\pi_{r+1}(X)$ it is trivial according to the useful Theorem 5.1 below (or to Corollary 5.2). Since h is surjective, $\pi_*\circ\alpha_* = 0$ and $H_{r+1}(\mathrm{Sub}_3 X) = 0$ as desired. □

5. The three-fold finite subset space

There are many subtle points that come up in the study of finite subset spaces. We illustrate several of them through the study of the pair $(\mathrm{Sub}_3 X, X)$. The three-fold subset space has been studied in [**17, 19, 23**] for the case of the circle and in [**24**] for topological surfaces.

Again all spaces below are assumed to be connected. We say a map is weakly contractible (or weakly trivial) if it induces the trivial map on all homotopy groups. The following is based on a cute argument well explained in [**9**] or ([**3**, §3.4]).

Theorem 5.1 ([**9**])**.** $\mathrm{Sub}_k(X)$ *is weakly contractible in* $\mathrm{Sub}_{2k+1}(X)$.

Caveat 1. A map $f\colon A \longrightarrow Y$ being weakly contractible does not generally imply that f is null homotopic. Indeed let T be the torus and consider the projection $T \longrightarrow S^2$ which collapses the one-skeleton. Then this map induces an isomorphism on H_2 but is trivial on homotopy groups since $T = K(\mathbb{Z}^2, 1)$. Of course, if $A = S^k$ is a sphere, then "weakly trivial" and "null-homotopic" are the same since the map $A \longrightarrow Y$ represents the zero element in $\pi_k Y$. For example, in ([**6**, Lemma 3]), the authors explicitly construct an extension of the inclusion $S^n \hookrightarrow \mathrm{Sub}_3(S^n)$ to the disk $B^{n+1} \longrightarrow \mathrm{Sub}_3(S^n)$, $\partial B^{n+1} = S^n$. This section argues that this implication does not generally hold for non-suspensions.

Caveat 2. When comparing symmetric products to finite subset spaces, one has to watch out for the fact that the basepoint inclusion $\mathrm{SP}^k(X) \longrightarrow \mathrm{SP}^{k+1}(X)$ *does not commute* via the projection maps with the inclusion $\mathrm{Sub}_k(X) \longrightarrow \mathrm{Sub}_{k+1}(X)$. This has already been pointed out in Example 2.3 and is further illustrated in the corollary below.

Corollary 5.2. *The composite*

$$SP^k(X) \longrightarrow SP^{2k+1}(X) \longrightarrow \mathrm{Sub}_{2k+1}(X)$$

is weakly trivial.

Proof. This map is equivalent to the composite

$$\mathrm{SP}^k(X) \longrightarrow \mathrm{Sub}_k(X) \xrightarrow{\mu} \mathrm{Sub}_{k+1}(X, x_0) \hookrightarrow \mathrm{Sub}_{2k+1}(X), \tag{8}$$

where $\mu(\{x_1,\dots,x_k\}) = \{x_0, x_1, \dots, x_k\}$, x_0 is the basepoint of X and $\mathrm{Sub}_{k+1}(X, x_0)$ is the subspace of $\mathrm{Sub}_{k+1}(X)$ of all subsets containing this basepoint. Note that μ is not an embedding as pointed out in [**24**] but is one-to-one away from the fat diagonal. The key point here is again ([**9**, Theorem 4.1]) which asserts that the inclusion

$$\mathrm{Sub}_{k+1}(X, x_0) \hookrightarrow \mathrm{Sub}_{2k+1}(X, x_0)$$

is weakly contractible. This in turn implies that the last map in (8) is weakly trivial as well and the claim follows. □

Caveat 3. For $n \geqslant 2$, one can embed $X \hookrightarrow \mathrm{Sub}_n(X)$ in several ways. There is of course the natural inclusion j giving X as the subspace of singletons. There is also, for any choice of $x_0 \in X$, the embedding $j_{x_0}\colon x \mapsto \{x, x_0\}$. Any two such embeddings for different choices of x_0 are equivalent when X is path-connected (any choice of a path between x_0 and x_0' gives a homotopy between j_{x_0} and $j_{x_0'}$). It turns out, however, that j and j_{x_0} are fundamentally different. The simplest example was already pointed out for S^1, where $\mathrm{Sub}_2(S^1)$ was the Möbius band with j being the embedding of the boundary circle while j_{x_0} is the embedding of an equator.

One might ask the question whether it is true that j is null-homotopic if and only if j_{x_0} is null-homotopic? This is at least true for suspensions as the next lemma illustrates.

Recall that a co-H space X is a space whose diagonal map factors up to homotopy through the wedge; that is there exists a δ such that the composite

$$X \xrightarrow{\delta} X \vee X \hookrightarrow X \times X$$

is homotopic to the diagonal $\Delta\colon X \longrightarrow X \times X, x \mapsto (x, x)$. A cogroup X is a co-H space that is co-associative with a homotopy inverse. This latter condition means there is a map $c\colon X \longrightarrow X$ such that $X \xrightarrow{\delta} X \vee X \xrightarrow{\nabla(c \vee 1)} X$ is null-homotopic. This is in fact the definition of a left inverse but it implies the existence of a right inverse as well [**2**]. If X is a cogroup, then for every based space Y, the set of based homotopy classes of based maps $[X, Y]$ is a group. The suspension of a space is a cogroup and there exist several interesting cogroups that are not suspensions ([**2**, §4]).

Write $j_{x_0}\colon X \hookrightarrow \mathrm{Sub}_3(X, x_0)$ for the map $x \mapsto \{x, x_0\}$. Its continuation to $\mathrm{Sub}_3(X)$ is also written j_{x_0}.

Lemma 5.3. *Suppose X is a cogroup. Then the embeddings $j_{x_0}\colon X \hookrightarrow \mathrm{Sub}_3(X, x_0)$ and $j\colon X \hookrightarrow \mathrm{Sub}_3(X)$ are null-homotopic.*

Proof. The argument in [**9**] extends to this situation. We deal with j_{x_0} first. This is a based map at x_0. Its homotopy class $[j_{x_0}]$ lives in the group $G = [X, \mathrm{Sub}_3(X, x_0)]$. The following composite is checked to be again j_{x_0}:

$$j_{x_0}\colon X \xrightarrow{\Delta} X \times X \xrightarrow{j_{x_0} + j_{x_0}} \mathrm{Sub}_3(X, x_0).$$

This factors up to homotopy through the wedge

$$\iota\colon X \xrightarrow{\delta} X \vee X \xrightarrow{j_{x_0} \vee j_{x_0}} \mathrm{Sub}_3(X, x_0).$$

Of course $[\iota] = [j_{x_0}]$, but observe that $[\iota] = 2[j_{x_0}]$ by definition of the additive structure of G. This means that $[j_{x_0}] = 2[j_{x_0}]$; thus $[j_{x_0}] = 0$ and j_{x_0} is trivial (through a homotopy fixing x_0)

Let us now apply this to the inclusion $j\colon X \hookrightarrow \mathrm{Sub}_3(X)$ which is assumed to be based at x_0. We also denote the composite $X \xrightarrow{j_{x_0}} \mathrm{Sub}_3(X, x_0) \longrightarrow \mathrm{Sub}_3 X$ by j_{x_0}. Using the co-H structure as before, we get the homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle j+j} \\ X \vee X & \xrightarrow{j_{x_0} \vee j_{x_0}} & \mathrm{Sub}_3(X). \end{array}$$

Since j_{x_0} was just shown to be null homotopic, then so is $j = (j + j) \circ \Delta$. $\square$

Let us now turn to the second part of Theorem 1.2.

5.1. The space $\mathrm{Sub}_3(X, x_0)$

The preceding discussion shows the usefulness of looking at the based finite subset space $\mathrm{Sub}_n(X, x_0)$. We start with a key computation. Write Δ for the diagonal $X \longrightarrow \mathrm{SP}^2 X$, $x \mapsto [x, x]$, and identify the image of $j_*\colon H_*(X) \hookrightarrow H_*(\mathrm{SP}^2(X))$ with $H_*(X)$ by the Steenrod homological splitting (7).

Lemma 5.4. *Let X be a compact cell complex. Then*

$$H_*(\mathrm{Sub}_3(X, x_0)) = H_*(\mathrm{SP}^2 X)/I$$

where I is the submodule generated by $\Delta_* c - c, c \in H_*(X) \hookrightarrow H_*(\mathrm{SP}^2 X)$.

Proof. Start with the map $\alpha\colon \mathrm{SP}^2(X) \longrightarrow \mathrm{Sub}_3(X, x_0), [x, y] \mapsto \{x, y, x_0\}$, which is surjective and generically one-to-one (i.e. one-to-one on the subspace of points $[x, y]$ with $x \neq y$). Observe that $\alpha([x, x]) = \alpha([x, x_0])$. This implies that $\mathrm{Sub}_3(X, x_0)$ is homeomorphic to the identification space

$$\mathrm{SP}^2(X)/\sim, \quad [x, x] \sim [x, x_0], \quad \forall x \in X. \tag{9}$$

In order to compute the homology of this quotient we will replace it with the following space:

$$\begin{gathered} W_2(X) := \mathrm{SP}^2(X) \sqcup X \times I/\sim, \\ [x, x] \sim (x, 1), \quad [x, x_0] \sim (x, 0), \quad [x_0, x_0] \sim (x_0, t). \end{gathered} \tag{10}$$

It is not hard to see that (9) and (10) are homotopy equivalent. We can easily see that these spaces are homology equivalent as follows (this is enough for our purpose): There is a well-defined map

$$g\colon W_2(X) \longrightarrow \mathrm{SP}^2(X)/\sim$$

sending $[x, y] \mapsto [x, y], (x, t) \mapsto [x, x_0]$. The inverse image $g^{-1}([x, y]) = [x, y]$ if $x \neq y$ and both points are different from x_0. The inverse image of $[x, x]$ or $[x, x_0]$ is an interval when $x \neq x_0$, hence contractible, and it is a point when $x = x_0$. In all cases, preimages under g are acyclic and hence g is a homology equivalence by the Begle-Vietoris theorem. The homology structure of $\mathrm{Sub}_3(X, x_0)$ can be made much more apparent using the form (10) and this is why we have introduced it.

Let $(C_*(\mathrm{SP}^2(X)), \partial)$ be a chain complex for $\mathrm{SP}^2(X)$ containing $C_*(X)$ as a subcomplex and for which the diagonal map $X \longrightarrow \mathrm{SP}^2 X$ is cellular. Associate to

$c \in C_i(X)$ a chain $|c|$ in degree $i+1$ representing $I \times c \in C_{i+1}(I \times X)$ if $c \neq x_0$ (the 0-chain representing the basepoint). We write $|C_*(X)|$ for the set of all such chains. The geometry of our construction gives a chain complex for $W_2(X)$ as follows:

$$C_*(W_2(X)) = C_*(\mathrm{SP}^2(X)) \oplus |C_*(X)| \tag{11}$$

with boundary d such that $d(c) = \partial c$ and

$$d|c| = c - \Delta_*(c) - |\partial c|.$$

This comes from the formula for the boundary of the product of two cells which is in general given by $\partial(\sigma_1 \times \sigma_2) = \partial(\sigma_1) \times \sigma_2 + (-1)^{|\sigma_1|}\sigma_1 \times \partial(\sigma_2)$. We check indeed that $d \circ d = 0$. To compute the homology we need to understand cycles and boundaries in this chain complex. Write a general element of (11) as $\alpha + |c|$. The boundary of this element is $\partial\alpha + c - \Delta_*(c) - |\partial c|$ and it is zero, if and only if, $\partial\alpha = \Delta_*(c) - c$ and $|\partial c| = 0$. That is, if and only if, c is a cycle and $\Delta_*(c) - c$ is a boundary. This means that in $H_*(\mathrm{SP}^2(C))$, $\Delta_*(c) = c$. We claim this is not possible unless $c = 0$. Indeed, if c is a positive dimensional (homology) class, then $\Delta_*(c) = c \otimes 1 + \sum c' \otimes c'' + 1 \otimes c$ in $H_*(X \times X)$ and hence in $H_*(\mathrm{SP}^2(C))$, $\Delta_*(c) = 2c + \sum c' * c''$ where by definition $c' * c'' = q_*(c' \otimes c'')$ and $q\colon X \times X \longrightarrow \mathrm{SP}^2(X)$ is the projection. This can never be equal to c since $\sum c' * c'' \in H_*(\mathrm{SP}^2 X, X)$.

To recapitulate, $\alpha + |c|$ is a cycle if and only if α is a cycle and $c = 0$. The only cycles in $C_*(W_2(X))$ are those that are already cycles in the first summand $C_*(\mathrm{SP}^2(X))$. On the other hand, among these classes the only boundaries consist of boundaries in $C_*(\mathrm{SP}^2(X))$ and those of the form $\Delta_*(c) - c$ with c a cycle in $C_*(X)$ (in particular the only 0-cycle is represented by x_0). This proves our claim. □

Remark 5.5 (Added in revision). We could have noticed alternatively the existence of a pushout diagram

$$\begin{array}{ccc} X \vee X & \xrightarrow{f} & \mathrm{SP}^2 X \\ \big\downarrow{\scriptstyle \mathrm{fold}} & & \big\downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{j_{x_0}} & \mathrm{Sub}_3(X, x_0), \end{array}$$

where $f(x, x_0) = [x, x]$ is the diagonal and $f(x_0, x) = [x, x_0]$. We can in fact deduce Lemma 5.4 from this pushout. We can also deduce that $\mathrm{Sub}_3(X, x_0)$ is simply connected if X is. This useful fact we use to establish Proposition 5.6 next.

Note that Lemma 5.4 above says that $H_*(\mathrm{Sub}_3(X, x_0))$ only depends on $H_*(X)$ and on its coproduct (i.e. on the cohomology of X). When X is a suspension the situation becomes simpler. The following result is a nice combination of Lemmas 5.3 and 5.4.

Proposition 5.6. *There is a homotopy equivalence* $\mathrm{Sub}_3(\Sigma X, x_0) \simeq \overline{\mathrm{SP}}^2(\Sigma X)$.

Proof. When X is a suspension, all classes are primitive so that $\Delta_*(c) = 2c$ for all $c \in H_*(X)$. Combining Steenrod's splitting (7),

$$H_*(\mathrm{SP}^2 X) \cong H_*(X) \oplus H_*(\mathrm{SP}^2 X, X),$$

with Lemma 5.4, we deduce immediately that $H_*(\mathrm{Sub}_3(\Sigma X, x_0)) \cong H_*(\overline{\mathrm{SP}}^2(\Sigma X))$. Both spaces are simply connected (by Remark 5.5 and Theorem 4.1) and so it is enough to exhibit a map between them that induces this homology isomorphism. Consider the map $\alpha\colon \mathrm{SP}^2(\Sigma X) \longrightarrow \mathrm{Sub}_3(\Sigma X, x_0)$, $[x,y] \mapsto \{x,y,x_0\}$ as in the proof of Lemma 5.4. Its restriction to ΣX is null-homotopic according to Lemma 5.3 and hence it factors through the quotient $\overline{\mathrm{SP}}^2(\Sigma X) \longrightarrow \mathrm{Sub}_3(\Sigma X, x_0)$. By inspection of the proof of Lemma 5.4 we see that this map induces an isomorphism on homology. □

Example 5.7. A description of $\overline{\mathrm{SP}}^2(S^k)$ is given in ([**10**, Example 4K.5]) from which we infer that

$$\mathrm{Sub}_3(S^k, x_0) \simeq \Sigma^{k+1}\mathbb{R}P^{k-1}, \qquad k \geqslant 1.$$

This generalizes the calculation in [**24**] that $\mathrm{Sub}_3(S^2, x_0) \simeq S^4$.

5.2. Homology calculations

We determine the homology of $\mathrm{Sub}_3(T, x_0)$ and $\mathrm{Sub}_3(T)$ where T is the torus $S^1 \times S^1$. Symmetric products of surfaces are studied in various places (see [**13, 24**] and references therein). Their homology is torsion free and hence particularly simple to describe. We will write $q\colon X^n \longrightarrow \mathrm{SP}^n X$ throughout for the quotient map and

$$q_*(a_1 \otimes \ldots \otimes a_n) = a_1 * a_2 * \cdots * a_n$$

for its induced effect in homology. (Since our spaces are torsion free we identify $H_*(X \times Y)$ with $H_*(X) \otimes H_*(Y)$.)

Corollary 5.8. *The inclusion $j\colon \mathrm{Sub}_2(T, x_0) \hookrightarrow \mathrm{Sub}_3(T, x_0)$ is essential.*

Proof. We will show that j_* is non-trivial on $H_2(\mathrm{Sub}_2(T, x_0)) = H_2(T) = \mathbb{Z}$. Here $H_*(T)$ is generated by e_1, e_2 in dimension one, and by the orientation class $[T]$ in dimension two. The groups $H_*(\mathrm{SP}^2 T)$ are given as follows [**13**] (the generators are indicated between brackets):

$$\tilde{H}_*(\mathrm{SP}^2 T) = \begin{cases} \mathbb{Z}\{\gamma_2\}, & \dim 4 \\ \mathbb{Z}\{e_1 * [T], e_2 * [T]\}, & \dim 3 \\ \mathbb{Z}\{[T], e_1 * e_2\}, & \dim 2 \\ \mathbb{Z}\{e_1, e_2\}, & \dim 1, \end{cases} \tag{12}$$

where γ_2 is the orientation class $[\mathrm{SP}^2 T]$ ($\mathrm{SP}^2(T)$ is a compact complex surface). Then $[T] * [T] = 2\gamma_2$. Let Δ be the diagonal into the symmetric square

$$X \xrightarrow{\Delta} X \times X \xrightarrow{q} \mathrm{SP}^2(X).$$

Since

$$\Delta_*([T]) = [T] \otimes 1 + e_1 \otimes e_2 - e_2 \otimes e_1 + 1 \otimes [T],$$

$$q_*([T] \otimes 1) = q_*(1 \otimes [T]) = [T]$$

and

$$q_*(e_1 \otimes e_2) = -q_*(e_2 \otimes e_1) = e_1 * e_2,$$

we see that

$$\Delta_*([T]) = 2[T] + 2e_1 * e_2. \tag{13}$$

We can consider the composite

$$j_{x_0}\colon T \xrightarrow{\Delta} \mathrm{SP}^2T \xrightarrow{\alpha} \mathrm{Sub}_3(T, x_0) = \mathrm{SP}^2T/\sim\,,$$

where α is as in the proof of Lemma 5.4. According to Lemma 5.4, using the expression of the diagonal in (13), there are classes $a = \alpha_*[T], b = \alpha_*(e_1 * e_2)$ with $a = -2b \neq 0$. But $(j_{x_0})_*[T] = (\alpha \circ \Delta)_*[T] = \alpha_*([T]) = a$, and this is non-zero as desired. □

Remark 5.9. We can of course complete the calculation of $H_*(\mathrm{Sub}_3(T, x_0))$ from Lemma 5.4. Under α_*, $e_i \mapsto 0$ (primitive classes map to 0), $e_1 * e_2 \mapsto b$, $[T] \mapsto a = -2b$, $e_i * [T] \mapsto c_i$, and $\gamma_2 \mapsto d$, so that

$$H_1 = 0, \quad H_2 = \mathbb{Z}\{a\}, \quad H_3 = \mathbb{Z}\{c_1, c_2\}, \quad H_4 = \mathbb{Z}\{d\}.$$

It is equally easy to write down the homology groups for $\mathrm{Sub}_3(S, x_0)$ for any genus $g \geqslant 1$ surface, orientable or not.

Next we analyze the inclusion $T \hookrightarrow \mathrm{Sub}_3 T$ in the case of the torus (compare [**24**]). The starting point is the pushout (3) and the associated Mayer-Vietoris sequence

$$\cdots \longrightarrow H_*(T \times T) \xrightarrow{q_* \oplus i_*} H_*(\mathrm{SP}^2T) \oplus H_*(\mathrm{SP}^3T) \xrightarrow{g_* - \pi_*} H_*(\mathrm{Sub}_3 T) \longrightarrow H_{*-1}(T \times T) \longrightarrow \cdots,$$

where $q\colon T \times T \longrightarrow \mathrm{SP}^2T$ is the quotient map, $i(x,y) = x^2y$, $g\colon \mathrm{SP}^2T \hookrightarrow \mathrm{Sub}_3 T$ is the inclusion (here we have identified SP^2T with $\mathrm{Sub}_2 T$) and $\pi\colon \mathrm{SP}^3T \longrightarrow \mathrm{Sub}_3 T$ is the projection. We focus on degree 2 and follow [**13**] for the next computations.

We have $H_2(T \times T) = \mathbb{Z}^2$ generated by $[T] \otimes 1$ and $1 \otimes [T]$, $H_2(\mathrm{SP}^2T) = \mathbb{Z}^2 = H_2(\mathrm{SP}^3T)$ generated by a class of the same name $[T] = q_*([T] \otimes 1) = q_*(1 \otimes [T])$ and by $e_1 * e_2$; see (12). To describe the effect of i_* we write it as a composite

$$i\colon T \times T \xrightarrow{\Delta \times 1} T \times T \times T \xrightarrow{q} \mathrm{SP}^3T.$$

This gives $i_*([T] \otimes 1) = 2[T] + 2e_1 * e_2$ as in (13), while $i_*(1 \otimes [T]) = [T]$. The Mayer-Vietoris then looks like

$$\begin{aligned}\cdots \longrightarrow \mathbb{Z}^2 &\xrightarrow{q_* \oplus i_*} \mathbb{Z}^2 \oplus \mathbb{Z}^2 \xrightarrow{g_* - \pi_*} H_2(\mathrm{Sub}_3 T) \longrightarrow H_1(T \times T) \longrightarrow \cdots \\ (1,0) &\longmapsto ((1,0),(2,2)) \\ (0,1) &\longmapsto ((1,0),(1,0)).\end{aligned}$$

This sequence is exact. Observe that the class $((2,2),(0,0))$ is not in the kernel of $g_* - \pi_*$ because it cannot be in the image of $q_* \oplus i_*$. This means that $g_*(2,2) \neq 0$. This is all we need to derive the non-nullity of the map $j\colon X \hookrightarrow \mathrm{Sub}_3 X$.

Corollary 5.10. $j_*([T]) \neq 0$.

Proof. The inclusion j is the composite

$$j\colon T \xrightarrow{\Delta} T \times T \xrightarrow{\pi} \mathrm{SP}^2T \xrightarrow{g} \mathrm{Sub}_3 T$$

so that $j_*([T]) = g_*(2,2)$, and this is non-trivial as asserted above. □

6. The top dimension

Using facts about orientability of configuration spaces of closed manifolds ([**11**] for example), we slightly elaborate on [**9**] and ([**24**, Theorem 3]).

Proposition 6.1. *Suppose M is a closed manifold of dimension $d \geqslant 2$. Then*

$$H_{nd}(SP^nM;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } d \text{ even and } M \text{ orientable} \\ 0, & \text{if } d \text{ odd or } M \text{ non-orientable.} \end{cases}$$

For mod-2 coefficients, $H_{nd}(SP^nM;\mathbb{F}_2) = \mathbb{F}_2$. In all cases, the map

$$H_{nd}(SP^nM) \longrightarrow H_{nd}(\mathrm{Sub}_n M)$$

is an isomorphism (Corollary 3.4).

Proof. When $d = 2$ the claim is immediate since, as is well known, SP^nM is a closed manifold (orientable if and only if M is; see [**26**]). Generally our statement follows from the fact that $\mathrm{SP}^n(X)$ is an orbifold with codimension > 1 singularities, and hence its top homology group is that of a manifold. More explicitly, in our case, let us denote by $B(M, n)$ the configuration space of finite sets of cardinality n in M; that is

$$B(M, n) = \mathrm{SP}^nM - \boldsymbol{\Delta}^n = \mathrm{Sub}_n M - \mathrm{Sub}_{n-1} M,$$

where $\boldsymbol{\Delta}^n$ is the singular set consisting of tuples with at least one repeated entry (the image of the fat diagonal as defined in Section 2). By Poincaré duality suitably applied ([**11**, Lemma 3.5])

$$H^i(B(M, n); \pm\mathbb{Z}) \cong H_{nd-i}(\mathrm{SP}^nM, \boldsymbol{\Delta}^n; \mathbb{Z}), \tag{14}$$

where $\pm\mathbb{Z}$ is the orientation sheaf. By definition,

$$H^i(B(M, n), \pm\mathbb{Z}) = H^i(\mathrm{Hom}_{Br_n(M)}(C_*(\tilde{B}(M, n)), \mathbb{Z}))$$

where $Br_n(M) = \pi_1(B(M, n))$ is the braid group of M, $\tilde{B}(M, n)$ is the universal cover of $B(M, n)$ and the action of the class of a loop on $\mathbb{Z}$ is multiplication by ± 1 according to whether the loop preserves or reverses orientation. It is known that $B(M, n)$ is orientable if and only if M is orientable and even-dimensional ([**11**, Lemma 2.6]). That is, we can replace $\pm\mathbb{Z}$ by $\mathbb{Z}$ if M is orientable and d is even.

Since $\boldsymbol{\Delta}^n$ is a subcomplex of codimension d in SP^nM, we have

$$H_{nd-i}(\mathrm{SP}^nM, \boldsymbol{\Delta}^n) \cong H_{nd-i}(\mathrm{SP}^nM) \quad \text{for} \quad i < d - 1.$$

In particular, for $i = 0$ we obtain

$$H^0(B(M, n); \pm\mathbb{Z}) \cong H_{nd}(\mathrm{SP}^nM; \mathbb{Z}). \tag{15}$$

If M is even-dimensional and orientable, then

$$H^0(B(M, n); \pm\mathbb{Z}) \cong H^0(B(M, n); \mathbb{Z}) = \mathbb{Z},$$

since $B(M, n)$ is connected if $\dim M \geqslant 2$. If $\dim M$ is odd or M is non-orientable, then $B(M, n)$ is not orientable and $H^0(B(M, n); \pm\mathbb{Z}) = 0$, because $H^0(B(M, n); \pm\mathbb{Z})$ is the subgroup $\{m \in \mathbb{Z} \mid gm = m, \forall g \in \mathbb{Z}[\pi_1(B(M, n)]\}$. This establishes the claim for the symmetric products and hence for the finite subset spaces according to Corollary 3.4. □

Example 6.2. For $k \geqslant 2$ we have $H_{2k}(\mathrm{SP}^2 S^k) = H_{2k}(\overline{\mathrm{SP}}^2 S^k) = H_{k-1}(\mathbb{R}P^{k-1})$ (see Example 5.7) and this is $\mathbb{Z}$ or 0 depending on whether k is even or odd as predicted by Proposition 6.1.

6.1. The case of the circle

When $M = S^1$, Proposition 6.1 is not true anymore since $\mathrm{SP}^n S^1 \simeq S^1$ for all $n \geqslant 1$, while $\mathrm{Sub}_n(S^1)$ is either S^n or S^{n-1} depending on whether n is odd or even [**15, 23**]. In this case, it is still possible to explicitly describe the quotient map $\mathrm{SP}^n(S^1) \longrightarrow \mathrm{Sub}_n(S^1)$.

A beautiful theorem of Morton asserts that the multiplication map

$$\mathrm{SP}^{n+1}(S^1) \longrightarrow S^1$$

is an n-disc bundle η_n over S^1 which is orientable if and only if n is even [**16**]. A close scrutiny of Morton's proof shows that the sphere bundle associated to η_n consists of the image of the fat diagonal $\boldsymbol{\Delta}^{n+1}$, i.e. the singular set. If $\mathrm{Th}(\eta_n)$ is the Thom space of η_n, then

$$\mathrm{Th}(\eta_n) = \mathrm{SP}^{n+1}(S^1)/\boldsymbol{\Delta}^{n+1} = \mathrm{Sub}_{n+1} S^1 / \mathrm{Sub}_n S^1. \tag{16}$$

Since η_n is trivial when $n = 2k$ is even, it follows that

$$\mathrm{Th}(\eta_{2k}) = S^{2k} \wedge S^1_+ = S^{2k+1} \vee S^{2k}. \tag{17}$$

However, as pointed out above, $\mathrm{Sub}_{2k+1}(S^1) \simeq S^{2k+1}$. The map

$$\mathrm{SP}^{2k+1}(S^1) \longrightarrow \mathrm{Sub}_{2k+1}(S^1)$$

factors through the Thom space (17) and the top cell maps to the top cell. Combining (16) and (17), it is immediate to see that

Lemma 6.3. *The map* $\mathrm{Th}(\eta_{2k}) \longrightarrow \mathrm{Sub}_{2k+1}(S^1)$*, restricted to the first wedge summand in (17), induces a map* $S^{2k+1} \longrightarrow \mathrm{Sub}_{2k+1}(S^1)$ *which is a homotopy equivalence.*

7. Manifold structure

In this last section we prove Theorem 1.3. We distinguish three cases: when the dimension of the manifold is $d > 2$, $d = 2$ or $d = 1$.

Lemma 7.1. *Suppose X is a manifold of dimension $d > 2$. Then* $\mathrm{Sub}_n X$ *is never a manifold if $n \geqslant 2$.*

Proof. Consider the projection $X^n \longrightarrow \mathrm{Sub}_n X$ given by identifying tuples whose sets of coordinates are the same. This projection restricts to an $n!$ regular covering between the complements $\pi_n : X^n - \Lambda^n \longrightarrow \mathrm{Sub}_n X - \mathrm{Sub}_{n-1} X$, where Λ^n as before is the fat diagonal in X^n. Suppose $\mathrm{Sub}_n X$ is a manifold of dimension nd (necessarily). Pick a point in $\mathrm{Sub}_{n-1} X$ and an open chart U around it. Now $U \cong \mathbb{R}^{nd}$ and

$Y = U \cap \mathrm{Sub}_{n-1} X$ is a closed subset in U. We can apply Alexander duality to the pair (Y, U) and obtain

$$H_{nd-i-1}(U - Y) \cong H^i(Y).$$

But $Y \subset \mathrm{Sub}_{n-1}(X)$ is an open subspace in a simplicial complex of dimension $(n-1)d$; therefore $H^{nd-2}(Y) = 0$ (since $d > 2$) and so $H_1(U - Y) = 0$. We can now use an elementary observation of Mostovoy [**17**] to the effect that since $U - Y$ is covered by $\pi_n^{-1}(U - Y)$, a connected étale cover of degree $n!$, then it is impossible for $H_1(U - Y)$ to be trivial since the monodromy gives a surjection $\pi_1(U - Y) \longrightarrow \mathfrak{S}_n$, and hence a non-trivial map $H_1(U - Y) \longrightarrow \mathbb{Z}_2$. □

Theorem 2.4 of [**26**] shows that our Lemma 7.1 is valid if $d = 2$ and $n > 2$ as well. As opposed to the geometric approach of Wagner, we provide below a short homological proof of this result.

Lemma 7.2. *Suppose X is a closed topological surface. Then $\mathrm{Sub}_n X$ is a manifold if and only if $n = 2$.*

Proof. We will show that if $n \geqslant 3$, then $\mathrm{Sub}_n(X)$ cannot even have the homotopy type of a closed manifold by showing that it does not satisfy Poincaré duality. We rely on results of [**13**] that give a simple description of a CW-decomposition of a space $\widehat{\mathrm{SP}}^n X$ homotopy equivalent to $\mathrm{SP}^n X$ when X is a two-dimensional complex. Since X is a closed two-dimensional manifold, it has a cell structure of the form $X = \bigvee^r S^1 \cup D^2$ where D^2 is a two-dimensional cell attached to a bouquet of circles. Each circle corresponds in the cellular chain complex for $\widehat{\mathrm{SP}}^n X$ to a one-dimensional cell generator e_i, $1 \leqslant i \leqslant r$, while the two-dimensional cell is represented by D. This chain complex has a concatenation product $*\colon C_*(\widehat{\mathrm{SP}}^r X) \otimes C_*(\widehat{\mathrm{SP}}^s X) \longrightarrow C_*(\widehat{\mathrm{SP}}^{r+s} X)$ under which these cells map to product cells. The full cell complex for $\widehat{\mathrm{SP}}^n X$ is made up of all products of the form

$$e_{i_1} * \cdots * e_{i_\ell} * \mathrm{SP}^k D, \qquad i_1 + \cdots + i_\ell + k \leqslant n,$$

where $i_r \neq i_s$ if $r \neq s$, and where $\mathrm{SP}^k D$ is a $2k$-dimensional cell represented geometrically by the k-th symmetric product of D^2. The boundary ∂ is a derivation and is completely determined on generators by $\partial e_i = 0$ and $\partial \mathrm{SP}^n D = \partial D * \mathrm{SP}^{n-1} D$.

If $X = \bigvee^r S^1 \cup D$ is a closed manifold, then in mod-2 homology, $\partial D = 0$ (the top cell). This implies of course that $\partial \mathrm{SP}^n D = 0$ (the top cell of $\mathrm{SP}^n X$), while $H_{2n-1}(\mathrm{SP}^n X, \mathbb{Z}_2) \cong \mathbb{Z}_2^r$ with generators $e_i * \mathrm{SP}^{n-1} D$. This shows, in particular, that $H_{2n-1}(\mathrm{SP}^n X; \mathbb{Z}_2) \neq 0$ if $r \geqslant 1$, that is if X is not the two sphere. Observe that this calculation is compatible with Theorem 2 of [**24**].

Now we know that $\mathrm{Sub}_n X$ is simply connected if $n \geqslant 3$. Suppose $\mathrm{Sub}_n X$ is a closed manifold, then by Poincaré duality, $H_{2n-1}(\mathrm{Sub}_n X; \mathbb{Z}_2) = H_1(\mathrm{Sub}_n X; \mathbb{Z}_2) = 0$. But recall the pushout diagram (2) and its associated Mayer-Vietoris exact sequence

$$\begin{aligned} H_{2n-1}(\Delta_n) \longrightarrow H_{2n-1}(\mathrm{Sub}_{n-1} X) &\oplus H_{2n-1}(\mathrm{SP}^n X) \\ &\longrightarrow H_{2n-1}(\mathrm{Sub}_n X) \longrightarrow H_{2n-2}(\Delta_n) \longrightarrow \cdots . \end{aligned}$$

Since Δ_n and $\mathrm{Sub}_{n-1} X$ are $(2n-2)$-dimensional subcomplexes of $\mathrm{Sub}_n X$, their homology in degree $2n-1$ vanishes. The sequence above becomes

$$0 \longrightarrow H_{2n-1}(\mathrm{SP}^n X) \longrightarrow H_{2n-1}(\mathrm{Sub}_n X) \longrightarrow H_{2n-2}(\Delta_n) \longrightarrow \cdots$$

and $H_{2n-1}(\mathrm{SP}^n X)$ injects into $H_{2n-1}(\mathrm{Sub}_n X)$. When $H_1(X) \neq 0$, that is when X is not the sphere, $H_{2n-1}(\mathrm{Sub}_n X)$ is non-trivial contradicting Poincaré duality.

We are left with the case $\mathrm{Sub}_n(S^2)$ and $n \geqslant 3$. Here we have to rely on a calculation of Tuffley [**24**] who shows that

$$H_{2n-2}(\mathrm{Sub}_n(S^2)) = \mathbb{Z} \oplus \mathbb{Z}_{n-1}. \tag{18}$$

But $\mathrm{Sub}_n(S^2)$ is 2-connected according to Theorem 1.1 and Poincaré duality is violated in this case as well. □

Remark 7.3. A computation of the homology of $\mathrm{Sub}_n(S^2)$ for all n and various field coefficients will appear in [**12**]. It is however straightforward using the Mayer-Vietoris sequence for the pushout (3) to show that

$$\tilde{H}_*(\mathrm{Sub}\, 3S^2) \cong \begin{cases} \mathbb{Z}, & * = 6 \\ \mathbb{Z} \oplus \mathbb{Z}_2, & * = 4. \end{cases} \tag{19}$$

Similar computations appear in [**5, 22, 24**].

Finally we address the case $d = 1$. Write $I = [0,1]$, $\dot{I} = (0,1)$. First of all $\mathrm{SP}^n(I) \cong I^n$. In fact, this is precisely the n-simplex since any point of $\mathrm{SP}^n(I)$ can be written uniquely as an n-tuple $(x_1, \ldots, x_n)$ with $0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant 1$. The quotient map $q_2 \colon \mathrm{SP}^2(I) \longrightarrow \mathrm{Sub}_2(I)$ is a homeomorphism and hence every interior point of $\mathrm{Sub}_2(I)$ has a manifold neighborhood. The same for $n = 3$ since $\mathrm{SP}^3(I)$ is the three simplex

$$\{(x_1, x_2, x_3) \mid 0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant 1\}$$

with four faces: $F_1 : \{x_1 = 0\}$, $F_2 : \{x_1 = x_2\}$, $F_3 : \{x_2 = x_3\}$ and $F_4 : \{x_3 = 1\}$, and the quotient map $q_3 \colon \mathrm{SP}^3(I) \to \mathrm{Sub}_3(I)$ identifies the faces F_2 and F_3. Such an identification gives again I^3 and $\mathrm{Sub}_3(\dot{I})$ is this simplex with two faces removed [**19**]. For $n > 3$, the corresponding map q_n identifies various faces of the simplex $\mathrm{SP}^n(I)$ to obtain $\mathrm{Sub}_n(I)$, but this fails to give a manifold structure on the quotient for there are just too many "branches" that come together at a single point in the image of the boundary of this simplex. This is made precise below.

Lemma 7.4. $\mathrm{Sub}_n(S^1)$ *is a closed manifold if and only if* $n = 1, 3$.

Observe that if n is even, then $\mathrm{Sub}_n S^1$ cannot be a closed manifold for a simple reason: no closed manifold of dimension n can be homotopic to a sphere of dimension $n - 1$.

*Proof of Lemma 7.4 following [**26**, Theorem 2.3].* Let M be a manifold and D a disc neighborhood of a point $x \in M$. Then an open neighborhood of $x \in \mathrm{Sub}_n(M)$ is $\mathrm{Sub}_n(D)$. So if $\mathrm{Sub}_n(D)$ is not a manifold, then neither is $\mathrm{Sub}_n(M)$. To prove Lemma 7.4 we will argue as in [**26**] that $\mathrm{Sub}_n(\mathbb{R})$ is not a manifold for $n \geqslant 4$.

For a metric space X (with metric d), non-empty subsets $S,T\subset X$, and fixed elements $s\in S, t\in T$, we define

$$d(s,T)=\inf\{d(s,t)\mid t\in T\},$$
$$d(S,t)=\inf\{d(s,t)\mid s\in S\}.$$

Then the Hausdorff metric D on $\mathrm{Sub}_n(X)$ is defined to be

$$D(S,T):=\sup\{d(s,T),d(t,S)\mid s\in S,t\in T\}.$$

Thus $D(S,T)<\epsilon$ means that each $s\in S$ is within an ϵ-neighborhood of some point in T and each $t\in T$ is within an ϵ-neighborhood of some point in S.

We wish to show that $\mathrm{Sub}_n(\mathbb{R})$ for $n\geqslant 4$ is not homemorphic to $\mathbb{R}^n$. Pick $S=\{1,2,\dots,n-1\}$ in $\mathrm{Sub}_{n-1}(\mathbb{R})$ and for each i consider the open set C_i (in the Hausdorff metric) of all subsets $\{p_1,\dots,p_{n-1},q_i\}\in\mathrm{Sub}_n(\mathbb{R})$ such that $p_j\in(j-\frac{1}{2},j+\frac{1}{2})$ and $q_i\in(i-\frac{1}{2},i+\frac{1}{2})$. We then see that C_i is the subset with one or two points in the $\frac{1}{2}$-neighborhood of i and a single point in the $\frac{1}{2}$-neighborhood of j for $i\neq j$. Note that $C_i\subset U$ where $U=\{T\in\mathrm{Sub}_n(\mathbb{R})\mid D(S,T)<1/2\}$. Observe that

$$C_1=\mathrm{Sub}_2\left(\frac{1}{2},\frac{3}{2}\right)\times\left(\frac{3}{2},\frac{5}{2}\right)\times\cdots\times\left(n-1-\frac{1}{2},n-1+\frac{1}{2}\right).$$

This is an n-dimensional manifold with boundary $V=U\cap\mathrm{Sub}_{n-1}(\mathbb{R})$, and in fact one has

$$C_i=\left\{T\in U: T\cap\left(i-\frac{1}{2},i+\frac{1}{2}\right)\text{ has 1 or 2 points}\right\}\cup V.$$

Clearly $C_1\cup C_2\cup\cdots\cup C_{n-1}=U$ and, more importantly, all these open sets have a common boundary at V; i.e. $C_i\cap C_j=V$. If $n\geqslant 4$, we can choose at least three such C_i, say C_1,C_2,C_3. Then $C_1\cup C_2$ is an open n-dimensional manifold (union over the common boundary V). It must be contained in the interior of $\mathrm{Sub}_n(\mathbb{R})$ and hence must be open there if $\mathrm{Sub}_n(\mathbb{R})$ were to be an n-dimensional manifold. But $C_1\cup C_2$ is not open in $\mathrm{Sub}_n(\mathbb{R})$ since every neighborhood of $\{1,2,\dots,n-1\}$ must meet C_3-V which is disjoint from $C_1\cup C_2$ (i.e. "too many" branches come together at that point). □

We conclude this paper with the following cute theorem of Bott, which is the most significant early result on the subject:

Corollary 7.5 (Bott)**.** *There is a homeomorphism* $\mathrm{Sub}_3(S^1)\cong S^3$.

Proof. It has been known since Seifert that the Poincaré conjecture holds for Seifert manifolds; that is, if a Seifert 3-manifold is simply connected then it is homeomorphic to S^3.[1] Clearly $\mathrm{Sub}_3(S^1)$ is a Seifert manifold where the action of S^1 on a subset is by multiplication on elements of that subset. Since it is simply connected (Corollary 2.2), the claim follows. Note that the S^1-action has two exceptional fibers consisting of the orbits of $\{1,-1\}$ and $\{1,j,j^2\}$ where $j=e^{2\pi i/3}$ (compare [**23**]). □

[1]We thank Peter Zvengrowski for reminding us of this fact.

References

[1] S. Albeverio, Y. Kodratiev, and M. Rockner, Analysis and geometry on configuration spaces, *J. Functional Analysis* **154** (1998), no. 2, 444–500.

[2] M. Arkowitz, Co-H-spaces, *Handbook of algebraic topology*, North-Holland, Amsterdam (1995), 1143–1173.

[3] A. Beilinson and V. Drinfeld, *Chiral algebras*, AMS Colloq. Publications **51**, Amer. Math. Soc., Providence, RI, 2004.

[4] K. Borsuk and S. Ulam, On symmetric products of spaces, *Bull. Amer. Math. Soc.* **37** (1931), no. 12, 875–882.

[5] R. Biro, The homotopy cyclic configuration spaces and the homotopy type of Kontsevich's orientation space, Stanford thesis (1994).

[6] D. Curtis and N. Nhu, Hyperspaces of finite subsets which are homeomorphic to $\aleph_0$-dimensional linear metric spaces, *Topology Appl.* **19** (1985), no. 3, 251–260.

[7] A. Dold, Homology of symmetric products and other functors of complexes, *Ann. of Math. (2)* **68** (1958), no. 1, 54–80.

[8] B. Dwyer, Classifying spaces and homology decompositions, in *Homotopy theoretic methods in group cohomology*, Advanced Courses in Math, CRM Barcelona, Birkhäuser (2001).

[9] D. Handel, Some homotopy properties of spaces of finite subsets of topological spaces, *Houston J. Math.* **26** (2000), no. 4, 747–764.

[10] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge (2002).

[11] S. Kallel, Symmetric products, duality and homological dimension of configuration spaces, in *Groups, homotopy and configuration spaces*, Geometry and Topology Monographs **13** (2008), 499–527.

[12] S. Kallel, Finite subset spaces and a spectral sequence of Biro, work in progress.

[13] S. Kallel and P. Salvatore, Symmetric products of two dimensional complexes, *Contemp. Math.* **407** (2006), 147–161.

[14] S.D. Liao, On the topology of cyclic products of spheres, *Transactions Amer. Math. Soc.* **77** (1954), 520–551.

[15] R.J. Milgram and R.C. Penner, Riemann's moduli space and the symmetric groups, in *Mapping class groups and moduli spaces of Riemann surfaces* (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math. **150** (1993), 247–290.

[16] H.R. Morton, Symmetric products of the circle, *Proc. Cambridge Philos. Soc.* **63** (1967), 349–352.

[17] J. Mostovoy, Geometry of truncated symmetric products and real roots of real polynomials, *Bull. London Math. Soc.* **30** (1998), no. 2, 159–165.

[18] M. Nakaoka, Cohomology of symmetric products, *J. Institute of Polytechnics*, Osaka City University **8** (1957), no. 2, 121–140.

[19] S. Rose, A hyperbolic approach to $\exp_3(S^1)$, `arXiv:0708.2085`.

[20] R. Seroul, Anneau de cohomologie entière et KU-théorie d'un produit symétrique d'une surface de Riemann, *Publ. Dép. Math. (Lyon)* **9** (1972), 27–66.

[**21**] P.A. Smith, Manifolds with abelian fundamental group, *Annals of Math. (2)* **37** (1936), no. 3, 526–532.

[**22**] W. Taamallah, thesis in progress, University of Tunis.

[**23**] C. Tuffley, Finite subset spaces of S^1, *Algebraic Geom. Topol.* **2** (2002), 1119–1145.

[**24**] C. Tuffley, Finite subset spaces of graphs and punctured surfaces, *Algebr. Geom. Topol.* **3** (2003), 873–904.

[**25**] C. Tuffley, Connectivity of finite subset spaces of cell complexes, *Pacific J. Math.* **217** (2004), no. 1, 175–179.

[**26**] C. Wagner, Symmetric, cyclic and permutation products of manifolds, dissertationes mathematicae, Polska Akademia Nauk, Warszawa (1980).

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