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A COHOMOLOGICAL INTERPRETATION OF BRION'S FORMULA

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Abstract

A subset P of $\mathbb{R}^n$ gives rise to a formal Laurent series with monomials corresponding to lattice points in P. Under suitable hypotheses, this series represents a rational function $R(P)$; this happens, for example, when P is bounded in which case $R(P)$ is a Laurent polynomial. Michel Brion [2] has discovered a surprising formula relating the Laurent polynomial $R(P)$ of a lattice polytope P to the sum of rational functions corresponding to the supporting cones subtended at the vertices of P. The result is re-phrased and generalised in the language of cohomology of line bundles on complete toric varieties. Brion's formula is the special case of an ample line bundle on a projective toric variety. The paper also contains some general remarks on the cohomology of torus-equivariant line bundles on complete toric varieties, valid over arbitrary commutative ground rings.

1. Introduction

The main result of this paper is a generalisation of a formula discovered by Brion, relating the lattice point enumerator of a rational polytope to the lattice point enumerators of supporting cones subtended at its vertices [2, §2.2]. (See [1] for an introduction to the theory and an elementary proof based on "irrational decompositions".) In spirit the proof of the generalisation is similar to Brion's original exposition, but avoids the use of equivariant K-theory in favour of a more elementary treatment of cohomology of line bundles on complete toric varieties.

Since line bundles are encoded by support functions defined on a fan, the result can be re-formulated in combinatorial terms. This has been done for upper convex support functions (corresponding to line bundles which are generated by global sections) by Ishida [6, Theorem 2.3], generalising the original result of Brion. The present paper goes one step further and includes the case of arbitrary, non-convex support functions.

We will give a precise formulation of the result below. Roughly speaking, we prove that a sum of certain rational functions, all given by infinite Laurent series,

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degenerates to a Laurent polynomial, and interpret the coefficients of the occurring monomials as homogeneous parts of equivariant Euler characteristics of the sheaf cohomology of a torus-equivariant (algebraic) line bundle.

The proof relies on a new non-standard computation of the cohomology of line bundles on complete toric varieties (Theorem 2.2) which is similar to, but easier than, the standard result as given by Oda [7, Theorem 2.6] and Danilov [4, Theorem 7.2]. This computation in turn depends on a variant of Čech cohomology (Proposition 2.1) which should be well-known. Since it seems not to be well-documented in available publications, we include a proof at the end of the paper (§4).

Notational conventions and the main result

We have to introduce some notation first. Let $M \cong \mathbb{Z}^n$ be a lattice of rank n. We call the set of maps $S = \mathrm{map}(M, \mathbb{C})$ the set of *formal Laurent series*. Given an element $\mathbf{b} \in M$, we let $x^{\mathbf{b}} \in S$ denote the map which is zero on $M \setminus \{\mathbf{b}\}$, and takes the value 1 on $\mathbf{b}$. We call $x^{\mathbf{b}}$ the *Laurent monomial* with exponent $\mathbf{b}$.

The terminology can be justified. Given a choice of basis $e_1, e_2, \ldots, e_n$ of M, we can write every element $\mathbf{b} \in M$ uniquely as $\mathbf{b} = \sum_j b_j e_j$ with $b_j \in \mathbb{Z}$. Then for $f \in S$ the formal sum

$$\sum_{\mathbf{b} \in M} f(\mathbf{b}) \cdot x_1^{b_1} x_2^{b_2} \ldots\ x_n^{b_n}$$

is a Laurent series in the indeterminates $x_1, x_2, \ldots, x_n$. The map $x^{\mathbf{b}}$ corresponds to the product $x_1^{b_1} x_2^{b_2} \ldots\ x_n^{b_n}$, i.e., a series with a single non-trivial summand.

Let $P \subset S$ denote the subset of maps with finite support; in particular, it contains the maps $x^{\mathbf{b}}$ defined above. After choosing a basis of M, we can identify P with the ring of Laurent polynomials in n indeterminates. On the level of maps, the product is given by a convolution formula. The same formula equips S with the usual structure of a P-module.

Set $M_{\mathbb{R}} = M \otimes \mathbb{R} \cong \mathbb{R}^n$. We consider M as a subset of $M_{\mathbb{R}}$ using the natural identification $M = M \otimes 1$. Given a subset $K \subseteq M_{\mathbb{R}}$ and an element $b \in M_{\mathbb{R}}$, we define

$$b + K = \{b + x \,|\, x \in K\} \qquad \text{and} \qquad -K = \{-x \,|\, x \in K\}.$$

Definition 1.1. For a subset $K \subseteq M_{\mathbb{R}}$ we define the formal Laurent series

$$R[K] = \sum_{\mathbf{a} \in M \cap K} x^{\mathbf{a}} \ \in S.$$

A straightforward calculation shows $R[\mathbf{b} + K] = x^{\mathbf{b}} R[K]$ for any $\mathbf{b} \in M$.

In favourable cases, for example, when K is a pointed rational polyhedral cone in $M_{\mathbb{R}}$, the series $R[K]$ represents a rational function (an element in the quotient field $Q(P)$ of P) which we will denote $R(K) \in Q(P)$.

As an explicit example, for

$$K = \mathbb{R}_{\leqslant 2} = 2 + \mathbb{R}_{\leqslant 0} \subset \mathbb{R}$$

we have $R[K] = x^2 R[\mathbb{R}_{\leqslant 0}] = x^2 \sum_{a \leqslant 0} x^a$, so $R(K) = x^2/(1 - x^{-1})$. See [1] for more examples.

Let $N = \hom_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathbb{Z}^n$ be the dual lattice of M. Then $N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{R}^n$ is naturally the dual of the $\mathbb{R}$-vector space $M_{\mathbb{R}}$. The duals of N and $N_{\mathbb{R}}$ are canonically isomorphic to M and $M_{\mathbb{R}}$, respectively.

Let Σ be a finite complete fan in $N_{\mathbb{R}}$, consisting of strongly convex rational polyhedral cones, and denote by X_Σ the associated toric variety defined over $\mathbb{C}$. (See [7] for details on cones, fans, and the relation to varieties.) Let $h\colon N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be a support function on Σ. On each cone $\sigma \in \Sigma$, it coincides with a linear function $h_\sigma \otimes \mathrm{id}_{\mathbb{R}}$ for some $h_\sigma \in \hom_{\mathbb{Z}}(N, \mathbb{Z}) = M$. Define the rational function

$$R(\Sigma, h) = \sum_{\substack{\sigma \in \Sigma \\ \dim \sigma = n}} R(-h_\sigma + \sigma^\vee)$$

where $\sigma^\vee = \{x \in M_{\mathbb{R}} \mid \forall y \in \sigma : \langle x, y\rangle \geqslant 0\}$ is the dual cone, defined using the standard evaluation pairing $\langle x, y\rangle = y(x)$. Denote the torus-equivariant line bundle on X_Σ associated to h by L_h; see §2.2 below for an explicit description. The torus $\operatorname{Spec} \mathbb{C}[M]$ acts naturally on the cohomology vector spaces $H^k(X_\Sigma; L_h)$ which consequently acquire an M-grading (note that M is the character group of the torus). Given a vector $\mathbf{a} \in M$, we write $H^k(X_\Sigma; L_h)_{\mathbf{a}}$ for the homogeneous part of degree $\mathbf{a}$ of the k-th sheaf cohomology of L_h. See §2.2 below for an elementary description.

Theorem 1.2. *The rational function $R(\Sigma, h)$ is a Laurent polynomial. The coefficient of the monomial $x^{\mathbf{a}}$ in the polynomial $R(\Sigma, h)$ is the Euler characteristic of $H^*(X_\Sigma; L_h)_{\mathbf{a}}$, so it is given by the alternating sum*

$$\chi(L_h)_{\mathbf{a}} = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{C}} H^k(X_\Sigma; L_h)_{\mathbf{a}}.$$

In short, we have the equality

$$R(\Sigma, h) = \sum_{\mathbf{a} \in M} \chi(L_h)_{\mathbf{a}} \cdot x^{\mathbf{a}}. \tag{1}$$

Brion's formula for lattice polytopes

For an n-dimensional polytope $K \subset M_{\mathbb{R}}$ with vertices in M, let Σ_K denote the inner normal fan of K. The support function of K, given by

$$h_K\colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad x \mapsto -\inf\{\langle p, x\rangle \mid p \in K\}$$

defines a support function on Σ_K. Explicit calculation of sheaf cohomology shows

$$H^k(X_{\Sigma_K}; L_{h_K}) = \begin{cases} 0 & \text{if } k \neq 0 \\ \bigoplus_{M \cap K} \mathbb{C} & \text{if } k = 0. \end{cases}$$

(See [7, Corollary 2.9], or [5, Theorem 2.5.3] for an elementary proof.) If σ is an n-dimensional cone in Σ_K, and h_σ is the linear function associated to h_K and σ, then $-h_\sigma + \sigma^\vee$ is the support cone of K subtended at the vertex corresponding to σ. Thus Theorem 1.2 reduces to the original theorem of Brion [2, §2.2]: The rational function $R(\Sigma_K, h_K)$ is a Laurent polynomial with terms corresponding to the integral points of K.

Similarly, by considering the support function $-h_K$, and using the calculation

$$H^k(X_{\Sigma_K}; L_{-h_K}) = \begin{cases} 0 & \text{if } k \neq n \\ \bigoplus_{M\cap \operatorname{int}(-K)} \mathbb{C} & \text{if } k = n \end{cases}$$

(see [4, §11.12.4], or [5, Theorem 2.5.3] for an elementary proof), we see that the rational function $R(\Sigma_K, -h_K)$ is a Laurent polynomial with summands corresponding to the integral points in the interior of $-K$, up to a factor of $(-1)^n$ [2, §2.5].

Finally, by considering a globally linear support function $h = \mathbf{a} \in M$, so that $L_{\mathbf{a}} \cong \mathcal{O}_{X_{\Sigma_K}}$, we see that Theorem 1.2, together with the calculation

$$H^k(X_{\Sigma_K}; L_{\mathbf{a}}) = \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{C} & \text{if } k = 0 \end{cases}$$

with H^0 concentrated in homogeneous degree $\mathbf{a}$ (see [4, Corollary 7.4], or [5, Theorem 2.5.3] for an elementary treatment), gives $R(\Sigma_K, \mathbf{a}) = x^{\mathbf{a}}$ (cf. [6, Corollary 2.4]; see [3, Proposition 3.1] for a Laurent series version).

The cohomology calculations in this subsection can be done with the aid of Theorem 2.2 below; in essence, one has to check that certain subcomplexes of the sphere S^{n-1} are contractible. This is what is behind the calculations in the paper [5] which, however, uses a dual point of view, using the fact that the fans considered above are normal fans of polytopes. We omit the details.

Ishida's formula

If the support function h is upper convex (equivalently, if the associated line bundle is generated by global sections), then the negatives of the linear functions h_σ for n-dimensional cones $\sigma \in \Sigma$ span a polytope Q in $M_{\mathbb{R}}$ with vertices in M. Since

$$H^k(X_{\Sigma_P}; L_h) = \begin{cases} 0 & \text{if } k \neq 0 \\ \bigoplus_{M\cap Q} \mathbb{C} & \text{if } k = 0 \end{cases}$$

(see [7, Corollary 2.9]), our Theorem 1.2 specialises to [6, Theorem 2.3] for complete fans.

An explicit example

Example 1.3. We consider the case $n = 2$, $N = \mathbb{Z}^2$ and $N_{\mathbb{R}} = \mathbb{R}^2$. Let Σ be the unique complete fan in $N_{\mathbb{R}}$ whose 1-cones are generated by the following four vectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Let $X, Y \in M = \hom_{\mathbb{Z}}(N, \mathbb{Z})$ denote the dual of the standard basis of $N = \mathbb{Z}^2$. Let $h\colon N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be the support function specified by the values

$$h(v_1) = 0, \qquad h(v_2) = -2, \qquad h(v_3) = 0, \qquad h(v_4) = -2,$$

given by extending linearly over cones. For example, on $\sigma = \operatorname{cone}(v_1, v_2)$ it agrees with the linear function $h_\sigma = 2X - 2Y \in M$ which corresponds to a Laurent monomial written x^2y^{-2}. Using Theorem 2.2 we can explicitly compute the cohomology of L_h (see the end of §2 below). It turns out that $H^0(X_\Sigma; L_h) = 0$, and that

$\dim H^2(X_\Sigma; L_h) = 1$ concentrated in homogeneous degree $-Y \in M$. The vector space $H^1(X_\Sigma; L_h)$ is 4-dimensional, with a 1-dimensional contribution coming from degrees 0, $-X+Y$, Y and $X+Y$. The right-hand side of equation (1) thus is (denoting the indeterminates again by x and y)

$$-1 - x^{-1}y - y - xy + y^{-1}.$$

The left-hand side is worked out easily as well. For example, the summand corresponding to $\sigma = \mathrm{cone}(v_1, v_2)$ is the rational function represented by the lattice point enumerator of the shifted cone

$$-h_\sigma + \sigma^\vee = (-2X + 2Y) + \mathrm{cone}\,(\,X, -X+Y\,) \;\subset M_\mathbb{R}$$

or, in coordinates of $M_\mathbb{R} \cong \mathbb{R}^2$,

$$-h_\sigma + \sigma^\vee = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \mathrm{cone}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

This lattice point enumerator is given by

$$\frac{x^{-2}y^2}{(1-x^{-1}y)(1-x)}.$$

In total, the left-hand side of equation (1) equals

$$\frac{x^{-2}y^2}{(1-x^{-1}y)(1-x)} + \frac{x^2y^2}{(1-x^{-1})(1-xy)} + \frac{x^{-2}y^{-2}}{(1-x^{-1})(1-x^{-1}y^{-1})} + \frac{x^2y^{-2}}{(1-x)(1-xy^{-1})},$$

which coincides with the Laurent polynomial $-1 - x^{-1}y - y - xy + y^{-1}$, as an explicit calculation shows.

2. Cohomology of line bundles

2.1. Čech cohomology of quasi-coherent sheaves

Let Σ be a finite complete fan in $N_\mathbb{R}$. By taking intersection of positive-dimensional cones with the unit sphere S^{n-1} (defined with respect to any inner product), the fan induces the structure of a regular CW-complex on S^{n-1}. Given a cone $\sigma \in \Sigma$, we write $\bar\sigma = \sigma \cap S^{n-1}$ for the corresponding cell of S^{n-1}. This includes the case of the empty cell $\bar 0$. We fix once and for all orientations of the cells and write $[\bar\sigma : \bar\tau]$ for the incidence number of $\bar\sigma$ and $\bar\tau$. By convention, we have $[\bar\tau : \bar 0] = 1$ for all 1-dimensional cones $\tau \in \Sigma$. Regularity of the CW-decomposition implies that $[\bar\sigma : \bar\tau] \in \{-1, 0, 1\}$ for all cones $\sigma, \tau \in \Sigma$. Note that in the (augmented) cellular chain complex of S^{n-1} the empty cell corresponds to the augmentation (concentrated in degree -1).

The computation of Čech cohomology does not depend on using the complex numbers as a ground field, so let A denote a commutative ring, and let X_Σ denote the toric A-scheme associated to Σ. It is obtained by gluing the affine A-schemes $\mathrm{Spec}\, A[M \cap \sigma^\vee]$ where σ varies over the elements of Σ.

Proposition 2.1. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X_Σ. Then we can compute the cohomology modules $H^k(X_\Sigma;\mathcal{F})$ as the cohomology of the Čech cochain complex $\mathcal{C}^\bullet = (\mathcal{C}^\bullet, d)$ which is defined by*

$$\mathcal{C}^d = \bigoplus_{\substack{\sigma\in\Sigma \\ \operatorname{codim}\sigma=d}} \mathcal{F}^\sigma,$$

with differential defined on direct summands by

$$\mathcal{F}^\sigma \xrightarrow{[\bar\sigma:\bar\tau]} \mathcal{F}^\tau.$$

Here $\mathcal{F}^\sigma$ denotes the module of sections of $\mathcal{F}$ over the affine open subset of X_Σ determined by the cone σ. In particular, $\mathcal{F}^\sigma$ is an $A[M\cap\sigma^\vee]$-module. The family $(\mathcal{F}^\sigma)_{\sigma\in\Sigma}$ of modules determines $\mathcal{F}$ completely.

This variant of Čech cohomology should be well-known, but unfortunately there seems to be no published proof available. For the reader's convenience we give a proof in §4 below.

2.2. Torus-equivariant line bundles

We will apply Proposition 2.1 in the case where $\mathcal{F}$ is a torus-equivariant line bundle on X_Σ. Recall [7, §2] that such a sheaf is specified by a support function $h\colon N_\mathbb{R} \longrightarrow \mathbb{R}$ which is linear on each cone, and takes integral values on N. In other words, for each $\sigma\in\Sigma$ there exists $h_\sigma \in \hom_\mathbb{Z}(N,\mathbb{Z}) = M$ such that $h|_\sigma = (h_\sigma \otimes \mathrm{id}_\mathbb{R})\,|_\sigma$. The linear function h_σ is well-defined up to the addition of a linear function which vanishes on $N\cap\sigma$; cf. [4, §6.2].

The line bundle L_h corresponding to a support function h has a very explicit description: On the affine open set corresponding to $\sigma\in\Sigma$ the space of sections is the free $A[M\cap\sigma^\vee]$-module of rank 1 with basis $-h_\sigma$. Note that all these modules are contained in the free A-module $A[M]$ with basis M; hence we may consider them as M-graded A-modules.

We can apply Proposition 2.1 to the line bundle L_h. The resulting cochain complex of A-modules has a natural M-grading, and the differentials are homogeneous of degree 0 with respect to this grading (in the language of free modules, all the terms in $\mathcal{C}^\bullet$ have a basis consisting of a subset of M, and all structure maps are induced by inclusion of subsets.) Hence the cohomology modules $H^k(X_\Sigma;L_h)$ have a direct sum decomposition

$$H^k(X_\Sigma;L_h) = \bigoplus_{\mathbf{b}\in M} H^k(X_\Sigma;L_h)_\mathbf{b}$$

with $H^k(X_\Sigma;L_h)_\mathbf{b}$ being isomorphic to the cohomology of the degree-$\mathbf{b}$ sub-cochain complex $\mathcal{C}^\bullet_\mathbf{b} = (\mathcal{C}^\bullet_\mathbf{b}, d)$ of $\mathcal{C}^\bullet$.

The cochain complex $\mathcal{C}^\bullet_\mathbf{b}$ itself admits a simple description: It is given by

$$\mathcal{C}^d_\mathbf{b} = \bigoplus_{\substack{\sigma\in\Sigma,\ \operatorname{codim}\sigma=d \\ \mathbf{b}+h_\sigma\in\sigma^\vee}} A$$

with differential induced by incidence numbers as before. Now if τ is a face of

$\sigma \in \Sigma$, then $\sigma^\vee \subseteq \tau^\vee$, so $\mathbf{b} + h_\sigma \in \sigma^\vee$ implies $\mathbf{b} + h_\sigma \in \tau^\vee$. Hence $\mathbf{b} + h_\tau \in \tau^\vee$ (for $h_\tau - h_\sigma \in \tau^\vee$, and $\tau^\vee$ is closed under addition since it is a convex cone). Thus the set

$$S(h, \mathbf{b}) := \bigcup_{\substack{\sigma \in \Sigma \\ \mathbf{b} + h_\sigma \in \sigma^\vee}} \bar{\sigma} \tag{2}$$

is a sub-complex of S^{n-1}, and $\mathcal{C}^\bullet_{\mathbf{b}}$ is nothing but the augmented cellular chain complex of $S(h, \mathbf{b})$, re-indexed suitably as a cochain complex. In other words, we have shown:

Theorem 2.2. *Suppose Σ is a complete fan in $N_\mathbb{R}$, and $h\colon N_\mathbb{R} \longrightarrow \mathbb{R}$ is a support function on Σ. Let L_h denote the torus-equivariant line bundle on X_Σ associated to h, and define the space $S(h, \mathbf{b})$ as in (2). For all $\mathbf{b} \in M$ there is an isomorphism of A-modules*

$$H^k(X_\Sigma; L_h)_\mathbf{b} \cong \tilde{H}_{n-1-k}(S(h, \mathbf{b}); A)$$

where $\tilde{H}_d(\,\cdot\,; A)$ denotes reduced cellular (or singular) homology with coefficients in A.

For this to make sense, it is imperative to consider the augmented cellular chain complex to compute $\tilde{H}_d$ with augmentation concentrated in degree -1. In other words, $\tilde{H}_{-1}(\emptyset) = A$ by convention, while $\tilde{H}_{-1}(X) = 0$ whenever $X \neq \emptyset$.

The advantage of Theorem 2.2 over the standard result as given in [7, Theorem 2.6] is that the former deals with the cell complex $S(h, \mathbf{b})$ arising as the intersection of a sub-fan of Σ with S^{n-1}, whereas the latter relies on computing certain subsets of $N_\mathbb{R}$ with a rather more delicate combinatorial structure.

The theorem leads immediately to some general observations (which could also be verified using Serre duality). For example, the remark following Theorem 2.2 implies:

Corollary 2.3. *The top-dimensional cohomology is given by*

$$H^n(X_\Sigma; L_h)_\mathbf{b} = \begin{cases} 0 & \text{if there exists } \sigma \in \Sigma,\ \sigma \neq \{0\} \text{with } \mathbf{b} + h_\sigma \in \sigma^\vee, \\ A & \text{otherwise.} \end{cases}$$

Moreover, if $H^n(X_\Sigma; L_h)_\mathbf{b} = A$, then $H^k(X_\Sigma; L_h)_\mathbf{b} = 0$ for all $k \neq n$.

Suppose now that K is a subcomplex of S^{n-1}. Then $\tilde{H}_{n-1}(K; A) \neq 0$ if and only if $K = S^{n-1}$. Indeed, if $K \neq S^{n-1}$, then K misses an $(n-1)$-dimensional cell of S^{n-1}; i.e., there exists an n-dimensional cone $\sigma \in \Sigma$ such that K is contained in $S^{n-1} \setminus \operatorname{int} \bar{\sigma}$. Now $S^{n-1} \setminus \operatorname{int} \bar{\sigma}$ is contractible; hence it has trivial reduced homology. The homology long exact sequence of the pair $(K, S^{n-1} \setminus \operatorname{int} \bar{\sigma})$ proves the assertion.

If there exists $\mathbf{b} \in M$ such that $\mathcal{S}(h, \mathbf{b}) = S^{n-1}$, then $\mathbf{b}$ is contained in the intersection of the closed half-spaces $-h_\rho + \rho^\vee$ where ρ varies over the 1-dimensional cones in Σ. Since Σ is complete, this implies that for all $\mathbf{a} \in \mathbb{Z}^n$ there exists $\rho \in \Sigma$ with $\mathbf{a} \in -h_\rho + \rho^\vee$; thus $\mathcal{S}(h, \mathbf{a}) \neq \emptyset$. Conversely, if $\mathcal{S}(h, \mathbf{b}) = \emptyset$ for some $\mathbf{b} \in M$ then there is no $\mathbf{a} \in M$ with $\mathcal{S}(h, \mathbf{a}) = S^{n-1}$ (in fact, there is a 1-dimensional cone $\rho \in \Sigma$ with $\mathbf{a} \notin -h_\rho + \rho^\vee$). Together with Theorem 2.2, this shows that the line bundle L_h cannot have global sections and n-th cohomology at the same time:

Corollary 2.4. *At least one of the A-modules $H^0(X_\Sigma; L_h)$ and $H^n(X_\Sigma; L_h)$ is trivial.*

2.3. On Example 1.3

Recall the notation from Example 1.3; we will use the field $A = \mathbb{C}$ of complex numbers. To work out the complex $\mathcal{S}(h, \mathbf{b}) \subseteq S^1$ for given $\mathbf{b} \in M$, one can start from a sketch of the halfspace arrangement $-h_{\rho_j} + \rho_j^\vee$, $j = 1, 2, 3, 4$ given by the shifted duals of the 1-dimensional cones in Σ. Furthermore, it is enough to consider those $\mathbf{b}$ which are contained in some bounded region of the resulting decomposition of $M_\mathbb{R}$ since $H^*(X_\Sigma; L_h)$ is finite-dimensional.

In our example, this leaves us to check contributions from five elements of M only. We will use coordinate notation for this paragraph. It is easily verified that $\mathcal{S}(h, (0, -1)^t) = \emptyset$, so $H^2(X_\Sigma; L_h)_{(0,-1)^t} = \tilde{H}_{-1}(\emptyset) = \mathbb{C}$. If $\mathbf{b}$ is one of the vectors $\{(0, 0)^t,\ (-1, 1)^t,\ (0, 1)^t,\ (1, 1)^t\}$, then $\mathcal{S}(h, \mathbf{b})$ is a 0-sphere corresponding to the intersection of the cones spanned by v_1 and v_3 with the unit sphere in $N_\mathbb{R} = \mathbb{R}^2$. Thus $H^1(X_\Sigma; L_h)_\mathbf{b} = \tilde{H}_0(S^0) = \mathbb{C}$ in these cases.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 proceeds by verifying a Laurent series identity first. Let as before $h \colon N_\mathbb{R} \longrightarrow \mathbb{R}$ be a support function, and choose corresponding linear functions $h_\sigma \in M$ for $\sigma \in \Sigma$ (§2). Define a formal Laurent power series

$$R[\Sigma, h] = \sum_{\sigma \in \Sigma} (-1)^{\operatorname{codim} \sigma} R[-h_\sigma + \sigma^\vee]. \tag{3}$$

Fix $\mathbf{a} \in M$; we want to consider the coefficient of $x^\mathbf{a}$ in $R[\Sigma, h]$. The summand corresponding to $\sigma \in \Sigma$ contributes 0 if $\mathbf{a} + h_\sigma \notin \sigma^\vee$, and it contributes $(-1)^{\operatorname{codim} \sigma}$ otherwise. Since $0^\vee = \mathbb{R}^n$, we get a contribution of $(-1)^n$ for $\sigma = 0$ always. In other words, the coefficient of $x^\mathbf{a}$ is the Euler characteristic of the chain complex $\mathcal{C}^\bullet_\mathbf{a}$ (§2.2); using $A = \mathbb{C}$ again:

$$\sum_{k=0}^{n} (-1)^k \dim_\mathbb{C} \mathcal{C}^k_\mathbf{a} = \chi(\mathcal{C}^\bullet_\mathbf{a}).$$

The Euler characteristic can be computed using the cohomology groups of the cochain complex as well. Since $H^k(\mathcal{C}^\bullet_\mathbf{a}) = \tilde{H}_{n-1-k}(S(h, \mathbf{a}))$ (cf. §2.2), the coefficient of $x^\mathbf{a}$ is given by

$$\chi(\mathcal{C}^\bullet_\mathbf{a}) = \sum_{k=0}^{n} (-1)^k \dim_\mathbb{C} \tilde{H}_{n-1-k} S(h, \mathbf{a}). \tag{4}$$

Using Theorem 2.2, we see that this is equal to $\sum_{k=0}^n (-1)^k \dim_\mathbb{C} H^k(X_\Sigma; L_h)_\mathbf{a}$. Since the cohomology of L_h is finitely generated (the variety X_Σ is complete by hypothesis), we see that this coefficient is zero for almost all $\mathbf{a} \in M$. In particular, $R[\Sigma, h]$ is a Laurent polynomial.

Let Π denote the P-submodule of S generated by the rational functions corresponding to rational polyhedral cones. According to [6, Theorem 1.2], there is a unique P-linear homomorphism $\rho\colon \Pi \longrightarrow Q(P)$ (here $Q(P)$ denotes the quotient field of P as before) with $\rho(R[b+\sigma]) = R(b+\sigma)$ for all $b \in M_{\mathbb{R}}$ and all pointed rational polyhedral cones $\sigma \subset M_{\mathbb{R}}$ (see also [1, Theorem 2.4]). Note that ρ preserves Laurent polynomials as they are finite sums of Laurent power series associated to sets of the form $\mathbf{a} + \{0\}$. In particular, $\rho(x^{\mathbf{b}}) = x^{\mathbf{b}} \in P \subset Q(P)$ for all $\mathbf{b} \in M$. If the rational polyhedral cone σ contains a line, then it can be shown that $\rho(K[\sigma]) = 0$; cf. [6, Lemma 2.1] or [1, Lemma 2.5].

We now apply the homomorphism ρ to the Laurent power series $R[\Sigma, h]$. On the one hand, we have

$$\begin{aligned}\rho(R[\Sigma,h]) &= \sum_{\sigma\in\Sigma}(-1)^{\operatorname{codim}\sigma}\rho(R[-h_\sigma+\sigma^\vee]) \\ &= \sum_{\substack{\sigma\in\Sigma\\ \dim\sigma=n}}(-1)^{\operatorname{codim}\sigma}\rho(R[-h_\sigma+\sigma^\vee]) \\ &= \sum_{\substack{\sigma\in\Sigma\\ \dim\sigma=n}} R(-h_\sigma+\sigma^\vee) \\ &= R(\Sigma,h).\end{aligned}$$

(The second equality comes from the fact that if $\operatorname{codim}\sigma > 0$, then the dual cone $\sigma^\vee$ contains a line.) On the other hand, we have already seen that $R[\Sigma, h]$ is a Laurent polynomial. Hence $R(\Sigma, h) = \rho(R[\Sigma, h]) = R[\Sigma, h]$ is a Laurent polynomial as well, and, as seen before, the coefficient of $x^{\mathbf{a}}$ is given by $\chi(H^*(X_\Sigma; L_h)_{\mathbf{a}})$. This finishes the proof.

As a final remark, we can also use equation (4) to identify the coefficients of the monomials in $R(\Sigma, h)$ as this is an intermediate step in the above proof. The result then reads:

Corollary 3.1. *The coefficient of $x^{\mathbf{a}}$ in $R(\Sigma, h)$ is equal to $(-1)^{n-1}\tilde{\chi}(S(h, \mathbf{a}))$, the reduced Euler characteristic of the cell complex $S(h, \mathbf{a})$, up to sign. In other words,*

$$R(\Sigma,h) = (-1)^{n-1}\sum_{\mathbf{a}\in M}\tilde{\chi}(S(h,\mathbf{a}))\cdot x^{\mathbf{a}}.$$

4. Proof of Proposition 2.1

Let A denote a commutative ring with unit. For a complete fan Σ in $N_{\mathbb{R}}$ we let X_Σ denote the associated toric scheme defined over A. A quasi-coherent sheaf of modules $\mathcal{F}$ on X_Σ determines, by evaluation on affine pieces, a diagram of A-modules

$$D(\mathcal{F}) = D\colon \Sigma^{\mathrm{op}} \longrightarrow A\text{-Mod}, \quad \sigma \mapsto D^\sigma = \mathcal{F}^\sigma$$

(where as before $\mathcal{F}^\sigma$ denotes the A-module of sections of $\mathcal{F}$ over the open affine subset of X_Σ determined by σ, cf. Proposition 2.1). The functor $\mathcal{F} \mapsto D(\mathcal{F})$ is exact: A short exact sequence of quasi-coherent sheaves yields a short exact sequence of diagrams. We need the fact that we can compute sheaf cohomology by higher derived limits of the associated diagram:

Lemma 4.1. *There are canonical isomorphisms*

$$H^j(X_\Sigma; \mathcal{F}) \cong \lim_{\leftarrow}{}^j D(\mathcal{F}).$$

Proof. Given a cone $\sigma \in \Sigma$, write U_σ for the open affine subset of X_Σ determined by σ. Then by construction $D(\mathcal{F})^\sigma = \Gamma(U_\sigma; \mathcal{F})$, and the case $j = 0$ of the lemma is just the sheaf axiom: A global section is uniquely determined by a collection of compatible local section.

Recall now that sheaf cohomology can be computed with flasque resolutions. That is, considering $\mathcal{F}$ as a sheaf of abelian groups, choose a resolution

$$\mathcal{F} \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_1 \longrightarrow \cdots \tag{5}$$

with all the $\mathcal{G}_i$ being flasque. Let $U \subseteq X_\Sigma$ be an open subset; then $\mathcal{G}_i|_U$ is flasque, and $H^j(U; \mathcal{F})$ is isomorphic to the cohomology groups of the cochain complex

$$\Gamma(U; \mathcal{G}_0|_U) \longrightarrow \Gamma(U; \mathcal{G}_1|_U) \longrightarrow \cdots . \tag{6}$$

Passing to associated diagrams of abelian groups, the resolution (5) gives rise to a cochain complex

$$D(\mathcal{F}) \longrightarrow D(\mathcal{G}_0) \longrightarrow D(\mathcal{G}_1) \longrightarrow \cdots . \tag{7}$$

We claim that this is in fact a resolution of $D(\mathcal{F})$, considered as a diagram of abelian groups. Indeed, given a cone $\sigma \in \Sigma$ the cochain complex

$$D(\mathcal{G}_0)^\sigma \longrightarrow D(\mathcal{G}_1)^\sigma \longrightarrow \cdots$$

is nothing but the cochain complex (6) for $U = U_\sigma$. Hence its j-th cohomology group is isomorphic to $H^j(U_\sigma; \mathcal{F})$, but U_σ is affine and $\mathcal{F}$ quasi-coherent, so these groups vanish for $j \geqslant 1$, proving the claim.

We observe that the resolution (7) is flasque in the sense that the canonical restriction maps

$$D(\mathcal{G}_i)^\sigma \longrightarrow \lim_{\tau \subset \sigma} D(\mathcal{G}_i)^\tau \tag{8}$$

are surjective. Indeed, using the definition of associated diagrams, the map (8) corresponds to the restriction map

$$\Gamma(U_\sigma; \mathcal{G}_i) \longrightarrow \Gamma(\bigcup_{\tau \subset \sigma} U_\tau; \mathcal{G}_i)$$

which is surjective since $\mathcal{G}_i$ is flasque. Hence we can use the resolution (7) to compute higher derived inverse limits of $D(\mathcal{F})$ by applying the functor lim to (7), then taking cohomology groups. However, applying lim to (7) yields precisely the cochain complex (6) for $U = X_\Sigma$, which computes $H^j(X_\Sigma; \mathcal{F})$. Taking into account the well-known fact that the higher derived inverse limits of a diagram of A-modules can be computed in the category of diagrams of abelian groups, we have thus proved the lemma. $\square$

The proof of Proposition 2.1 thus reduces to proving the following claim:

Proposition 4.2. *Let $D\colon \Sigma^{\mathrm{op}} \longrightarrow A\text{-Mod}$, $\sigma \mapsto D^\sigma$ be a diagram of A-modules, where A is an arbitrary ring with unit. Form the cochain complex $\mathcal{C}^\bullet = \mathcal{C}(D)^\bullet$ by setting*

$$\mathcal{C}^k = \mathcal{C}(D)^k = \bigoplus_{\substack{\sigma\in\Sigma \\ \operatorname{codim}\sigma = k}} D^\sigma,$$

with differential defined on direct summands by

$$D^\sigma \xrightarrow{[\bar\sigma:\bar\tau]} D^\tau.$$

Then the Čech cohomology modules $\check{H}^k(D) = h^k(\mathcal{C}(D)^\bullet)$ are naturally isomorphic to the higher derived inverse limits $\lim^k(D)$.

The proof of the proposition will occupy the rest of this section. First, for $n = 1$ we know that Σ consists of the zero-cone, $\mathbb{R}_{\geqslant 0}$ and $\mathbb{R}_{\leqslant 0}$. The cochain complex $\mathcal{C}^\bullet$ has the form

$$D^{\mathbb{R}_{\geqslant 0}} \oplus D^{\mathbb{R}_{\leqslant 0}} \xrightarrow{+} D^{\{0\}},$$

and the result is well-known in this case.

We can thus restrict to the case $n \geqslant 2$. We extend the cell structure on S^{n-1} introduced in §2.1 to a regular cell structure on B^n with a single n-cell denoted $\bar B$. The canonical maps $\lim(D) \longrightarrow D^\sigma$, modified by the incidence numbers $[\bar B : \bar\sigma]$, assemble to a map

$$\iota\colon \lim(D) \longrightarrow \bigoplus_{\substack{\sigma\in\Sigma \\ \dim\sigma = n}} D^\sigma = \mathcal{C}^0$$

which is, by the properties of incidence numbers, a co-augmentation of the cochain complex $\mathcal{C}^\bullet$.

Lemma 4.3. *The map ι is injective and induces an isomorphism* $\lim(D) \cong \check{H}^0(D)$.

Proof. An element of $\lim(D)$ is determined by its images in the D^σ where σ ranges over all n-dimensional cones of Σ. Conversely, an element of $\mathcal{C}^0$ lies in the kernel of $\mathcal{C}^0 \longrightarrow \mathcal{C}^1$ if and only if its components in D^σ and D^τ agree in $D^{\sigma\cap\tau}$ where $\sigma, \tau \in \Sigma$ are n-dimensional cones with $(n-1)$-dimensional intersection. Such an element thus determines a unique element of $\lim D$ mapping to the given element of $\mathcal{C}^0$. □

Observe now that the functor $D \mapsto \check{H}^*(D)$ is a δ-functor [8, §2.1]. Indeed, a short exact sequence of diagrams

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0 \tag{9}$$

gives rise to a short exact sequence of cochain complexes, hence by the Snake Lemma to an associated natural long exact sequence in cohomology. Since $D \mapsto \lim^*(D)$ is a universal δ-functor [8, §2.1 and §2.5], it follows that we have uniquely determined natural maps

$$\nu_k\colon \lim^k(D) \longrightarrow \check{H}^k(D)$$

such that ν_0 is the isomorphism of Lemma 4.3, and such that the ν_k give rise to a commutative ladder diagram in cohomology for every short exact sequence of the form (9).

To prove that the maps ν_k are isomorphisms, we consider a decreasing filtration of the diagram D. For $0 \leqslant j \leqslant n$ we write

$$\kappa_j D\colon \Sigma^{\mathrm{op}} \longrightarrow A\text{-Mod}, \quad \sigma \mapsto \begin{cases} D^\sigma & \text{if } \operatorname{codim}\sigma \leqslant j \\ 0 & \text{else.} \end{cases}$$

Lemma 4.4. *The maps ν_k are isomorphisms for all diagrams of the form $\kappa_0 D$.*

Proof. The diagram $\kappa_0 D$ has non-zero values only on n-dimensional cones; hence $\check{H}^k(\kappa_0 D) = 0$ for $k > 0$. It is easy to check that $\lim^k(\kappa_0 D) = 0$ for $k > 0$ (for example, examine the cochain complex of [8, Vista 3.5.12] which computes higher derived inverse limits). Thus $\nu_k\colon \lim^k(\kappa_0 D) \longrightarrow \check{H}^k(\kappa_0 D)$ is an isomorphism for all k. $\square$

We proceed by induction on j and state the **induction hypothesis:** *The maps $\nu_k\colon \lim^k(\kappa_{j-1}D) \longrightarrow \check{H}^k(\kappa_{j-1}D)$ are isomorphisms for all $k \geqslant 0$ and all diagrams D.* The case $j = 1$ is covered by the previous lemma.

We have a sequence of epimorphisms of diagrams

$$D = \kappa_n D \xrightarrow{e_n} \kappa_{n-1} D \xrightarrow{e_{n-1}} \ldots \xrightarrow{e_1} \kappa_0 D$$

and consequently a collection of short exact sequences (for $1 \leqslant j \leqslant n$)

$$0 \longrightarrow \ker(e_j) \longrightarrow \kappa_j D \longrightarrow \kappa_{j-1} D \longrightarrow 0\ .$$

Consider the associated ladder diagram for some fixed $k \geqslant 1$:

$$\begin{tikzcd}
\lim^{k-1}(\kappa_{j-1}D) \arrow[r] \arrow[d, "\nu_{k-1}"] & \lim^k(\ker e_j) \arrow[r] \arrow[d, "\nu_k"] & \lim^k(\kappa_j D) \arrow[r] \arrow[d, "\nu_k"] & \lim^k(\kappa_{j-1}D) \arrow[r] \arrow[d, "\nu_k"] & \lim^{k+1}(\ker e_j) \arrow[d, "\nu_{k+1}"] \\
\check{H}^{k-1}(\kappa_{j-1}D) \arrow[r] & \check{H}^k(\ker e_j) \arrow[r] & \check{H}^k(\kappa_j D) \arrow[r] & \check{H}^k(\kappa_{j-1}D) \arrow[r] & \check{H}^{k+1}(\ker e_j).
\end{tikzcd}$$

By our induction hypothesis we know that the first and fourth vertical arrows are isomorphisms. In view of the Five Lemma it is enough to show that the second and fifth vertical arrows are isomorphisms as well. Since $\kappa_n D = D$, this proves the assertion of Proposition 4.2.

We are left to show that the maps $\nu_k\colon \lim^k(\ker e_j) \longrightarrow \check{H}^k(\ker e_j)$ are isomorphisms for all k, and all $j \geqslant 1$. Now the diagram $\ker e_j$ has non-trivial entries only on cones of codimension j, and can thus be written as a direct sum of *atomic* diagrams with a single non-trivial entry. Since both $\lim^k$ and $\check{H}^k$ commute with direct sums of atomic diagrams (the former by [8, Vista 3.5.12], the latter by direct inspection), the induction step is completed if we can verify the following assertion:

Lemma 4.5. *Let C be an A-module. Let $\tau \in \Sigma$ be a cone of codimension $j > 0$, and let C_τ denote the atomic Σ^{op}-diagram with non-trivial value C attained at τ. Then the maps $\nu_k\colon \lim^k(C_\tau) \longrightarrow \check{H}^k(C_\tau)$ are isomorphisms for all $k \geqslant 0$.*

(Note that by uniqueness of the natural maps ν_k the direct sum decomposition of the diagram $\ker e_j$ carries over to a direct sum decomposition of the corresponding ν_k.)

The lemma will follow from a brute-force calculation. By construction, $\check{H}^k(C_\tau) = 0$ for $k \neq j$, and $\check{H}^j(C_\tau) = C$. We have $\lim(C_\tau) = 0$ since τ has positive codimension. To compute the higher derived inverse limits, we embed the diagram C_τ into a short exact sequence

$$0 \longrightarrow C_\tau \longrightarrow C_{\geqslant\tau} \longrightarrow C_{>\tau} \longrightarrow 0 \tag{10}$$

where we write

$$C_{\geqslant\tau}\colon \Sigma^{\mathrm{op}} \longrightarrow A\text{-Mod}, \quad \sigma \mapsto \begin{cases} C & \text{if } \sigma \supseteq \tau \\ 0 & \text{else} \end{cases}$$

(with non-trivial structure maps identities), and the diagram $C_{>\tau}$ is defined similarly.

Lemma 4.6. *The higher derived inverse limits remain unchanged when the diagram $C_{\geqslant\tau}$ is restricted to the subcategory $\tau \downarrow \Sigma = \{\sigma \in \Sigma \,|\, \sigma \supseteq \tau\}$. More precisely, the canonical restriction maps $\lim^k_{\Sigma^{\mathrm{op}}} C_{\geqslant\tau} \longrightarrow \lim^k_{(\tau\downarrow\Sigma)^{\mathrm{op}}} (C_{\geqslant\tau}|_{(\tau\downarrow\Sigma)})$ are isomorphisms.*

Similarly, the higher derived inverse limits remain unchanged when the diagram $C_{>\tau}$ is restricted to the subcategory $\tau \Downarrow \Sigma = \{\sigma \in \Sigma \,|\, \sigma \supset \tau\}$.

Proof. This can be read off from the usual cochain complex computing higher derived inverse limits as given in [8, Vista 3.5.12]. □

Definition 4.7 (Oda [7, Corollary 1.7])**.** Given Σ and τ as before, we define the j-dimensional *quotient fan* Σ/τ as the complete fan in the vector space $N_{\mathbb{R}}/\mathrm{span}(\tau)$ with cones given by the images of $\sigma \in \tau \downarrow \Sigma$ under the map $N_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}/\mathrm{span}(\tau)$.

Note that the posets $\tau \downarrow \Sigma$ and Σ/τ are isomorphic. Similarly, $\tau \Downarrow \Sigma$ is isomorphic as a poset to $(\Sigma/\tau) \setminus \{0\}$.

Lemma 4.8. *If P is any poset, regarded as a category, and $F\colon P^{\mathrm{op}} \longrightarrow A$-Mod is a constant functor with value C, then $\lim^k(F) \cong H^k(NP; C)$ for all $k \geqslant 0$, where NP denotes the nerve of P, and $H^k(NP;C)$ is the cohomology of NP with coefficients in C.*

Proof. The cochain complex computing $\lim^k(F)$ given in [8, Vista 3.5.12] is identical to the usual cochain complex used to compute $H^k(NP;C)$. The latter is the cochain complex associated to the cosimplicial A-module $\hom_{\mathbb{Z}}(\mathbb{Z}[NP], C)$, where $\mathbb{Z}[NP]$ is the simplicial free abelian group obtained from NP by applying the functor $\mathbb{Z}[\,\cdot\,]$ in each simplicial degree. □

Lemma 4.9. *We have* $\lim(C_{\geqslant\tau}) = C$ *and* $\lim^k(C_{\geqslant\tau}) = 0$ *for* $k \geqslant 1$.

Proof. By Lemmas 4.6 and 4.8 we have isomorphisms

$$\lim^k(C_{\geqslant\tau}) \cong H^k(N(\tau \downarrow \Sigma)^{\mathrm{op}}; C)$$

for all $k \geqslant 0$, where N denotes the nerve of the category. But $\tau \downarrow \Sigma$ has an initial object; hence it is contractible. □

Lemma 4.10. *We have* $\lim^k(C_{>\tau}) = H^k(S^{j-1}; C)$. *In particular,* $\lim^k(C_{>\tau}) = 0$ *for* $k \neq 0$, $j-1$.

Proof. By Lemmas 4.6 and 4.8 we have isomorphisms

$$\lim^k(C_{>\tau}) \cong H^k(N(\tau \Downarrow \Sigma)^{\mathrm{op}}; C)$$

for all $k \geqslant 0$, where N denotes the nerve of the category. By Definition 4.7, the poset $\tau \Downarrow \Sigma$ is isomorphic to the j-dimensional fan Σ/τ with the 0-cone removed. Clearly $N((\Sigma/\tau) \setminus \{0\}) = S^{j-1}$, which implies the lemma. □

Lemma 4.11. *We have* $\lim^k(C_\tau) = 0$ *for* $k \neq j$, *and* $\lim^j(C_\tau) = C$.

Proof. The long exact sequence associated to the short exact sequence (10) and the calculations in the previous two lemmas give a short exact sequence

$$0 \longrightarrow C \xrightarrow{f} H^0(S^{j-1}) \longrightarrow \lim^1(C_\tau) \longrightarrow 0.$$

In case $j = 1$, we have $H^0(S^0) = C \oplus C$, and the map marked f above is the diagonal map. Hence $\lim^1(C_\tau) = C$ in this case. If $j \geqslant 2$ we have $H^0(S^{j-1}) = C$ and $f = \mathrm{id}$, so $\lim^1(C_\tau) = 0$ in this case.

The long exact sequence associated to the short exact sequence (10) and the previous two lemmas also yield isomorphisms $\lim^k(C_\tau) \cong \lim^{k-1}(C_{>\tau})$ for $k \geqslant 2$. Together with Lemma 4.10 again this proves the assertion. □

As a consequence we are reduced to consider the case $k = j > 0$ in Lemma 4.5.

Lemma 4.12. *We have* $\check{H}^k(C_{\geqslant\tau}) = 0$ *for* $k > 0$.

Proof. In short, this follows from the fact that the cochain complex $\mathcal{C}(C_{\geqslant\tau})^\bullet$ is a re-indexed variant of the cellular chain complex computing the reduced homology of a $(j-1)$-sphere with coefficients in C, so $\check{H}^k(C_{\geqslant\tau}) \cong \tilde{H}_{j-1-k}(S^{j-1}; C)$.

In more detail, recall that the poset $\tau \downarrow \Sigma$ is isomorphic to the j-dimensional quotient fan Σ/τ; cf. Definition 4.7. The fan Σ/τ induces a cell structure on some unit sphere S^{j-1} in $N_{\mathbb{R}}/\mathrm{span}(\tau)$, and taking the incidence numbers coming from the fan Σ as defined before, we see that $\mathcal{C}(C_{\geqslant\tau})^\bullet$ is, up to re-indexing, an augmented cellular chain complex of S^{j-1}. This chain complex is slightly non-standard: The augmentation maps are given by id_C or $-\mathrm{id}_C$, depending on the incidence numbers $[\bar{\sigma} : \bar{\tau}]$. However, it is not difficult to show that this chain complex is isomorphic to a standard chain complex for any choice of orientations of the cones in Σ/τ, with the required isomorphism being constructed by induction on the dimensions of the cones, starting with τ. We omit the details. □

We are now ready to prove Lemma 4.5. Consider the following piece of the ladder diagram relating $\lim^*$ and $\check{H}^*$:

$$\begin{tikzcd}
\lim^{j-1}(C_{\geqslant\tau}) \arrow[r] \arrow[d, "\nu_{j-1}", "\cong"'] & \lim^{j-1}(C_{>\tau}) \arrow[r, "f"] \arrow[d, "\nu_{j-1}", "\cong"'] & \lim^{j}(C_{\tau}) \arrow[r] \arrow[d, "\nu_j"] & \lim^{j}(C_{\geqslant\tau}) = 0 \\
\check{H}^{j-1}(C_{\geqslant\tau}) \arrow[r] & \check{H}^{j-1}(C_{>\tau}) \arrow[r, "g"] & \check{H}^{j}(C_{\tau}) \arrow[r] & \check{H}^{j}(C_{\geqslant\tau}) = 0.
\end{tikzcd} \tag{11}$$

Both rows are exact. The entries on the right are trivial by Lemmas 4.9 and 4.12, respectively. The first vertical map is an isomorphism. For $j = 1$ it is the map ν_0, and for $j > 1$ it follows from Lemmas 4.9 and 4.12 that the source and target are trivial. The second vertical map is an isomorphism in view of our induction hypothesis (note that $C_{>\tau} = \kappa_{j-1}C_{>\tau}$). From the Five Lemma we conclude that the third vertical map is an isomorphism as desired. This finishes the proof.

References

[1] M. Beck, C. Haase, and F. Sottile, *Theorems of Brion, Lawrence, and Varchenko on rational generating functions for cones.* arXiv:math.CO/0506466.

[2] M. Brion, *Points entiers dans les polyèdres convexes*, Ann. Sci. École Norm. Sup. **21** (1988), 653–663.

[3] M. Brion and M. Vergne, *Lattice points in simple polytopes*, J. Amer. Math. Soc. **10** (1997), 371–392.

[4] V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk **33** (1978), 85–134, 247.

[5] T. Hüttemann, *K-theory of non-linear projective toric varieties.* To appear in Forum Math.

[6] M.-N. Ishida, *Polyhedral Laurent series and Brion's equalities*, Internat. J. Math. **1** (1990), 251–265.

[7] T. Oda, *Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties*, Ergeb. Math. Grenzgeb. (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988. Translated from the Japanese.

[8] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

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