Homology, Homotopy and Applications, vol. 10(1), 2008, pp.53–95

ON AFFINE MORPHISMS OF HOPF ALGEBROIDS

GEOFFREY M.L. POWELL

(*communicated by Donald M. Davis*)

Abstract

The paper considers the FPQC stacks which are associated to affine groupoid schemes. Using a formulation of a descent datum in terms of morphisms of affine groupoid schemes, explicit arguments are given which avoid appeal to the general principle of faithfully flat descent. This theory is applied to consider the notion of affine morphism.

1. Introduction

Groupoid schemes are a natural generalization of group schemes, in which the notion of an abstract group is replaced by that of a small groupoid; as such, they appear in the work of Demazure and Gabriel [**1**, Chapitre III, Section 2] on algebraic groups. An affine groupoid scheme is one in which the objects and morphisms are represented by affine schemes. Such objects arise for example when considering deformations of algebraic group schemes.

Affine groupoid schemes arise naturally in algebraic topology, where the equivalent structure in commutative rings (given by passage to the coordinate ring) is known as a Hopf algebroid. Namely, if E is a flat ring spectrum which represents the homology theory $E_*(-)$, there is an associated Hopf algebroid (E_*, E_*E), where E_*E denotes the E_*-homology cooperations given by the E-homology of the spectrum E.

Comodules over the Hopf algebroid (E_*, E_*E) arise in the calculation of the E_2-term of the Adams-Novikov spectral sequence, which is usually expressed in terms of the Ext groups in the category of comodules. It is a fundamental observation, going back to the work of Morava ([**10**] and much unpublished work), that these Ext groups can be considered in terms of the cohomology of a stack associated to the Hopf algebroid: in particular, a comodule for a flat Hopf algebroid is equivalent to a quasi-coherent module on the associated stack.

The interest of the stack theoretic point of view has been underlined by the work of Hopkins, Rezk, Strickland and others on elliptic cohomology and the theory of topological modular forms (see the unpublished notes [**7**], for example). Foundational work has been carried out on a category of algebraic stacks suitable for the application to stable homotopy theory by Hopkins and Miller [**8**], Goerss [**3**], Pribble in his thesis [**12**], Naumann [**11**] and by others. There is related foundational work on

Received June 27, 2007, revised October 18, 2007; published on Month Day, Year.
2000 Mathematics Subject Classification: Primary 14A20; Secondary 55N20, 55S05, 14L99.
Key words and phrases: stack, Hopf algebroid, FPQC topology, affine morphism.
This article is available at `http://intlpress.com/HHA/v10/n1/a4`

stacks from the homotopy theoretic or derived viewpoint by Hollander in [**6, 5, 4**]. The algebraic stacks which are considered differ from the usual notion from algebraic geometry, in that there are no finiteness assumptions. Moreover, the natural Grothendieck topology is the flat topology (FPQC: *fidèlement plat quasi-compact*), essentially because a faithfully flat extension of coefficient rings does not affect the Bousfield class of a ring spectrum. For the remainder of the paper, all stacks considered are defined on the category of affine S-schemes, for a fixed affine scheme S (which can be taken to be $\mathrm{Spec}(\mathbb{Z})$), with respect to the FPQC topology.

To state the relation between algebraic stacks and Hopf algebroids requires the notion of an affine morphism between stacks, more particularly, the condition that the diagonal morphism of a stack is affine. A morphism $f\colon \mathscr{M} \to \mathscr{N}$ of stacks is affine if, for any morphism $U \to \mathscr{N}$, where U is a stack represented by an affine S-scheme, the stack 2-fibre product $\mathscr{M} \times_{\mathscr{N}} U$ is equivalent to the stack represented by an affine scheme. In particular, a stack $\mathscr{M}$ is said to be affine if the canonical morphism $\mathscr{M} \to S$ is an affine morphism; this is equivalent to the existence of an affine S-scheme V such that $\mathscr{M}$ is equivalent to the stack represented by V. Affine morphisms between stacks are important for a number of reasons; for example, they induce continuous morphisms between the flat sites associated to suitable stacks (see [**12**], for example).

The category of affine groupoid schemes is equivalent to the category of stacks over affine schemes with respect to the FPQC topology, subject to the conditions that the diagonal morphism is affine and the choice of a surjective morphism (a form of presentation) from an affine scheme to the stack. Hence, there is a contravariant equivalence between the category of Hopf algebroids and this category of algebraic stacks.

The purpose of this paper is to study the condition for a morphism of Hopf algebroids to induce an affine morphism of stacks. This requires an understanding of the construction of the associated stack. An explicit construction of the associated stack is given by analysing descent data in terms of morphisms of affine groupoid schemes; this allows most of the analysis to be carried out at the level of affine schemes. The arguments considering affine morphisms are explicit (constructive), without appeal to a general faithfully flat descent argument. The treatment of descent data given here is related to the homotopy theoretic approach to stacks, as in the work of Hollander [**6, 5, 4**].

An affine groupoid scheme $X_\bullet$ is defined by its underlying pair of affine schemes (X_0, X_1) together with its structure morphisms. The source and target morphisms $X_1 \rightrightarrows X_0$ give rise to the affine scheme $\overline{X}$, which is defined as the coequalizer in affine schemes. Moreover, there is a canonical associated morphism $X_1 \to X_0 \times_{\overline{X}} X_0$. This structure is familiar to topologists: an affine groupoid scheme $X_\bullet$ corresponds to a Hopf algebroid (A, Γ), where A is the coordiate ring $\mathscr{O}(X_0)$ and Γ the ring $\mathscr{O}(X_1)$, a correspondence which can be indicated by writing $X_\bullet = \mathrm{Spec}(A, \Gamma)$. The scheme $\overline{X}$ identifies with $\mathrm{Spec}(A^\Gamma)$, where A^Γ is the equalizer of the left and right units $A \rightrightarrows \Gamma$. This appears as the zero cohomology of the Hopf algebroid, and thus in the Adams-Novikov spectral sequence.

To state the results, the category of sheaves of sets for the FPQC topology on the category of affine S-schemes is written as $\mathsf{Shv}_{\mathrm{fpqc}}$; the stack associated to an affine groupoid scheme $X_\bullet$ is denoted by $[X_\bullet]$.

Theorem 1.1. *Let $X_\bullet$ be an affine groupoid scheme. The associated stack $[X_\bullet]$ is affine if and only if the following equivalent conditions are satisfied.*

1. *The canonical morphism $[X_\bullet] \to \overline{X}$ is an equivalence of stacks.*
2. *The following conditions are satisfied*
   (a) *$X_0 \to \overline{X}$ induces a surjection in $\mathsf{Shv}_{\mathrm{fpqc}}$;*
   (b) *$X_1 \to X_0 \times_{\overline{X}} X_0$ is an isomorphism of affine S-schemes.*

A morphism of affine groupoid schemes $X_\bullet \to Y_\bullet$ is said to be affine if the induced morphism of associated stacks $[X_\bullet] \to [Y_\bullet]$ is affine. Theorem 1.1 gives rise to an explicit criterion for a morphism of affine groupoid schemes to be affine.

Affine groupoid schemes form a 2-category and this 2-category admits 2-fibre products. In particular, a morphism of affine groupoid schemes $X_\bullet \to Y_\bullet$ gives rise to a 2-fibre product $X_\bullet \times_{Y_\bullet} Y_0$, which has underlying affine schemes $(Y_1 \times_{Y_0} X_0, Y_1 \times_{Y_0} X_1)$; there are source and target structure morphisms s, t which are defined explicitly in Section 8.2. Let

$$Y_1 \times_{Y_0} X_1 \underset{t}{\overset{s}{\rightrightarrows}} Y_1 \times_{Y_0} X_0 \longrightarrow W$$

denote the coequalizer in the category of affine S-schemes, Aff/S. The morphisms s, t induce a natural morphism

$$Y_1 \times_{Y_0} X_1 \to (Y_1 \times_{Y_0} X_0) \times_W (Y_1 \times_{Y_0} X_0).$$

Theorem 1.2. *A morphism $X_\bullet \to Y_\bullet$ of affine groupoid schemes is affine if and only if the following two conditions are satisfied:*

1. *The morphism $Y_1 \times_{Y_0} X_0 \to W$ induces a surjection in $\mathsf{Shv}_{\mathrm{fpqc}}$.*
2. *The induced morphism*

$$Y_1 \times_{Y_0} X_1 \to (Y_1 \times_{Y_0} X_0) \times_W (Y_1 \times_{Y_0} X_0)$$

is an isomorphism of affine S-schemes.

The notion of affine morphism depends on the choice of Grothendieck topology on affine S-schemes which is used in defining the stacks. There is a stronger condition on a morphism of affine groupoid scheme which is independent of the Grothendieck topology.

The result is stated using the notion of a split coequalizer, which is recalled in Definition 7.4. The morphism $\tilde{\epsilon}\colon Y_1 \times_{Y_0} X_0 \to Y_1 \times_{Y_0} X_1$ is induced by the unit morphism $\epsilon\colon X_0 \to X_1$.

Corollary 1.3. *Let $X_\bullet \to Y_\bullet$ be a morphism of affine groupoid schemes such that*

1. *the diagram*

$$Y_1 \times_{Y_0} X_0 \xrightarrow{\tilde{\epsilon}} Y_1 \times_{Y_0} X_1 \underset{t}{\overset{s}{\rightrightarrows}} Y_1 \times_{Y_0} X_0$$

is an equalizer in Aff/S;

2. *the coequalizer diagram in* Aff/S

$$Y_1 \times_{Y_0} X_1 \underset{t}{\overset{s}{\rightrightarrows}} Y_1 \times_{Y_0} X_0 \longrightarrow W$$

is a split coequalizer.

Then the morphism $X_\bullet \to Y_\bullet$ *is affine.*

The category of Hopf algebroids is independent of any Grothendieck topology, whereas the category of stacks with presentation seemingly depends on the choice of the FPQC topology. It is therefore reassuring to note that the fact that the diagonal morphism of the stack $[X_\bullet]$ associated to the affine groupoid scheme $X_\bullet$ is affine is independent of the Grothendieck topology by the following observation.

Proposition 1.4. *Let* $X_\bullet$ *be an affine groupoid scheme. Then the diagonal morphism* $X_\bullet \to X_\bullet \times_S X_\bullet$ *satisfies the hypotheses of Corollary 1.3.*

Organization of the paper

Section 2 is devoted to a survey of the background material which is required in the paper. The key definition of the paper, that of an affine morphism, is given in Section 3.

Section 4 develops an explicit description of the stack associated to an affine groupoid scheme by reconsidering descent data; namely a descent datum for an affine groupoid scheme is shown to be equivalent to a morphism of affine groupoid schemes of a certain form and a similar description of morphisms between descent data is available. These results are used in the paper to make explicit certain arguments which are frequently given by appeal to faithfully flat descent techniques.

The structure of the 2-fibre products in the 2-category of affine groupoid schemes is explained in Section 5. This material is fundamental for the consideration of affine morphisms between affine groupoid schemes.

Section 6 develops the theory of the affine groupoid schemes which have discrete associated stack (a stack is discrete if it is isomorphic to the stack associated to a sheaf). An explicit argument is given to show how this behaves with respect to faithfully flat descent. Section 7 applies this material to consider the affine groupoid schemes which are affine.

Section 8 applies the previous results to consider affine morphisms. This gives an explicit criterion for a morphism of affine groupoid schemes to induce an affine morphism of stacks. However, this condition seems to be difficult to check in practice.

The appendix applies the material developed within the paper to sketch a proof of the equivalence between the 2-category of affine groupoid schemes and a suitable 2-category of stacks with presentation. This recovers the result of Pribble [**12**] and Naumann [**11**], which is also contained in the work of Hollander [**4**].

Acknowledgements

The author would like to thank the referee of this paper for their comments and suggestions, in particular for the observation that the reference in the first version of

this paper to the associated stack construction in [**9**] obscured the explicit nature of the results.

Contents

2. Background

This section reviews the foundational material on affine groupoid schemes, the FPQC topology, and stacks; references to the literature are given for the basic notions.

2.1. Affine schemes and the flat topology

Throughout this paper, R denotes a fixed commutative, associative ring with unit and S denotes the associated affine scheme, $\mathrm{Spec}(R)$. The category Aff/S of affine schemes over S is equivalent to the opposite of the category of R-algebras; the R-algebra associated to an affine S-scheme U will be written $\mathscr{O}(U)$. The inverse functor associates to an R-algebra A the affine scheme $\mathrm{Spec}(A)$ equipped with the morphism to S induced by $R \to A$. (An introduction suitable for the study of affine groupoid schemes is given in the introduction to affine group schemes by Waterhouse [**15**].)

Remark 2.1. The category Aff/S is a full subcategory of the category of schemes over S and has fibre products, coproducts, products and a final object S. (The latter two exist because the scheme S is taken to be affine.)

Definition 2.2. The FPQC topology (*fidèlement plat quasi-compact*) on the category Aff/S is the Grothendieck topology with covers which are finite families of flat morphisms $\{X_i \to X\}$ in Aff/S such that the morphism $\amalg_i X_i \to X$ is faithfully flat.

Remark 2.3. This is the correct definition when working with affine schemes, noting that a morphism between affine schemes is quasi-compact. The general definition in schemes is more subtle, since Zariski covers should be FPQC covers; a suitable definition is given in [**14**, Section 2.3.3].

A presheaf of sets on Aff/S can be considered either as a functor $(\mathsf{Aff}/S)^{\mathrm{op}} \to \mathsf{Set}$ or as a functor $R - \mathsf{Alg} \to \mathsf{Set}$. Thus, the presheaf represented by an affine S-scheme X can be considered as $\mathrm{Hom}_{\mathsf{Aff}/S}(-, X)$ or as $\mathrm{Hom}_{R-\mathsf{Alg}}(\mathscr{O}(X), -)$.

Every such representable presheaf is a sheaf, by the following

Proposition 2.4. *The FPQC topology is subcanonical.*

The category of sheaves of sets on Aff/S for the FPQC topology will be denoted by $\mathsf{Shv}_{\mathrm{fpqc}}$; this can be considered as a category of spaces (cf. [**9**, Chapitre 1]). The Yoneda embedding provides a fully faithful embedding

$$\mathsf{Aff}/S \hookrightarrow \mathsf{Shv}_{\mathrm{fpqc}},$$

by the above proposition; it also implies the following

Lemma 2.5. *A morphism of affine S-schemes $f\colon U \to V$ induces an isomorphism in $\mathsf{Shv}_{\mathrm{fpqc}}$ if and only if f is an isomorphism in Aff/S.*

Definition 2.6. Let $\mathscr{P}$ be a class of morphisms in Aff/S. (Such a class will always be taken implicitly to be closed under composition with isomorphisms.)

1. The class $\mathscr{P}$ is closed under base change if, for $p\colon X \to Y$ in $\mathscr{P}$ and $A \to Y$ a morphism in Aff/S, the morphism obtained by base change, $p'\colon A \times_Y X \to A$, is in $\mathscr{P}$.

2. The class $\mathscr{P}$ is local for the FPQC topology if $p\colon X \to Y$ lies in $\mathscr{P}$ if and only if there exists a faithfully flat morphism $Y' \to Y$ such that $p'\colon X \times_Y Y' \to Y'$ lies in $\mathscr{P}$.

Faithfully flat descent is a fundamental tool when considering the FPQC topology (see the notes on descent theory by Vistoli [**14**, Chapter 4], where the fundamental result is [**14**, Theorem 4.33]). However the arguments of this paper seek to be constructive, making all descent arguments explicit.

2.2. Affine groupoid schemes

An affine groupoid scheme is the structure in Aff/S which represents a presheaf with values in (small) groupoids. This corresponds to a pair of affine S-schemes (X_0, X_1), equipped with structure morphisms $s, t\colon X_1 \rightrightarrows X_0$, $m\colon X_1 \times_{X_0} X_1 \to X_1$, $\epsilon\colon X_0 \to X_1$ and $i\colon X_1 \to X_1$ which satisfy the usual axioms (cf. [**9**, Définition 2.4.3] and [**1**, Section III, §2]). Here X_0 represents the objects and X_1 represents the morphisms of the groupoid. The morphism $m\colon X_1 \times_{X_0} X_1 \to X_1$ corresponds to the composition, the fibre product being taken in the sense $X_1 \times_{t,X_0,s} X_1$. The relevant fibre product is usually clear from the context and, in this case, such precision is omitted from the notation.

Notation 2.7. An affine groupoid scheme as above will frequently be denoted simply by $X_\bullet$.

Proposition 2.8. *(Cf. [**11**]) Affine groupoid schemes form a 2-category, denoted* Gpd_S*, such that:*

1. *a* 1*-morphism* $f_\bullet\colon X_\bullet \to Y_\bullet$ *is a pair of morphisms* $(f_0\colon X_0 \to Y_0, f_1\colon X_1 \to Y_1)$ *which induce a morphism of groupoid-valued presheaves;*
2. *a* 2*-morphism between morphisms* $f_\bullet, g_\bullet\colon X_\bullet \to Y_\bullet$ *is a morphism* $\alpha\colon X_0 \to Y_1$ *such that* $s\alpha = f_0$*,* $t\alpha = g_0$ *and the following diagram is commutative:*

$$
\begin{array}{ccc}
X_1 & \xrightarrow{(g_1, \alpha s_X)} & Y_1 \times_{Y_0} Y_1 \\
{\scriptstyle (\alpha t_X, f_1)}\downarrow & & \downarrow{\scriptstyle m_Y} \\
Y_1 \times_{Y_0} Y_1 & \xrightarrow[m_Y]{} & Y_1.
\end{array}
$$

Finite coproducts and products exist in the category of affine groupoid schemes. Fibre products in the sense of 2-categories also exist; these are considered in Section 5.

Proposition 2.9. *The category of affine groupoid schemes has finite coproducts and products. Let* $X_\bullet$*,* $Y_\bullet$ *be affine groupoid schemes; then:*

1. *the product* $X_\bullet \times_S Y_\bullet$ *has underlying affine schemes* $(X_0 \times_S Y_0, X_1 \times_S Y_1)$*, with structure morphisms given by the product of the structure morphisms of* $X_\bullet$ *and* $Y_\bullet$*;*
2. *the coproduct* $X_\bullet \amalg Y_\bullet$ *has underlying affine schemes* $(X_0 \amalg Y_0, X_1 \amalg Y_1)$*, with structure morphisms given by the coproduct of the structure morphisms of* $X_\bullet$ *and* $Y_\bullet$*.*

Proof. The product (respectively the disjoint union) of two groupoids is defined by the usual product (resp. disjoint union) of categories. These constructions can be performed pointwise for a presheaf of small groupoids; in the representable case, the constructions are represented by the given structures. The only part of the argument which is not entirely formal is the identification of the fibre products given by the natural isomorphism

$$(X_1 \times_{X_0} X_1) \times (Y_1 \times_{Y_0} Y_1) \cong (X_1 \times Y_1) \times_{X_0 \times Y_0} (X_1 \times Y_1)$$

and the respective isomorphism in the category of R-algebras. □

The 2-category of affine groupoid schemes in Aff/S is equivalent to the opposite of the 2-category of R-Hopf algebroids. An R-Hopf algebroid is given by a pair of R-algebras (A, Γ) together with structure morphisms $\eta_L, \eta_R \colon A \to \Gamma$, $\epsilon \colon \Gamma \to A$, $\chi \colon \Gamma \to \Gamma$ and $\Delta \colon \Gamma \to \Gamma \otimes_A \Gamma$ (cf. [**13**, Appendix A1]). As above, the diagonal represents the composition of the groupoid and the tensor product $\Gamma \otimes_A \Gamma$ is taken in the sense $\Gamma \otimes_{\eta_R, A, \eta_L} \Gamma$. The 1-morphisms and 2-morphisms of the category of R-Hopf algebroids are defined as above (up to change of variance).

Definition 2.10.

1. An affine groupoid scheme $X_\bullet$ is flat if the morphism $s \colon X_1 \to X_0$ is flat (equivalently $t \colon X_1 \to X_0$ is flat).
2. An R-Hopf algebroid (A, Γ) is flat if the associated affine groupoid scheme is flat; this is equivalent to the morphism of R-algebras $\eta_L \colon A \to \Gamma$ being flat (equivalently $\eta_R \colon A \to \Gamma$ being flat).

The following lemma is clear.

Lemma 2.11. *There is a fully faithful embedding*

$$\mathsf{Aff}/S \hookrightarrow \mathsf{Gpd}/S$$

which associates to an affine S-scheme U the affine groupoid scheme (U, U) with all structure morphisms being the identity.

Notation 2.12. The affine groupoid scheme (U, U) associated to an affine S-scheme U will be denoted simply by U.

The following simple example arises naturally in considering the factorization of a morphism of affine schemes in Example 2.36 below. It is of fundamental importance in descent theory, as exhibited in Section 4.

Example 2.13. Let $f \colon U \to V$ be a morphism of Aff/S. There is an associated affine groupoid scheme with underlying affine schemes $(V, V \times_U V)$ and with structure morphisms given as follows. The morphisms s, t are given by the projections, ϵ is the diagonal morphism $V \to V \times_U V$, the inverse $i \colon V \times_U V \to V \times_U V$ is given by transposition of factors and the composition

$$m \colon (V \times_U V) \times_V (V \times_U V) \cong V \times_U V \times_U V \to V \times_U V$$

is induced by projection onto the outer factors. There are canonical morphisms of affine groupoid schemes

$$V \to (V, V \times_U V) \to U.$$

This affine groupoid scheme is flat if and only if the morphism $V \to U$ is flat.

2.3. Stacks and prestacks

One can consider prestacks and stacks for the FPQC topology on Aff/S. The standard reference for algebraic stacks is the monograph of Laumon and Moret-Bailly [**9**], and a useful introduction sufficient for the current purposes is the introductory text by Vistoli [**14**]; the reader is referred to these texts for further details.

Recall that a category $\mathscr{G}$ fibred in groupoids over Aff/S is a functor $\pi\colon \mathscr{G} \to \mathsf{Aff}/S$ which has fibres which are groupoids and which satisfies a base-change condition (see [**9**, Chapitre 2]). The fibre $\mathscr{G}_X$ over an affine S-scheme X is the inverse image of the discrete subcategory of Aff/S with single object X. The base-change condition provides, for each morphism $V \to U$ of Aff/S, a functor $\varphi^*\colon \mathscr{G}_U \to \mathscr{G}_V$.

Notation 2.14. Henceforth the only stacks (respectively prestacks) considered will be (pre)stacks over Aff/S for the FPQC topology; these will be referred to simply as (pre)stacks (over Aff/S).

Definition 2.15. A prestack over Aff/S is a category $\mathscr{G}$ fibred in groupoids over Aff/S such that, for any affine S-scheme U and objects x, y of $\mathscr{G}_U$, the presheaf on Aff/U which associates to a morphism $V \to U$ the set $\mathrm{Hom}_{\mathscr{G}_V}(\varphi^* x, \varphi^* y)$ is a sheaf for the FPQC topology.

Definition 2.16. Let $\mathscr{G}$ be a category fibred in groupoids over Aff/S.

1. A descent datum for $\mathscr{G}$ is a triple $(p\colon U' \to U, x', \psi')$, where $p\colon U' \to U$ is a faithfully flat morphism in Aff/S, x' is an object of $\mathscr{G}_{U'}$ and ψ' is an isomorphism $\psi\colon p_1^* x' \xrightarrow{\cong} p_2^* x'$ in $\mathscr{G}_{U' \times_U U'}$, where $p_1, p_2\colon U' \times_U U' \rightrightarrows U'$ are the projections, such that the cocycle condition holds in $\mathscr{G}_{U' \times_U U' \times_U U'}$:
$$p_{1,3}^* \psi' = (p_{2,3}^* \psi') \circ (p_{1,2}^* \psi')$$
where $p_{i,j}\colon U' \times_U U' \times_U U' \to U' \times_U U'$ is the projection onto the ith and jth factors.
2. The descent datum $(p\colon U' \to U, x', \psi')$ is effective if there exists an object $x \in \mathscr{G}_U$ and an isomorphism $\gamma\colon p^* x \xrightarrow{\cong} x'$ which is compatible with the descent datum
$$p_2^* \gamma = \psi' \circ p_1^* \gamma$$
(observing that $p \circ p_1 = p \circ p_2$).

Definition 2.17. A stack over Aff/S is a prestack for which every descent datum is effective.

Remark 2.18. Stacks and prestacks over Aff/S form 2-categories; the category of morphisms is written as **Hom**.

Definition 2.19. [**9**, Section 2.2] A 1-morphism $g\colon \mathscr{G} \to \mathscr{H}$ of categories fibred in groupoids over Aff/S is

1. a monomorphism if the functor on fibres $\mathscr{G}_X \to \mathscr{H}_X$ is fully faithful for each affine S-scheme X;
2. an equivalence if the functor on fibres $\mathscr{G}_X \to \mathscr{H}_X$ is an equivalence of categories for each affine S-scheme X.

These definitions apply to the 2-category of prestacks over Aff/S, by considering the underlying category fibred in groupoids.

Definition 2.20. A 1-morphism $f\colon \mathscr{M} \to \mathscr{N}$ of stacks over Aff/S is a monomorphism if the underlying 1-morphism of categories fibred in groupoids is a monomorphism.

The notion of 1-epimorphism between stacks is local for the FPQC topology:

Definition 2.21. A 1-morphism $f\colon \mathscr{M} \to \mathscr{N}$ of stacks over Aff/S is an epimorphism if, for each section $z \in \mathscr{N}_X$, X an affine S-scheme, there exists a faithfully flat morphism $\varphi\colon X' \to X$ and a section $y \in \mathscr{M}_{X'}$ such that $f(y)$ is isomorphic to $\varphi^* z$ in the groupoid $\mathscr{M}_{X'}$.

Definition 2.22. Let $\mathscr{X}$ be a prestack. An associated stack for $\mathscr{X}$ is a 1-morphism $\iota\colon \mathscr{X} \to \tilde{\mathscr{X}}$ of prestacks such that, for every stack $\mathscr{Y}$, the functor

$$\mathbf{Hom}(\tilde{\mathscr{X}}, \mathscr{Y}) \xrightarrow{\iota^*} \mathbf{Hom}(\mathscr{X}, \mathscr{Y})$$

is an equivalence of categories.

An explicit formulation of the existence of the associated stack is given in Section 4. Existence and uniqueness is stated in the following result.

Proposition 2.23. *(Cf. [**9**, Lemme 3.2]) Let $\mathscr{X}$ be a prestack.*

1. *There exists an associated stack $\iota\colon \mathscr{X} \to \tilde{\mathscr{X}}$.*
2. *The 1-morphism ι is a monomorphism of prestacks.*
3. *If $\mathscr{X}$ is a stack, then ι is an equivalence of stacks.*
4. *If $\iota'\colon \mathscr{X} \to \tilde{\mathscr{X}}'$ is a second associated stack, then the stacks $\tilde{\mathscr{X}}$ and $\tilde{\mathscr{X}}'$ are equivalent.*

A thorough exposition of the construction of an associated stack is given in the monograph of Giraud on non-abelian cohomology, [**2**, Chapitre II, Section 2]. This reference treats the more general case of the stack associated to a category fibred in groupoids and also is more explicit on the functorial nature of the construction.

Example 2.24. An affine S-scheme X represents a stack on Aff/S which will be denoted abusively by X. The underlying category fibred in groupoids has fibre X_V, for V an affine S-scheme, the discrete category with objects the set of sections $\mathrm{Hom}_{\mathsf{Aff}/S}(V, X)$ of the sheaf associated to X.

Example 2.25. An affine groupoid scheme $X_\bullet$ defines a prestack $[X_\bullet]'$ on Aff/S which has underlying category fibred in groupoids with fibre $[X_\bullet]'_V$ (V an affine S-scheme) the groupoid $\mathrm{Hom}_{\mathsf{Aff}/S}(V, X_\bullet)$. This is a prestack because the FPQC topology is subcanonical.

The associated stack construction given in Section 4 provides a 1-monomorphism $[X_\bullet]' \hookrightarrow [X_\bullet]$ and $[X_\bullet]$ is referred to as the stack associated to the affine groupoid scheme $X_\bullet$.

There is a canonical 1-epimorphism of stacks $X_0 \twoheadrightarrow [X_\bullet]$. Moreover, there is a 2-commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{t} & X_0 \\ {\scriptstyle s}\downarrow & & \downarrow \\ X_0 & \twoheadrightarrow & [X_\bullet]. \end{array}$$

Example 2.26. For (A, Γ) an R-Hopf algebroid, the above constructions give rise to a stack $\mathscr{M}_{(A,\Gamma)}$, equipped with a 1-epimorphism $\mathrm{Spec}(A) \twoheadrightarrow \mathscr{M}_{(A,\Gamma)}$. If $(A, \Gamma) \to (B, \Sigma)$ is a morphism of R-Hopf algebroids, there is a (strictly) commutative diagram of 1-morphisms of stacks

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathscr{M}_{(B,\Sigma)} & \longrightarrow & \mathscr{M}_{(A,\Gamma)}. \end{array}$$

The fact that the diagram can be taken to be strictly commutative depends on the functorial nature of the associated stack construction.

Remark 2.27. The stacks arising from affine groupoid schemes have two fundamental properties (as will be explained below): the diagonal morphism $\Delta\colon \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is affine and there is a canonical presentation $X \twoheadrightarrow \mathscr{X}$ which is a 1-epimorphism from the stack represented by an affine scheme X.

It should be noted that such stacks are not in general algebraic in the usual sense of algebraic geometry. In particular, no finiteness or smoothness hypothesis is placed on the presentation $X \twoheadrightarrow \mathscr{X}$ of the stack. Moreover, algebraic spaces do not intervene explicitly in the theory developed here: the only algebraic spaces considered are those represented by affine schemes.

Morphisms from a stack to an affine scheme are understood by the following result.

Lemma 2.28. *Let $\mathscr{M}$ be a prestack and $\mathscr{M} \to \tilde{\mathscr{M}}$ be an associated stack. For an affine S-scheme U, the categories $\mathbf{Hom}(\tilde{\mathscr{M}}, U)$ and $\mathbf{Hom}(\mathscr{M}, U)$ are discrete and there is a bijection of the underlying sets*

$$\mathbf{Hom}(\tilde{\mathscr{M}}, U) \xrightarrow{\cong} \mathbf{Hom}(\mathscr{M}, U)$$

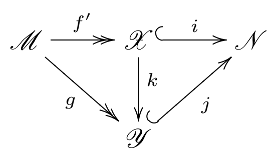
induced by $\mathscr{M} \to \tilde{\mathscr{M}}$.

2.4. The epi-mono factorization

A 1-morphism of stacks over Aff/S factorizes canonically as a 1-epimorphism followed by a 1-monomorphism by the following result. If $f\colon \mathscr{M} \to \mathscr{N}$ is a 1-morphism of stacks, an object z of $\mathscr{N}_U$ is said to be locally in the essential image of f if there exists a faithfully flat morphism $\varphi\colon U' \to U$ in Aff/S, an object y of $\mathscr{M}_{U'}$ and an isomorphism between $f(y)$ and $\varphi^*(z)$ in $\mathscr{N}_{U'}$.

Proposition 2.29. *[**9**, Proposition 3.7] Let $f\colon \mathscr{M} \to \mathscr{N}$ be a 1-morphism of stacks over Aff/S.*

1. *The full subcategory $\mathscr{X}$ of $\mathscr{N}$ of objects which are locally in the essential image of f is a substack of $\mathscr{N}$.*
2. *The morphism f factorizes canonically as*
$$\mathscr{M} \stackrel{f'}{\twoheadrightarrow} \mathscr{X} \stackrel{i}{\hookrightarrow} \mathscr{N}$$
where f' is an epimorphism and i is the canonical monomorphism of stacks.
3. *If $\mathscr{M} \stackrel{g}{\twoheadrightarrow} \mathscr{Y} \stackrel{j}{\hookrightarrow} \mathscr{N}$ is a factorization of f as an epimorphism followed by a monomorphism, then there exists a 2-commutative diagram*



in which k is an equivalence of stacks.

This implies the following fundamental result.

Corollary 2.30. *[**9**, Corollaire 3.7.1] A 1-morphism of stacks is an equivalence if and only if it is a monomorphism and an epimorphism of stacks.*

2.5. Groupoid spaces and induced groupoids

The Yoneda embedding of Aff/S in $\mathsf{Shv}_{\mathrm{fpqc}}$ allows the definition of an affine groupoid scheme to be generalized to a groupoid space as follows.

Definition 2.31. A groupoid space is a presheaf on Aff/S with values in small groupoids such that the pair of presheaves (Y_0, Y_1) corresponding to the objects and morphisms respectively belong to $\mathsf{Shv}_{\mathrm{fpqc}}$.

Remark 2.32.

1. A groupoid space is defined by the pair of sheaves (Y_0, Y_1) together with the usual structure morphisms. An affine groupoid scheme is a representable groupoid space.
2. A groupoid space $Y_\bullet$ induces a prestack $[Y_\bullet]'$ and an associated stack $[Y_\bullet]$, as in the representable case.

Lemma 2.33. *Let $X_\bullet$ be an affine groupoid scheme and $Y_\bullet$ be a groupoid space. A pair of morphisms of sheaves $f_0\colon X_0 \to Y_0$, $f_1\colon X_1 \to Y_1$ induces a morphism of affine groupoid spaces if and only if the following identities hold*

1. *$s_Y f_1 = f_0 s_X$ and $t_Y f_1 = f_0 t_X$ as morphisms $X_1 \to Y_0$;*
2. *$m_Y(f_1 \times f_1) = f_1 m_X$ as morphisms $X_1 \times_{X_0} X_1 \to Y_1$.*

Proof. The forward implication is immediate. For the converse, suppose that the morphisms satisfy the given conditions. It is necessary to establish the identities

1. $f_1\epsilon_X = \epsilon_Y f_0$, as morphisms $X_0 \to Y_1$;
2. $f_1 i_X = i_Y f_1$ as morphisms $X_1 \to Y_1$.

These conditions correspond to the condition on the unit and on the inverse respectively. By Yoneda's lemma, the first condition is a property of the groupoid $[Y_\bullet]'_{X_0}$ and the second condition is a property of the groupoid $[Y_\bullet]'_{X_1}$.

This reduces the proof to a property of morphisms of (small) groupoids: a map between the sets of morphisms of two groupoids defines a morphism of groupoids if and only if it induces a map on the set of objects which is compatible with the source and target of morphisms and the map commutes with composition of morphisms. □

Discrete stacks are considered in Section 6 (the definition of discrete is given in Definition 6.1). The following result can be proved using the methods of Section 6; alternatively, a direct approach using the definition of the 2-fibre product in stacks can be used.

Lemma 2.34. *Let $Y_1 \to \mathscr{M} \leftarrow Y_2$ be a diagram of 1-morphisms, where Y_1, Y_2 are stacks associated to sheaves. The 2-fibre product $Y_1 \times_{\mathscr{M}} Y_2$ is a stack associated to a sheaf.*

The following general result gives a way of defining an induced groupoid space.

Proposition 2.35. *[**9**, Proposition 3.8] Let $\mathscr{Y}$ be a stack and $f\colon Y \to \mathscr{Y}$ be a 1-morphism of stacks, where Y is the stack associated to a sheaf.*

1. *The pair $(Y_0 := Y, Y_1 := Y \times_{\mathscr{Y}} Y)$ has the structure of a groupoid space, where the morphisms $s, t\colon Y \times_{\mathscr{Y}} Y \rightrightarrows Y$ are given by the canonical projections, the morphism $i\colon Y \times_{\mathscr{Y}} Y \to Y \times_{\mathscr{Y}} Y$ is induced by transposition of the factors Y and the morphism $\epsilon\colon Y \to Y \times_{\mathscr{Y}} Y$ is induced by the diagonal. The composition*
$$Y_1 \times_{Y_0} Y_1 \cong (Y \times_{\mathscr{Y}} Y) \times_Y (Y \times_{\mathscr{Y}} Y) \to Y \times_{\mathscr{Y}} Y$$
is induced by the projection on the two outer factors and the composition of morphisms in $\mathscr{Y}$.
2. *There is a canonical 1-monomorphism of stacks $\Phi\colon [Y_\bullet] \hookrightarrow \mathscr{Y}$ and a factorization of f as*
$$Y \twoheadrightarrow [Y_\bullet] \overset{\Phi}{\hookrightarrow} \mathscr{Y}.$$
3. *The morphism Φ is an equivalence of stacks if and only if f is an epimorphism.*

Example 2.36. A morphism of affine S-schemes $V \to U$ induces a morphism between representable stacks. The previous construction defines a factorization
$$V \twoheadrightarrow [V_\bullet] \hookrightarrow U$$
where $V_\bullet$ is the groupoid space represented by the affine groupoid scheme $(V, V \times_U V)$, as in Example 2.13.

The stack $[V_\bullet]$ is equivalent to the affine stack U if and only if the morphism of representable sheaves $V \to U$ is surjective for the FPQC topology. The latter condition is equivalent to the existence of a morphism $U' \to V$ of Aff/S such that the composite morphism $U' \to V \to U$ is faithfully flat. This example is fundamental in the theory of faithfully flat descent.

3. Definition of affine morphisms

The notion of an affine morphism between FPQC stacks is fundamental. In particular, any property of morphisms of Aff/S which is local for the FPQC topology and stable by base change gives rise to a property of affine morphisms between FPQC stacks.

Moreover, stacks with affine diagonal morphism have good properties; for example, there is a standard way to associate a (small) FPQC site to a stack with affine diagonal (see [**8, 12**]). An affine morphism between FPQC stacks (with affine diagonal) induces a continuous morphism between the associated small FPQC sites.

3.1. Affine morphisms of stacks

The notion of an affine morphism is important since the fundamental geometric objects which are considered here are affine S-schemes.

Definition 3.1.

1. A 1-morphism $f\colon \mathscr{M} \to \mathscr{N}$ is affine if, for any 1-morphism $U \to \mathscr{N}$ with U represented by an affine scheme, the 2-fibre product $\mathscr{M} \times_{\mathscr{N}} U$ is isomorphic to a stack represented by an affine scheme.
2. A stack $\mathscr{M}$ is affine (or an affine stack) if the canonical morphism $\mathscr{M} \to S$ is affine (so that $\mathscr{M}$ is equivalent to the stack represented by an affine scheme).

Remark 3.2. It should be stressed that if $\mathscr{N} = \tilde{\mathscr{H}}$ for a prestack $\mathscr{H}$, the section represented by $U \to \tilde{\mathscr{H}}$ need not correspond to a 1-morphism of prestacks $U \to \mathscr{H}$. This is where descent data enter the picture (see Section 4).

Remark 3.3. The notion of affine morphism is analogous to that of *morphisme schématique* used in the theory of algebraic stacks, which is a special case of the standard notion of a representable morphism. The definition of a representable morphism is similar to that of an affine morphism, but replacing affine scheme by algebraic space. Here, no use is made of algebraic spaces (which are the basic geometric objects in the study of algebraic stacks, defined with respect to the étale topology) and the only schemes which are considered are affine.

The following result is analogous to general results on representable morphisms of stacks (cf. [**9**, Lemme 3.11] and [**9**, Lemme 3.12]).

Lemma 3.4.

1. *Let* $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$ *be* 1*-morphisms of stacks. If* f, g *are affine, then* $g \circ f$ *is affine.*
2. *Let* $\mathscr{X} \xrightarrow{p} \mathscr{Z} \xleftarrow{q} \mathscr{Y}$ *be* 1*-morphisms of stacks and*

$$\begin{array}{ccc} \mathscr{X} \times_{\mathscr{Z}} \mathscr{Y} & \longrightarrow & \mathscr{X} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \mathscr{Y} & \xrightarrow[q]{} & \mathscr{Z} \end{array}$$

be the associated 2*-cartesian diagram. If* p *is affine, then the morphism* p' *is affine.*

Definition 3.5. Let $\mathscr{P}$ be a class of morphisms of Aff/S which is local for the FPQC topology and stable by base change. An affine 1-morphism of stacks $f\colon \mathscr{M} \to \mathscr{N}$ has property $\mathscr{P}$ if, for each 1-morphism of the form $V \to \mathscr{N}$, with V represented by an affine scheme, the 1-morphism $\mathscr{M} \times_{\mathscr{N}} V \to V$ defined by the 2-fibre product has property $\mathscr{P}$.

The following descent property for affine morphisms of stacks is useful and gives a partial converse to the second property of Lemma 3.4.

Proposition 3.6. *[**12**, Proposition 3.16] Let $\mathscr{X} \overset{p}{\twoheadrightarrow} \mathscr{Z} \overset{f}{\leftarrow} \mathscr{Y}$ be a diagram of 1-morphisms of stacks, where p is an epimorphism, and let*

$$\begin{array}{ccc} \mathscr{X} \times_{\mathscr{Z}} \mathscr{Y} & \xrightarrow{f'} & \mathscr{X} \\ \downarrow & & \downarrow p \\ \mathscr{Y} & \xrightarrow[f]{} & \mathscr{Z} \end{array}$$

be the associated 2-cartesian diagram.

1. *If f' is affine, then f is affine.*
2. *If f' is affine and possesses the stable and local property $\mathscr{P}$, then f is affine and possesses the stable and local property $\mathscr{P}$.*

3.2. Stacks with affine diagonal

The stack associated to the affine scheme S is terminal in stacks over Aff/S. In particular, if $\mathscr{M}$ is a stack, then there is a canonical 1-morphism $\mathscr{M} \to S$ of stacks and hence a diagonal morphism

$$\Delta_{\mathscr{M}}\colon \mathscr{M} \to \mathscr{M} \times_S \mathscr{M}.$$

The condition that the diagonal morphism of a stack be affine is a fundamental property.

Proposition 3.7. *(Cf. [**12**, Proposition 3.14]) Let $\mathscr{M}$ be a stack over Aff/S. The stack $\mathscr{M}$ has affine diagonal if and only if each 1-morphism of the form $U \to \mathscr{M}$, with U an affine stack, is affine.*

Proof. Let $U \to \mathscr{M}$, $V \to \mathscr{M}$ be 1-morphisms of stacks with U, V affine. There is a product 1-morphism of stacks

$$U \times V \to \mathscr{M} \times_S \mathscr{M}$$

and the stacks $(U \times V)_{\mathscr{M} \times_S \mathscr{M}} \mathscr{M}$ and $U \times_{\mathscr{M}} V$ are isomorphic. Hence, if the diagonal is affine, then $U \times_{\mathscr{M}} V$ is affine.

Conversely, suppose that each 1-morphism $U \to \mathscr{M}$ is affine when U is affine; this implies that $U \times_{\mathscr{M}} U$ is affine. Consider a 1-morphism of stacks $U \to \mathscr{M} \times_S \mathscr{M}$; this morphism factorizes canonically as $U \to U \times U \to \mathscr{M} \times_S \mathscr{M}$, where the second morphism is the product of the two projections. The 2-fibre product $\mathscr{M} \times_{\mathscr{M} \times_S \mathscr{M}} U$

is equivalent to the 2-fibre product

$$\begin{array}{ccc} U \times_{U\times U} ((U \times U) \times_{\mathscr{M}\times_S \mathscr{M}} \mathscr{M}) & \longrightarrow & U \\ \downarrow & & \downarrow \\ (U \times U) \times_{\mathscr{M}\times_S \mathscr{M}} \mathscr{M} & \longrightarrow & U \times U. \end{array}$$

The stack $(U \times U) \times_{\mathscr{M}\times_S \mathscr{M}} \mathscr{M}$ is equivalent to $U \times_{\mathscr{M}} U$ as above, hence is affine, by hypothesis. Hence the fibre product above is affine, as required. □

4. Descent and the associated stack

The definition of a stack is phrased in terms of descent data, as recalled in Definition 2.17, and the standard construction of the stack associated to a prestack, as sketched in [**9**, Lemme 3.2], uses descent data. In the interest of making the arguments explicit at the level of affine groupoid schemes, a descent datum for an affine groupoid scheme is interpreted in this section as a morphism of groupoid schemes. This gives an explicit construction of the stack associated to an affine groupoid scheme.

4.1. Explicit construction of the associated stack

Let $X_\bullet$ denote an affine groupoid scheme in Aff/S. Tautologically, $X_\bullet$ defines a presheaf of groupoids on Aff/S, which is a prestack since the FPQC topology is subcanonical; this prestack is denoted $[X_\bullet]'$.

The key step in constructing an associated stack for $X_\bullet$ is the definition of the fibre category $[X_\bullet]_U$ over an affine S-scheme U; this is provided by the following lemma.

Lemma 4.1. *Let U be an affine S-scheme and $X_\bullet$ be an affine groupoid scheme over S. There is a category $[X_\bullet]_U$ with objects pairs $(p' \colon U' \to U, \Psi')$, where p' is a faithfully flat morphism in Aff/S and Ψ' is a 1-morphism of affine groupoid schemes*

$$(U', U' \times_U U') \to X_\bullet .$$

A morphism between objects (p', Ψ') and (p'', Ψ'') is a morphism $f \colon U' \times_U U'' \to X_1$ which makes the following diagrams commute:

1.

$$\begin{array}{ccccc} U' & \xleftarrow{p_1} & U' \times_U U'' & \xrightarrow{p_2} & U'' \\ \downarrow & & \downarrow f & & \downarrow \\ X_0 & \xleftarrow[s]{} & X_1 & \xrightarrow[t]{} & X_0, \end{array}$$

in which p_1 and p_2 denote the projections;

2.

$$\begin{array}{ccc} U' \times_U U'' & \xleftarrow{\pi_l} & (U' \times_U U') \times_{U'} (U' \times_U U'') \\ f \downarrow & & \downarrow \Psi' \times f \\ X_1 & \xleftarrow[m]{} & X_1 \times_{X_0} X_1 \end{array}$$

where π_l is induced by the projection onto the outer factors;

3.

$$\begin{array}{ccc} (U' \times {}_U U'') \times_{U''} (U'' \times {}_U U'') & \xrightarrow{\pi_r} & U' \times {}_U U'' \\ \downarrow{\scriptstyle f\times\Psi''} & & \downarrow{\scriptstyle f} \\ X_1 \times_{X_0} X_1 & \xrightarrow[m]{} & X_1 \end{array}$$

where π_r is induced by the projection onto the outer factors.

Remark 4.2. Some remarks are in order upon the definition implicit in the statement of Lemma 4.1.

1. The definition depends only upon the small FPQC site of U.
2. The condition (1) states that f is a morphism in the groupoid $[X_\bullet]'_{U'\times_U U''}$ between the pullbacks $p_1^*(p', \Psi')$ and $p_2^*(p'', \Psi'')$.
3. The conditions (2), (3) give compatibility with the morphisms associated to the objects (p', Ψ') and (p'', Ψ''). There is an alternative way to view these diagrams, in terms of groupoid actions.
   The affine groupoid scheme $(U', U' \times {}_U U')$ acts on the left upon $U' \times_U U''$; the structure morphism is analogous to the definition of the structure of the groupoid $(U', U' \times {}_U U')$ given in Example 2.13, namely
   $$(U' \times {}_U U') \times_{U'} (U' \times {}_U U'') \to (U' \times {}_U U''),$$
   the morphism induced by the projection onto the two outer factors. Similarly the affine groupoid scheme $(U'', U'' \times {}_U U'')$ acts upon the right upon $U' \times_U U''$; moreover, these left and right actions commute.
   Similarly, the morphism Ψ' induces a left action of $(U', U' \times {}_U U')$ upon X_1 and the morphism Ψ'' induces a right action of $(U'', U'' \times {}_U U'')$ upon X_1, and these two actions commute. The compatibility conditions are equivalent to the condition that $f\colon U' \times {}_U U'' \to X_1$ is a morphism of left $U' \times {}_U U'$ and right $U'' \times {}_U U''$ groupoid actions.

Proof. It is necessary to show that there is a well-defined composition operation, which defines the structure of a category. For this, let

$$\Psi_j\colon (U_j, U_j \times_U U_j) \to X_\bullet$$

be morphisms of affine groupoid schemes, for $j \in \{1, 2, 3\}$, where $U_j \to U$ are faithfully flat morphisms. Suppose further that $f_{12}\colon U_1 \times_U U_2 \to X_1$ and $f_{23}\colon U_2 \times_U U_3 \to X_1$ satisfy the compatibility conditions of the statement of the lemma with respect to (Ψ_1, Ψ_2) and (Ψ_2, Ψ_3) respectively. The composite should be a morphism $U_1 \times_U U_3 \to X_1$. The diagram

$$\begin{array}{ccc} (U_1 \times_U U_2) \times_{U_2} (U_2 \times_U U_3) & \xrightarrow{\cong} & U_1 \times_U U_2 \times_U U_3 \\ \downarrow{\scriptstyle f_{12}\times f_{23}} & & \downarrow{\scriptstyle F} \\ X_1 \times_{X_0} X_1 & \xrightarrow[m]{} & X_1 \end{array}$$

defines a morphism $F\colon U_1 \times_U U_2 \times_U U_3 \to X_1$. One uses faithfully flat descent to show that this descends to the required morphism; this uses the elementary fact that the presheaf represented by X_1 is a sheaf for the FPQC topology, since the topology is subcanonical. Explicitly, it is sufficient to show that the two composites in the diagram

$$U_1 \times_U U_2 \times_U U_2 \times_U U_3 \underset{q_2}{\overset{q_1}{\rightrightarrows}} U_1 \times_U U_2 \times_U U_3 \xrightarrow{F} X_1$$

coincide, where the morphisms q_1, q_2 are induced by the projections

$$U_2 \times_U U_2 \rightrightarrows U_2.$$

This fact follows from the compatibility conditions for f_{12} and f_{23} by using the following diagram:

$$\begin{array}{ccc} (U_1 \times_U U_2) \times_{U_2} (U_2 \times_U U_2) \times_{U_2} (U_2 \times_U U_3) & \xrightarrow{\cong} & U_1 \times_U U_2 \times_U U_2 \times_U U_3 \\ \downarrow{\scriptstyle f_{12} \times \Psi_2 \times f_{23}} & & \downarrow\downarrow{\scriptstyle F\circ q_2,\ F\circ q_1} \\ X_1 \times_{X_0} X_1 \times_{X_0} X_1 & \xrightarrow[m^{\circ 2}]{} & X_1, \end{array}$$

where $m^{\circ 2}$ corresponds to the iterated product. The diagram commutes in the sense that all composites are the same; this follows by associativity of the product m together with the fact that f_{12} satisfies condition (3) and f_{23} satisfies condition (2) of Lemma 4.1, these conditions corresponding to the respective right and left groupoid actions by $(U_2, U_2 \times_U U_2)$. This implies that the morphism F factorizes canonically across the required morphism

$$f_{23} \circ f_{12} \colon U_1 \times_U U_3 \to X_1.$$

From the construction, it can be shown that this satisfies the required compatibility conditions, hence the composition is defined.

It remains to show that the composition satisfies the axioms for a category. This is straightforward, upon noting that the identity morphism for an object represented by a morphism of affine groupoid schemes $(U', U' \times_U U') \to X_\bullet$ is the underlying morphism $U' \times_U U' \to X_1$. □

Lemma 4.3. *For $X_\bullet$ an affine groupoid scheme in Aff/S, there is an associated category fibred in groupoids $[X_\bullet]$ with*

1. *fibre category over an object U of Aff/S the category $[X_\bullet]_U$ of Lemma 4.1;*
2. *base change for a morphism $\varphi\colon V \to U$ of affine S-schemes the functor $\varphi^*\colon [X_\bullet]_U \to [X_\bullet]_V$ induced by the base change from the small FPQC site of U to the small FPQC site of V.*

Proof. The result is standard. Namely, if $U' \to U$ is a faithfully flat morphism, then base change along $\varphi\colon V \to U$ gives a faithfully flat morphism $V' := U' \times_U V \to V$. If $U'' \to U$ is a second faithfully flat morphism and $V'' \to V$ the morphism obtained by base change, then there is an induced morphism $V' \times_V V'' \to U' \times_U U''$, which is compatible with the projections. In particular, these observations are sufficient to define a base change functor $\varphi^*\colon [X_\bullet]_U \to [X_\bullet]_V$; the fact that this gives rise to a category fibred in groupoids follows from general considerations, as in [**14**, Chapter 3]. □

Recall that $X_\bullet$ defines a presheaf of groupoids $[X_\bullet]'$, which is a prestack since the FPQC topology is subcanonical. The relationship between the cocycle condition given in the definition of a descent datum and morphisms of affine groupoid schemes is given by the following lemma.

Lemma 4.4. *Let $p' \colon U' \to U$ be a faithfully flat morphism, x' be an object of $[X_\bullet]'_{U'}$ represented by a morphism $x' \colon U' \to X_0$, and $\psi' \colon U' \times_U U' \to X_1$ represent a morphism between $p_1^* x'$ and $p_2^* x'$, where $p_1, p_2 \colon U' \times_U U' \rightrightarrows U'$ are the projections. Then ψ' satisfies the cocycle condition if and only if the morphisms (x', ψ') induce a morphism of affine groupoid schemes*

$$(U', U' \times_U U') \to X_\bullet.$$

Proof. Consider the forward implication; by hypothesis the morphisms x' and ψ' are compatible with the source and target morphisms of the affine groupoid scheme $X_\bullet$. The cocycle condition implies that the morphism ψ' is compatible with composition. This is sufficient to show that x', ψ' induce a morphism of affine groupoid schemes, by Lemma 2.33.

Conversely, the compatibility with composition implies the cocycle condition. □

The following lemma introduces notation used below and is important in transitivity arguments involving descent data.

Lemma 4.5. *Let $U'' \to U' \to U$ be morphisms in Aff/S (not necessarily faithfully flat). The canonical morphism induced by the universal property of fibre products*

$$\iota \colon U'' \times_{U'} U'' \to U'' \times_U U''$$

induces a morphism of affine groupoid schemes which factorizes canonically as

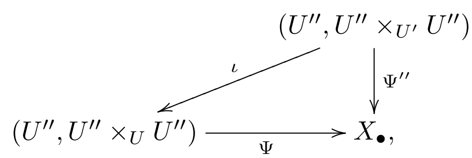
$$(U'', U'' \times_{U'} U'') \to (U'' \times_U U', (U'' \times_U U') \times_{U'} (U'' \times_U U')) \to (U'', U'' \times_U U'')$$

in which the second morphism is induced by pullback along $U' \to U$.

Proof. Straightforward. □

The key step in the proof that $[X_\bullet]$ is a stack is isolated in the following lemma.

Lemma 4.6. *Let $X_\bullet$ be an affine groupoid scheme in Aff/S. A descent datum for $X_\bullet$ of the form $(p' \colon U' \to U, x' \in [X_\bullet]_{U'}, \psi')$, where x' is defined by a morphism of affine groupoid schemes $\Psi'' \colon (U'', U'' \times_{U'} U'') \to X_\bullet$, where $q \colon U'' \to U'$ is a faithfully flat morphism, is equivalent to a commutative diagram of morphisms of affine groupoid schemes*



(1)

associated to faithfully flat morphisms $U'' \xrightarrow{q} U' \xrightarrow{p'} U$.

Proof. Suppose given a descent datum as in the statement of the lemma; by the definition of the category $[X_\bullet]$, the object x' is defined by a morphism of affine groupoid schemes $\Psi'' \colon (U'', U'' \times_{U'} U'') \to X_\bullet$, where $q \colon U'' \to U'$ is a faithfully flat morphism. The morphism ψ' is a morphism between the restrictions $p_1^* x'$, $p_2^* x'$ of x' to $[X_\bullet]_{U' \times_U U'}$ along the projections $p_1, p_2 \colon U' \times_U U' \rightrightarrows U'$, such that ψ' satisfies the cocycle condition.

The restriction of x' along the projection $p_1 \colon U' \times_U U' \to U'$ is defined as in Lemma 4.3, using the pullback

$$\begin{array}{ccc} U'' \times_U U' & \xrightarrow{\tilde{p}_1} & U'' \\ \downarrow & & \downarrow q \\ U' \times_U U' & \xrightarrow[p_1]{} & U'. \end{array}$$

Thus, the underlying object in $[X_\bullet]_{U'' \times_U U'}$ of $p_1^* x'$ is defined by the composite

$$U'' \times_U U' \xrightarrow{\tilde{p}_1} U'' \xrightarrow{\Psi''} X_0.$$

Similarly, the restriction of x' along p_2 is defined using the pullback

$$\begin{array}{ccc} U' \times_U U'' & \xrightarrow{\tilde{p}_2} & U'' \\ \downarrow & & \downarrow q \\ U' \times_U U' & \xrightarrow[p_2]{} & U'. \end{array}$$

It follows from the definition of morphisms in the category $[X_\bullet]_{U' \times_U U'}$ that ψ' is defined by a morphism

$$(U'' \times_U U') \times_{U' \times_U U'} (U' \times_U U'') \to X_1$$

which satisfies the conditions of Lemma 4.1. The fibre product on the left is canonically isomorphic to $U'' \times_U U''$, hence this gives a morphism

$$\psi \colon U'' \times_U U'' \to X_1.$$

The cocycle condition for ψ' implies that ψ defines a descent datum with respect to the cover $U'' \to U$, hence by Lemma 4.4 defines an object x of $[X_\bullet]_U$, which is represented by a morphism

$$\Psi \colon (U'', U'' \times_U U'') \to X_\bullet$$

of affine groupoid schemes. It remains to establish that Ψ'' factorizes across Ψ via the morphism ι.

The morphism ψ is defined by the morphism ψ' in the category $[X_\bullet]_{U' \times_U U'}$, hence satisfies the condition (3) of Lemma 4.1, which corresponds to a commutative diagram

$$\begin{array}{ccc} (U'' \times_U U'') \times_{U''} (U'' \times_{U'} U'') & \xrightarrow{P} & U'' \times_U U'' \\ \downarrow \psi \times \Psi'' & & \downarrow \psi \\ X_1 \times_{X_0} X_1 & \xrightarrow[m]{} & X_1 \end{array} \qquad (2)$$

in which P is induced by projection on the outer factors.

The projection $p_1\colon U'' \times_{U'} U'' \to U''$ and the diagonal $U'' \stackrel{\mathrm{diag}}{\to} U'' \times_U U''$ induce a morphism $\delta\colon U'' \times_{U'} U'' \to U'' \times {}_U U''$ and hence a morphism

$$\sigma = \delta \times 1\colon U'' \times_{U'} U'' \to (U'' \times {}_U U'') \times_{U''} (U'' \times_{U'} U'')$$

such that the composite $P \circ \sigma$ is the morphism $\iota\colon U'' \times_{U'} U'' \to U'' \times {}_U U''$.

The composite

$$U'' \stackrel{\mathrm{diag}}{\to} U'' \times {}_U U'' \stackrel{\psi}{\to} X_1$$

coincides with the composite $U'' \stackrel{\Psi''}{\to} X_0 \stackrel{\epsilon}{\to} X_1$, since Ψ is a morphism of affine groupoid schemes. Composing the diagram (2) with the morphism σ, one concludes that the composite $\psi \circ \iota$ is equal to the morphism Ψ''. This establishes the commutativity of the diagram (1) of morphisms of affine groupoid schemes.

Conversely, given a commutative diagram (1) of morphisms of affine groupoid schemes, the morphism Ψ defines the required descent morphism ψ' for the object represented by Ψ''. □

Remark 4.7. It is important to note that the diagram 1 of Lemma 4.6 does *not* correspond to the identification of the object x' with the pullback $(p')^*x$, along the morphism $p'\colon U' \to U$, of the object x represented by Ψ. This is the point of the factorization which is provided by Lemma 4.5.

Proposition 4.8. *Let $X_\bullet$ be an affine groupoid scheme in Aff/S. Then the category fibred in groupoids $[X_\bullet]$ over Aff/S is a stack and there is a canonical* 1-*monomorphism of prestacks:*

$$[X_\bullet]' \hookrightarrow [X_\bullet].$$

Proof. The fact that $[X_\bullet]$ is a prestack and that $[X_\bullet]' \hookrightarrow [X_\bullet]$ is a fully faithful functor (and hence a monomorphism) is a straightforward verification. For example, to show that $[X_\bullet]$ is a prestack, observe that the set of morphisms between two objects $(p'\colon U' \to U, \Psi')$ and $(p''\colon U'' \to U, \Psi'')$ is defined as an equalizer of a diagram of the form

$$\mathrm{Hom}(U' \times {}_U U'', X_1) \rightrightarrows \mathrm{Hom}(U' \times {}_U U'' \times {}_U U'', X_1) \times \mathrm{Hom}(U' \times {}_U U' \times {}_U U'', X_1)$$
$$\times \mathrm{Hom}(U', X_0) \times \mathrm{Hom}(U'', X_0)$$

where the morphisms are defined in terms of (p', Ψ') and (p'', Ψ''). This extends to a diagram of sheaves on the small FPQC site of U by base change. If follows that the equalizer is a sheaf, hence that $[X_\bullet]$ is a prestack, as required.

It remains to verify that $[X_\bullet]$ is a stack; this is largely a question of interpreting the definitions. Recall from Definition 2.17 that a prestack is a stack if and only if every descent datum is effective. A descent datum for $[X_\bullet]$ is given by a faithfully flat morphism $p'\colon U' \to U$ in Aff/S, an object x' of $[X_\bullet]_{U'}$ and a morphism ψ' between the restrictions p_1^*x', p_2^*x' of x' to $[X_\bullet]_{U' \times_U U'}$ along the projections $p_1, p_2\colon U' \times {}_U U' \rightrightarrows U'$, such that ψ' satisfies the cocycle condition. Lemma 4.6 shows that this is equivalent to a commutative diagram of morphisms of affine groupoid schemes (1). In particular, the morphism Ψ defines an object x of $[X_\bullet]_U$.

It remains to verify that x restricts up to isomorphism to the object x' of $[X_\bullet]_{U'}$ and satisfies the coherence condition of Definition 2.16 (2). This follows from the commutative diagram (1) of Lemma 4.6 together with the factorization provided by Lemma 4.5.

That is, an isomorphism between x' and $(p')^*x$ is defined by a morphism h which fits into a commutative diagram

$$\begin{array}{ccccc} U''\times_U U' & \longleftarrow & (U''\times_U U')\times_{U'} U'' & \longrightarrow & U'' \\ \downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\ X_0 & \xleftarrow[s]{} & X_1 & \xrightarrow[t]{} & X_0. \end{array}$$

There is a canonical isomorphism $(U''\times_U U')\times_{U'} U'' \cong U''\times_U U''$, hence h is a morphism of the form $U''\times_U U'' \to X$.

The isomorphism is given by $\psi\colon U''\times_U U'' \to X$. The conditions of Definition 2.16 follow from the factorization provided by Lemma 4.5. $\square$

The sequence of results above, culminating in Proposition 4.8, has established that there is a functorial construction of a stack $[X_\bullet]$ associated to the affine groupoid scheme $X_\bullet$, which is equipped with a canonical 1-morphism of prestacks:

$$[X_\bullet]' \hookrightarrow [X_\bullet].$$

Theorem 4.9. *Let $X_\bullet$ be an affine groupoid scheme in Aff/S. Then the 1-monomorphism of prestacks*

$$[X_\bullet]' \hookrightarrow [X_\bullet]$$

is an associated stack construction for $[X_\bullet]'$.

Proof. (Outline) The theorem asserts that the 1-morphism has the universal property which characterizes the associated stack up to equivalence. Suppose that $[X_\bullet]' \to \mathscr{M}$ is a 1-morphism of prestacks, where $\mathscr{M}$ is a stack. Let $x' = (p'\colon U' \to U, \Psi')$, $x'' = (p''\colon U'' \to U, \Psi'')$ be two objects of $[X_\bullet]_U$; these give rise to descent data for the stack $\mathscr{M}$, and hence to sections x_1, x_2 of $\mathscr{M}_U$ respectively together with the coherence data, since $\mathscr{M}$ is a stack. This is used to define a 1-morphism $[X_\bullet] \to \mathscr{M}$ on objects.

It remains to define the 1-morphism on morphisms. The key point is the verification that a morphism f between (p', Ψ') and (p'', Ψ'') in $[X_\bullet]_U$ induces a morphism between x_1 and x_2 in $\mathscr{M}_U$. By definition, the morphism f is induced by a morphism $U' \times_U U'' \to X_1$ and hence gives rise to a morphism of $\mathscr{M}_{U'\times_U U''}$. The canonical morphism $U'\times_U U'' \to U$ is faithfully flat, hence the required morphism in $\mathscr{M}_U$ can be constructed by using the fact that $\mathscr{M}$ is a prestack.

This depends upon an analysis based on the following diagram, which is induced from the coequalizer diagrams $U'\times_U U' \rightrightarrows U' \to U$ and $U''\times_U U'' \rightrightarrows U'' \to U$ by forming pullbacks:

$$\begin{array}{ccccc} U'\times_U U'\times_U U''\times_U U'' & \Rrightarrow & U'\times_U U''\times_U U'' & \longrightarrow & U''\times_U U'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ U'\times_U U'\times_U U'' & \rightrightarrows & U'\times_U U'' & \longrightarrow & U'' \\ \downarrow & & \downarrow & & \downarrow \\ U'\times_U U' & \rightrightarrows & U' & \longrightarrow & U, \end{array}$$

where all morphisms are induced by one of p', p'' on a single factor of the product. Commutativity of the upper left hand square should be understood as the existence of two commuting squares which correspond to the two projections $(U' \times_U U'') \times_U (U' \times_U U'') \rightrightarrows U' \times_U U''$ (after reordering factors).

The descent data together with the morphism f give rise to a morphism $\pi^* x_1 \to \pi^* x_2$ in $\mathscr{M}_{U' \times_U U''}$, where $\pi \colon U' \times_U U'' \to U$ denotes the common projection. The respective compatibility conditions of the morphism f with the descent data and of the objects x_1 and x_2 with the descent data for x' and x'' can be used to show that this morphism descends as required to a unique morphism $\overline{f} \colon x_1 \to x_2$, using the fact that $\mathscr{M}$ is a prestack. $\square$

4.2. Generalization to arbitrary prestacks

The techniques of the previous section can be generalized to arbitrary prestacks; the purpose of this section is to indicate the necessary modifications.

Lemma 4.10. *Let $U' \to \mathscr{M}$ be a 1-morphism of prestacks, where U' is represented by an affine S-scheme. Then there exists an associated groupoid space $(U', U' \times_{\mathscr{M}} U')$.*

Proof. This result is analogous to Proposition 2.35. If the existence of an associated stack is assumed, then the result can be proved as a consequence of the proposition, as follows.

Proposition 2.35 gives a groupoid space $(U', U' \times_{\tilde{\mathscr{M}}} U')$, where $\tilde{\mathscr{M}}$ is the stack associated to $\mathscr{M}$. The sheaf $U' \times_{\tilde{\mathscr{M}}} U'$ is isomorphic to the sheaf defined by the 2-fibre product $U' \times_{\mathscr{M}} U'$, hence there is a groupoid space of the form $(U', U' \times_{\mathscr{M}} U')$. $\square$

The associated stack construction for the prestack $\mathscr{M}$ proceeds as in Section 4.1; the essential part of the construction is to specify the fibre category $\tilde{\mathscr{M}}_U$ for an affine S-scheme U. This is the category with objects given by pairs $(p' \colon U' \to U, \Psi')$, where $p' \colon U' \to U$ is a faithfully flat morphism in Aff/S and Ψ' is a 1-morphism of groupoid spaces

$$(U', U' \times_U U') \to (U', U' \times_{\mathscr{M}} U').$$

A morphism between objects $(p' \colon U' \to U, \Psi')$ and $(p'' \colon U'' \to U, \Psi'')$ is a morphism of sheaves $U' \times_U U'' \to U' \times_{\mathscr{M}} U''$ for which the following diagram commutes

$$\begin{array}{ccccc} U' & \longleftarrow & U' \times_U U'' & \longrightarrow & U'' \\ \Vert & & \downarrow & & \Vert \\ U' & \longleftarrow & U' \times_{\mathscr{M}} U'' & \longrightarrow & U'' \end{array}$$

and which satisfies the analogues of the conditions (2) and (3) of Lemma 4.1.

5. Fibre products of affine groupoid schemes

This section is devoted to an explicit identification of the 2-fibre product in the category of affine groupoid schemes. Section 5.2 establishes a technical result which is used in later descent arguments.

5.1. Affine groupoid schemes and 2-fibre products

The 2-category of affine groupoid schemes admits 2-fibre products; for a diagram of morphisms of affine groupoid schemes

$$X_\bullet \to Z_\bullet \leftarrow Y_\bullet, \tag{3}$$

the 2-fibre product is denoted by $X_\bullet \times_{Z_\bullet} Y_\bullet$. The existence and structure is established below by Lemma 5.1 and Proposition 5.2.

Lemma 5.1. *The affine groupoid scheme $X_\bullet \times_{Z_\bullet} Y_\bullet$ has underlying affine S-schemes $(X_0 \times_{Z_0} Z_1 \times_{Z_0} Y_0, X_1 \times_{Z_0} Z_1 \times_{Z_0} Y_1)$.*

There are canonical morphisms of affine groupoid schemes $X_\bullet \leftarrow X_\bullet \times_{Z_\bullet} Y_\bullet \to Y_\bullet$ and the diagram

$$\begin{array}{ccc} X_\bullet \times_{Z_\bullet} Y_\bullet & \longrightarrow & Y_\bullet \\ \downarrow & & \downarrow \\ X_\bullet & \longrightarrow & Z_\bullet \end{array}$$

is 2-commutative.

Proof. It is a straightforward exercise to give the structure morphisms of this affine groupoid scheme explicitly, by considering the structure which it represents as a category fibred over Aff/S. □

Proposition 5.2. *Let $X_\bullet \to Z_\bullet \leftarrow Y_\bullet$ be morphisms of affine groupoid schemes. The affine groupoid scheme $X_\bullet \times_{Z_\bullet} Y_\bullet$ is a 2-fibre product of the diagram.*

Proof. This result is essentially a consequence of Yoneda's lemma. The universal property can be established explicitly as follows.

Suppose that

$$\begin{array}{ccc} A_\bullet & \longrightarrow & Y_\bullet \\ \downarrow & & \downarrow \\ X_\bullet & \longrightarrow & Z_\bullet \end{array}$$

is a 2-commutative diagram of morphisms of affine groupoid schemes. Namely, there are morphisms $(f_0, f_1)\colon (A_0, A_1) \to (X_0, X_1)$ and $(g_0, g_1)\colon (A_0, A_1) \to (Y_0, Y_1)$ such that the composite morphisms $(A_0, A_1) \rightrightarrows (Z_0, Z_1)$ are 2-isomorphic via a morphism $\alpha\colon A_0 \to Z_1$ which satisfies the conditions given in Proposition 2.8. In particular, there is a morphism $A_1 \to Z_1$ which is given as the composite

$$\hat{\alpha}\colon A_1 \stackrel{g_1, \alpha s_X}{\longrightarrow} Y_1 \times_{Y_0} Y_1 \stackrel{m_Y}{\to} Y_1$$

(or, equivalently, as the composite $A_1 \stackrel{\alpha t_X, f_1}{\longrightarrow} Y_1 \times_{Y_0} Y_1 \stackrel{m_Y}{\to} Y_1$).

The required morphism $(A_0, A_1) \to (X_0 \times_{Z_0} Z_1 \times_{Z_0} Y_0, X_1 \times_{Z_0} Z_1 \times_{Z_0} Y_1)$ is defined by the morphisms

$$A_0 \xrightarrow{f_0 \times \alpha \times g_0} X_0 \times_{Z_0} Z_1 \times_{Z_0} Y_0$$

$$A_1 \xrightarrow{f_1 \times \hat{\alpha} \times g_1} X_1 \times_{Z_0} Z_1 \times_{Z_0} Y_1.$$

The verification that this defines a morphism of affine groupoid schemes is left to the reader. □

The diagram (3) gives rise to a diagram of prestacks

$$[X_\bullet]' \to [Z_\bullet]' \leftarrow [Y_\bullet]'.$$

The 2-fibre product $[X_\bullet]' \times_{[Z_\bullet]'} [Y_\bullet]'$ (defined in the category of groupoids fibred over Aff/S) is equivalent to the prestack associated to the affine groupoid scheme $X_\bullet \times_{Z_\bullet} Y_\bullet$.

Corollary 5.3. *Let $X_\bullet \to Z_\bullet \leftarrow Y_\bullet$ be morphisms of affine groupoid schemes. The stack $[X_\bullet] \times_{[Z_\bullet]} [Y_\bullet]$ is equivalent to the stack $[X_\bullet \times_{Z_\bullet} Y_\bullet]$.*

Proof. This result is a special case of a general result on 2-fibre products of diagrams of 1-morphisms of prestacks. An explicit approach is available by using the results of Section 4; this allows the results to be expressed in terms of morphisms of affine groupoid schemes. The key point is to verify that a morphism of descent data gives rise to an appropriate 2-morphism in the 2-category of groupoids.

There is a canonical 1-morphism of stacks

$$[X_\bullet \times_{Z_\bullet} Y_\bullet] \to [X_\bullet] \times_{[Z_\bullet]} [Y_\bullet] \tag{4}$$

which is a 1-monomorphism of stacks. (The fact that this is a monomorphism before passage to the associated stacks is clear; the associated stack construction preserves monomorphisms.) It is therefore sufficient to show that the morphism is a 1-epimorphism of stacks. This is seen as follows.

Consider descent data $(U', U' \times_U U') \to X_\bullet$ and $(U'', U'' \times_U U'') \to Y_\bullet$, equipped with a 2-morphism between the descent data provided by composition with $X_\bullet \to Z_\bullet$ and $Y_\bullet \to Z_\bullet$ respectively. It is sufficient to show that the corresponding section of $([X_\bullet] \times_{[Z_\bullet]} [Y_\bullet])_U$ is in the image of the morphism (4).

The projections $U' \leftarrow U' \times_U U'' \to U''$ give rise to morphisms of affine groupoid schemes

$$(U', U' \times_U U') \leftarrow (U' \times_U U'', U' \times_U U'' \times_U U' \times_U U'') \to (U'', U'' \times_U U'').$$

The morphism of descent data is defined by a morphism $\alpha\colon U' \times_U U'' \to Z_1$. It remains to verify that this defines a 2-morphism between the two composite morphisms

$$f_\bullet\colon (U' \times_U U'', U' \times_U U'' \times_U U' \times_U U'') \to (U', U' \times_U U') \to X_\bullet \to Z_\bullet$$
$$g_\bullet\colon (U' \times_U U'', U' \times_U U'' \times_U U' \times_U U'') \to (U'', U'' \times_U U'') \to Y_\bullet \to Z_\bullet.$$

To show that the composition with the source and target morphisms $s, t\colon Z_1 \rightrightarrows Z_0$ behaves correctly is straightforward. The remaining condition is to verify that the following diagram is commutative:

$$\begin{array}{ccc} W_1 & \xrightarrow{(g_1, \alpha s_X)} & Z_1 \times_{Z_0} Z_1 \\ {\scriptstyle (\alpha t_X, f_1)}\downarrow & & \downarrow{\scriptstyle m_Z} \\ Z_1 \times_{Z_0} Z_1 & \xrightarrow[m_Z]{} & Z_1, \end{array}$$

where $W_\bullet = (U' \times_U U'', U' \times_U U'' \times_U U' \times_U U'')$.

The conditions (2), (3) of Lemma 4.1 upon α show that both composites coincide with the composite morphism

$$U' \times_U U'' \times_U U' \times_U U'' \xrightarrow{\pi} U' \times_U U'' \xrightarrow{\alpha} Z_1,$$

where π denotes the projection onto the first and the last factors of $U' \times_U U'' \times_U U' \times_U U''$. In particular, the square is commutative, as required. □

5.2. A technical lemma

Recall that a morphism of affine S-schemes $U' \to U$ has factorization as a morphism of affine groupoid schemes $U' \to (U', U' \times_U U') \to U$. If the morphism $U' \to U$ induces a surjection of FPQC sheaves (for example, if $U' \to U$ is faithfully flat), then the morphism $(U', U' \times_U U') \to U$ induces an equivalence of stacks.

Lemma 5.4. *Let $U' \to U$ be a faithfully flat morphism of affine S-schemes. The commutative diagram*

$$\begin{array}{ccccc} & & U' & = = = & U' \\ & & \downarrow & & \downarrow \\ X_\bullet & \longrightarrow & (U', U' \times_U U') & \longrightarrow & U \end{array}$$

of morphisms of affine groupoid schemes induces an isomorphism

$$Y_\bullet := X_\bullet \times_{(U', U' \times_U U')} U' \xrightarrow{\cong} X_\bullet \times_U U'$$

on the associated 2-fibre products.

Proof. By definition, the affine schemes Y_0 and Y_1 identify as follows:

$$Y_0 \cong X_0 \times_{U'} (U' \times_U U') \times_{U'} U' \cong X_0 \times_U U'$$
$$Y_1 \cong X_1 \times_{U'} (U' \times_U U') \times_{U'} U' \cong X_1 \times_U U'.$$

Correspondingly, the affine groupoid scheme $X_\bullet \times_U U'$ has objects represented by $X_0 \times_U U'$ and morphisms represented by $X_1 \times_U U'$. The result follows by verifying that the induced morphism corresponds to the isomorphisms given above. □

6. Discrete stacks

This section analyses the condition that a stack is discrete and shows that the condition is amenable to explicit descent arguments.

6.1. The discreteness condition

A presheaf of sets $\mathfrak{X}$ on Aff/S naturally gives rise to a prestack, with underlying fibre category above an affine S-scheme U the set of sections $\mathfrak{X}_U$, regarded as a discrete category. This is a stack if $\mathfrak{X}$ is a sheaf.

Definition 6.1. A stack $\mathscr{X}$ is discrete if it is equivalent to a stack associated to a sheaf $\mathfrak{X}$.

Proposition 6.2. *[**9**, (3.4.1)] Let $\mathscr{X}$ be a stack on Aff/S. The stack $\mathscr{X}$ is discrete if and only if the diagonal morphism $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is a monomorphism.*

Remark 6.3. The condition provided by Proposition 6.2 can be interpreted in terms of the fibre categories. Namely, the stack $\mathscr{X}$ is discrete if and only if, for each pair of sections z, z' of $\mathscr{X}_U$, for $U \in \mathrm{Obj}\mathsf{Aff}/S$, the cardinality of $\mathrm{Hom}_{\mathscr{X}_U}(z, z')$ is at most one. Equivalently, using the fact that all morphisms are invertible, $\mathrm{Aut}_{\mathscr{X}_U}(z) = \{1_z\}$, for each section z of $\mathscr{X}_U$.

This gives rise to a straightforward criterion for a groupoid scheme to induce a discrete stack.

Notation 6.4. For $X_\bullet$ an affine groupoid scheme, let $\overline{X}$ be the affine S-scheme defined by the coequalizer in Aff/S:

$$X_1 \rightrightarrows X_0 \to \overline{X}.$$

If $X_\bullet = \mathrm{Spec}(A, \Gamma)$ for a Hopf algebroid (A, Γ), then $\overline{X} = \mathrm{Spec}(A^\Gamma)$, where A^Γ is the equalizer of the left and right units $A \rightrightarrows \Gamma$.

The universal property of the fibre product implies the following:

Lemma 6.5. *Let $X_\bullet$ be an affine groupoid scheme. The morphisms $s, t \colon X_1 \rightrightarrows X_0$ induce a morphism*

$$X_1 \to X_0 \times_{\overline{X}} X_0.$$

In terms of the associated Hopf algebroid (A, Γ), the morphism is of the form $A \otimes_{A^\Gamma} A \to \Gamma$, induced by the left and right units.

Remark 6.6. The canonical morphism $X_0 \times_{\overline{X}} X_0 \hookrightarrow X_0 \times X_0$ is a closed immersion of affine schemes. This corresponds to the surjection $A \otimes A \twoheadrightarrow A \otimes_{A^\Gamma} A$ of rings.

Lemma 6.7. *The morphism $X_1 \to X_0 \times_{\overline{X}} X_0$ induces a morphism of affine groupoid schemes*

$$X_\bullet \to (X_0, X_0 \times_{\overline{X}} X_0).$$

Proof. Straightforward. □

In terms of the Hopf algebroid (A, Γ), this morphisms corresponds to a morphism $(A, A \otimes_{A^\Gamma} A) \to (A, \Gamma)$.

For a stack associated to an affine groupoid scheme, the criterion of Proposition 6.2 simplifies to give the following.

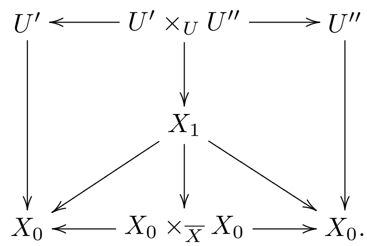
Proposition 6.8. *Let $X_\bullet$ be an affine groupoid scheme. The stack $[X_\bullet]$ is discrete if and only if the following equivalent conditions are satisfied*

1. *the morphism* $X_1 \to X_0 \times_{\overline{X}} X_0$ *induces a monomorphism of presheaves on* Aff/S*;*
2. *the morphism* $X_1 \to X_0 \times_{\overline{X}} X_0$ *induces a monomorphism in* $\mathsf{Shv}_{\mathrm{fpqc}}$.

Proof. The equivalence of the two numbered conditions follows from the fact that the FPQC topology is subcanonical and the inclusion of the category of sheaves in presheaves is left exact.

Hence it suffices to prove the result using the presheaf condition (1). That this condition is necessary follows from Remark 6.3.

Sufficiency follows from general considerations for the passage from a prestack to the associated stack. This can be seen explicitly here using the results of Section 4. Namely, using the notation of Lemma 4.1, a morphism between descent data is equivalent to a commutative diagram



The hypothesis that $X_1 \to X_0 \times_{\overline{X}} X_0$ induces a monomorphism of presheaves implies that this diagram is uniquely determined by the morphisms $U' \to X_0$ and $U'' \to X_0$. Hence there exists at most one morphism between the corresponding sections. □

Proposition 6.8 has an alternative formulation in terms of an equalizer diagram.

Proposition 6.9. *The stack* $[X_\bullet]$ *associated to an affine groupoid scheme* $X_\bullet$ *is discrete if and only if the diagram*

$$X_0 \xrightarrow{\epsilon} X_1 \overset{s}{\underset{t}{\rightrightarrows}} X_0$$

is an equalizer diagram.

Proof. The result follows from Proposition 6.8, by using the final observation of Remark 6.3. □

Definition 6.10. An affine groupoid scheme $X_\bullet$ is said to be discrete if it satisfies the equivalent conditions of Proposition 6.9.

This condition can be translated in terms of R-Hopf algebroids.

Corollary 6.11. *The stack* $\mathscr{M}_{(A,\Gamma)}$ *associated to an* R*-Hopf algebroid* (A, Γ) *is discrete if and only if the diagram*

$$A \overset{\eta_L}{\underset{\eta_R}{\rightrightarrows}} \Gamma \xrightarrow{\epsilon} A$$

is a coequalizer diagram in R*-algebras.*

6.2. An explicit descent argument

The following result can be seen directly by the construction of the 2-fibre product. However the purpose here is to give a reduction to an argument in affine groupoid schemes.

Proposition 6.12. *The stack $[X_\bullet]$ associated to an affine groupoid scheme $X_\bullet$ is discrete if and only if there exists an affine stack U, a 1-morphism $[X_\bullet] \to U$ and a faithfully flat morphism $U' \to U$ of S-schemes such that $[X_\bullet] \times_U U'$ is discrete.*

Proof. The result follows by using the criterion provided by Proposition 6.9 in conjunction with Lemmas 6.13 and 6.14 below. □

Recall that $\overline{X}$ denotes the coequalizer of $X_1 \rightrightarrows X_0$ in Aff/S.

Lemma 6.13. *Let $X_\bullet$ be an affine groupoid scheme and U an affine S-scheme.*

1. *There is an equivalence of discrete categories*

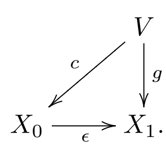
$$\mathbf{Hom}([X_\bullet], U) \cong \mathbf{Hom}_{\mathsf{Gpd}_S}(X_\bullet, U) \cong \mathrm{Hom}_{\mathsf{Aff}/S}(\overline{X}, U).$$

2. *If $V \to U$ is a morphism of affine S-schemes, the stack 2-fibre product $[X_\bullet] \times_U V$ is equivalent to the stack associated to the affine groupoid scheme $(X_\bullet \times_U V)$.*

Proof. The first statement follows from Lemma 2.28 and standard considerations. The second statement is a special case of the result on 2-fibre products of groupoids in Section 5 given in Corollary 5.3. (Alternatively, it can be proved directly by elementary considerations.) □

Lemma 6.14. *Let $X_\bullet$ be an affine groupoid scheme and $\overline{X} \to U$ be a morphism of affine schemes. The diagram $X_0 \to X_1 \rightrightarrows X_0$ is an equalizer if and only if there exists a fully faithful morphism $U' \to U$ such that $X_0 \times_U U' \to X_1 \times_U U' \rightrightarrows X_0 \times_U U'$ is an equalizer.*

Proof. It is sufficient to show that, for any morphism $g\colon V \to X_1$ which equalizes $X_1 \rightrightarrows X_0$, the common composite $c\colon V \to X_0$ induces a commutative triangle



This can be checked after faithfully flat base change. □

6.3. Identifying the sheaf

Proposition 6.15. *Let $X_\bullet$ be a discrete affine groupoid scheme. The stack $[X_\bullet]$ is equivalent to the stack associated to the sheaf $\mathfrak{X}$ which is defined by the coequalizer diagram in sheaves:*

$$X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0 \longrightarrow \mathfrak{X}.$$

Proof. Let W be the coequalizer in presheaves of the diagram $X_1 \rightrightarrows X_0$. There is a canonical 1-morphism of prestacks $[X_\bullet]' \to W$ which is induced by the morphism $X_0 \to W$. The hypothesis that $X_\bullet$ is discrete implies that the morphism is a monomorphism.

Passage to the associated stacks yields a monomorphism of stacks $[X_\bullet] \hookrightarrow \mathfrak{X}$ where $\mathfrak{X}$ is induced by the sheaf associated to W. The composite morphism $X_0 \to [X_\bullet] \to \mathfrak{X}$ is an epimorphism of stacks, hence $[X_\bullet] \to \mathfrak{X}$ is an epimorphism as well. Thus the morphism $[X_\bullet] \to \mathfrak{X}$ is an equivalence, as required. □

7. Affine stacks

The concept of an affine stack is introduced and studied in this section. The main result, Theorem 7.16, gives an explicit criterion for a stack to be affine.

7.1. The sheaf criterion

Recall that the stack associated to an affine groupoid scheme $X_\bullet$ is discrete if and only if the diagram

$$X_0 \xrightarrow{\epsilon} X_1 \rightrightarrows X_0 \tag{5}$$

is an equalizer in Aff/S. Moreover, $\overline{X}$ denotes the coequalizer in Aff/S of the diagram $X_1 \rightrightarrows X_0$, so that there is a coequalizer diagram

$$X_1 \rightrightarrows X_0 \to \overline{X}. \tag{6}$$

Proposition 7.1. *Let $X_\bullet$ be an affine groupoid scheme. The stack $[X_\bullet]$ is affine if and only if the following two conditions hold:*

1. *the diagram (5) is an equalizer in Aff/S;*
2. *the diagram (6) defines a coequalizer in the category of FPQC sheaves on Aff/S.*

Proof. The first condition is necessary and sufficient for $[X_\bullet]$ to be discrete, by Proposition 6.12. Moreover Proposition 6.15 implies that, if $[X_\bullet]$ is discrete, it is isomorphic to the stack induced by the sheaf coequalizer $\mathfrak{X}$ of $X_1 \rightrightarrows X_0$. Hence, $[X_\bullet]$ is affine if and only if $\mathfrak{X}$ is representable.

The Yoneda lemma shows that, if $\mathfrak{X}$ is representable, it is represented by $\overline{X}$. Thus, the discrete stack $[X_\bullet]$ is affine if and only if $X_1 \rightrightarrows X_0 \to \overline{X}$ is a coequalizer in sheaves. □

7.2. Explicit descent for affine stacks

The following lemma gives an explicit descent argument for a coequalizer diagram in affine schemes to induce a coequalizer diagram in sheaves.

Lemma 7.2. *The coequalizer $X_1 \rightrightarrows X_0 \to \overline{X}$ in Aff/S induces a coequalizer in sheaves if and only if there exists a morphism of affine schemes $\overline{X} \to V$ and a faithfully flat morphism $V' \to V$ such that the diagram in Aff/S given by faithfully flat base change*

$$X_1 \times_V V' \rightrightarrows X_0 \times_V V' \to \overline{X} \times_V V'$$

induces a coequalizer diagram in $\mathsf{Shv}_{\mathrm{fpqc}}$.

Proof. The forward implication is trivial, hence consider the reverse implication.

The diagram

$$X_1 \times_V V' \rightrightarrows X_0 \times_V V' \to \overline{X} \times_V V'$$

is associated to the affine groupoid scheme $X_\bullet \times_V V'$ which is given by faithfully flat base change. It is a coequalizer in Aff/S, since $V' \to V$ is faithfully flat.

Consider a sheaf $\mathfrak{X}$; we are required to prove that the diagram $\mathfrak{X}(\overline{X}) \to \mathfrak{X}(X_0) \rightrightarrows \mathfrak{X}(X_1)$ is an equalizer in sets. There is a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{X}(\overline{X}) & \longrightarrow & \mathfrak{X}(X_0) & \rightrightarrows & \mathfrak{X}(X_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{X}(\overline{X} \times_V V') & \hookrightarrow & \mathfrak{X}(X_0 \times_V V') & \rightrightarrows & \mathfrak{X}(X_1 \times_V V') \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathfrak{X}(\overline{X} \times_V V' \times_V V') & \hookrightarrow & \mathfrak{X}(X_0 \times_V V' \times_V V') & \rightrightarrows & \mathfrak{X}(X_1 \times_V V' \times_V V')
\end{array}$$

in which the lower two rows are equalizers, following from the hypothesis, and the vertical diagrams are equalizers using the fact that $\mathfrak{X}$ is a sheaf applied to the coequalizer diagrams of the following form in Aff/S

$$(Y \times_V V') \times_Y (Y \times_V V') \cong Y \times_V V' \times_V V' \rightrightarrows Y \times_V V' \to Y,$$

(natural in the affine S-scheme Y), for $Y \in \{\overline{X}, X_0, X_1\}$.

Observe that the above diagram implies that the morphism $\mathfrak{X}(\overline{X}) \to \mathfrak{X}(X_0)$ is a monomorphism. The result now follows by a diagram chase. Namely, a section $x \in \mathfrak{X}(X_0)$ which maps to the same element under the morphisms $\mathfrak{X}(X_0) \rightrightarrows \mathfrak{X}(X_1)$ defines a section x' of $\mathfrak{X}(\overline{X} \times_V V')$, using the fact that the middle row is an equalizer.

The section x' is in the image of the monomorphism $\mathfrak{X}(\overline{X}) \hookrightarrow \mathfrak{X}(\overline{X} \times_V V')$ since the left hand column is an equalizer, using the fact that the morphism $\mathfrak{X}(\overline{X} \times_V V' \times_V V') \hookrightarrow \mathfrak{X}(X_0 \times_V V' \times_V V')$ is a monomorphism, the commutativity of the left hand lower square (as a diagram of equalizers) and the construction of x' from x. It is straightforward to verify that this lift maps to x in $\mathfrak{X}(X_0)$, as required. □

The previous lemma yields the following explicit descent criterion:

Proposition 7.3. *The stack $[X_\bullet]$ is affine if and only if there exists a 1-morphism of stacks $[X_\bullet] \to V$, where V is affine, and a faithfully flat morphism $V' \to V$ such that $[X_\bullet] \times_V V'$ is affine.*

Proof. The forward implication is clear, hence consider the reverse one. Proposition 6.12 implies that a stack is discrete if and only if it is discrete after faithfully flat base change. Hence suppose that there exist morphisms $[X_\bullet] \to V$, $V' \to V$ having the stated properties.

There is a canonical morphism $[X_\bullet] \to \overline{X}$ induced by the morphism of affine groupoid schemes $X_\bullet \to \overline{X}$ and the morphism $[X_\bullet] \to V$ factorizes naturally as $[X_\bullet] \to \overline{X} \to V$.

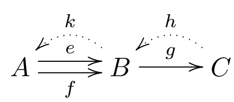
The 2-fibre product $[X_\bullet] \times_V V'$ is the stack associated to the affine groupoid scheme given by base change, $(X_0 \times_V V', X_1 \times_V V')$, using the fact that all structure morphisms of $X_\bullet$ are defined in $\mathsf{Aff}/\overline{X}$.

The result follows from Lemma 7.2, by Proposition 7.1. □

7.3. A splitting criterion

A coequalizer diagram in Aff/S does not in general induce a coequalizer diagram in $\mathsf{Shv}_{\mathrm{fpqc}}$. The stronger criterion of having a coequalizer diagram in the category of presheaves of sets is equivalent to the notion of having a split coequalizer, when the diagram corresponds to an equivalence relation. This is the case in the example of interest, where the diagram corresponds to $s, t \colon X_1 \rightrightarrows X_0$ and, for example, the transitivity of the relation is induced by the composition.

Definition 7.4. A diagram



is a split coequalizer if there exists morphisms $h \colon C \to B$ and $k \colon B \to A$ such that $gh = 1_C$, $fk = 1_B$ and $ek = hg \colon B \to B$.

When considering the sheaf theoretic analogue, it is necessary to take into account the topology on Aff/S. The basic argument is the following:

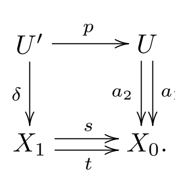
Lemma 7.5. *Let $X_\bullet$ be an affine groupoid scheme and let*

$$X_1 \rightrightarrows X_0 \to \mathfrak{X}$$

be the coequalizer diagram in the category $\mathsf{Shv}_{\mathrm{fpqc}}$. Two sections a_1, a_2 in $X_0(U)$ have the same image in $\mathfrak{X}(U)$ if and only if there exists a faithfully flat morphism $p \colon U' \to U$ of affine schemes and a section $\delta \in X_1(U')$ such that $s\delta = a_1'$ and $t\delta = a_2'$, where $a_1', a_2' \in X_0(U')$ are the images of the sections a_1, a_2 and $s, t \colon X_1(U') \to X_0(U')$ are the structure morphisms.

Proof. The result follows from the construction of the sheaf associated to the presheaf coequalizer of $X_1 \rightrightarrows X_0$. □

Remark 7.6. The data in the lemma corresponds, by Yoneda's lemma, to a commutative diagram of the form



Here, commutativity has to be interpreted in the appropriate way: namely, the diagram corresponds to a pair of commutative squares. (This convention will be adopted in the following without further comment.)

The following lemma is useful in determining whether a given diagram is a coequalizer.

Lemma 7.7. *Let $X \rightrightarrows Y \to Z$ be a coequalizer diagram in a category $\mathscr{D}$ and let*

$$A \rightrightarrows Y \to Z \tag{7}$$

be a diagram in which the composites are equal.

The diagram (7) is a coequalizer if there exists a categorical surjection $\tilde{X} \twoheadrightarrow X$ and a morphism $\tilde{X} \to A$ which makes the following diagram commute:

$$\begin{array}{ccc} \tilde{X} & \twoheadrightarrow & X \\ \downarrow & & \downdownarrows \\ A & \rightrightarrows & Y. \end{array}$$

Proof. The hypothesis that $\tilde{X} \twoheadrightarrow X$ is a categorical surjection implies that, for a morphism $Y \to W$ such that the composites $A \rightrightarrows Y \to W$ are equal, the composites $X \rightrightarrows Y \to W$ are equal. The required canonical factorization follows from the fact that $X \rightrightarrows Y \to Z$ is a coequalizer. □

A fundamental result in the theory of stacks is the following, which is essentially contained within the statement of Proposition 2.35. A direct proof is given here, so as to maintain the explicit nature of the arguments.

Lemma 7.8. *Let $X \to Y$ be a morphism of affine S-schemes which induces a surjection of FPQC sheaves. Then the diagram of affine S-schemes*

$$X \times_Y X \rightrightarrows X \to Y \tag{8}$$

induces a coequalizer diagram in $\mathsf{Shv}_{\mathrm{fpqc}}$.

Proof. The hypothesis that $X \to Y$ induces a surjection of flat sheaves is equivalent to the existence of a morphism $Y' \to X$ such that the composite $Y' \to X \to Y$ is faithfully flat.

The result now follows by the faithfully flat base change argument which is used in the proof of Proposition 7.3. Namely, it is straightforward to verify that the diagram

$$X' \times_{Y'} X' \rightrightarrows X' \to Y'$$

is a split coequalizer in affine schemes, and hence induces a coequalizer in sheaves, where $X' = X \times_Y Y'$ and this diagram corresponds to the base change via the faithfully flat morphism $Y' \to Y$ of the diagram (8), using the isomorphism $X' \times_{Y'} X' \cong (X \times_Y X) \times_Y Y'$. The argument used in the proof of Proposition 7.3 now applies to this situation. □

Remark 7.9. In the case that $X \to Y$ is a faithfully flat morphism of affine S-schemes, this result is tautological.

Lemma 7.10. *Let $X_\bullet$ be an affine groupoid scheme and let $X_1 \rightrightarrows X_0 \to \overline{X}$ be the associated coequalizer in affine S-schemes.*

There is a canonical morphism $X_1 \to X_0 \times_{\overline{X}} X_0$ *induced by the morphisms* $X_1 \rightrightarrows X_0$ *such that a diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X_1 \\ {\scriptstyle\beta}\downarrow & & \downarrow\downarrow \\ X_0 \times_{\overline{X}} X_0 & \rightrightarrows & X_0 \end{array}$$

is commutative if and only if the triangle

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X_1 \\ {\scriptstyle\beta}\downarrow & \swarrow & \\ X_0 \times_{\overline{X}} X_0 & & \end{array}$$

is commutative.

Proof. A direct consequence of the universal property of the fibre product. □

Definition 7.11. Let $X_\bullet$ be an affine groupoid scheme. The coequalizer diagram $X_1 \rightrightarrows X_0 \to \overline{X}$ is FPQC split if the following conditions are satisfied.

1. There exists a faithfully flat morphism $p\colon C \to \overline{X}$ and a morphism $h\colon C \to X_0$ which makes the following diagram commute:

$$\begin{array}{ccc} & & C \\ & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle p} \\ X_0 & \xrightarrow[q]{} & \overline{X}. \end{array}$$

2. There exists a faithfully flat morphism $U \twoheadrightarrow X_0 \times_{\overline{X}} X_0$ and a morphism $U \to X_1$ which makes the following diagram commute:

$$\begin{array}{ccc} U & \twoheadrightarrow & X_0 \times_{\overline{X}} X_0 \\ \downarrow & & \downarrow\downarrow \\ X_1 & \rightrightarrows & X_0. \end{array}$$

Remark 7.12. The definition of an FPQC split coequalizer can be given an equivalent formulation in terms of surjectivity of certain morphisms between representable sheaves (cf. Lemma 7.13 below).

The previous condition has been maintained because of its relation to the notion of a split coequalizer diagram, which corresponds to the presheaf version. Namely, to consider presheaves, one must replace the faithfully flat morphisms by isomorphisms; then the morphism $X_0 \to \overline{X}$ admits a section h and this gives rise to a morphism $\tilde{\Delta}\colon X_0 \to X_0 \times_{\overline{X}} X_0$, induced by the identity on X_0 and the composite $X_0 \to \overline{X} \to X_0$. The second part of the hypothesis now provides a morphism $X_0 \times_{\overline{X}} X_0 \to X_1$ and the composite morphism

$$X_0 \xrightarrow{\tilde{\Delta}} X_0 \times_{\overline{X}} X_0 \to X_1$$

plays the role of k in the definition of a split coequalizer.

Lemma 7.13. *The diagram $X_1 \rightrightarrows X_0 \to \overline{X}$ is FPQC split if and only if the morphisms*

$$X_0 \to \overline{X}$$
$$X_1 \to X_0 \times_{\overline{X}} X_0$$

induce surjective morphisms in $\mathsf{Shv}_{\mathrm{fpqc}}$.

Proof. The result follows from the usual criterion for a morphism of affine S-schemes to induce a surjection of FPQC sheaves. □

Proposition 7.14. *Let $X_\bullet$ be an affine groupoid scheme. The coequalizer diagram $X_1 \rightrightarrows X_0 \to \overline{X}$ induces a coequalizer in sheaves if and only if the following equivalent conditions hold:*

1. *the diagram is an FPQC split coequalizer in affine S-schemes;*
2. *the morphisms $X_0 \to \overline{X}$ and $X_1 \to X_0 \times_{\overline{X}} X_0$ induce surjections of FPQC sheaves.*

Proof. The equivalence of the numbered conditions is proved in Lemma 7.13. Hence it suffices to prove that the diagram induces a coequalizer diagram in sheaves if and only if it is an FPQC split coequalizer diagram.

First, suppose that the diagram is an FPQC split coequalizer diagram. The first hypothesis implies, by Lemma 7.13 and Lemma 7.8, that the coequalizer diagram in affine S-schemes

$$X_0 \times_{\overline{X}} X_0 \rightrightarrows X_0 \to \overline{X}$$

induces a coequalizer in $\mathsf{Shv}_{\mathrm{fpqc}}$.

This fact, together with the second hypothesis, means that Lemma 7.7 can be applied to deduce that the diagram $X_1 \rightrightarrows X_0 \to \overline{X}$ is a coequalizer in sheaves.

For the reverse implication, suppose that the diagram induces a coequalizer in FPQC sheaves. Yoneda's lemma implies that the diagram is a coequalizer in Aff/S, by restriction to the representable sheaves. The remainder of the proof relies on the explicit construction of the sheaf coequalizer of the diagram $X_1 \rightrightarrows X_0$ as the sheaf associated to the presheaf coequalizer (cf. Lemma 7.5). Let $\mathfrak{X}$ denote this explicit sheaf, which is isomorphic to the sheaf represented by $\overline{X}$, by hypothesis.

The first condition for an FPQC split coequalizer is equivalent to the fact that the morphism $X_0 \to \overline{X}$ induces a surjection of sheaves, as in Lemma 7.13.

For the second condition for an FPQC split coequalizer, consider the sections $p_1, p_2 \in X_0(X_0 \times_{\overline{X}} X_0)$ which are represented by the projections $p_1, p_2 \colon X_0 \times_{\overline{X}} X_0 \to X_0$. It is clear that these define the same element in $\mathfrak{X}(X_0 \times_{\overline{X}} X_0) = \overline{X}(X_0 \times_{\overline{X}} X_0)$. Hence, Lemma 7.5 applies to give the required diagram, since $\mathfrak{X}$ is the sheaf coequalizer, by definition. □

Proposition 7.14 gives an explicit criterion for the coequalizer diagram in affine S-schemes

$$X_1 \rightrightarrows X_0 \to \overline{X}$$

to induce a coequalizer in FPQC sheaves. The hypothesis that $X_0 \to \overline{X}$ is surjective in sheaves is clearly necessary, however it is not sufficient. This problem is already evident in the consideration of homogeneous spaces for algebraic groups, as indicated by the following example.

Example 7.15. [**15**, Section 16.2] Fix a field $\mathbb{K}$ and consider the following algebraic groups over $\mathbb{K}$: the general linear group GL_2 and $H \subset GL_2$ the subgroup corresponding to the upper triangular matrices. Thus, the group H occurs in a split extension

$$\mathbb{G}_a \to H \to \mathbb{G}_m \times \mathbb{G}_m,$$

where $\mathbb{G}_a$ is the additive group and $\mathbb{G}_m$ the multiplicative group.

The coordinate rings are given by $\mathscr{O}(GL_2) = \mathbb{K}[x_{11}, x_{22}, x_{12}, x_{21}](\det)^{\pm 1}$ and $\mathscr{O}(H) = \mathbb{K}[x_{11}^{\pm 1}, x_{22}^{\pm 1}, x_{12}]$ respectively, where $\det = x_{11}x_{22} - x_{12}x_{21}$. There is a surjective morphism of Hopf algebras $\mathscr{O}(GL_2) \to \mathscr{O}(H)$ corresponding to the inclusion of the subgroup, and the action of H on G corresponds to the coaction

$$\mathscr{O}(GL_2) \to \mathscr{O}(GL_2) \otimes \mathscr{O}(H).$$

This leads to a split Hopf algebroid $(\mathscr{O}(GL_2), \mathscr{O}(GL_2) \otimes \mathscr{O}(H))$, corresponding to the split affine groupoid scheme $(G, G \times H)$.

From the construction it is clear that the stack associated to $(G, G \times H)$ is discrete. However, the equalizer of the diagram $\mathscr{O}(GL_2) \rightrightarrows \mathscr{O}(GL_2) \otimes \mathscr{O}(H)$ can be seen to be $\mathbb{K}$. Hence, in the notation of Proposition 7.14, the surjection $X_0 \to \overline{X}$ corresponds to the canonical morphism $GL_2 \to S = \mathrm{Spec}(\mathbb{K})$, which has a section induced by the identity section, hence induces a surjection of FPQC sheaves.

However, the sheaf equalizer corresponds to the sheaf represented by the projective line $\mathbb{P}^1$. The difficulty in this example is clearly the restriction to the affine setting, as explained in [**15**, Chapter 16].

7.4. Affine stacks

The results of Section 6 and Section 7.3 give rise to the following criterion for a stack to be affine. Recall that $\overline{X}$ denotes the coequalizer in Aff/S of $X_1 \rightrightarrows X_0$.

Theorem 7.16. *Let $X_\bullet$ be an affine groupoid scheme. The associated stack $[X_\bullet]$ is affine if and only if the following equivalent conditions are satisfied:*

1. *The canonical morphism $[X_\bullet] \to \overline{X}$ is an equivalence of stacks.*
2. *The following conditions are satisfied:*
   (a) *$X_0 \to \overline{X}$ induces a surjection in $\mathsf{Shv}_{\mathrm{fpqc}}$;*
   (b) *$X_1 \to X_0 \times_{\overline{X}} X_0$ is an isomorphism of affine S-schemes.*

Proof. The equivalence with the first condition follows from the argument used in the proof of Proposition 7.1. The equivalence with the second condition follows by combining Proposition 6.8 and Proposition 7.14, together with Lemma 2.5 to pass from an isomorphism of flat sheaves to an isomorphism of affine schemes. □

The second condition can be restated to give the following:

Corollary 7.17. *Let $X_\bullet$ be an affine groupoid scheme. The stack $[X_\bullet]$ is affine if and only if the following conditions are satisfied:*

1. *the morphism $X_0 \to \overline{X}$ induces a surjection of flat sheaves;*

2. *the morphism of affine groupoid schemes* $X_\bullet \to (X_0, X_0 \times_{\overline{X}} X_0)$ *is an isomorphism.*

Remark 7.18. This result can be expressed in terms of Hopf algebroids as follows: the stack $\mathscr{M}_{(A,\Gamma)}$ associated to a R-Hopf algebroid (A, Γ) is affine if and only if the following two conditions are satisfied:

1. there exists a faithfully flat morphism $\alpha\colon A^\Gamma \to B$ and a commutative diagram in R-algebras

$$\begin{array}{ccc} A^\Gamma & \xrightarrow{i} & A \\ & \searrow^{\alpha} & \downarrow \\ & & B \end{array}$$

in which $i\colon A^\Gamma \hookrightarrow A$ is the canonical inclusion;

2. there is an isomorphism of Hopf algebroids

$$(A, \Gamma) \cong (A, A \otimes_{A^\Gamma} A).$$

Remark 7.19. It is possible to give a more direct proof of Corollary 7.17, combining elements of the proof together with the criterion for a morphism of affine groupoid schemes to induce an equivalence of the associated stacks. The approach given here, which treats separately the condition for having a discrete stack and the condition for a coequalizer diagram of sheaves, has been preferred so as to allow the explicit consideration of the respective descent questions.

Example 7.20. Let $k \hookrightarrow K$ be a field extension. The previous results imply that, up to isomorphism, the only Hopf algebroid of the form (K, Γ) which gives a model for $\mathrm{Spec}(k)$, up to equivalence for the associated stack $\mathscr{M}_{(K,\Gamma)}$, is the Hopf algebroid $(K, K \otimes_k K)$.

8. Affine morphisms of affine groupoid schemes

This section applies the results of Section 7 to derive a criterion for a morphism of affine groupoid schemes to be affine. The criterion given depends upon the flat topology. A stronger criterion, depending on the existence of a splitting in affine schemes, is given.

8.1. Affine morphisms

Definition 8.1. A morphism $X_\bullet \to Y_\bullet$ of affine groupoid schemes is affine if the morphism of stacks $[X_\bullet] \to [Y_\bullet]$ is affine.

Proposition 8.2. *A morphism of affine groupoid schemes* $X_\bullet \to Y_\bullet$ *is affine if and only if the stack associated to* $X_\bullet \times_{Y_\bullet} Y_0$ *is affine.*

Proof. The forward implication is clear, so we consider the reverse implication. Any morphism of affine groupoid schemes $V \to Y_\bullet$ factorizes canonically as

$$V \to Y_0 \to Y_\bullet.$$

Hence, the hypothesis implies that the stack associated to $X_\bullet \times_{Y_\bullet} V$ is affine for every such morphism $V \to Y_\bullet$.

Consider a 1-morphism of stacks $U \to [Y_\bullet]$; this is induced by a morphism of affine groupoid schemes

$$(U', U' \times_U U') \to Y_\bullet$$

for some faithfully flat morphism $U' \to U$, by Theorem 4.9 (and Lemma 4.1). It is necessary to show that the stack associated to $X_\bullet \times_{Y_\bullet} (U', U' \times_U U')$ is affine.

Consider the 2-commutative diagram of morphisms of affine groupoid schemes

$$\begin{array}{ccc} X_\bullet \times_{Y_\bullet} U' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ X_\bullet \times_{Y_\bullet} (U', U' \times_U U') & \longrightarrow & (U', U' \times_U U') \\ \downarrow & & \downarrow \\ X_\bullet & \longrightarrow & Y_\bullet \end{array} \tag{9}$$

in which the squares are given by 2-fibre products.

Lemma 5.4 implies that the top square is equivalent to the 2-Cartesian diagram

$$\begin{array}{ccc} X_\bullet \times_{Y_\bullet} U' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ X_\bullet \times_{Y_\bullet} (U', U' \times_U U') & \longrightarrow & U \end{array}$$

and the hypothesis implies that the stack associated to $X_\bullet \times_{Y_\bullet} U'$ is affine.

The explicit descent result, Proposition 7.3, therefore implies that the stack associated to $X_\bullet \times_{Y_\bullet} (U', U' \times_U U')$ is affine, as required. □

8.2. Explicit criteria

A morphism of affine groupoid schemes $X_\bullet \to Y_\bullet$ gives rise to a 2-fibre product $X_\bullet \times_{Y_\bullet} Y_0$, which has underlying affine schemes $(Y_1 \times_{Y_0} X_0, Y_1 \times_{Y_0} X_1)$. The structure morphisms $s, t \colon Y_1 \times_{Y_0} X_1 \rightrightarrows Y_1 \times_{Y_0} X_0$ are induced respectively by the morphism $s \colon X_1 \to X_0$ and by the composite

$$Y_1 \times_{Y_0} X_1 \to Y_1 \times_{Y_0} Y_1 \times_{Y_0} X_0 \xrightarrow{m_Y \times 1_{X_0}} Y_1 \times_{Y_0} X_0,$$

where the first morphism is induced by the identity on Y_1 and the morphism $X_1 \to Y_1 \times_{Y_0} X_0$ induced by the morphism $X_\bullet \to Y_\bullet$.

Theorem 7.16 leads to the following result, in which

$$Y_1 \times_{Y_0} X_1 \underset{t}{\overset{s}{\rightrightarrows}} Y_1 \times_{Y_0} X_0 \longrightarrow W$$

denotes the coequalizer in Aff/S. The morphisms s, t induce a natural morphism

$$Y_1 \times_{Y_0} X_1 \to (Y_1 \times_{Y_0} X_0) \times_W (Y_1 \times_{Y_0} X_0).$$

Theorem 8.3. *A morphism $X_\bullet \to Y_\bullet$ of affine groupoid schemes is affine if and only if the following two conditions are satisfied:*

1. *The morphism $Y_1 \times_{Y_0} X_0 \to W$ induces a surjection in $\mathsf{Shv}_{\mathrm{fpqc}}$.*

2. *The induced morphism*

$$Y_1 \times_{Y_0} X_1 \to (Y_1 \times_{Y_0} X_0) \times_W (Y_1 \times_{Y_0} X_0)$$

is an isomorphism of affine S-schemes.

Observe that the unit morphism $\epsilon\colon X_0 \to X_1$ induces a morphism $\tilde{\epsilon}\colon Y_1 \times_{Y_0} X_0 \to Y_1 \times_{Y_0} X_1$.

The following weaker result can be easier to apply.

Corollary 8.4. *Let $X_\bullet \to Y_\bullet$ be a morphism of affine groupoid schemes such that*

1. *the diagram*

$$Y_1 \times_{Y_0} X_0 \xrightarrow{\tilde{\epsilon}} Y_1 \times_{Y_0} X_1 \underset{t}{\overset{s}{\rightrightarrows}} Y_1 \times_{Y_0} X_0$$

is an equalizer in Aff/S;

2. *the coequalizer diagram in Aff/S*

$$Y_1 \times_{Y_0} X_1 \underset{t}{\overset{s}{\rightrightarrows}} Y_1 \times_{Y_0} X_0 \longrightarrow W$$

is a split coequalizer.

Then the morphism $X_\bullet \to Y_\bullet$ is affine.

Proof. Split coequalizers are preserved under passage to sheaves. □

Remark 8.5. The condition given in the corollary is not necessary. For this, the split coequalizer hypothesis should be replaced by an FPQC-split coequalizer hypothesis.

Appendix A. Stacks and affine groupoid schemes

This section sketches the proof of the equivalence between the 2-category of affine groupoid schemes and a suitable 2-category of stacks with presentation. This result is well known to the experts, and versions of the result are given by Pribble in [**12**], Naumann in [**11**] and in the work of Hollander [**4**]. However, these references restrict to the situation corresponding to flat affine groupoid schemes.

The proof outlined does not make any usage of general abstract faithfully flat descent arguments.

A.1. A good 2-category of stacks

The relationship between affine groupoid schemes and stacks is made explicit by introducing a suitable category of stacks with presentation.

Definition A.1. Let $\mathsf{Stack}_S^{\Delta,\mathrm{surj}}$ denote the 2-category which has the following structure:

1. Objects are 1-epimorphisms $P_X\colon X \twoheadrightarrow \mathscr{X}$, where X is an affine stack and $\mathscr{X}$ has affine diagonal.

2. A 1-morphism $(f_0, f)\colon (P_X\colon X \twoheadrightarrow \mathscr{X}) \to (P_Y\colon Y \twoheadrightarrow \mathscr{Y})$ is a 2-commutative diagram of 1-morphisms of stacks

$$\begin{array}{ccc} X & \xrightarrow{f_0} & Y \\ {\scriptstyle P_X}\downarrow & & \downarrow{\scriptstyle P_Y} \\ \mathscr{X} & \xrightarrow[f]{} & \mathscr{Y}. \end{array}$$

3. A 2-morphism $(f_0, f) \to (g_0, g)$ is a 2-morphism

$$\mathscr{X} \underset{g}{\overset{f}{\genfrac{}{}{0pt}{}{\longrightarrow}{\Downarrow\ \longrightarrow}}} \mathscr{Y}$$

in the 2-category of stacks.

Remark A.2. The 1-epimorphism $P_X\colon X \to \mathscr{X}$ can be considered as a presentation of the stack $\mathscr{X}$ with affine diagonal. However, no flatness condition is imposed, so this must not be confused with terminology used elsewhere in the literature.

A.2. The groupoid associated to a stack

Recall that Gpd_S denotes the 2-category of affine groupoid schemes over S.

Consider $P_X\colon X \twoheadrightarrow \mathscr{X}$, an object of the 2-category $\mathsf{Stack}_S^{\Delta,\mathrm{surj}}$. The hypothesis that $\mathscr{X}$ has affine diagonal implies that the sheaf $X_1 := X \times_{\mathscr{X}} X$ is represented by an affine S-scheme. Proposition 2.35 implies that $(X_0 := X, X_1 := X \times_{\mathscr{X}} X)$ has the structure of an affine groupoid scheme. Moreover, there is a canonical 1-morphism of stacks

$$[X_\bullet] \xrightarrow{\cong} \mathscr{X}$$

which is an equivalence of stacks, by the hypothesis that the morphism P_X is an epimorphism.

The morphism of affine groupoid schemes $X_0 \to X_\bullet$ induces a surjective 1-morphism of stacks

$$X \twoheadrightarrow [X_\bullet] \cong \mathscr{X},$$

which is equivalent to the 1-morphism P_X.

Proposition A.3. *There is a 2-functor*

$$\mathsf{Stack}_S^{\Delta,\mathrm{surj}} \to \mathsf{Gpd}_S$$

which is induced by $(P_X\colon X \twoheadrightarrow \mathscr{X}) \mapsto (X, X \times_{\mathscr{X}} X)$.

Proof. (The proof of this statement is contained in the paper [**11**], where no usage is made of the ambient flatness hypothesis.) The functoriality of the construction of the associated affine groupoid scheme implicit in Proposition 2.35 shows that the above definition is a 1-functor, hence it remains to check that the construction defines a 2-functor.

A 2-morphism $\alpha\colon (f_0, f) \to (g_0, g)$ induces a 2-morphism between the composites $X \stackrel{f_0}{\to} Y \stackrel{P_Y}{\to} \mathscr{Y}$ and $X \stackrel{g_0}{\to} Y \stackrel{P_Y}{\to} \mathscr{Y}$, by using the 2-morphisms which make commutative the squares defining the morphisms (f_0, f), (g_0, g). Hence, by the universal property of the 2-fibre product, there exists a morphism

$$X_0 = X \to Y_1 = Y \times_{\mathscr{Y}} Y$$

which corresponds to this 2-morphism. Moreover, this morphism is canonically defined.

It is straightforward to verify that the morphism $X_0 \to Y_1$ is a 2-morphism between the associated morphisms of groupoid schemes and that the construction defines a 2-functor. □

A.3. The stack associated to an affine groupoid scheme

There are a number of proofs available of the fact that the stack associated to an affine groupoid scheme has affine diagonal. The proof presented below is chosen so as to make the arguments completely explicit.

Proposition A.4. *Let $X_\bullet$ be an affine groupoid scheme. Then the associated stack $[X_\bullet]$ has affine diagonal.*

Proof. By Corollary 8.4, it is sufficient to establish the following conditions concerning the structure of the 2-fibre product $X_\bullet \times_{X_\bullet \times X_\bullet} (X_0 \times X_0)$.

1. The diagram

$$X_0 \times_{X_0 \times X_0} (X_1 \times X_1) \to X_1 \times_{X_0 \times X_0} (X_1 \times X_1) \rightrightarrows X_0 \times_{X_0 \times X_0} (X_1 \times X_1)$$

   is an equalizer diagram.
2. There is a split coequalizer diagram

$$X_1 \times_{X_0 \times X_0} (X_1 \times X_1) \rightrightarrows X_0 \times_{X_0 \times X_0} (X_1 \times X_1) \to X_1.$$

   (To exhibit a splitting of the above diagram will establish that it is a coequalizer.)

Both of these conditions can be checked after passage to the category of presheaves of sets on Aff/S. In this case the statements become elementary statements concerning groupoids. The details are left to be supplied by the reader. □

Corollary A.5. *There is a 2-functor*

$$\mathsf{Gpd}_S \to \mathsf{Stack}_S^{\Delta,\mathrm{surj}}$$

which is induced by $X_\bullet \mapsto (X \twoheadrightarrow [X_\bullet])$.

Proof. A morphism of affine groupoid schemes $X_\bullet \to Y_\bullet$ induces a commutative diagram of 1-morphisms of stacks

$$\begin{array}{ccc} X_0 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ {[X_\bullet]} & \longrightarrow & {[Y_\bullet]} \end{array}$$

and this construction defines a functor.

A 2-morphism between morphisms of groupoids $X_\bullet \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} Y_\bullet$ naturally induces a 2-morphism of prestacks $[X_\bullet]' \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} [Y_\bullet]'$, by Yoneda. The passage to the associated stack is a 2-functor, hence this shows that the functor is indeed a 2-functor, as required. □

A.4. The equivalence

Theorem A.6. *There is an equivalence of* 2*-categories:*

$$\mathsf{Gpd}_S \cong \mathsf{Stack}_S^{\Delta,\mathrm{surj}}.$$

Proof. The equivalence is defined by the 2-functors given in Corollary A.5 and Proposition A.3. The key point of the proof is the fact that a 2-commutative diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & Y_0 \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow \\ \mathscr{M} & \longrightarrow & \mathscr{N} \end{array}$$

of 1-morphisms of stacks, where $\mathscr{M}$, $\mathscr{N}$ have affine diagonal gives rise to a canonical diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ [X_\bullet] & \longrightarrow & [Y_\bullet] \\ \cong\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow\cong \\ \mathscr{M} & \longrightarrow & \mathscr{N} \end{array}$$

in which the upper square is strictly commutative and the lower square is commutative up to canonical 2-isomorphism. □

References

[**1**] Michel Demazure and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970. With Appendix *Corps de classes local* by Michiel Hazewinkel.

[**2**] Jean Giraud, *Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften, Band 179, Springer-Verlag, Berlin, 1971.

[**3**] Paul G. Goerss, (Pre-)sheaves of ring spectra over the moduli stack of formal group laws, in *Axiomatic, enriched and motivic homotopy theory*, Volume 131, *NATO Sci. Ser. II Math. Phys. Chem.*, Kluwer Acad. Publ., Dordrecht, 2004, pp. 101–131.

[**4**] Sharon Hollander, A homotopy theory for stacks, *Israel J. Math.* To appear.

[**5**] Sharon Hollander, Characterizing algebraic stacks, *Proc. Amer. Math. Soc.*, 2007. To appear. DOI: 10.1090/S0002-9939-07-08832-6

[**6**] Sharon Hollander, Descent for quasi-coherent sheaves of stacks, *Algebr. Geom. Topol.* **7** (2007), 411–437.

[**7**] M. J. Hopkins, Complex oriented cohomology theories and the language of stacks, Notes on a course given by Mike Hopkins (1999). Available at `http://www.math.rochester.edu/u/faculty/doug/otherpapers/coctalos.pdf`.

[**8**] M. J. Hopkins and H.R. Miller, Elliptic curves and stable homotopy theory I, (Pre-release version, 29 April 1999), 1999.

[**9**] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Volume 39, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, Springer-Verlag, Berlin, 2000.

[**10**] Jack Morava, Noetherian localisations of categories of cobordism comodules. *Ann. of Math. (2)* **121**(1) (1985), 1–39.

[**11**] Niko Naumann, The stack of formal groups in stable homotopy theory, *Adv. Math.* **215**(2) (2007), 569–600.

[**12**] Ethan Pribble, *Algebraic stacks for stable homotopy theory and the algebraic chromatic convegence theorem*, PhD thesis, Northwestern University, 2004.

[**13**] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Volume 121, *Pure and Applied Mathematics*, Academic Press Inc., Orlando, FL, 1986.

[**14**] Angelo Vistoli, Grothendieck topologies, fibered categories and descent theory, in *Fundamental algebraic geometry*, Volume 123, *Math. Surveys Monogr.*, Amer. Math. Soc., Providence, RI, 2005, pp. 51–104.

[**15**] William C. Waterhouse, *Introduction to affine group schemes*, Volume 66, *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1979.

Geoffrey M.L. Powell `powell@math.univ-paris13.fr`

Laboratoire Analyse, Géométrie et Applications, UMR 7539
Institut Galilée, Université Paris 13, 93430 Villetaneuse, France