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THREE-CROSSED MODULES

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(*communicated by Jean-Louis Loday*)

Abstract

We introduce the notion of a 3-crossed module, which extends the notions of a 1-crossed module (Whitehead) and a 2-crossed module (Conduché). We show that the category of 3-crossed modules is equivalent to the category of simplicial groups having a Moore complex of length 3. We make explicit the relationship with the cat^3-groups (Loday) and the 3-hypercomplexes (Cegarra-Carrasco), which also model algebraically homotopy 4-types.

1. Introduction

Crossed modules (or 1-crossed modules) were first defined by Whitehead in [**25**]. They model connected homotopy 2-types. Conduché [**12**] in 1984 described the notion of a 2-crossed module as a model of connected 3-types. More generally, Loday [**20**] gave the foundation of a theory of another algebraic model, which is called cat^n-groups, for connected $(n+1)$-types. Ellis-Stein [**17**] showed that cat^n-groups are equivalent to crossed n-cubes. A link between simplicial groups and crossed n-cubes were given by Porter [**23**]. Conduché [**13**] gave a relation between crossed 2-cubes (i.e., crossed squares) and 2-crossed modules. 2-crossed modules were known to be equivalent to that of simplicial groups whose Moore complex has length 2. In [**4, 5**], Baues introduced a related notion of a quadratic module. The first author and Ulualan [**2**] also explored some relations among these algebraic models for (connected) homotopy 3-types.

The most general investigation into the extra structure of the Moore complex of a simplicial group was given by Carrasco-Cegarra in [**9**] to construct the non-abelian version of the classical Dold-Kan theorem. A much more general context of their work was given by Bourn in [**6**]. Carrasco and Cegarra arrived at a notion of hypercrossed complexes and proved that the category of such hypercrossed complexes is equivalent to that of simplicial groups. If one truncates hypercrossed complexes at level n, throwing away terms of higher dimension, then the resulting n-hypercrossed complexes form a category equivalent to the n-hyper groupoids of groups given by Duskin [**15**] and Glenn [**18**], and give algebraic models for n-types. For $n = 1$, a 1-hypercrossed complex gives a crossed module, whilst a subcategory of the category of hypercrossed 2-complexes is equivalent to Conduche's category of 2-crossed modules.

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Mutlu-Porter [**22**] introduced a Peiffer pairing structure within the Moore complexes of a simplicial group. They applied this structure to the study of algebraic models for homotopy types.

In this article we will define the notion of a 3-crossed module as a model for homotopy 4-types. The methods we use are based on ideas of Conduché given in [**12**] and a Peiffer pairing structure within the Moore complexes of a simplicial group. We prove that the category of 3-crossed modules is equivalent to that of simplicial groups with Moore complex of length 3 which is equivalent to that of 3-hypercrossed complexes. The main problem with the 3-hypercrossed complex is difficult to handle intuitively.

The advantages of the notion of a 3-crossed module are the following:

(i) It provides a new algebraic model for (connected) homotopy 4-types;

(ii) It is easy to handle with respect to other models such as the 3-hypercrossed complex;

(iii) It gives a possible way of generalising n-crossed modules (or equivalently n-groups (see [**24**])) which is analogous to a n-hypercrossed complex.

(iv) In [**5**], Baues points out that a "nilpotent" algebraic model for 4-types is not known. 3-crossed modules go some way toward that aim.

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2. Simplicial groups, Moore complexes, Peiffer pairings

We refer the reader to [**14**] and [**21**] for the basic properties of simplicial structures.

2.1. Simplicial groups

A simplicial group $\mathbf{G}$ consists of a family of groups $\{G_n\}$ together with face and degeneracy maps $d_i^n\colon G_n \to G_{n-1}$, $0 \leqslant i \leqslant n$, $(n \neq 0)$ and $s_i^n\colon G_{n-1} \to G_n$, $0 \leqslant i \leqslant n$, satisfying the usual simplicial identities given in [**14, 21**]. The category of simplicial groups will be denoted by **SimpGrp**.

Let Δ denote the category of finite ordinals. For each $k \geqslant 0$ we obtain a subcategory $\Delta_{\leqslant k}$ determined by the objects $[i]$ of Δ with $i \leqslant k$. A k-truncated simplicial group is a functor from $\Delta^{\mathrm{op}}_{\leqslant k}$ to **Grp** (the category of groups). We will denote the category of k-truncated simplicial groups by $\mathbf{Tr}_k\mathbf{SimpGrp}$. By a *$k$-truncation of a simplicial group*, we mean a k-truncated simplicial group $\mathbf{tr}_k\mathbf{G}$ obtained by forgetting dimensions of order $> k$ in a simplicial group $\mathbf{G}$. Then we have the adjoint situation

$$\mathbf{SimpGrp} \underset{\mathbf{st}_k}{\overset{\mathbf{tr}_k}{\rightleftarrows}} \mathbf{Tr}_k\mathbf{SimpGrp},$$

where $\mathbf{st}_k$ is called the k-skeleton functor. For detailed definitions see [**15**].

2.2. The Moore complex

The Moore complex $\mathbf{NG}$ of a simplicial group $\mathbf{G}$ is defined to be the normal chain complex $(\mathbf{NG}, \partial)$ with

$$NG_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} d_i$$

and with the differential $\partial_n \colon NG_n \to NG_{n-1}$ induced from d_n by restriction.

The nth *homotopy group* $\pi_n(\mathbf{G})$ of $\mathbf{G}$ is the nth homology of the Moore complex of $\mathbf{G}$; i.e.,

$$\pi_n(\mathbf{G}) \cong H_n(\mathbf{NG}, \partial) = \bigcap_{i=0}^{n} \ker d_i^n / d_{n+1}^{n+1}\Big(\bigcap_{i=0}^{n} \ker d_i^{n+1}\Big).$$

We say that the Moore complex $\mathbf{NG}$ of a simplicial group $\mathbf{G}$ is of *length* k if $\mathbf{NG}_n = 1$ for all $n \geqslant k+1$. We denote the category of simplicial groups with Moore complex of length k by $\mathbf{SimpGrp}_{\leqslant k}$.

The Moore complex, $\mathbf{NG}$, carries a hypercrossed complex structure (see Carrasco [**9**]) from which $\mathbf{G}$ can be rebuilt. We briefly recall some of the aspects of this reconstruction that we will need later.

2.3. The poset of surjective maps

The following notation and terminology is derived from [**10**].

For the ordered set $[n] = \{0 < 1 < \cdots < n\}$, let $\alpha_i^n \colon [n+1] \to [n]$ be the increasing surjective map given by

$$\alpha_i^n(j) = \begin{cases} j & \text{if } j \leqslant i, \\ j-1 & \text{if } j > i. \end{cases}$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various α_i^n by composition. The composition of these generating maps is subject to the following rule:

$$\alpha_j \alpha_i = \alpha_{i-1} \alpha_j, j < i.$$

This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_r}$ with $0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) : 0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly, the only element of $S(n, 0)$ is

$$(n-1, n-2, \ldots, 0).$$

For all $n \geqslant 0$, let

$$S(n) = \bigcup_{0 \leqslant r \leqslant n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_s, \ldots, j_1)$ in $S(n)$ if $i_1 = j_1, \ldots, i_k = j_k$ but $i_{k+1} > j_{k+1}$, $(k \geqslant 0)$, or if $i_1 = j_1, \ldots, i_r = j_r$ and $r < s$. This makes $S(n)$ an ordered set. For example,

$$\begin{aligned}
S(2) &= \{\phi_2 < (1) < (0) < (1,0)\},\\
S(3) &= \{\phi_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\},\\
S(4) &= \{\phi_4 < (3) < (2) < (3,2) < (1) < (3,1) < (2,1) < (3,2,1)\\
&\quad < (0) < (3,0) < (2,0) < (3,2,0) < (1,0) < (3,1,0) < (2,1,0) < (3,2,1,0)\}.
\end{aligned}$$

2.4. The semidirect decomposition of a simplicial group

The fundamental idea behind the semidirect decomposition of a simplicial group can be found in Conduché [**12**]. A detailed investigation of this construction for the case of simplicial groups is given in Carrasco and Cegarra [**9**].

Given a split extension of groups

$$1 \longrightarrow K \longrightarrow G \underset{s}{\overset{d}{\rightleftarrows}} P \longrightarrow 1,$$

we write $G \cong K \rtimes s(P)$, the semidirect product of the normal subgroup, K, with the image of P under the splitting s.

Proposition 2.1. *If $\boldsymbol{G}$ is a simplicial group, then for any $n \geqslant 0$*

$$\begin{aligned}
G_n \cong\ & (\cdots(NG_n \rtimes s_{n-1}NG_{n-1}) \rtimes \cdots \rtimes s_{n-2}\cdots s_0NG_1)\\
& \rtimes (\cdots(s_{n-2}NG_{n-1} \rtimes s_{n-1}s_{n-2}NG_{n-2}) \rtimes \cdots \rtimes s_{n-1}s_{n-2}\cdots s_0NG_0).
\end{aligned}$$

Proof. This is done by the repeated use of the following lemma. □

Lemma 2.2. *Let $\boldsymbol{G}$ be a simplicial group. Then G_n can be decomposed as a semidirect product:*

$$G_n \cong \operatorname{Ker} d_n^n \rtimes s_{n-1}^{n-1}(G_{n-1}).$$

The bracketing and the order of terms in this multiple semidirect product are generated by the sequence

$$\begin{aligned}
G_1 &\cong NG_1 \rtimes s_0NG_0\\
G_2 &\cong (NG_2 \rtimes s_1NG_1) \rtimes (s_0NG_1 \rtimes s_1s_0NG_0)\\
G_3 &\cong ((NG_3 \rtimes s_2NG_2) \rtimes (s_1NG_2 \rtimes s_2s_1NG_1))\\
&\quad \rtimes ((s_0NG_2 \rtimes s_2s_0NG_1) \rtimes (s_1s_0NG_1 \rtimes s_2s_1s_0NG_0))
\end{aligned}$$

and

$$\begin{aligned}
G_4 \cong\ & (((NG_4 \rtimes s_3NG_3) \rtimes (s_2NG_3 \rtimes s_3s_2NG_2)) \rtimes ((s_1NG_3 \rtimes s_3s_1NG_2)\\
& \rtimes (s_2s_1NG_2 \rtimes s_3s_2s_1NG_1))) \rtimes s_0(\text{decomposition of } G_3).
\end{aligned}$$

Note that the term corresponding to $\alpha = (i_r, \ldots, i_1) \in S(n)$ is

$$s_\alpha(NG_{n-\#\alpha}) = s_{i_r\cdots i_1}(NG_{n-\#\alpha}) = s_{i_r}\cdots s_{i_1}(NG_{n-\#\alpha}),$$

where $\#\alpha = r$. Hence any element $x \in G_n$ can be written in the form

$$x = y \prod_{\alpha\in S(n)} s_\alpha(x_\alpha) \qquad \text{with} \quad y \in NG_n \quad \text{and} \quad x_\alpha \in NG_{n-\#\alpha}.$$

2.5. Hypercrossed complex pairings

In the following we recall from [**22**] hypercrossed complex pairings. The fundamental idea behind these can be found in Carrasco and Cegarra (cf. [**9**]). The construction depends on a variety of sources, mainly Conduché [**12**] and Mutlu and Porter [**22**]. Define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_r, \ldots, i_1), \beta = (j_s, \ldots, j_1) \in S(n)$. The pairings that we will need,

$$\{F_{\alpha,\beta} \colon NG_{n-\sharp\alpha} \times NG_{n-\sharp\beta} \to NG_n : (\alpha, \beta) \in P(n), n \geqslant 0\},$$

are given as composites by the diagram

$$\begin{array}{ccc} NG_{n-\#\alpha} \times NG_{n-\#\beta} & \xrightarrow{F_{\alpha,\beta}} & NG_n \\ {\scriptstyle s_\alpha \times s_\beta}\downarrow & & \uparrow{\scriptstyle p} \\ G_n \times G_n & \xrightarrow[\mu]{} & G_n, \end{array}$$

where

$$s_\alpha = s_{i_r}, \ldots, s_{i_1} \colon NG_{n-\sharp\alpha} \to G_n,$$
$$s_\beta = s_{j_s}, \ldots, s_{j_1} \colon NG_{n-\sharp\beta} \to G_n x,$$

and $p \colon G_n \to NG_n$ is defined by the composite projections $p(x) = p_{n-1} \cdots p_0(x)$, where $p_j(z) = z s_j d_j(z)^{-1}$ with $j = 0, 1, \ldots, n-1$. $\mu \colon G_n \times G_n \to G_n$ is given by a commutator map and $\sharp\alpha$ is the number of the elements in the set of α, similarly for $\sharp\beta$. Thus

$$\begin{aligned} F_{\alpha,\beta}(x_\alpha, y_\beta) &= p\mu[(s_\alpha \times s_\beta)(x_\alpha, x_\beta)] \\ &= p[(s_\alpha x_\alpha \times s_\beta x_\beta)]. \end{aligned}$$

Let N_n be the normal subgroup of G_n generated by elements of the form

$$F_{\alpha,\beta}(x_\alpha, y_\beta),$$

where $x_\alpha \in NG_{n-\sharp\alpha}$ and $y_\beta \in NG_{n-\sharp\beta}$.

We illustrate this subgroup for $n = 3$ and $n = 4$ as follows:

For $n = 3$, the possible Peiffer pairings are the following:

$$F_{(1,0)(2)}, F_{(2,0)(1)}, F_{(0)(2,1)}, F_{(0)(2)}, F_{(1)(2)}, F_{(0)(1)}.$$

For all $x_1 \in NG_1, y_2 \in NG_2$, the corresponding generators of N_3 are

$$F_{(1,0)(2)}(x_1, y_2) = [s_1 s_0 x_1, s_2 y_2][s_2 y_2, s_2 s_0 x_1],$$
$$F_{(2,0)(1)}(x_1, y_2) = [s_2 s_0 x_1, s_1 y_2][s_1 y_2, s_2 s_1 x_1][s_2 s_1 x_1, s_2 y_2][s_2 y_2, s_2 s_0 x_1],$$

and for all $x_2 \in NG_2, y_1 \in NG_1$,

$$F_{(0)(2,1)}(x_2, y_1) = [s_0 x_2, s_2 s_1 y_1][s_2 s_1 y_1, s_1 x_2][s_2 x_2, s_2 s_1 y_1],$$

whilst for all $x_2, y_2 \in NG_2$,

$$\begin{aligned} F_{(0)(1)}(x_2,y_2) &= [s_0x_2, s_1y_2][s_1y_2, s_1x_2][s_2x_2, s_2y_2],\\ F_{(0)(2)}(x_2,y_2) &= [s_0x_2, s_2y_2],\\ F_{(1)(2)}(x_2,y_2) &= [s_1x_2, s_2y_2][s_2y_2, s_2x_2]. \end{aligned}$$

For $n = 4$, the key pairings are thus the following:

$$\begin{array}{lllll} F_{(0)(3,2,1)}, & F_{(3,2,0)(1)}, & F_{(3,1,0)(2)}, & F_{(2,1,0)(3)}, & F_{(3,0)(2,1)},\\ F_{(2,0)(3,1)}, & F_{(1,0)(3,2)}, & F_{(1)(3,2)}, & F_{(0)(3,2)}, & F_{(0)(3,1)},\\ F_{(0)(2,1)}, & F_{(3,1)(2)}, & F_{(2,1)(3)}, & F_{(3,0)(2)}, & F_{(3,0)(1)},\\ F_{(2,0)(3)}, & F_{(2,0)(1)}, & F_{(1,0)(3)}, & F_{(1,0)(2)}, & F_{(2)(3)},\\ F_{(1)(3)}, & F_{(0)(3)}, & F_{(1)(2)}, & F_{(0)(2)}, & F_{(0)(1)}. \end{array}$$

For $x_1, y_1 \in NG_1$, $x_2, y_2 \in NG_2$ and $x_3, y_3 \in NG_3$ the generator element of the normal subgroup N_4 can be easily written down from Lemma 2.5.

Theorem 2.3 ([**22**]). *For $n = 2, 3$ and 4, let $\mathbf{G}$ be a simplicial group with Moore complex $\mathbf{NG}$ in which $G_n = D_n$ is the normal subgroup of G_n generated by the degenerate elements in dimension n. Then*

$$\partial_n(NG_n) = \prod_{I,J} [K_I, K_J]$$

for $I, J \subseteq [n-1]$ with

$$\begin{aligned} I \cup J &= [n-1],\\ I &= [n-1] - \{\alpha\}\\ J &= [n-1] - \{\beta\}, \end{aligned}$$

where $(\alpha, \beta) \in P(n)$.

Remark 2.4. In [**22**], Mutlu and Porter defined the normal subgroup $\partial_n(NG_n \cap D_n)$ by $F_{\alpha,\beta}$ elements which were first defined by Carrasco in [**9**]. In [**11**], Castiglioni and Ladra gave a general proof for the inclusions partially proved by Arvasi and Porter in [**1**], Arvasi and Akça in [**3**] and Mutlu and Porter in [**22**]. Their approach to the problem was different from that of cited works. They have succeeded with a proof, for the case of algebras, over an operad by introducing a different description of the adjoint inverse of the normalization functor $\mathbf{N}\colon \mathrm{Ab}^{\Delta^{\mathrm{op}}} \to \mathrm{Ch}_{\geqslant 0}$. For the case of groups, they then adapted the construction for the adjoint inverse used for algebras to get a simplicial group $G \boxtimes \Lambda$ from the Moore complex of a simplicial group G.

Following the theorem named as **Theorem B** in [**22**], we have

Lemma 2.5. *Let $\mathbf{G}$ be a simplicial group with Moore complex $\mathbf{NG}$ of length 3. Then for the $n = 4$ case, the images of $F_{\alpha,\beta}$ elements under ∂_4, given in the table on the next page, are trivial.*

Proof. Since $NG_4 = 1$, by Theorem B in [**22**] the result is trivial. □

$$\begin{aligned}
d_4(F_{(0)(3,2,1)}(x_3,x_1)) &= [s_0d_3x_3, s_2s_1x_1]\,[s_2s_1x_1, s_1d_3x_3]\,[s_2d_3x_3, s_2s_1x_1]\,[s_2s_1x_1, x_3]\\
d_4(F_{(3,2,0)(1)}(x_1,x_3)) &= [s_2s_0x_1, s_1d_3x_3]\,[s_1d_3x_3, s_2s_1x_1]\,[s_2s_1x_1, s_2d_3x_3]\\
&\quad [s_2d_3x_3, s_2s_0x_1]\,[s_2s_0x_1, x_3]\,[x_3, s_2s_1x_1]\\
d_4(F_{(3,1,0)(2)}(x_1,x_3)) &= [s_1s_0x_1, s_2d_3x_3]\,[s_2d_3x_3, s_2s_0x_1]\,[s_2s_0x_1, x_3]\,[x_3, s_1s_0x_1]\\
d_4(F_{(2,1,0)(3)}(x_1,x_3)) &= [s_2s_1s_0d_1x_1, x_3]\,[x_3, s_1s_0x_1]\\
d_4(F_{(3,0)(2,1)}(x_2,y_2)) &= [s_0x_2, s_2s_1d_2y_2]\,[s_2s_1d_2y_2, s_1x_2]\,[s_2x_2, s_2s_1d_2y_2]\,[s_1y_2, s_2x_2]\\
&\quad [s_1x_2, s_1y_2]\,[s_1y_2, s_0x_2]\\
d_4(F_{(2,0)(3,1)}(x_2,y_2)) &= [s_2s_0d_2x_2, s_1y_2]\,[s_1y_2, s_2s_1d_2x_2]\,[s_2s_1d_2x_2, s_2y_2]\\
&\quad [s_2y_2, s_2s_0d_2x_2]\,[s_0x_2, s_2y_2]\,[s_2y_2, s_1x_2]\\
&\quad [s_1x_2, s_1y_2]\,[s_1y_2, s_0x_2]\\
d_4(F_{(1,0)(3,2)}(x_2,y_2)) &= [s_1s_0d_2x_2, s_2y_2]\,[s_2y_2, s_2s_0d_2x_2]\,[s_0x_2, s_2y_2]\\
d_4(F_{(1)(3,2)}(x_3,x_2)) &= [s_1d_3x_3, s_2x_2]\,[s_2x_2, s_2d_3x_3]\,[x_3, s_2x_2]\\
d_4(F_{(0)(3,2)}(x_3,x_2)) &= [s_0d_3x_3, s_2x_2]\\
d_4(F_{(3,1)(2)}(x_3,x_2)) &= [s_0d_3x_3, s_1x_2]\,[s_1x_2, s_1d_3x_3]\,[s_2d_3x_3, s_2x_2]\,[s_2x_2, x_3]\\
d_4(F_{(0)(2,1)}(x_3,x_2)) &= [s_0d_3x_3, s_2s_1d_2x_2]\,[s_2s_1d_2x_2, s_1d_3x_3]\\
&\quad [s_2d_3x_3, s_2s_1d_2x_2]\,[s_1x_2, x_3]\\
d_4(F_{(3,1)(2)}(x_2,x_3)) &= [s_1x_2, s_2d_3x_3]\,[s_2d_3x_3, s_2x_2]\,[s_2l, x_3]\,[x_3, s_1x_2]\\
d_4(F_{(2,1)(3)}(x_2,x_3)) &= [s_2s_1d_2x_2, x_3]\,[x_3, s_1x_2]\\
d_4(F_{(3,0)(2)}(x_2,x_3)) &= [s_0x_2, s_2d_3x_3]\,[x_3, s_0x_2]\\
d_4(F_{(3,0)(1)}(x_2,x_3)) &= [s_0x_2, s_1d_3x_3]\,[s_1d_3x_3, s_1x_2]\,[s_2x_2, s_2d_3x_3]\,[x_3, s_2x_2]\\
d_4(F_{(2,0)(3)}(x_2,x_3)) &= [s_2s_0d_2x_2, x_3]\,[x_3, s_0x_2]\\
d_4(F_{(2,0)(1)}(x_2,x_3)) &= [s_2s_0d_2x_2, s_1d_3x_3]\,[s_1d_3x_3, s_2s_1d_2x_2]\,[s_2s_1d_2x_2, s_2d_3x_3]\\
&\quad [s_2d_3x_3, s_2s_0d_2x_2]\,[s_0x_2, x_3]\,[x_3, s_1x_2]\\
d_4(F_{(1,0)(3)}(x_2,x_3)) &= [s_1s_0d_2x_2, x_3]\\
d_4(F_{(1,0)(2)}(x_2,x_3)) &= [s_1s_0d_2x_2, s_2d_3x_3]\,[s_2d_3x_3, s_2s_0d_2x_2]\,[s_0x_2, x_3]\\
d_4(F_{(2)(3)}(x_3,y_3)) &= [s_2d_3x_3, y_3]\,[y_3, x_3]\\
d_4(F_{(1)(3)}(x_3,y_3)) &= [s_1d_3x_3, y_3]\\
d_4(F_{(0)(3)}(x_3,y_3)) &= [s_0d_3x_3, y_3]\\
d_4(F_{(1)(2)}(x_3,y_3)) &= [s_1d_3x_3, s_2d_3y_3]\,[s_2d_3y_3, s_2d_3x_3]\,[x_3, y_3]\\
d_4(F_{(0)(2)}(x_3,y_3)) &= [s_0d_3x_3, s_2d_3y_3]\\
d_4(F_{(0)(1)}(x_3,y_3)) &= [s_0d_3x_3, s_1d_3y_3]\,[s_1d_3y_3, s_1d_3x_3]\,[s_2d_3x_3, s_2d_3y_3]\,[y_3, x_3]
\end{aligned}$$

where $x_3, y_3 \in NG_3, x_2, y_2 \in NG_2, x_1 \in NG_1$.

3. 2-crossed modules

The notion of a crossed module is an efficient algebraic tool to handle connected spaces with only the first homotopy groups nontrivial, up to homotopy.

A *crossed module* is a group homomorphism $\partial\colon M \to P$ together with an action of P on M, written ${}^{p}m$ for $p \in P$ and $m \in M$, satisfying the conditions:

CM1) ∂ is P-equivariant; i.e., for all $p \in P$, $m \in M$,

$$\partial({}^{p}m) = p\partial(m)p^{-1}.$$

CM2) (Peiffer identity). For all $m, m' \in M$,

$${}^{\partial m}m' = mm'm^{-1}.$$

We will denote such a crossed module by (M, P, ∂).

A *morphism of a crossed module* from (M, P, ∂) to (M', P', ∂') is a pair of group homomorphisms

$$\phi\colon M \longrightarrow M', \psi\colon P \longrightarrow P'$$

such that $\phi({}^{p}m) = {}^{\psi(p)}\phi(m)$ and $\partial'\phi(m) = \psi\partial(m)$.

We thus get a category **XMod** of crossed modules.

Examples of crossed modules:

1) Any normal subgroup $N \trianglelefteq P$ gives rise to a crossed module namely the inclusion map, $i\colon N \hookrightarrow P$. Conversely, given any crossed module $\partial\colon M \longrightarrow P$, $\operatorname{Im}\partial$ is a normal subgroup of P.

2) Given any P-module M, the trivial map$1\colon M \longrightarrow P$, which maps everything to 1 in P, is a crossed module. Conversely, if $\partial\colon M \to P$ is a crossed module, then $\ker\partial$ is central in M and inherits a natural P-module structure from the P-action on M.

The following definition of a 2-crossed module is equivalent to that given by Conduché ([**12**]).

A 2-crossed module of groups consists of a complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with

(a) actions of N on M and L so that ∂_2, ∂_1 are morphisms of N-groups, and

(b) an N-equivariant function

$$\{\quad,\quad\}\colon M \times M \longrightarrow L$$

called a *Peiffer lifting*.

This data must satisfy the following axioms:

2CM1) $\partial_2\{m,m'\} = \left({}^{\partial_1 m}m'\right) mm'^{-1}m^{-1},$

2CM2) $\{\partial_2 l, \partial_2 l'\} = [l', l],$

2CM3) $(i)\quad \{mm', m''\} = {}^{\partial_1 m}\{m', m''\}\{m, m'm''m'^{-1}\},$

$(ii)\quad \{m, m'm''\} = \{m, m'\}^{mm'm^{-1}}\{m, m''\},$

2CM4) $\{m, \partial_2 l\}\{\partial_2 l, m\} = {}^{\partial_1 m}ll^{-1}$

2CM5) ${}^{n}\{m, m'\} = \{{}^{n}m, {}^{n}m'\}$

for all $l, l' \in L$, $m, m', m'' \in M$ and $n \in N$.

Here we have used ${}^{m}l$ as a shorthand for $\{\partial_2 l, m\}l$ in condition **2CM3**)(ii) where l is $\{m, m''\}$ and m is $mm'(m)^{-1}$. This gives a new action of M on L. Using this notation, we can split **2CM4**) into two pieces, the first of which is tautologous:

2CM4) $(a)\quad \{\partial_2 l, m\} = {}^{m}l(l)^{-1},$

$(b)\quad \{m, \partial_2 l\} = ({}^{\partial_1 m}l)({}^{m}l^{-1}).$

The old action of M on L, via ∂_1 and the N-action on L, is, in general, distinct from this second action with $\{m, \partial_2 l\}$ measuring the difference (by **2CM4**)(b)). An easy argument using **2CM2**) and **2CM4**)(b) shows that with this action, ${}^{m}l$, of M on L, (L, M, ∂_2) becomes a crossed module. A morphism of 2-crossed modules can be defined in an obvious way. We thus define the category of 2-crossed modules denoting it by $\mathbf{X_2Mod}$.

A crossed square as defined by D. Guin-Waléry and J.-L. Loday in [**19**] (see also [**8, 20**]) can be seen as a mapping cone in [**13**]. Furthermore, 2-crossed modules are related to simplicial groups. This relation can be found in [**12, 22**].

Theorem 3.1. *The category* $\mathbf{X_2Mod}$ *of* 2*-crossed modules is equivalent to the category of* $\mathbf{SimpGrp}_{\leqslant 2}$ *simplicial groups with Moore complex of length* 2*.*

4. 3-crossed modules

In the following we will define the category of 3-crossed modules. First of all we adapt ideas from Conduché's method given in [**12**]. He gave some equalities by using the semi-direct decomposition of a simplicial group, but these are exactly the images of Peiffer pairings $F_{\alpha,\beta}$ under ∂_3 for $n = 3$ case defined in [**22**]. The difference of our method is to use $F_{\alpha,\beta}$ instead of semi-direct decomposition. Thus we will define similar equalities for $n = 4$ and get the axioms of a 3-crossed module.

Let $\mathbf{G}$ be a simplicial group with Moore complex of length 3 and $NG_0 = N$, $NG_1 = M$, $NG_2 = L$, $NG_3 = K$. Thus we have a group complex

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N.$$

Let the actions of N on K, L, M, M on L, K and L on K be as follows:

$$\begin{aligned} {}^{n}m &= s_0n\,(m)\,s_0n^{-1}, \\ {}^{n}l &= s_1s_0n\,(l)\,s_1s_0n^{-1}, \\ {}^{n}k &= s_2s_1s_0n\,(k)\,s_2s_1s_0n^{-1} \\ {}^{m}l &= s_1m\,(l)\,s_1m^{-1}, \\ {}^{m}k &= s_2s_1m\,(k)\,s_2s_1m^{-1}, \\ l\cdot k &= s_2l\,(k)\,s_2l^{-1}. \end{aligned} \tag{1}$$

Using the table on page 167, since

$$\begin{aligned} [s_1s_0ms_2s_1\partial_1m, k] &= 1, \\ [s_1ls_2s_1\partial_2l, k] &= 1, \\ \left[k', k^{-1}s_2\partial_3k\right] &= 1, \end{aligned}$$

we get

$$\begin{aligned} {}^{\partial_1m}k &= s_1s_0m\,(k)\,s_1s_0m^{-1}, \\ {}^{\partial_2l}k &= s_1l\,(k)\,s_1l^{-1}, \\ \partial_3k\cdot k' &= k\,(k')\,k^{-1}, \end{aligned}$$

and using the simplicial identities we get

$$\partial_3(l\cdot k) = \partial_3(s_2l\,(k)\,s_2l^{-1}) = \partial_3s_2l\,(\partial_3k)\,s_2l^{-1} = l\,(\partial_3k)\,l^{-1}.$$

Thus $\partial_3\colon K\to L$ is a crossed module.

The Peiffer liftings given in the definition below are the $F_{\alpha,\beta}$ pairings for the case $n=3$ defined in [**9**].

Definition 4.1. Let $K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ be a group complex defined above. We define the Peiffer liftings as follows:

$$\begin{array}{llcl}
\{\ ,\ \}\colon & M\times M & \longrightarrow & L \\
 & \{m,m'\} & = & [s_1m, s_1m']\,[s_1m', s_0m] \\
\{\ ,\ \}_{(1)(0)}\colon & L\times L & \longrightarrow & K \\
 & \{l,l'\}_{(1)(0)} & = & [s_2l', s_2l]\,[s_1l, s_1l']\,[s_1l', s_0l] \\
\{\ ,\ \}_{(2)(1)}\colon & L\times L & \longrightarrow & K \\
 & \{l,l'\}_{(2)(1)} & = & [s_2l, s_2l']\,[s_2l', s_1l] \\
\{\ ,\ \}_{(0)(2)}\colon & L\times L & \longrightarrow & K \\
 & \{l,l'\}_{(0)(2)} & = & [s_2l', s_0l] \\
\{\ ,\ \}_{(1,0)(2)}\colon & M\times L & \longrightarrow & K \\
 & \{m,l'\}_{(1,0)(2)} & = & [s_2s_0m, s_2l']\,[s_2l', s_1s_0m] \\
\{\ ,\ \}_{(2,0)(1)}\colon & M\times L & \longrightarrow & K \\
 & \{m,l'\}_{(2,0)(1)} & = & [s_2s_0m, s_2l']\,[s_2l', s_2s_1m]\,[s_2s_1m, s_1l']\,[s_1l', s_2s_0m] \\
\{\ ,\ \}_{(0)(2,1)}\colon & L\times M & \longrightarrow & K \\
 & \{l',m\}_{(0)(2,1)} & = & [s_2s_1m, s_2l']\,[s_1l', s_2s_1m]\,[s_2s_1m, s_0l']
\end{array}$$

where m, $m'\in M$, l, $l'\in L$.

Then using the table on page 167 we get the following identities:

$$
\begin{aligned}
\{m,\partial_3 k\}_{(1,0)(2)} &= \{m,\partial_3 k\}_{(2,0)(1)}\ {}^{m}(k)^{\partial_1 m}(k^{-1})\\
\{\partial_3 k,m\}_{(0)(2,1)} &= {}^{m}(k)k^{-1}\\
\{m,\partial_3 k\}_{(1,0)(2)} &= \{m,\partial_3 k\}_{(2,0)(1)}\ \{\partial_3 k,m\}_{(0)(2,1)}\ k\ {}^{\partial_1 m}(k^{-1})\\
\{l',\partial_2 l\}_{(0)(2,1)} &= \{l,l'\}^{-1}_{(2)(1)}\{l',l\}_{(1)(0)}\\
\{\partial_2 l,l'\}_{(2,0)(1)} &= \{l,l'\}^{-1}_{(0)(2)}\ {}^{[l',\,l]}(\{l,l'\}_{(2)(1)})\{l,l'\}_{(1)(0)}\\
\{\partial_2 l,l'\}_{(1,0)(2)} &= (\{l,l'\}_{(0)(2)})^{-1}\\
\{l,l'l''\}_{(2)(1)} &= \{l,l'\}_{(2)(1)}{}^{\partial l}l'.\{l,l''\}_{(2)(1)}\\
\{ll',l''\}_{(2)(1)} &= l.\{l',l''\}_{(2)(1)}\{l,{}^{\partial l'}\,l''\}_{(2)(1)}\\
\partial_3(\{l,l'\}_{(1)(0)}) &= [l,l']\,\{\partial_2 l,\partial_2 l'\}\\
\partial_3(\{l,l'\}_{(2)(1)}) &= ll'l^{-1}({}^{\partial_2 l}l')^{-1}\\
\partial_3(\{l,l'\}_{(0)(2)}) &= \partial_3(\{\partial_2 l,l'\}_{(1,0)(2)})^{-1}\\
\partial_3\{l,m\}_{(0)(2,1)} &= {}^{m}ll^{-1}\{\partial_2 l,m\}\\
\partial_3\{m,l\}_{(2,0)(1)} &= \partial_3\{m,l\}_{(1,0)(2)}\ {}^{\partial_1 m}l\ {}^{m}(l^{-1})\{m,\partial_2 l\}\\
\{\partial_3 k,l\}_{(2)(1)}\{l,\partial_3 k\}_{(2)(1)} &= k\left({}^{\partial_2 l}(k^{-1})\right)\\
\{\partial_3 k,l\}_{(1)(0)}\{l,\partial_3 k\}_{(1)(0)} &= 1\\
\{\partial_3 k,\partial_3 k'\}_{(2)(1)} &= [k,k']\\
\{\partial_3 k,\partial_3 k'\}_{(1)(0)} &= [k',k]\\
\{\partial_3 k,l'\}_{(0)(2)} &= 1\\
\{\partial_2 l,\partial_3 k\}_{(1,0)(2)} &= \{l,\partial_3 k\}^{-1}_{(0)(2)}\\
\{\partial_2 l,\partial_3 k\}_{(2,0)(1)} &= \{l,\partial_3 k\}_{(0)(2)}k\left({}^{\partial_2 l}(k^{-1})\right)\\
\{\partial_3 k,\partial_2 l\}_{(0)(2,1)} &= {}^{\partial_2 l}k\ k^{-1}
\end{aligned}
$$

Table 1

$$
\begin{aligned}
{}^{n}\{m,m'\} &= \{{}^{n}m,{}^{n}m'\}\\
{}^{n}\{l,l'\}_{(1)(0)} &= \{{}^{n}l,{}^{n}l'\}_{(1)(0)}\\
{}^{n}\{l,l'\}_{(2)(1)} &= \{{}^{n}l,{}^{n}l'\}_{(2)(1)}\\
{}^{n}\{l,l'\}_{(0)(2)} &= \{{}^{n}l,{}^{n}l'\}_{(0)(2)}\\
{}^{n}\{m,l'\}_{(1,0)(2)} &= \{{}^{n}m,{}^{n}l'\}_{(1,0)(2)}\\
{}^{n}\{m,l't\}_{(2,0)(1)} &= \{{}^{n}m,{}^{n}l'\}_{(2,0)(1)}\\
{}^{n}\{l',m\}_{(0)(2,1)} &= \{{}^{n}l',{}^{n}m\}_{(0)(2,1)}
\end{aligned}
$$

Table 2

$$
\begin{array}{lcl}
{}^{m}\{m', m''\} & = & {}^{m}\{m', {}^{m}m''\} \\
{}^{m}\{l, l'\}_{(1)(0)} & = & {}^{m}l, {}^{m}l'\}_{(1)(0)} \\
{}^{m}\{l, l'\}_{(2)(1)} & = & \{{}^{m}l, {}^{m}l'\}_{(2)(1)} \\
{}^{m}\{l, l'\}_{(0)(2)} & = & \{{}^{m}l, {}^{m}l'\}_{(0)(2)} \\
{}^{m}\{m, l'\}_{(1,0)(2)} & = & \{{}^{m}m, {}^{m}l'\}_{(1,0)(2)} \\
{}^{m}\{m, l'\}_{(2,0)(1)} & = & \{{}^{m}m, {}^{m}l'\}_{(2,0)(1)} \\
{}^{m}\{l', m\}_{(0)(2,1)} & = & \{{}^{m}l', {}^{m}m\}_{(0)(2,1)}
\end{array}
$$

Table 3

where m, $m', m'' \in M$, l, $l' \in L, k, k' \in K$. From these results, all liftings are N- and M-equivariant.

Definition 4.2. A 3-*crossed module* consists of a complex of groups

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with an action of N on K, L and M, an action of M on K and L, an action of L on K so that ∂_3, ∂_2, ∂_1 are morphisms of N, M-groups, and M, N-equivariant liftings

$$\{\ ,\ \}_{(1)(0)} : L \times L \longrightarrow K, \qquad \{\ ,\ \}_{(0)(2)} : L \times L \longrightarrow K, \qquad \{\ ,\ \}_{(2)(1)} : L \times L \longrightarrow K,$$

$$\{\ ,\ \}_{(1,0)(2)} : M \times L \longrightarrow K, \quad \{\ ,\ \}_{(2,0)(1)} : M \times L \longrightarrow K,$$

$$\{\ ,\ \}_{(0)(2,1)} : L \times M \longrightarrow K, \quad \{\ ,\ \} : M \times M \longrightarrow L$$

called 3-*dimensional Peiffer liftings*. This data must satisfy the following axioms:

3CM1) $K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M$ is a 2-crossed module with the Peiffer lifting $\{\ ,\ \}_{(2,1)}$

3CM2) $\{m, \partial_3 k\}_{(1,0)(2)} = \{m, \partial_3 k\}_{(2,0)(1)}\, {}^{m}(k)^{\partial_1 m}(k^{-1})$

3CM3) $\{\partial_3 k, m\}_{(0)(2,1)} = {}^{m}(k)k^{-1}$

3CM4) $\{m, \partial_3 k\}_{(1,0)(2)} = \{m, \partial_3 k\}_{(2,0)(1)}\ \{\partial_3 k, m\}_{(0)(2,1)}\ k^{\partial_1 m}(k^{-1})$

3CM5) $\{l', \partial_2 l\}_{(0)(2,1)} = \{l, l'\}^{-1}_{(2)(1)}\{l', l\}_{(1)(0)}$

3CM6) $\{\partial_2 l, l'\}_{(2,0)(1)} = \{l, l'\}^{-1}_{(0)(2)}\ {}^{[l',\, l]}(\{l, l'\}_{(2)(1)})\{l, l'\}_{(1)(0)}$

3CM7) $\{\partial_2 l, l'\}_{(1,0)(2)} = (\{l, l'\}_{(0)(2)})^{-1}$

3CM8) $\partial_3(\{l, l'\}_{(1)(0)}) = [l, l']\ \{\partial_2 l, \partial_2 l'\}$

3CM9) $\partial_3(\{l, l'\}_{(0)(2)}) = \partial_3(\{\partial_2 l, l'\}_{(1,0)(2)})^{-1}$

3CM10) $\partial_3\{l, m\}_{(0)(2,1)} = {}^{m}l\, l^{-1}\{\partial_2 l, m\}$

3CM11) $\partial_3\{m, l\}_{(2,0)(1)} = \partial_3\{m, l\}_{(1,0)(2)}\ {}^{\partial_1 m}l\, {}^{m}(l^{-1})\{m, \partial_2 l\}$

3CM12a) $\{\partial_3 k, l\}_{(1)(0)} = ({}^{l}k)k^{-1}$

3CM12b) $\{l, \partial_3 k\}_{(1)(0)} = k({}^{l}k)^{-1}$

3CM13) $\{\partial_3 k, \partial_3 k'\}_{(1)(0)} = [k', k]$

3CM14) $\{\partial_3 k, l'\}_{(0)(2)} = 1$

3CM15) $\{\partial_2 l, \partial_3 k\}_{(1,0)(2)} = \{l, \partial_3 k\}^{-1}_{(0)(2)}$

3CM16) $\{\partial_2 l, \partial_3 k\}_{(2,0)(1)} = \{l, \partial_3 k\}_{(0)(2)} k \left({}^{\partial_2 l}(k^{-1})\right)$

3CM17) $\{\partial_3 k, \partial_2 l\}_{(0)(2,1)} = {}^{\partial_2 l}k\ k^{-1}$

3CM18) $\partial_2\{m, m'\} = mm'm^{-1}({}^{\partial_1 m}m')^{-1}.$

We denote such a 3-crossed module by $(K, L, M, N, \partial_3, \partial_2, \partial_1)$.

A *morphism of 3-crossed modules* of groups may be pictured by the diagram

$$\begin{array}{ccccccc} L_3 & \xrightarrow{\partial_3} & L_2 & \xrightarrow{\partial_2} & L_1 & \xrightarrow{\partial_1} & L_0 \\ {\scriptstyle f_3}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow \\ L'_3 & \xrightarrow[\partial'_3]{} & L'_2 & \xrightarrow[\partial'_2]{} & L'_1 & \xrightarrow[\partial'_1]{} & L'_0, \end{array}$$

where

$$f_1({}^n m) = {}^{(f_0(n))}f_1(m),\ f_2({}^n l) = {}^{(f_0(n))}f_2(l),\ f_3({}^n k) = {}^{(f_0(n))}f_3(k).$$

We require the following equations to hold: for $\{\ ,\ \}_{(0)(2)}, \{\ ,\ \}_{(2)(1)}$ and $\{\ ,\ \}_{(1)(0)}$,

$$\{\ ,\ \}f_2 \times f_2 = f_3\{\ ,\ \};$$

for $\{\ ,\ \}_{(1,0)(2)}$ and $\{\ ,\ \}_{(2,0)(1)}$,

$$\{\ ,\ \}f_1 \times f_2 = f_3\{\ ,\ \};$$

for $\{\ ,\ \}_{(0)(2,1)}$,

$$\{\ ,\ \}f_2 \times f_1 = f_3\{\ ,\ \};$$

and for $\{\ ,\ \}$,

$$\{\ ,\ \}f_1 \times f_1 = f_2\{\ ,\ \}$$

for all $k \in K, l \in L, m \in M$ and $n \in N$. These compose in an obvious way. We thus can define the category of 3-crossed modules, denoting it by $\mathbf{X_3Mod}$.

5. Applications

5.1. Simplicial groups

As an application we consider in detail the relation between simplicial groups and 3-crossed modules.

Proposition 5.1. *Let* **G** *be a simplicial group with Moore complex* **NG**. *Then the group complex*

$$NG_3/\partial_4(NG_4 \cap D_4) \xrightarrow{\overline{\partial}_3} NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0$$

is a 3-crossed module with the Peiffer liftings defined below:

$$
\begin{array}{rcl}
\{\ ,\ \}\colon NG_1 \times NG_1 & \longrightarrow & NG_2 \\
\{x_1, y_1\} & \longmapsto & [s_0x_1, s_1y_1][s_1y_1, s_1x_1], \\
\{\ ,\ \}_{(1)(0)}\colon NG_2 \times NG_2 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \\
\{x_2, y_2\} & \longmapsto & \overline{([s_0x_2, s_1y_2][s_1y_2, s_1x_2][s_2x_2, s_2y_2])}, \\
\{\ ,\ \}_{(2)(1)}\colon NG_2 \times NG_2 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \\
\{x_2, y_2\} & \longmapsto & \overline{([s_1x_2, s_2y_2][s_2y_2, s_2x_2])}, \\
\{\ ,\ \}_{(0)(2)}\colon NG_2 \times NG_2 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \\
\{x_2, y_2\} & \longmapsto & \overline{([s_0x_2, s_2y_2])}, \\
\{\ ,\ \}_{(1,0)(2)}\colon NG_1 \times NG_2 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \\
\{x_1, y_2\} & \longmapsto & \overline{([s_1s_0x_1, s_2y_2][s_2y_2, s_2s_0x_1])}, \\
\{\ ,\ \}_{(2,0)(1)}\colon NG_1 \times NG_2 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D)4) \\
\{x_1, y_2\} & \longmapsto & \overline{([s_2s_0x_1, s_1y_2][s_1y_2, s_2s_1x_1][s_2s_1x_1, s_2y_2][s_2y_2, s_2s_0x_1])}, \\
\{\ ,\ \}_{(0)(2,1)}\colon NG_2 \times NG_1 & \longrightarrow & NG_3/\partial_4(NG_4 \cap D_4) \\
\{y_2, x_1\} & \longmapsto & \overline{([s_0y_2, s_2s_1x_1][s_2s_1x_1, s_1y_2])[s_2y_2, s_2s_1x_1]|}.
\end{array}
$$

(The elements denoted by $\overline{[\ ,\ ]}$ *are cosets in* $NG_3/\partial_4(NG_4 \cap D_4)$ *and are given by the elements in* NG_3*.)*

Proof. The proof is given in Appendix A.. □

Theorem 5.2. *The category of 3-crossed modules is equivalent to the category of simplicial groups with Moore complex of length 3.*

Proof. Let **G** be a simplicial group with Moore complex of length 3. In the above proposition we showed that the group complex

$$NG_3 \xrightarrow{\partial_3} NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0$$

is a 3-crossed module. Since the Moore complex is of length 3, $NG_4 \cap D_4 = 1$, so $\partial_4(NG_4 \cap D_4) = 1$. Thus we can take NG_3 instead of $NG_3/\partial_4(NG_4 \cap D_4))$. Finally, there is a functor

$$\Im_3\colon \mathbf{SimpGrp}_{\leqslant 3} \longrightarrow \mathbf{X_3Mod}$$

from the category of simplicial groups with Moore complex of length 3 to the category of 3-crossed modules. Conversely, let

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

be a 3-crossed module. Let $H_0 = N$. By the action of N on M obtain the $H_1 = M \rtimes N$

semidirect product. For $(m,n) \in M \rtimes N$, define the degeneracy and face maps as

$$\begin{array}{llll} d_0\colon & M \rtimes N & \longrightarrow & N \\ & (m,n) & \longmapsto & n \\ d_1\colon & M \rtimes N & \longrightarrow & N \\ & (m,n) & \longmapsto & (\partial_1(m))n \\ s_0\colon & N & \longrightarrow & M \rtimes N \\ & n & \longmapsto & (1,n). \end{array}$$

Now by the actions of M and N on L we obtain the $H_2 = (L \rtimes M) \rtimes (M \rtimes N)$ semidirect product. For $l \in L, m, m' \in M, n \in N$, define the degeneracy and face maps as

$$\begin{array}{lll} d_0\colon (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (M \rtimes N) \\ \quad (l,m,m',n) & \longmapsto & (m',n), \\ d_1\colon (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (M \rtimes N) \\ \quad (l,m,m',n) & \longmapsto & (mm',n), \\ d_2\colon (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (M \rtimes N) \\ \quad (l,m,m',n) & \longmapsto & (\partial_2(l)m, \partial_1(m')n), \\ s_0\colon (M \rtimes N) & \longrightarrow & (L \rtimes M) \rtimes (M \rtimes N) \\ \quad (m',n) & \longmapsto & (1,1,m',n), \\ s_1\colon (M \rtimes N) & \longrightarrow & (L \rtimes M) \rtimes (M \rtimes N), \\ \quad (m',n) & \longmapsto & (1,m',1,n). \end{array}$$

Since $\{\ ,\ \}_{(2)(1)}$ is a 2-crossed module there is an action of L on K defined as

$${}^{l}k = \{\partial_3 k, l\}_{(2)(1)} k^{-1}$$

for $l \in L$, $k \in K$. Using this action we obtain a semidirect product $K \rtimes L$. The action of $(l,m) \in L \rtimes M$ on $(k,l) \in K \rtimes L$ can be expressed as

$$\begin{aligned} {}^{(1,m)}(k,l') &= ({}^{m}({}^{1}k), {}^{m}({}^{1}l')) \\ &= ({}^{m}(k), {}^{m}(l')), \\ {}^{(l,1)}(k,l') &= ({}^{1}({}^{l}k), {}^{1}({}^{l}l')) \\ &= ({}^{l}k, {}^{l}l') \\ &= ({}^{\partial_2 l}k\{l, \partial_3 k\}_{(2)(1)}, ll'l^{-1}). \end{aligned}$$

After these definitions we have the semidirect product

$$H_3 = (K \rtimes L) \rtimes (L \rtimes M) \rtimes (M \rtimes N).$$

Define the degeneracy and face maps as:

$$\begin{array}{llcl}
d_0\colon & (K \rtimes L) \rtimes (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (L \rtimes M) \rtimes (M \rtimes N) \\
& (k, l, l', m, m', n) & \longmapsto & (l', m, m', n), \\
d_1\colon & (K \rtimes L) \rtimes (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (L \rtimes M) \rtimes (M \rtimes N) \\
& (k, l, l', m, m', n) & \longmapsto & (l, m, m', n), \\
d_2\colon & (K \rtimes L) \rtimes (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (L \rtimes M) \rtimes (M \rtimes N) \\
& (k, l, l', m, m', n) & \longmapsto & (ll', m, m', n), \\
d_3\colon & (K \rtimes L) \rtimes (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (L \rtimes M) \rtimes (M \rtimes N) \\
& (k, l, l', m, m', n) & \longmapsto & (\partial_3 kl, \partial_2 l' m, m', n), \\
s_0\colon & (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (K \rtimes L) \rtimes (L \rtimes M) \rtimes (M \rtimes N) \\
& (l, m, m', n) & \longmapsto & (1, l, 1, m, m', n), \\
s_1\colon & (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (K \rtimes L) \rtimes (L \rtimes M) \rtimes (M \rtimes N) \\
& (l, m, m', n) & \longmapsto & (1, 1, l, m, m', n), \\
s_2\colon & (L \rtimes M) \rtimes (M \rtimes N) & \longrightarrow & (K \rtimes L) \rtimes (L \rtimes M) \rtimes (M \rtimes N) \\
& (l, m, m', n) & \longmapsto & (1, l, 1, m, m', n).
\end{array}$$

Thus we have a 3-truncated simplicial group $\mathbf{H} = \{H_0, H_1, H_2, H_3\}$. Applying the 3-skeleton functor defined in Subsection 2.1 to 3-truncation gives us a simplicial group, which will again be denoted $\mathbf{H}$, and the result has Moore complex

$$\ker \partial_3 \to K \to L \to M \to N.$$

We set $\mathbf{H}' = \mathbf{st}_3\mathbf{H}$ and note that $NH'_p = D_p \cap NH_p$, where D_p is the subgroup of H_p generated by the degenerate elements, and so $NH'_p = 1$ if $p > 4$. We claim $NH'_4 = 1$. By Theorem B, case $n = 4$ (see [**22**]), $\partial_4(NH_4 \cap D_4)$ is the product of commutators. A direct calculation, using the descriptions of the actions and the face maps above, shows that these are all trivial, so $\partial_4(NH_4 \cap D_4) = 1$, but $\partial_4^{\mathbf{H}}$ is a monomorphism so NH'_4 is trivial as required. □

Proposition 5.3. *Let* $\mathbf{G}$ *be a simplicial group, let* π'_n *be the homotopy groups of its 3-crossed module and let* π_n *be the homotopy groups of the classifying space of* $\mathbf{G}$*; then we have* $\pi_n \cong \pi'_n$ *for* $n = 0, 1, 2, 3, 4$.

Proof. Let $\mathbf{G}$ be a simplicial group. The nth homotopy group of $\mathbf{G}$ is the nth homology of the Moore complex of $\mathbf{G}$; i.e.,

$$\pi_n(\mathbf{G}) \cong H_n(\mathbf{NG}) \cong \frac{\ker d_{n-1}^{n-1} \cap NG_{n-1}}{d_n^n(NG_n)}.$$

Thus the homotopy groups $\pi_n(\mathbf{G}) = \pi_n$ of $\mathbf{G}$ are

$$\pi_n = \begin{cases} NG_0/d_1(NG_1) & n = 1 \\ \dfrac{\ker d_{n-1}^{n-1} \cap NG_{n-1}}{d_n^n(NG_n)} & n = 2, 3, 4, \\ 0 & n = 0 \text{ for } n > 4 \end{cases}$$

and the homotopy groups π'_n of its 3-crossed module are

$$\pi'_n = \begin{cases} NG_0/\partial_1(M) & n=1 \\ \ker\partial_1/\mathrm{Im}(\partial_2) & n=2 \\ \ker\partial_2/\mathrm{Im}(\partial_3) & n=3 \\ \ker\partial_3 & n=4 \\ 0 & n=0 \text{ for } n>4. \end{cases}$$

The isomorphism $\pi_n \cong \pi'_n$ can be shown by a direct calculation. □

5.2. Crossed 3-cubes

Crossed squares (or crossed 2-cubes) were introduced by D. Guin-Waléry and J.-L. Loday [**19**]; see also [**8**] and [**20**].

Definition 5.4. A crossed square is a commutative diagram of group morphisms

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ N & \xrightarrow[g]{} & P \end{array}$$

with the action of P on every other group and a function $h\colon M\times N \to L$ such that:

(1) the maps f and u are P-equivariant and $g, v, v\circ f$ and $g\circ u$ are crossed modules,
(2) $f\circ h(x,y) = x^{g(y)}x^{-1}$, $u\circ h(x,y) = {}^{v(x)}yy^{-1}$,
(3) $h(f(z),y) = z^{g(y)}z^{-1}$, $h(x,u(z)) = {}^{v(x)}zz^{-1}$,
(4) $h(xx',y) = {}^{v(x)}h(x',y)h(x,y)$, $h(x,yy') = h(x,y)^{g(y)}h(x,y')$,
(5) $h({}^tx,{}^ty) = {}^th(x,y)$

for $x,x'\in M$, $y,y'\in N$, $z\in L$ and $t\in P$.

It is a consequence of the definition that $f\colon L\to M$ and $u\colon L\to N$ are crossed modules where M and N act on L via their images in P. A crossed square can be seen as a crossed module in the category of crossed modules.

A crossed square can be seen as a complex of crossed modules of length one; thus Conduché [**13**] gave a direct proof from crossed squares to 2-crossed modules. This construction is the following:

Let

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ N & \xrightarrow[g]{} & P \end{array}$$

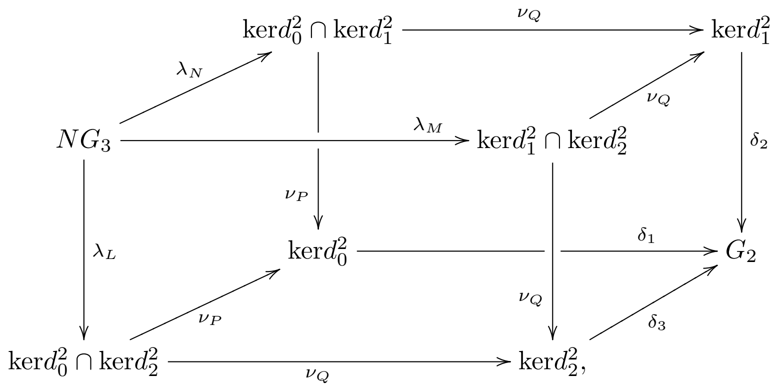
be a crossed square. Then seeing the horizontal morphisms as a complex of crossed modules, the mapping cone of this square is a 2-crossed module$L \xrightarrow{\partial_2} M\rtimes N \xrightarrow{\partial_1} P$, where $\partial_2(z) = (f(z)^{-1}, u(z))$ for $z\in L, \partial_1(x,y) = g(x)g(y)$ for $x\in M$ and $y\in N$, and the Peiffer lifting is given by $\{(x,y),(x',y')\} = h(x, yy'y^{-1})$.

Crossed squares were generalised by G. Ellis in [**16, 17**] and were called "Crossed n-cubes which were related to simplicial groups by T. Porter in [**23**]. Here we only consider this construction for $n = 3$ and look at the relation between crossed 3-cubes (see Appendix B) and 3-crossed modules.

Let

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

be a 3-crossed module and let $\mathbf{G}$ be the corresponding simplicial group. The crossed 3-cube associated to $\mathbf{G}$, defined by T. Porter in [**23**], is up to a canonical isomorphism

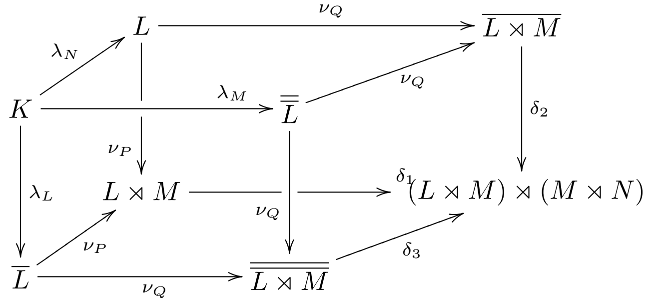


where λ_L , λ_M, λ_N are restrictions of d_3^3 and the others are inclusions. The h-maps are

$$\begin{array}{llll}
h_1\colon & \ker d_1^2 \times \ker d_0^2 \cap \ker d_2^2 & \longrightarrow & NG_3 \\
& (x,y) & \longmapsto & [s_1x_2s_0x_2^{-1}, s_2y_2^{-1}s_1y_2^{-1}], \\
h_2\colon & \ker d_0^2 \times \ker d_1^2 \cap \ker d_2^2 & \longrightarrow & NG_3 \\
& (x,y) & \longmapsto & [s_1x_2, s_2y_2s_1y_2^{-1}s_0y_2], \\
h_3\colon & \ker d_0^2 \cap \ker d_1^2 \times \ker d_2^2 & \longrightarrow & NG_3 \\
& (x,y) & \longmapsto & [s_2x_2, s_2y_2s_1y_2^{-1}], \\
h_7\colon & \ker d_0^2 \cap \ker d_2^2 \times \ker d_1^2 \cap \ker d_2^2 & \longrightarrow & NG_3 \\
& (x,y) & \longmapsto & h_2(ix,y) = h_2(x,y), \\
h_8\colon & \ker d_0^2 \cap \ker d_1^2 \times \ker d_0^2 \cap \ker d_2^2 & \longrightarrow & NG_3 \\
& (x,y) & \longmapsto & h_3(x,iy) = h_3(x,y), \\
h_9\colon & \ker d_0^2 \cap \ker d_1^2 \times \ker d_1^2 \cap \ker d_2^2 & \longrightarrow & NG_3 \\
& (x,y) & \longmapsto & h_3(x,iy) = h_3(x,y),
\end{array}$$

and the others are commutators. (The name of the maps are given with respect to the crossed 3-cube definition in [**16**].)

Then in terms of the 3-crossed module, this crossed cube can be written as



where

$$\begin{aligned} L \ltimes M &\cong \{(l,m,1,1)\colon l \in L, m \in M\}, \\ \overline{L \ltimes M} &= \{(l,m,m',1)\colon l \in L, m, m' \in M, mm' = 1, l \in L, m \in M\}, \\ \overline{\overline{L \ltimes M}} &= \{(l,m,m',n)\colon \partial_2(l) = 1, \partial_1(m')n = 1, l \in L, m \in M\}, \\ L &\cong \{(l,1,1,1) : l \in L\}, \\ \overline{L} &= \{(l,m,1,1) : \partial_2 lm = 1, l \in L, m \in M\} \\ &= \{(l, \partial_2(l^{-1}), 1, 1) : l \in L\}, \\ \overline{\overline{L}} &= \{(l,m,m',1) : mm' = 1, \partial_2(l)m = 1, \partial_1(m')n = 1, l \in L, m \in M, n \in N\} \\ &= \{(l, \partial_2(l^{-1}), \partial_2(l), 1) : l \in L\}. \end{aligned}$$

By Definition 2.7, given in [**13**], we have the mapping cone of this crossed 3-cube as

$$K \to (L \ltimes \overline{\overline{L}}) \ltimes \overline{L} \to (\overline{L \ltimes M})((L \ltimes M) \ltimes (\overline{\overline{L \ltimes M}})) \to (L \ltimes M) \ltimes (M \ltimes N).$$

Example 5.5. Let

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

be a 3-crossed module. If $M = \{1\}$, then, for $i = 1, 2, 3$, the commutative diagram

$$C_i \quad = \quad \left(\begin{array}{ccc} K & \xrightarrow{\partial_3} & L \\ \downarrow{\scriptstyle \partial'_3} & & \downarrow{\scriptstyle \mathrm{Id}} \\ L & \xrightarrow[\mathrm{Id}]{} & L \end{array} \right)$$

is a crossed square with the following h_i-maps:

$$\begin{aligned} h_1 &= \{,\}_{(2)(1)}\colon L \times L \to K, \\ h_2 &= \{,\}_{(0)(2)}\colon L \times L \to K, \\ h_3 &= \{,\}_{(0)(1)}\colon L \times L \to K, \\ &\quad (x,y) \mapsto \{y,x\}^{-1}_{(1)(0)}, \end{aligned}$$

where the action of L on itself is by conjugation.

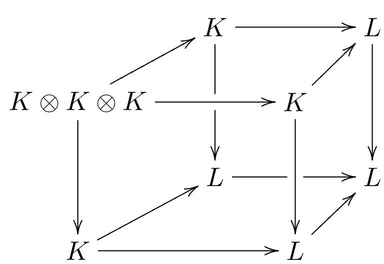
Since $M = \{1\}$, $\partial_2(l) = 1_M$ for all $l \in L$. Thus from the 3-crossed module axioms we find

$$\begin{aligned}
\{l, \partial_3 k\}_{(2)(1)} &= (l \cdot k)k^{-1}, \\
\{ll', l''\}_{(2)(1)} &= l \cdot \{l', l''\}_{(2)(1)}\{l, l''\}_{(2)(1)}, \\
\{l, l'l''\}_{(2)(1)} &= \{l, l'\}_{(2)(1)}\ {}^{l'}\{l, l''\}_{(2)(1)}, \\
\partial_3\{l, l'\}_{(2)(1)} &= l({}^{l'}l^{-1}), \\
(\{l', l\}_{(1)(0)})^{-1} &= \{l, l'\}_{(2)(1)}, \\
\{l, l'\}_{(0)(2)} &= 1
\end{aligned}$$

for all $l, l', l'' \in L$, $k, k' \in K$. Using these equalities and the 3-crossed module axioms, crossed square conditions can be easily verified.

In this example the result is trivial for the h-map h_1 from [**12**] since $\{,\}_{(2)(1)}$ is a 2-crossed module. Here the liftings $\{,\}_{(0)(2)}$, $\{,\}_{(0)(1)}$ are not 2-crossed modules but the associated h-maps are crossed squares.

Example 5.6. In the universal cube definition given in [**16**], take $P, R, N, M = L$, $S = M$ and $T_0 = K \otimes K \otimes K$. Then



is a universal crossed 3-cube with the crossed squares obtained by the Peiffer maps $\{,\}_{(0)(1)}$, $\{,\}_{(2)(1)}$, $\{,\}_{(0)(2)}$ given in the above proposition.

Appendix A.

Proof of Proposition 5.1.

3CM1)

$$\begin{aligned}
\overline{\partial}_3\left(\{x_2, y_2\}_{(2)(1)}\right) &= [x_2, y_2]\,[y_2, s_1\partial_2 x_2] \\
&= x_2 y_2 x_2^{-1} y_2^{-1} y_2 s_1\partial_2 x_2 y_2^{-1} s_1\partial_2 x_2^{-1} \\
&= x_2 y_2 x_2^{-1}({}^{\partial_2 x_2} y_2)^{-1}.
\end{aligned}$$

Since

$$d_4(F_{(1)(3,2)}(x_3, y_2)) = [s_1 d_3 x_3, s_2 y_2]\,[s_2 y_2, s_2 d_3 x_3]\,[x_3, s_2 y_2]\,,$$

we find

$$\begin{aligned}
\{\overline{\partial}_3 x_3, y_2\}_{(2)(1)} &= [s_1\overline{\partial}_3 x_3, s_2 y_2][s_2 y_2, s_2\overline{\partial}_3 x_3] \\
&\equiv [x_3, s_2 y_2] \qquad \bmod\ \partial_4(NG_4 \cap D_4) \\
&= x_3({}^{y_2}x_3)^{-1}.
\end{aligned}$$

Since

$$d_4(F_{(3,1)(2)}(x_2,y_3)) = [s_1x_2, s_2d_3y_3]\,[s_2d_3y_3, s_2x_2]\,[s_2x_2, y_3]\,[y_3, s_1x_2]\,,$$

we find

$$\begin{aligned}\{x_2,\overline{\partial}_3y_3\}_{(2)(1)} &= \left[s_2x_2, s_2\overline{\partial}_3y_3\right]\,\left[s_2\overline{\partial}_3y_3, s_1x_2\right]\\ &\equiv [s_2x_2,y_3]\,[y_3,s_1x_2] \qquad \bmod \partial_4(NG_4\cap D_4)\\ &= {}^{x_2}y_3s_1x_2y_3^{-1}s_1x_2^{-1}\\ &\equiv {}^{x_2}y_3({}^{\partial_2x_2}y_3)^{-1} \qquad \bmod \partial_4(NG_4\cap D_4),\end{aligned}$$

$$\begin{aligned}\{x_2y_2,z_2\}_{(2)(1)} &= [s_1(x_2y_2), s_2z_2]\,[s_2z_2, s_2(x_2y_2)]\\ &= s_1(x_2y_2)s_2z_2s_1(x_2y_2)^{-1}s_2(x_2y_2)s_2z_2^{-1}s_2(x_2y_2)^{-1}\\ &\equiv s_2(x_2y_2)s_2z_2^{-1}s_2(x_2y_2)^{-1}s_1(x_2y_2)\\ &\quad\ s_2z_2s_1(x_2y_2)^{-1} \quad \bmod \partial_4(NG_4\cap D_4)\\ &= \{x_2, y_2z_2y_2^{-1}\}_{(2)(1)}\ {}^{\partial_1x_2}\{y_2,z_2\}_{(2)(1)},\end{aligned}$$

and

$$\begin{aligned}\{x_2,y_2z_2\}_{(2)(1)} &= [s_1(x_2), s_2(y_2z_2)]\,[s_2(y_2z_2), s_2(x_2)]\\ &\equiv [s_2(x_2), s_2(y_2z_2)]\,[s_2(y_2z_2), s_1(x_2)]\\ &\quad (s_2(x_2)s_2(y_2)s_2(x_2)^{-1})s_1(x_2)s_2(y_2)s_1(x_2)^{-1} \quad \bmod \partial_4(NG_4\cap D_4)\\ &= (x_2y_2x_2^{-1})\cdot\{x_2,z_2\}_{(2)(1)}\{x_2,y_2\}_{(2)(1)}.\end{aligned}$$

3CM2) Since

$$\begin{aligned}d_4(F_{(3,2,0)(1)}(x_1,y_3)) &= [s_2s_0x_1, s_1d_3y_3]\,[s_1d_3y_3, s_2s_1x_1]\,[s_2s_1x_1, s_2d_3y_3]\\ &\quad [s_2d_3y_3,]\,[s_2s_0x_1, y_3]\,[y_3, s_2s_1x_1]\,,\\ d_4(F_{(3,1,0)(2)}(x_1,y_3)) &= [s_1s_0x_1, s_2d_3y_3]\,[s_2d_3y_3\ \ , s_2s_0x_1]\\ &\quad [s_2s_0x_1, y_3]\,[y_3, s_1s_0x_1]\end{aligned}$$

and

$$d_4(F_{(2,1,0)(3)}(x_1,y_3)) = [s_2s_1s_0d_1x_1, y_3]\,[y_3, s_1s_0x_1]\,,$$

we find

$$\{x_1,\overline{\partial}_3y_3\}_{(1,0)(2)} = \{x_1,\overline{\partial}_3y_3\}_{(2,0)(1)}\{\overline{\partial}_3y_3,x_1\}_{(0)(2,1)}y_3(\ {}^{\partial_1x_1}y_3)^{-1}.$$

3CM3) and **3CM4)** are left to the reader.

3CM5) Since

$$\begin{aligned}d_4(F_{(3,0)(2,1)}) &= [s_0x_2, s_2s_1\partial_2y_2][s_2s_1\partial_2y_2, s_1x_2][s_2x_2, s_2s_1\partial_2y_2]\\ &\quad [s_1y_2, s_2x_2][s_1x_2, s_1y_2][s_1y_2, s_0x_2],\end{aligned}$$

we find

$$\begin{aligned}\{x_2,\partial_2 y_2\}_{(0)(2,1)} &= [s_0x_2,s_2s_1\partial_2y_2][s_2s_1\partial_2y_2,s_1x_2][s_2x_2,s_2s_1\partial_2y_2]\\ &\equiv [s_1y_2,s_2x_2][s_1x_2,s_1y_2][s_1y_2,s_0x_2] \quad \bmod \partial_4(NG_4\cap D_4)\\ &= (\{y_2,x_2\}_{(1)(2)})^{-1}\{x_2,y_2\}_{(1)(0)}.\end{aligned}$$

3CM6) Since

$$\begin{aligned}d_4(F_{(2,0)(3,1)}(x_2,y_2)) = {} & [s_2s_0d_2x_2,s_1y_2][s_1y_2,s_2s_1d_2x_2]\\ & [s_2s_1d_2x_2,s_2y_2][s_2y_2,s_2s_0d_2x_2]\\ & [s_0x_2,s_2y_2][s_2y_2,s_1x_2]\\ & [s_1x_2,s_1y_2][s_1y_2,s_0x_2],\end{aligned}$$

we find

$$\begin{aligned}\{\partial_2x_2,y_2\}_{(2,0)(1)} &= [s_2s_0\partial_2x_2,s_1y_2][s_1y_2,s_2s_1\partial_2x_2]\\ & [s_2s_1\partial_2x_2,s_2y_2][s_2y_2,s_2s_0\partial_2x_2]\\ &\equiv [s_0x_2,s_2y_2][s_2y_2,s_1x_2]\\ &= \{x_2,y_2\}^{-1}_{(0)(2)}\ {}^{[y_2,x_2]}(\{x_2,y_2\}_{(2)(1)})\{x_2,y_2\}_{(1)(0)}.\end{aligned}$$

3CM7) Since

$$d_4(F_{(1,0)(3,2)}(x_2,y_2)) = [s_1s_0\overline{\partial}_2x_2,s_2y_2][s_2y_2,s_2s_0\overline{\partial}_2x_2][s_0x_2,s_2y_2],$$

we find

$$\begin{aligned}\{\partial_2x_2,y_2\}_{(1,0)(2)} &= [s_1s_0\overline{\partial}_2x_2,s_2y_2][s_2y_2,s_2s_0\overline{\partial}_2x_2]\\ &\equiv [s_0x_2,s_2y_2] \quad \bmod \ \partial_4(NG_4\cap D_4)\\ &= \{x_2,y_2\}^{-1}_{(0)(2)}.\end{aligned}$$

3CM8)

$$\begin{aligned}\overline{\partial}_3(\{x_2,y_2\}_{(1)(0)}) &= [x_2,y_2]\left[\overline{\partial}_3s_1x_2,\overline{\partial}_3s_1y_2\right]\left[\overline{\partial}_3s_1y_2,\overline{\partial}_3s_0x_2\right]\\ &= [x_2,y_2]\, s_1\,[\partial_2x_2,\partial_2y_2]\,[s_1\partial_2y_2,s_0\partial_2x_2]\\ &= [x_2,y_2]\,\{\partial_2x_2,\partial_2y_2\}.\end{aligned}$$

3CM9)

$$\overline{\partial}_3\left(\{x_2,y_2\}_{(0)(2)}\right) = \overline{\partial}_3\left(\{\partial_2x_2,y_2\}_{(1,0)(2)}\right)^{-1}.$$

3CM10)

$$\begin{aligned}\overline{\partial}_3\{x_2,y_1\}_{(0)(2,1)} &= \overline{\partial}_3\left([s_2s_1y_1,s_2x_2]\,[s_1x_2,s_2s_1y_1]\,[s_2s_1y_1,s_0x_2]\right)\\ &= [s_1y_1,x_2]\left[\overline{\partial}_3s_1x_2,s_1y_1\right]\left[s_1y_1,\overline{\partial}_3s_0x_2\right]\\ &= {}^{y_1}x_2x_2^{-1}\{\partial_2x_2,y_1\}.\end{aligned}$$

3CM11) Since

$$\begin{aligned}\overline{\partial}_3\{x_1,y_2\}_{(1,0)(2)} &= [s_0x_1,y_2]\left[y_2,\overline{\partial}_3s_1s_0x_1\right]\\ \overline{\partial}_3\{x_1,y_2\}_{(1,0)(2)}\left[\overline{\partial}_3s_1s_0x_1,y_2\right] &= [s_0x_1,y_2]\,,\\ \overline{\partial}_3\{x_1,y_2\}_{(2,0)(1)} &= [s_0x_1,y_2]\,[y_2,s_1x_1]\,[s_1x_1,\partial_3s_1y_2]\,[\partial_3s_1y_2,s_0x_1]\\ &= [s_0x_1,y_2]\,[y_2,s_1x_1]\,[s_1x_1,s_1\partial_2y_2]\,[s_1\partial_2y_2,s_0x_1]\\ &= [s_0x_1,y_2]\,[y_2,s_1x_1]\,\{x_1,\partial_2y_2\},\end{aligned}$$

we find

$$\begin{aligned}\overline{\partial}_3\{x_1,y_2\}_{(2,0)(1)} &= \overline{\partial}_3\{x_1,y_2\}_{(1,0)(2)}\left[\overline{\partial}_3s_1s_0x_1,y_2\right][y_2,s_1x_1]\,\{x_1,\partial_2y_2\}\\ &= \overline{\partial}_3\{x_1,y_2\}_{(1,0)(2)}\,[s_1s_0\partial_1x_1,y_2]\,[y_2,s_1x_1]\,\{x_1,\partial_2y_2\}\\ &= \overline{\partial}_3\{x_1,y_2\}_{(1,0)(2)}\ {}^{\partial_1x_1}y_2\ {}^{x_1}y_2\{x_1,\partial_2y_2\}.\end{aligned}$$

3CM12) Since

$$d_4(F_{(0)(3,1)}(x_3,y_2)) = [s_0d_3x_3,s_1y_2][s_1y_2,s_1d_3x_3][s_2d_3x_3,s_2y_2][s_2y_2,x_3],$$

we find

$$\begin{aligned}\{\overline{\partial}_3x_3,y_2\}_{(1)(0)} &= [s_0\overline{\partial}_3x_3,s_1y_2][s_1y_2,s_1\overline{\partial}_3x_3][s_2\overline{\partial}_3x_3,s_2y_2]\\ &\equiv [s_2y_2,x_3] \quad \bmod \partial_4(NG_4\cap D_4)\\ &= ({}^{y_2}x_3)x_3^{-1}.\end{aligned}$$

Since

$$d_4(F_{(3,0)(1)}(x_2,y_3)) = [s_0x_2,s_1d_3y_3]\,[s_1d_3y_3,s_1x_2]\,[s_2x_2,s_2d_3y_3]\,[y_3,s_2x_2]\,,$$

we find

$$\begin{aligned}\{x_2,\overline{\partial}_3y_3\}_{(1)(0)} &= \left[s_0x_2,s_1\overline{\partial}_3y_3\right]\left[s_1\overline{\partial}_3y_3,s_1x_2\right]\left[s_2x_2,s_2\overline{\partial}_3y_3\right]\\ &\equiv [y_3,s_2x_2] \quad \bmod \partial_4(NG_4\cap D_4)\\ &= y_3({}^{x_2}y_3)^{-1}.\end{aligned}$$

3CM13) is left to the reader.

3CM14) Since

$$d_4(F_{(0)(3,2)}(x_3,y_2)) = [s_0d_3x_3,s_2y_2],$$

we find

$$\begin{aligned}\{\overline{\partial}_3x_3,y_2\}_{(0)(2)} &= [s_0\overline{\partial}_3x_3,s_2y_2]\\ &\equiv 1 \quad \bmod \partial_4(NG_4\cap D_4).\end{aligned}$$

3CM15) Since

$$d_4(F_{(3,0)(2)}(x_2,y_3)) = [s_0x_2,s_2d_3y_3]\,[y_3,s_0x_2]$$

and

$$d_4(F_{(1,0)(2)}(x_2,y_3)) = [s_1s_0\partial_2x_2,s_2\partial_3y_3]\,[s_2\partial_3y_3,s_2s_0\partial_2x_2]\,[s_0x_2,y_3]\,,$$

we find

$$
\begin{aligned}
\{x_2, \overline{\partial}_3 y_3\}_{(0)(2)} &= \left[s_0 x_2, s_2 \overline{\partial}_3 y_3\right] \\
&\equiv [y_3, s_0 x_2] \quad \bmod \partial_4(NG_4 \cap D_4) \\
&\equiv \left[s_2 s_0 \partial_2 x_2, s_2 \overline{\partial}_3 y_3\right] \left[s_2 \overline{\partial}_3 y_3, s_1 s_0 \partial_2 x_2\right] \quad \bmod \partial_4(NG_4 \cap D_4) \\
&= \{\partial_2 (x_2), \overline{\partial}_3 (y_3)\}^{-1}_{(1,0)(2)}.
\end{aligned}
$$

3CM16) Since

$$
\begin{aligned}
d_4(F_{(2,0)(1)}(x_2, y_3)) = {} & [s_2 s_0 \partial_2 x_2, s_1 \partial_3 y_3] [s_1 \partial_3 y_3, s_2 s_1 \partial_2 x_2] \\
& [s_2 s_1 \partial_2 x_2, s_2 \partial_3 y_3] [s_2 \partial_3 y_3, s_2 s_0 \partial_2 x_2] \\
& [s_0 x_2, y_3] [y_3, s_1 x_1],
\end{aligned}
$$

we find

$$
\begin{aligned}
\{\partial_2 x_2, \overline{\partial}_3 y_3\}_{(2,0)(1)} = {} & \left[s_2 s_0 \partial_2 x_2, s_1 \overline{\partial}_3 y_3\right] \left[s_1 \overline{\partial}_3 y_3, s_2 s_1 \partial_2 x_2\right] \\
& \left[s_2 s_1 \partial_2 x_2, s_2 \overline{\partial}_3 y_3\right] \left[s_2 \overline{\partial}_3 y_3, s_2 s_0 \partial_2 x_2\right] \\
\equiv {} & [s_0 x_2, y_3] [y_3, s_1 x_1] \quad \bmod \partial_4(NG_4 \cap D_4) \\
\equiv {} & [s_0 x_2, y_3]\, y_3 ({}^{\partial_2 x_2} y_3)^{-1} \quad \bmod \partial_4(NG_4 \cap D_4) \\
\equiv {} & \left[s_1 s_0 \partial_2 x_2, s_2 \overline{\partial}_3 y_3\right] \left[s_2 \overline{\partial}_3 y_3, s_2 s_0 \partial_2 x_2\right] \quad \bmod \partial_4(NG_4 \cap D_4) \\
= {} & \{\partial_2 x_2, \overline{\partial}_3 y_3\}_{(1,0)(2)}.
\end{aligned}
$$

3CM17) Since

$$
\begin{aligned}
d_4(F_{(0)(2,1)}(x_3, y_2)) = {} & [s_0 d_3 x_3, s_2 s_1 d_2 y_2] [s_2 s_1 d_2 y_2, s_1 d_3 x_3] \\
& [s_2 d_3 x_3, s_2 s_1 d_2 y_2] [s_1 y_2, x_3]
\end{aligned}
$$

and

$$
d_4(F_{(0)(2,1)}(x_2, y_3)) = [s_2 s_1 d_2 x_2, y_3] [y_3, s_1 x_2],
$$

we find

$$
\begin{aligned}
\{\overline{\partial}_3 x_3, \partial_2 y_2\}_{(0)(2,1)} = {} & \left[s_0 \overline{\partial}_3 x_3, s_2 s_1 \partial_2 y_2\right] \left[s_2 s_1 \partial_2 y_2, s_1 \overline{\partial}_3 x_3\right] \\
& \left[s_2 \overline{\partial}_3 x_3, s_2 s_1 \partial_2 y_2\right] \\
\equiv {} & [s_1 y_2, x_3] \quad \bmod \partial_4(NG_4 \cap D_4) \\
\equiv {} & [x_3, s_2 s_1 \partial_2 y_2] \quad \bmod \partial_4(NG_4 \cap D_4) \\
= {} & x_3 ({}^{\partial_2 y_2} x_3)^{-1}.
\end{aligned}
$$

3CM18)

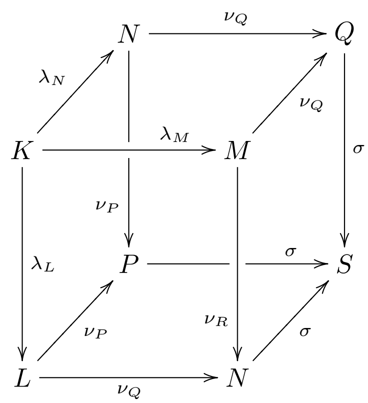
$$
\begin{aligned}
\partial_2 \{x_1, y_1\} &= [x_1, y_1][y_1, \partial_2 s_0 x_1] \\
&= x_1 y_1 x_1^{-1} (\partial_1 x_{1 y_1})^{-1}.
\end{aligned}
$$

□

Appendix B.

B.1. Crossed 3-cube

Given a commutative diagram of groups



in which there is a group action of S on each of the other seven groups (hence the eight groups act on each other via the action of S), and there are six functions

$$\begin{aligned} h_1\colon\ & Q \times L \longrightarrow K, \\ h_2\colon\ & P \times M \longrightarrow K, \\ h_3\colon\ & N \times R \longrightarrow K, \\ h_4\colon\ & P \times R \longrightarrow L, \\ h_5\colon\ & Q \times R \longrightarrow M, \\ h_6\colon\ & P \times Q \longrightarrow N, \end{aligned}$$

we say that this structure is a crossed 3-cube of groups if

1. Each of the nine squares

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ Q & \longrightarrow & S \end{array} \quad \begin{array}{ccc} K & \longrightarrow & M \\ \downarrow & & \downarrow \\ P & \longrightarrow & S \end{array} \quad \begin{array}{ccc} K & \longrightarrow & R \\ \downarrow & & \downarrow \\ N & \longrightarrow & S \end{array} \quad \begin{array}{ccc} L & \longrightarrow & R \\ \downarrow & & \downarrow \\ P & \longrightarrow & S \end{array} \quad \begin{array}{ccc} M & \longrightarrow & R \\ \downarrow & & \downarrow \\ Q & \longrightarrow & S \end{array}$$

$$\begin{array}{ccc} N & \longrightarrow & O \\ \downarrow & & \downarrow \\ P & \longrightarrow & S \end{array} \quad \begin{array}{ccc} K & \longrightarrow & M \\ \downarrow & & \downarrow \\ L & \longrightarrow & R \end{array} \quad \begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ N & \longrightarrow & P \end{array} \quad \begin{array}{ccc} K & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & Q \end{array}$$

is a crossed square; for the last three squares, the functions $h\colon L \times M \to K$, $h\colon N \times L \to K$, $h\colon N \times M \to K$ are respectively given by $h(l,m) = h(v_P l, n)$, $h(n,l) = h(n, v_R l)$, $h(n,m) = h((n, v_R m)$.

2. $$h((v_P n)(v_P l), m)h((v_Q m)(v_Q n), l) = h(n, (v_R l)(v_R m)).$$

3. $$^{q}h(h(p, q^{-1})^{-1}, r) = {}^{p}h(q, h(p^{-1}, r))^{r}h(p, h(q, r^{-1})^{-1}).$$

4.

$$\lambda_L h(p,m) = h(p, v_R m),$$
$$\lambda_L h(n,r) = h(v_P n, r),$$
$$\lambda_M h(q,l) = h(q, v_R l),$$
$$\lambda_M h(n,r) = h(v_Q n, r),$$
$$\lambda_N h(p,m) = h(p, v_Q m),$$
$$\lambda_N h(q,l) = h(v_P l, q)^{-1}.$$

5.
$$h(v_Q m, l) = h(v_P l, m)^{-1},$$
$$h(n, v_R l) = h(v_Q n, l),$$
$$h(n, v_R m) = h(v_P n, m),$$

for all $l \in L$, $m \in M$, $n \in N$, $p \in P$, $q \in Q$, $r \in R$.

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