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METRIZABLE SHAPE AND STRONG SHAPE EQUIVALENCES

L. STRAMACCIA

(*communicated by Walter Tholen*)

Abstract

In this paper we construct a functor $\Phi : \text{pro}\mathcal{T}op \to \text{pro}\mathcal{ANR}$ which extends Mardešić correspondence that assigns to every metrizable space its canonical $\mathcal{ANR}$-resolution. Such a functor allows one to define the strong shape category of prospaces and, moreover, to define a class of spaces, called strongly fibered, that play for strong shape equivalences the role that $\mathcal{ANR}$-spaces play for ordinary shape equivalences. In the last section we characterize SSDR-promaps, as defined by Dydak and Nowak, in terms of the strong homotopy extension property considered by the author.

Introduction

In ordinary Shape Theory there is a canonical way of associating with every topological space X an inverse system $\check{\mathtt{X}}$ of absolute neighborhood retracts, namely its Čech system [**15**]. It is an inverse system in the homotopy category $\text{ho}\mathcal{T}op$ of topological spaces, whose bonding morphisms are homotopy classes of maps. This gives a functor $\text{ho}\mathcal{T}op \to \text{pro}(\text{ho}\mathcal{ANR})$, where $\mathcal{ANR}$ is the category of absolute neighborhood retracts. In Strong Shape Theory [**14**] one associates with every space X an inverse system $\mathtt{X}$ in the category $\mathcal{T}op$ of topological spaces, bonded by continuous maps. In [**19**] S. Mardešić introduced the notion of $\mathcal{ANR}$-resolution and proved that every topological space X admits a canonically associated $\mathcal{ANR}$-resolution $M(X) \in \text{pro}\mathcal{ANR}$. However, the correspondence $X \mapsto M(X)$ does not give a functor $\mathcal{T}op \to \text{pro}\mathcal{ANR}$. In their 1991 paper [**8**], Dydak and Nowak tried to overcome such difficulties defining a Mardešić-like functor $\mathcal{T}op \to \text{pro}\mathcal{ANR}$ but, due to some technical error, their construction there does not work (see [**14**], [**9**]). In another more recent paper [**9**], the same authors correct their errors adopting a different point of view. In this paper we undertake the program above and construct a functor $\Phi : \text{pro}\mathcal{T}op \to \text{pro}\mathcal{ANR}$, from the category of prospaces (inverse systems of topological spaces) to the category of inverse systems of absolute neighborhood retracts, which has the following properties :

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i) the restriction of Φ to the category of metrizable spaces coincides with Mardešić's correspondence which assigns to every space its canonical $\mathcal{ANR}$-resolution [**13**],

ii) Φ has a reflective lifting $\mathrm{ho}(\Phi) : \mathrm{ho}(\mathrm{pro}\mathcal{T}op) \to \mathrm{ho}(\mathrm{pro}\mathcal{ANR})$ to the Steenrod homotopy categories,

iii) the restriction of $\mathrm{ho}(\Phi)$ to $\mathrm{ho}(\mathcal{T}op)$ is naturally equivalent to Cathey and Segal's functor $R : \mathrm{ho}(\mathcal{T}op) \to \mathrm{ho}(\mathrm{pro}\mathcal{ANR})$ [**5**],

iv) one can define the strong shape category of prospaces $sSh(\mathrm{pro}\mathcal{T}op)$ as a natural extension of the category $sSh(\mathcal{T}op)$, defined in [**5**], [**19**], considering the full image factorization of $\mathrm{ho}(\Phi)$. It is shown that $sSh(\mathrm{pro}\mathcal{T}op)$ can be obtained localizing $\mathrm{pro}\mathcal{T}op$ at the class of strong shape equivalences (cf.[**17**]).

Crucial for the definition of the functor Φ is the consideration of the metrizable proreflector $\mathcal{T}op \to \mathrm{pro}\mathcal{M}et$ which gives a reflector $\mathrm{ho}(\mathrm{pro}\mathcal{T}op) \to \mathrm{ho}(\mathrm{pro}\mathcal{M}et)$ and the fact that the strong shape theory of metrizable spaces is well settled in the literature.

The existence of the functor Φ allows one to characterize strong shape equivalences as those maps inducing bijections $f_Z^* : [Y, Z] \to [X, Z]$, between sets of homotopy classes, for every *strongly fibered space* Z (section 2). Hence, strongly fibered spaces play, for strong shape equivalences, the role that ANR-spaces play for ordinary shape equivalences. Such a result was already stated by Dydak and Nowak in [**8**] and corrected in [**9**], where $\mathrm{SSDR}_{\mathcal{T}op}$- fibrant spaces were introduced. We compare our results with those of [**9**] in last section. In particular we prove that SSDR-promaps of [**9**] coincide with the class of level cofibrations that are strong shape equivalences. As a fundamental tool we use a generalization of the SHEP (strong homotopy extension property), introduced in [**21**].

1. Procategories and Localizations.

Let $\mathcal{C}$ be any category. The category $\mathrm{pro}\mathcal{C}$ of inverse system in $\mathcal{C}$ has objects the contravariant functors $\mathtt{X} : \Lambda \to \mathcal{C}$, where $\Lambda = (\Lambda, \leqslant)$ is a directed set. An inverse system in $\mathcal{C}$ will be explicitly denoted by $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$, where $X_\lambda = \mathtt{X}(\lambda)$ and $x_{\lambda\lambda'} = \mathtt{X}(\lambda \leqslant \lambda')$.

We refer to [**15**] for all details concerning the definition of $\mathrm{pro}\mathcal{C}$, but it will be useful to recall the following facts :

- a morphism $\mathtt{x} : X \to \mathtt{X}$, where $X \in \mathcal{C}$, is a family $\mathtt{x} = \{x_\lambda : X \to X_\lambda \mid \lambda \in \Lambda\}$ of morphisms of $\mathcal{C}$, with the property that $x_{\lambda\lambda'} \circ x_{\lambda'} = x_\lambda$, for all $\lambda \leqslant \lambda'$.

- given a morphism $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ in $\mathrm{pro}\mathcal{C}$, it is always possible to assume, up to isomorphisms, that Λ is cofinite (that is: every $\lambda \in \Lambda$ has only finitely many predecessors), that $\mathtt{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ is indexed over the same directed set as $\mathtt{X}$ and that $\mathtt{f}$ is a level morphism, that is given by a family $\{f_\lambda : X_\lambda \to Y_\lambda \mid \lambda \in \Lambda\}$ of morphisms of $\mathcal{C}$, with $y_{\lambda\lambda'} \circ f_{\lambda'} = f_\lambda \circ x_{\lambda\lambda'}$, for $\lambda \leqslant \lambda'$ ([**15**], Thm.3.1). Note that a level morphism is actually a natural transformation of functors.

1.1. A full subcategory $\mathcal{K}$ of $\mathcal{C}$ is *proreflective* in $\mathcal{C}$ ([**16**], [**20**], [**22**]) if, for every $X \in \mathcal{C}$, there exists an inverse system $\mathtt{X} \in \mathrm{pro}\mathcal{K}$ and a morphism $\mathtt{x} : X \to \mathtt{X}$ in

pro$\mathcal{C}$, which is universal (initial) with respect to every other morphism $\mathtt{f} : X \to \mathtt{K}$, with $\mathtt{K} \in$ pro$\mathcal{K}$. In such a case $\mathtt{x} : X \to \mathtt{X}$ is called a $\mathcal{K}$-*expansion* for X. It is clear that a $\mathcal{K}$-expansion for X is uniquely determined up to isomorphisms in pro$\mathcal{K}$. This fact allows one to define a functor $P : \mathcal{C} \to$ pro$\mathcal{K}$, $X \mapsto \mathtt{X}$, which is is called the *proreflector*.
Let $\mathcal{B}$ be any category having inverse limits. Every functor $F : \mathcal{C} \to \mathcal{B}$ has an extension $F^* :$ pro$\mathcal{C} \to \mathcal{B}$ which is defined by $F^* = \lim \cdot \, \mathrm{pro}F$ where pro$F :$ pro$\mathcal{C} \to$ pro$\mathcal{B}$ is the functor such that pro$F(\mathtt{X}) = (F(X_\lambda), F(x_{\lambda\lambda'}), \Lambda)$, while $\lim :$ pro$\mathcal{B} \to \mathcal{B}$ is the inverse limit functor. We give now a construction for the functor F^* in the case $\mathcal{B} =$ pro$\mathcal{K}$, for some category $\mathcal{K}$.

Let $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda) \in$ pro$\mathcal{C}$ and let $F(X_\lambda) = (K_i^\lambda, k_{ii'}^\lambda, I_\lambda)$, for every $\lambda \in \Lambda$. Then, $(F(X_\lambda), F(x_{\lambda\lambda'}), \Lambda)$ is an inverse system in pro$\mathcal{K}$, whose inverse limit is the system

$$F^*(\mathtt{X}) = (K_i^\lambda, k_{ii'}^{\lambda\lambda'}, \Gamma),$$

where $\Gamma = \bigcup\{\Lambda \times I_\lambda \mid \lambda \in \Lambda\}$ is directed by the relation

$$(\lambda, i) \leqslant (\lambda', i') \Leftrightarrow \begin{cases} \lambda \leqslant \lambda' \text{ in } \Lambda, \text{ and} \\ k_{ii'}^{\lambda\lambda'} : K_{i'}^{\lambda'} \to K_i^\lambda \text{ is part of } F(x_{\lambda\lambda'}). \end{cases}$$

Let $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ be a level morphism in pro$\mathcal{C}$, with $\mathtt{Y} = (Y_\lambda, y_{\lambda\lambda'}, \Lambda)$ and $\mathtt{f} = \{f_\lambda : X_\lambda \to Y_\lambda \mid \lambda \in \Lambda\}$. If we assume, as it is possible, that each $F(f_\lambda) : F(X_\lambda) \to F(Y_\lambda)$, $\lambda \in \Lambda$, is a level morphism, then $F(Y_\lambda) = (H_i^\lambda, h_{ii'}^\lambda, I_\lambda)$, hence it follows that

$$F^*(\mathtt{Y}) = (H_i^\lambda, h_{ii'}^{\lambda\lambda'}, \Gamma),$$

while $F^*(\mathtt{f})$ is the level morphism given by

$$F^*(\mathtt{f}) = \{F(f_\lambda)_i : K_i^\lambda \to H_i^\lambda \mid (\lambda, i) \in \Gamma\}.$$

Note that, if $P : \mathcal{C} \to$ pro$\mathcal{K}$ is a proreflector, then $P^* :$ pro$\mathcal{C} \to$ pro$\mathcal{K}$ is actually a reflector [**20**], [**22**] .

1.2. Recall that, given a class Σ of morphisms in a category $\mathcal{C}$, the *localization of* $\mathcal{C}$ *at* Σ is a pair $(\mathcal{C}\,[\Sigma^{-1}], L_\Sigma)$, where $\mathcal{C}\,[\Sigma^{-1}]$ is a category (possibly in a larger universe) having the same objects as $\mathcal{C}$ and $L_\Sigma : \mathcal{C} \to \mathcal{C}\,[\Sigma^{-1}]$ is a functor which is the identity on objects, having the following properties :

- L_Σ inverts all morphisms of Σ, that is $L_\Sigma(s)$ is an isomorphism in $\mathcal{C}[\Sigma^{-1}]$, for all $s \in \Sigma$,

- L_Σ is universal (initial) among all functors $F : \mathcal{C} \to \mathcal{E}$ that invert all morphisms of Σ.

Σ is usually called the class of *weak equivalences* of $\mathcal{C}$.
If $\mathcal{D}$ is another category, endowed with a notion Δ of weak equivalences, then a functor $F : \mathcal{C} \to \mathcal{D}$ can be extended to a functor $\widetilde{F} : \mathcal{C}\,[\Sigma^{-1}] \to \mathcal{D}\,[\Delta^{-1}]$ if and only if F preserves weak equivalences, that is $F(s) \in \Delta$, for all $s \in \Sigma$. $\widetilde{F}$ is the unique functor satisfying $\widetilde{F} \circ L_\Delta = L_\Sigma \circ F$; it acts on objects as F does [**18**].

Let $\mathcal{C} = \mathcal{T}op$ be the category of topological spaces and let Σ be the class of homotopy equivalences. Then, $\mathcal{T}op\,[\Sigma^{-1}] = \mathrm{ho}(\mathcal{T}op)$ is the usual homotopy category of spaces. In general, if $\mathcal{C}$ has a Quillen model structure, with Σ the class of its weak equivalences, then $\mathrm{ho}\mathcal{C} = \mathcal{C}\,[\Sigma^{-1}]$. Moreover, pro$\mathcal{C}$ inherits a Quillen model structure and its (Steenrod) homotopy category is $\mathrm{ho}(\mathrm{pro}\mathcal{C}) = \mathrm{pro}\mathcal{C}\,[\Sigma^{*-1}]$, where Σ^* is the class of level weak equivalences, that is the class of those level morphisms which belong levelwise to Σ. Σ^* will usually be considered as the class of weak equivalences of pro$\mathcal{C}$ [**10**], [**16**].

Theorem 1.3. *(cf. [**19**]) Let $\mathcal{C}$ and $\mathcal{K}$ have classes of weak equivalences Σ and Π, respectively, and let $P : \mathcal{C} \to \mathrm{pro}\mathcal{K}$ be any functor. If P preserves weak equivalences, then also P^* preserves weak equivalences. If, moreover, P is a proreflector, then $\widetilde{P^*} : (\mathrm{pro}\mathcal{C})[\Sigma^{*-1}] \to (\mathrm{pro}\mathcal{C})[\Gamma^{*-1}]$ is a reflector.*

Proof. Let $\mathtt{f} \in \Sigma^*$, $\mathtt{f} = \{f_\lambda\}$. $P^*(\mathtt{f})$ has level components of the form $P(f_\lambda)_i$, $(\lambda, i) \in \Gamma$, which are all members of Π, by assumption. If P is a proreflector and $\Pi = \Sigma \cap \{\text{morphisms of } \mathcal{K}\}$, then P^* is a reflector, hence left adjoint to the embedding $E : \mathrm{pro}\mathcal{K} \to \mathrm{pro}\mathcal{C}$. Since both P^* and E preserve weak equivalences, the assertion follows from ([**2**], Thm.1.1). □

1.4. The usual cylinder functor on $\mathcal{T}op$ can be extended naturally to a cylinder functor on pro$\mathcal{T}op$: for every $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda) \in \mathrm{pro}\mathcal{T}op$, let $\mathtt{X} \times I = (X_\lambda \times I, x_{\lambda\lambda'} \times 1, \Lambda)$, where I is the unit interval. One obtains, as a consequence, a notion of (*global*) *homotopy* between promaps (that is: between morphisms of prospaces) and a corresponding notion of (*global*) *homotopy equivalence*. Two promaps $\mathtt{f}, \mathtt{g} : \mathtt{X} \to \mathtt{Y}$ are globally homotopic if there is a homotopy $\mathtt{H} : \mathtt{X} \times I \to \mathtt{Y}$ such that $\mathtt{H} \circ \mathtt{e}^0 = \mathtt{f}$ and $\mathtt{H} \circ \mathtt{e}^1 = \mathtt{g}$, where $\mathtt{e}^0, \mathtt{e}^1 : \mathtt{X} \to \mathtt{X} \times I$ are the obvious promaps. The quotient category of pro$\mathcal{T}op$ modulo global homotopy is denoted by $\pi(\mathrm{pro}\mathcal{T}op)$ and $\pi : \mathrm{pro}\mathcal{T}op \to \pi(\mathrm{pro}\mathcal{T}op)$ is the quotient functor. In general, the classes of global and level homotopy equivalences in pro$\mathcal{T}op$ do not coincide, as shown in ([**10**], pp.55-56); however, every global homotopy equivalence $\mathtt{X} \to \mathtt{Y}$ is a level homotopy equivalence, whenever the bonding morphisms of $\mathtt{X}$ are epi ([**19**], Cor. 1.3).

1.5. Let $F : \mathcal{C} \to \mathcal{K}$ be any functor and let $\mathcal{C}_F$ be the category having the same objects as $\mathcal{C}$ while a morphism in $\mathcal{C}_F(X, Y)$ is a triple $(1_X, u, 1_Y)$, where $u \in \mathcal{K}(F(X), F(Y))$. $\mathcal{C}_F$ is called the *full image* of F. There are functors $F^0 : \mathcal{C} \to \mathcal{C}_F$ and $F^1 : \mathcal{C}_F \to \mathcal{K}$, defined by $F^0(X) = X$ and $F^0(f) = (1_X, F(f), 1_Y)$, for $f : X \to Y$ in $\mathcal{C}$, and $F^1(X) = F(X)$, $F^1(1_X, u, 1_Y) = u$. They give a factorization $F = F^1 \circ F^0$ of F which is uniquely determined, up to an isomorphism, among all factorizations $F = H'' \circ H'$, where H' is bijective on objects and H'' is fully faithful. $F = F^1 \circ F^0$ is called the *full image factorization* of F [**18**]. Recall that, when F is a reflector and Σ_F is the class of morphisms of $\mathcal{C}$ inverted by F, then there is an isomorphism $\mathcal{K} \cong \mathcal{C}[\Sigma_F^{-1}]$ ([**18**], 19.3.1). Moreover, from the uniqueness of the full image factorization and since L_{Σ_F} is the identity on objects, one also obtains an isomorphism $\mathcal{C}[\Sigma_F^{-1}] \cong \mathcal{C}_F$. Let Σ_{F^0} denote the class of morphisms in $\mathcal{C}$ that are inverted by F^0. Then clearly $\Sigma_F = \Sigma_{F^0}$ holds.

In what follows we give a brief account of the construction of the Steenrod homotopy category ho(pro$\mathcal{T}op$) of pro$\mathcal{T}op$, following the point of view of [**5**].

Definition 1.6. Let $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ and $\mathtt{p} : \mathtt{E} \to \mathtt{B}$ be promaps. $\mathtt{f}$ has the left lifting property with respect to $\mathtt{p}$ (and $\mathtt{p}$ has the right lifting property with respect to $\mathtt{f}$) if every commutative square

$$\begin{tikzcd} \mathtt{X} \arrow[r, "\mathtt{a}"] \arrow[d, "\mathtt{f}"'] & \mathtt{E} \arrow[d, "\mathtt{p}"] \\ \mathtt{Y} \arrow[r, "\mathtt{b}"] & \mathtt{B} \end{tikzcd}$$

has a filler $\mathtt{h} : \mathtt{Y} \to \mathtt{E}$, such that $\mathtt{h} \circ \mathtt{f} = \mathtt{a}$ and $\mathtt{p} \circ \mathtt{h} = \mathtt{b}$.
Let Σ be a a class of morphisms in pro$\mathcal{T}op$, then

- a promap $\mathtt{p} : \mathtt{E} \to \mathtt{B}$ is a Σ-fibration if it has the right lifting property with respect to all $\mathtt{f} \in \Sigma$,

- a prospace $\mathtt{Z} = (Z_\mu, z_{\mu\mu'}, M)$ is Σ-fibrant iff the unique morphism $\mathtt{Z} \to *$ is a Σ-fibration, where $*$ denotes the final object in $\mathcal{T}op$,

- a Σ-fibrant prospace $\mathtt{Z}$ is said to be strongly Σ-fibrant if, moreover, for every $\mu^* \in M$, the unique map $z_{\mu^*} : Z_{\mu^*} \to \lim_{\mu<\mu^*} Z_\mu$, induced by the bonding maps of the system, is a Σ-fibration,

- a topological space Z is Σ-strongly fibered if it is the inverse limit of a strongly Σ-fibrant prospace $\mathtt{Z} \in$ pro$\mathcal{ANR}$.

In the homotopy theory of pro$\mathcal{T}op$, as defined in [**10**], a promap $\mathtt{f}$ is a trivial cofibration if it has the left lifting property with respect to every Hurewicz fibration $p : E \to B$ in $\mathcal{T}op$. This notion is a natural extension of that of trivial cofibration in $\mathcal{T}op$. On the other hand, it is clear that a map p having the right lifting property with respect to all trivial cofibrations $\mathtt{f}$ in pro$\mathcal{T}op$, has to be a Hurewicz fibration. In the sequel, for Σ the class of trivial cofibrations in pro$\mathcal{T}op$, we shall speak of (strongly) fibrant prospaces and strongly fibered spaces, omitting the reference to the class Σ.

1.7. There is a reflective functor $F : \pi(\text{pro}\mathcal{T}op) \to \pi(\text{pro}\mathcal{T}op)_f$ onto the full subcategory of fibrant prospaces [**5**], with unit of adjunction $[\mathtt{i_X}] : \mathtt{X} \to \widehat{\mathtt{X}}$, where $\mathtt{i_X}$ is a trivial cofibration. By ([**5**], Prop. 3.3) F has a reflective restriction $F : \pi(\text{pro}\mathcal{ANR}) \to \pi(\text{pro}\mathcal{ANR})_{sf}$, where $\pi(\text{pro}\mathcal{ANR})_{sf}$ is the full subcategory of strongly fibrant prospaces. For $\mathtt{Z} \in$ pro$\mathcal{ANR}$, $\mathtt{i_Z} : \mathtt{Z} \to \widehat{\mathtt{Z}}$ is called the strongly fibrant modification of $\mathtt{Z}$.
ho(pro$\mathcal{T}op$) is the full image of the functor F above and is equipped with the canonical functors $F^0 : \pi(\text{pro}\mathcal{T}op) \to \text{ho}(\text{pro}\mathcal{T}op)$ and $F^1 : \text{ho}(\text{pro}\mathcal{T}op) \to \pi(\text{pro}\mathcal{T}op)_f$. The functor $L = F^0 \circ \pi : \text{pro}\mathcal{T}op \to \text{ho}(\text{pro}\mathcal{T}op)$ is known to localize pro$\mathcal{T}op$ at the class of trivial cofibrations and also at the class of level homotopy equivalences [**10**], [**16**].

Remark 1.8. For $\mathtt{X}, \mathtt{Y} \in$ pro$\mathcal{T}op$, with $\mathtt{Y}$ fibrant, there is a natural bijection

$$\text{ho}(\text{pro}\mathcal{T}op)(\mathtt{X}, \mathtt{Y}) \cong [\mathtt{X}, \mathtt{Y}],$$

where $[\mathtt{X}, \mathtt{Y}]$ is the set of global homotopy classes of morphisms $\mathtt{X} \to \mathtt{Y}$. This is because every prospace $\mathtt{X}$ is in fact cofibrant in pro$\mathcal{T}op$ ([**10**], Prop.3.4.1, pag. 95).

2. The functor $\Phi : \mathrm{pro}\mathcal{T}op \to \mathrm{pro}\mathcal{ANR}$ and the category $sSh(\mathrm{pro}\mathcal{T}op)$.

The category $\mathcal{M}et$ of metrizable spaces is proreflective in $\mathcal{T}op$. In order to obtain the metrizable expansion $\mathtt{x} : X \to \mathtt{X}$ of a topological space (X, τ), let us consider the set Λ of all continuous pseudometrics on X, directed by the relation

$$\lambda \leqslant \lambda' \Longleftrightarrow \tau_\lambda \subset \tau_{\lambda'},$$

Here τ_λ denotes the topology induced on X by the pseudometric λ, while the continuity of λ means that $\tau_\lambda \subset \tau$ [**1**]. Let X_λ denote the metric identification of (X, τ_λ). For every $\lambda \in \Lambda$, let $x_\lambda : X \to X_\lambda$ be the identity map $(X, \tau) \to (X, \tau_\lambda)$ followed by the quotient map $(X, \tau_\lambda) \to X_\lambda$. Moreover, for $\lambda \leqslant \lambda'$, let $x_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ be the unique map induced on the quotients by the identity $(X, \tau_{\lambda'}) \to (X, \tau_\lambda)$. We note explicitly that in the inverse system $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$, the bonding morphisms $x_{\lambda\lambda'}$ are all surjective maps. We shall denote by $P_M : \mathcal{T}op \to \mathrm{pro}\mathcal{M}et,\ X \mapsto \mathtt{X}$, the metrizable proreflector.

Theorem 2.1. *(cf. [**19**]) The metrizable proreflector $P_M : \mathcal{T}op \to \mathrm{pro}\mathcal{M}et$ induces a reflector $\mathrm{ho}(P_M^*) : \mathrm{ho}(\mathrm{pro}\mathcal{T}op) \to \mathrm{ho}(\mathrm{pro}\mathcal{M}et)$.*

Proof. In view of Thm.1.3, it suffices to prove that P_M preserves weak equivalences. Let us note that P_M respects the cylinders, in the sense that, if $\mathtt{x} : X \to \mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$ is the metrizable expansion of the space X, then $\mathtt{x} \times 1 : X \times I \to \mathtt{X} \times I = (X_\lambda \times I, x_{\lambda\lambda'} \times 1, \Lambda)$ is the metrizable expansion of $X \times I$, see ([**19**], Thm.2.3). It follows that P_M takes homotopy equivalences to global homotopy equivalences. Since $\mathtt{X}$ has epi bonding morphisms, the proof is complete. The reflection morphism $\chi : \mathtt{X} \to P_M^*(\mathtt{X})$, for the prospace $\mathtt{X}$, is induced by the family $\{\mathtt{x}_\lambda : X_\lambda \to \mathtt{X}_\lambda \mid \lambda \in \Lambda\}$ of the metrizable expansions of each X_λ, following the construction given in the previous section (1.1). $\square$

In [**13**] S. Mardešić introduced the notion of $\mathcal{ANR}$-resolution for topological spaces and proved that every space X has a canonically associated $\mathcal{ANR}$-resolution $\mathtt{m}_X : X \to M(X)$. Although the correspondence $\mathcal{T}op \to \mathrm{pro}\mathcal{ANR},\ X \mapsto M(X)$, is not functorial in general, Cathey and Segal [**5**] proved that it induces a reflective functor between the Steenrod homotopy categories $R : \mathrm{ho}(\mathcal{T}op) \to \mathrm{ho}(\mathrm{pro}\mathcal{ANR})$. Moreover, they obtained the strong shape category $sSh(\mathcal{T}op)$ and the strong shape functor $sS_{\mathcal{T}op}$ by taking the full image factorization of R :

$$\begin{tikzcd}
\mathrm{ho}(\mathcal{T}op) \arrow[rr, "R"] \arrow[dr, "sS_{\mathcal{T}op}"'] & & \mathrm{ho}(\mathrm{pro}\mathcal{ANR}) \\
& sSh(\mathcal{T}op) \arrow[ur, "R^1"'] &
\end{tikzcd}$$

where $sS_{\mathcal{T}op} = R^0$ is the identity on objects, while R^1 is fully faithful.

The fact that R is a reflective functor means that, for every $X \in \mathcal{T}op$ and for every $\mathtt{K} \in \mathrm{pro}\mathcal{ANR}$, Mardešić's $\mathcal{ANR}$-resolution $\mathtt{m} : X \to M(X)$ induces a bijection $\mathrm{ho}(\mathrm{pro}\mathcal{T}op)(X, \mathtt{K}) \cong \mathrm{ho}(\mathrm{pro}\mathcal{ANR})(M(X), \mathtt{K})$.

Another feature of Mardešić's correspondence is that it becomes a functor

$$M : \mathcal{M}et \to \mathrm{pro}\mathcal{ANR}$$

when restricted to the category $\mathcal{M}et$ of metrizable spaces. This fact was pointed out in [**19**] and used to give an alternative description of the strong shape category of topological spaces. The same paper (Thm. 2.4) also gave a particularly simple construction for the $\mathcal{ANR}$-resolution $\mathtt{m}_X : X \to M(X)$ of a metrizable space X, which is actually an $\mathcal{ANR}$-expansion. In such a case $M(X)$ is the inverse system of all open neighborhoods of X in its convex hull $H(X)$ in the Banach space $C(X)$ of all real, bounded, continuous functions on X, while $\mathtt{m}_X$ is formed by all the inclusions.
Let us note that, lifting the functor $M : \mathcal{M}et \to \mathrm{pro}\mathcal{ANR}$ to the Steenrod homotopy categories, amounts to taking the restriction $\mathrm{ho}(M) : \mathrm{ho}(\mathcal{M}et) \to \mathrm{ho}(\mathrm{pro}\mathcal{ANR})$ of Cathey and Segal's functor R, to the homotopy subcategory of metrizable spaces. It follows that $M : \mathcal{M}et \to \mathrm{pro}\mathcal{ANR}$ has to preserve weak equivalences. By Thm.1.1, the functor $M^* : \mathrm{pro}\mathcal{M}et \to \mathrm{pro}\mathcal{ANR}$ also preserves weak equivalences and has a lifting

$$\mathrm{ho}(M^*) : \mathrm{ho}(\mathrm{pro}\mathcal{M}et) \to \mathrm{ho}(\mathrm{pro}\mathcal{ANR}).$$

Theorem 2.2. $\mathrm{ho}(M^*)$ *is a reflector.*

Proof. Let $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$ be an inverse system of metrizable spaces. The family $\{\mathtt{m}_\lambda : X_\lambda \to M(X_\lambda)\}$ of the $\mathcal{ANR}$-resolutions constructed above, gives a morphism $\mathtt{m}_\mathtt{X} : \mathtt{X} \to M^*(\mathtt{X})$ in $\mathrm{pro}\mathcal{M}et$. We have to prove that, for every $\mathtt{Z} \in \mathrm{pro}\mathcal{M}et$, it induces a bijection

$$\mathrm{ho}(\mathrm{pro}\mathcal{M}et)(\mathtt{X},\mathtt{Z}) \cong \mathrm{ho}(\mathrm{pro}\mathcal{ANR})(M^*(\mathtt{X}),\mathtt{Z}).$$

Let $\mathtt{i}_\mathtt{Z} : \mathtt{Z} \to \widehat{\mathtt{Z}}$ be the strongly fibrant modification of $\mathtt{Z}$. By the preceding remarks, one has $\mathrm{ho}(\mathrm{pro}\mathcal{M}et)(\mathtt{X},\mathtt{Z}) \cong [\mathtt{X},\widehat{\mathtt{Z}}]$ and $\mathrm{ho}(\mathrm{pro}\mathcal{ANR})(M^*(\mathtt{X}),\mathtt{Z}) \cong [M^*(\mathtt{X}),\widehat{\mathtt{Z}}]$. It follows that proving the formula above amounts to proving that $\mathtt{m}_\mathtt{X}$ induces a bijection

$$[M^*(\mathtt{X}),\widehat{\mathtt{Z}}] \cong [\mathtt{X},\widehat{\mathtt{Z}}].$$

This is a consequence of the fact that $\mathrm{ho}(M)$ is reflective and of the construction of $M^*(\mathtt{X})$, as recalled in the first section. □

Let us now define the functor

$$\Phi : \mathrm{pro}\mathcal{T}op \to \mathrm{pro}\mathcal{ANR}$$

as follows: for every $\mathtt{X} \in \mathrm{pro}\mathcal{T}op$, let $\Phi(\mathtt{X}) = M^*(P^*_M(\mathtt{X}))$. It is clear that $\mathrm{ho}(\Phi) : \mathrm{ho}(\mathrm{pro}\mathcal{T}op) \to \mathrm{ho}(\mathrm{pro}\mathcal{ANR})$ exists and can be written as $\mathrm{ho}(\Phi) = \mathrm{ho}(M^*) \circ \mathrm{ho}(P^*_M)$. By (1.7) we may assume, without restriction of generality, that $\Phi(\mathtt{X})$ is strongly fibrant in $\mathrm{pro}\mathcal{ANR}$. Moreover, by the results above, it follows that $\mathrm{ho}(\Phi)$ is a reflector. If $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$, the reflection morphism $\mu : \mathtt{X} \to \Phi(\mathtt{X})$ is the promap obtained as the composition of $\chi : \mathtt{X} \to P^*_M(\mathtt{X})$, $\mathtt{m}_{P^*_M(\mathtt{X})} : P^*_M(\mathtt{X}) \to M^*(P^*_M(\mathtt{X}))$ and the strongly fibrant modification of $M^*(P^*_M(\mathtt{X}))$.
The restriction of $\mathrm{ho}(\Phi)$ to $\mathrm{ho}(\mathcal{T}op)$ coincides with the functor R of Cathey and

Segal [**19**] and, consequently, it defines the same strong shape category for the class of topological spaces. The functor Φ is an extension of Mardešić 's functor defined on the subcategory of metrizable spaces. Let us define the strong shape category $sSh(\mathrm{pro}\mathcal{T}op)$ for inverse systems of topological spaces and the related strong shape functor sS, by taking the full image factorization of $\mathrm{ho}(\Phi)$, as illustrated in the commutative diagram

$$\begin{tikzcd}
\mathrm{ho}(\mathrm{pro}\mathcal{T}op) \arrow[rr, "\mathrm{ho}(\Phi)"] \arrow[dr, "sS"'] & & \mathrm{ho}(\mathrm{pro}\mathcal{ANR}) \\
& sSh(\mathrm{pro}\mathcal{T}op) \arrow[ur, "\mathrm{ho}(\Phi)^1"'] &
\end{tikzcd}$$

3. Shape and Strong Shape Equivalences.

A continuous map $f : X \to Y$ is said to be a (*strong*) *shape equivalence* if it becomes an isomorphism in the (strong) shape category of topological spaces, that is $sS(L(f))$ is an isomorphism in $sSh(\mathrm{pro}\mathcal{T}op)$. We refer to [**15**] and [**14**] for basic facts concerning shape and strong shape theory. In particular we recall that:

- *f is a shape equivalence iff it induces a bijection $f_K^* : [Y, A] \to [X, A]$ between sets of homotopy classes, for all $A \in \mathcal{ANR}$,*

- *a shape equivalence f is a strong shape equivalence iff, given maps $g, h : Y \to A$, $A \in \mathcal{ANR}$, and a homotopy $F : X \times I \to A$ connecting $g \circ f$ and $h \circ f$, there exists a homotopy $G : Y \times I \to A$ connecting g and h, such that $G \circ (f \times 1)$ is homotopic to F w.r.t. end maps.*

The notion of strong shape equivalence in $\mathrm{pro}\mathcal{T}op$ is the obvious generalization of the notion given previously [**8**], [**14**] : $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ is a strong shape equivalence whenever the following two conditions hold :

(SSE1) *for every $A \in \mathcal{ANR}$ and for every $\mathtt{h} : \mathtt{X} \to A$, there is a morphism $\mathtt{g} : \mathtt{Y} \to A$, such that $\mathtt{g} \circ \mathtt{f} \simeq \mathtt{h}$,*

(SSE2) *given morphisms $\mathtt{g}$, $\mathtt{h} : \mathtt{Y} \to A$, $A \in \mathcal{ANR}$, and a global homotopy $\mathtt{F} : \mathtt{X} \times I \to A$ joining $\mathtt{f} \circ \mathtt{g}$ and $\mathtt{f} \circ \mathtt{h}$, there exists a global homotopy $\mathtt{G} : \mathtt{Y} \times I \to A$ joining $\mathtt{g}$ and $\mathtt{h}$, such that $\mathtt{F}$ is homotopic to $\mathtt{G} \circ (\mathtt{f} \times 1)$ w.r.t. end morphisms.*

Notice that, if a composition $\mathtt{g} \circ \mathtt{f}$ satisfies (SSE1), then $\mathtt{f}$ satisfies (SSE1). In fact, that $\mathtt{f}$ satisfies (SSE1) amounts to saying that the induced map $\mathtt{f}^* : [\mathtt{Y}, A] \to [\mathtt{X}, A]$ is onto. On the other hand one has $(\mathtt{g} \circ \mathtt{f})^* = \mathtt{f}^* \circ \mathtt{g}^*$.

Theorem 3.1. *The morphism $\mu : \mathtt{X} \to \Phi(\mathtt{X})$ is a strong shape equivalence.*

Proof. This is almost obvious. $\mathrm{ho}(\Phi)(\mu)$ must be an isomorphism in $\mathrm{ho}(\mathrm{pro}\mathcal{ANR})$, because of the reflectivity. Since $\mathrm{ho}(\Phi) = \mathrm{ho}(\Phi)^1 \circ sS$ and $\mathrm{ho}(\Phi)^1$ is fully faithful, it follows that $sS(\mu)$ is an isomorphism in the strong shape category $sSh(\mathrm{pro}\mathcal{T}op)$, hence μ is a strong shape equivalence. $\square$

Corollary 3.2. *For prospaces* $\mathtt{X}$, $\mathtt{Y}$, *the following relation holds*

$$\mathrm{ho}(\mathrm{pro}\mathcal{T}op)(\mathtt{X},\ \mathtt{Y}) \cong [\Phi(\mathtt{X}), \Phi(\mathtt{Y})].$$

We need to consider now the following facts. Let $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$, $\mathtt{f} = \{f_\lambda \mid \lambda \in \Lambda\}$, be a level promap.

3.3. For every $\lambda \in \Lambda$, let $M(f_\lambda)$ be the mapping cylinder of f_λ [**12**], with canonical maps $\Pi_\lambda : X_\lambda \times I \to M(f_\lambda)$ and $j_\lambda : Y_\lambda \to M(f_\lambda)$, such that $\Pi_\lambda \circ e_{0,\lambda} = j_\lambda \circ f_\lambda$. Note that j_λ has a left inverse p_λ, such that $p_\lambda \circ \Pi_\lambda = f_\lambda \circ \sigma_\lambda$, where $\sigma_\lambda : X_\lambda \times I \to X_\lambda$ is the usual map. Then f_λ has a decomposition $f_\lambda = f^1_\lambda \circ f^0_\lambda$, where $f^0_\lambda : X_\lambda \to M(f_\lambda)$ is a cofibration and $f^1_\lambda : M(f_\lambda) \to Y_\lambda$ is a homotopy equivalence. Since such a decomposition is functorial [**11**], one can define (levelwise) the mapping cylinder decompositon of the promap $\mathtt{f}$, given by

$$\mathtt{X} \xrightarrow{\ \mathtt{f}\ } \mathtt{Y} \ = \ \mathtt{X} \xrightarrow{\ \mathtt{f}^0\ } M(\mathtt{f}) \xrightarrow{\ \mathtt{f}^1\ } \mathtt{Y}$$

where $\mathtt{f}^0 = \{f^0_\lambda \mid \lambda \in \Lambda\}$ is a level cofibration, $\mathtt{f}^1 = \{f^1_\lambda \mid \lambda \in \Lambda\}$ is a level homotopy equivalence and $M(\mathtt{f}) = (M(f_\lambda), m_{\lambda\lambda'}, \Lambda)$. The maps $m_{\lambda\lambda'}$ are obtained from the universal properties of the various mapping cylinders.

3.4. $DM(f_\lambda)$ denote the *double mapping cylinder* of f_λ [**12**], [**14**], [**21**], that is the adjunction space $(X_\lambda \times I) \cup_{f_\lambda} (Y_\lambda \times \partial I)$, equipped with canonical maps $K_\lambda : X_\lambda \times I \to DM(f_\lambda)$ and $j_{i,\lambda} : Y_\lambda \to DM(f_\lambda)$, $i = 0, 1$, such that $K_\lambda \circ e_i = j_{i,\lambda} \circ f_\lambda$, $i = 0, 1$. Since $DM(f_\lambda)$ is a colimit object, there is a unique map $V_\lambda : DM(f_\lambda) \to Y_\lambda \times I$, with the property that $V_\lambda \circ K_\lambda = f_\lambda \times 1$ and $V_\lambda \circ j_{i,\lambda} = e_i$, $i = 0, 1$. For every $\lambda \leqslant \lambda'$, there is a unique map $n_{\lambda\lambda'} : DM(f'_\lambda) \to DM(f_\lambda)$, such that $n_{\lambda\lambda'} \circ K_{\lambda'} = K_\lambda \circ (x_{\lambda\lambda'} \times 1)$ and $n_{\lambda\lambda'} \circ j_{i,\lambda'} = j_{i,\lambda} \circ y_{\lambda\lambda'}$, $i = 0, 1$. The situation is better illustrated by the following commutative diagram

$$\begin{tikzcd}
X_{\lambda'} \arrow[rr, "x_{\lambda\lambda'}"] \arrow[dr, shift left, "e_0"] \arrow[dr, shift right, "e_1"'] \arrow[dd, "f_{\lambda'}"] & & X_\lambda \arrow[dr, shift left, "e_0"] \arrow[dr, shift right, "e_1"'] \arrow[dd, "f_\lambda"] & & \\
& X_{\lambda'} \times I \arrow[rr, "x_{\lambda\lambda'} \times 1"'] \arrow[dd, "K_{\lambda'}"'] \arrow[dddr, "f_{\lambda'} \times 1"] & & X_\lambda \times I \arrow[dd, "K_\lambda"] \arrow[dddr, "f_\lambda \times 1"] & \\
Y_{\lambda'} \arrow[rr, "y_{\lambda\lambda'}"] \arrow[dr, shift left, "j_0"] \arrow[dr, shift right, "j_1"'] & & Y_\lambda \arrow[dr, shift left, "j_0"] \arrow[dr, shift right, "j_1"'] & & \\
& DM(f_{\lambda'}) \arrow[rr, "n_{\lambda\lambda'}"'] \arrow[dr, "V_{\lambda'}"'] & & DM(f_\lambda) \arrow[dr, "V_\lambda"'] & \\
& & Y_{\lambda'} \times I \arrow[rr, "y_{\lambda\lambda'} \times 1"'] & & Y_\lambda \times I
\end{tikzcd}$$

with the obvious meaning of the maps involved. It follows that there is an inverse system $DM(\mathtt{f}) = (DM(f_\lambda), n_{\lambda\lambda'}, \Lambda)$ and level maps $\mathtt{K} : \mathtt{X} \times I \to DM(\mathtt{f})$, $\mathtt{j}_0$, $\mathtt{j}_1 : \mathtt{Y} \to DM(\mathtt{f})$ and $\mathtt{V} : DM(\mathtt{f}) \to \mathtt{Y} \times I$, with $\mathtt{K} \circ \mathtt{e}_i = \mathtt{j}_i \circ \mathtt{f}$, $i = 0, 1$, and such that $\mathtt{f} \times 1 = \mathtt{K} \circ \mathtt{V}$ and $\mathtt{V} \circ \mathtt{j}_i = \mathtt{e}_i$, $i = 0, 1$.
We point out that, if $\mathtt{f}$ is a level cofibration, then $\mathtt{V}$ is one too [**12**].

Theorem 3.5. *The class of strong shape equivalences of* pro$\mathcal{T}op$ *has the following properties:*

1. *contains all level homotopy equivalences,*
2. *if two of* $\mathtt{f}$, $\mathtt{g}$, $\mathtt{g} \circ \mathtt{f}$ *are strong shape equivalences, so is the third,*
3. *a level promap* $\mathtt{f}$ *is a strong shape equivalence iff* $\mathtt{f}^0$ *is,*
4. *if* $\mathtt{f}$ *is a strong shape equivalence and a level cofibration, then for every* $\mathtt{g} : \mathtt{X} \to A$, $A \in \mathcal{ANR}$, *there is an* $\mathtt{h} : \mathtt{Y} \to A$ *such that* $\mathtt{h} \circ \mathtt{f} = \mathtt{g}$.

Proof. (1) is clear (see also [**8**], 4.1, 4.2). (2) depends on the fact that $Ss(\mathtt{g} \circ \mathtt{f}) = Ss(\mathtt{g}) \circ Ss(\mathtt{f})$. (3) follows from (2). (4) Since $\mathtt{f}$ is a level cofibration, there is a weak pushout diagram in pro$\mathcal{T}op$, with respect to $\mathcal{ANR}$

$$\begin{tikzcd} \mathtt{X} \arrow[r, "\mathtt{e}_0"] \arrow[d, "\mathtt{f}"'] & \mathtt{X} \times I \arrow[d, "\mathtt{f} \times 1"] \\ \mathtt{Y} \arrow[r, "\mathtt{e}_0"] & \mathtt{Y} \times I \end{tikzcd}$$

□

Given now promaps $\phi : \mathtt{Y} \to A$, $A \in \mathcal{ANR}$, and $\mathtt{F} : \mathtt{X} \times I \to A$ such that $\mathtt{F} \circ \mathtt{e}_0^X = \phi \circ \mathtt{f}$, there exists a $\lambda \in \Lambda$, such that the relative λ-diagram commutes. Therefore there is a homotopy $G_\lambda : Y_\lambda \times I \to A$ with $G_\lambda \circ (f_\lambda \times 1) = F_\lambda$ and $G_\lambda \circ e_{0,\lambda} = \phi_\lambda$. Such data define a homotopy $\mathtt{G} : \mathtt{Y} \times I \to A$ with $\mathtt{G} \circ (f \times 1) = \mathtt{F}$ and $\mathtt{G} \circ \mathtt{e}_0^Y = \phi$. At this point the assertion follows from ([**12**], 2.2.4).

In view of the theorem above, one can restrict the study of strong shape equivalences to those promaps that are level cofibrations.

In [**21**] *the strong homotopy extension property* (SHEP) for maps has been introduced, with respect to $\mathcal{ANR}$. This can be generalized to promaps in the following way: a promap $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ has the SHEP, w.r.t. $\mathcal{ANR}$, iff the following diagram

$$\begin{tikzcd} \mathtt{X} \arrow[r, shift left, "\mathtt{e}_0"] \arrow[r, shift right, "\mathtt{e}_1"'] \arrow[d, "\mathtt{f}"'] & \mathtt{X} \times I \arrow[d, "\mathtt{f} \times 1"] \\ \mathtt{Y} \arrow[r, shift left, "\mathtt{e}_0"] \arrow[r, shift right, "\mathtt{e}_1"'] & \mathtt{Y} \times I \end{tikzcd}$$

is a weak colimit in pro$\mathcal{T}op$, w.r.t. $\mathcal{ANR}$. This means that, for given promaps $\mathtt{u}, \mathtt{v} : \mathtt{Y} \to A$ and homotopy $\mathtt{H} : \mathtt{X} \times I \to A$, $A \in \mathcal{ANR}$, connecting $\mathtt{u} \circ \mathtt{f}$ and $\mathtt{v} \circ \mathtt{f}$, there exists a homotopy $\mathtt{G} : \mathtt{Y} \times I \to A$, connecting $\mathtt{u}$ and $\mathtt{v}$ and such that $\mathtt{H} = \mathtt{G} \circ (\mathtt{f} \times 1)$.

Theorem 3.6. *(cf. [**21**], sec.2) Let* $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ *be a level cofibration in* pro$\mathcal{T}op$ *having property (SSE1). The following are equivalent :*

1. $\mathtt{f}$ *is a strong shape equivalence,*
2. $\mathtt{f}$ *has the SHEP w.r.t.* $\mathcal{ANR}$*,*
3. $\mathtt{V}$ *has property (SSE1).*

Proof. (1) implies (2) : since $\mathtt{f}$ is a level cofibration, this follows from ([**3**], 7.2.5). (2) implies (3) : Let $\alpha : DM(\mathtt{f}) \to A,\ A \in \mathcal{ANR}$, and consider $\alpha \circ \mathtt{K} : \mathtt{X} \times I \to A$. It is a homotopy connecting $\alpha \circ \mathtt{j}_0 \circ \mathtt{f}$ to $\alpha \circ \mathtt{j}_1 \circ \mathtt{f}$, then there is a homotopy $\mathtt{T} : \mathtt{Y} \times I \to A$ such that $\mathtt{T} \circ (\mathtt{f} \times 1) = \alpha \circ \mathtt{K}$ and $\mathtt{T} \circ \mathtt{e}_i = \alpha \circ \mathtt{j}_i$. It follows that $\mathtt{T} \circ \mathtt{V} \circ \mathtt{K} = \mathtt{T} \circ (\mathtt{f} \times 1) = \alpha \circ \mathtt{K}$ and $\mathtt{T} \circ \mathtt{V} \circ \mathtt{j}_i = \mathtt{T} \circ \mathtt{e}_i = \alpha \circ \mathtt{j}_i,\ i = 0, 1$. From the universal property of the double mapping cylinder, one obtains that $\mathtt{T} \circ \mathtt{V} = \alpha$.
(3) implies (1) : Let $\mathtt{h}_0, \mathtt{h}_1 : \mathtt{Y} \to A,\ A \in \mathcal{ANR}$, be given together with a homotopy $\mathtt{H} : \mathtt{X} \times I \to A$ connecting $\mathtt{h}_0 \circ \mathtt{f}$ to $\mathtt{h}_1 \circ \mathtt{f}$. There is a unique $\gamma : DM(\mathtt{f}) \to A$ such that $\gamma \circ \mathtt{j}_0 = \mathtt{h}_0,\ \gamma \circ \mathtt{j}_1 = \mathtt{h}_1$ and $\gamma \circ \mathtt{K} = \check{H}$. Since we may write $\mathtt{V} = \mathtt{p}_\mathtt{V} \circ \Pi_\mathtt{V} \circ \mathtt{e}_0$ (see (3.3)), it follows that $\Pi_\mathtt{V} \circ \mathtt{e}_0$ satisfies (SSE1) too and is a level cofibration. Then there is a $\mathtt{G} : M(\mathtt{V}) \to A$ such that $\mathtt{G} \circ \Pi_\mathtt{V} \circ \mathtt{e}_0 = \gamma$. It turns out that $\mathtt{G} \circ \mathtt{j}_\mathtt{V}$ is a (global) homotopy connecting $\mathtt{h}^0$ and $\mathtt{h}^1$. Moreover, one has $\mathtt{G} \circ \mathtt{j}_\mathtt{V} \circ (\mathtt{f} \times 1) = \mathtt{G} \circ \mathtt{j}_\mathtt{V} \circ \mathtt{V} \circ \mathtt{K} = \gamma \circ \mathtt{K} = \mathtt{H}$. □

Corollary 3.7. $\mathtt{V}$ *is a shape equivalence whenever* $\mathtt{f}$ *is a strong shape equivalence and a level cofibration.*

Proof. We only have to show that $\mathtt{V}$ induces, for all $A \in \mathcal{ANR}$, an onto map $\mathtt{V}^*_A : [\mathtt{Y} \times I, A] \to [DM(\mathtt{f}), A]$. Let $\alpha : DM(\mathtt{f}) \to A$, then $\alpha \circ \mathtt{K} : \mathtt{X} \times I \to A$ is a homotopy connecting $\alpha \circ \mathtt{j}_0$ to $\alpha \circ \mathtt{j}_1$. Since $\alpha \circ \mathtt{K} \circ \mathtt{e}_i = \alpha \circ \mathtt{j}_i \circ \mathtt{f}$, there exists a homotopy $\mathtt{T} : \mathtt{Y} \times I \to A$, such that $\mathtt{T} \circ (\mathtt{f} \times 1) = \alpha \circ \mathtt{K}$ and $\mathtt{T} \circ \mathtt{e}_i = \alpha \circ \mathtt{j}_i$. It follows $\mathtt{T} \circ \mathtt{V} \circ \mathtt{K} = \mathtt{T} \circ (\mathtt{f} \times 1) = \alpha \circ \mathtt{K}$ and $\mathtt{T} \circ \mathtt{e}_i = \alpha \circ \mathtt{j}_i$. From the universal property of the double mapping cylinder, one has $\alpha = \mathtt{T} \circ \mathtt{V}$. □

We need to state the following technical result.

Lemma 3.8. *Let* $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ *be a strong shape equivalence and a level cofibration in* pro$\mathcal{T}op$. $\mathtt{f}$ *induces a bijection* $\mathtt{f}^*_\mathcal{D} : [\mathtt{Y}, \lim\ \mathcal{D}] \to [\mathtt{X}, \lim\ \mathcal{D}]$*, for every finite diagram* $\mathcal{D}$ *[**18**] in* $\mathcal{ANR}$*, having at most one arrow connecting every two vertices.*

Proof. Let $\mathcal{D}$ have vertices $D_i,\ i \in I$, and morphisms $D_u : D_i \to D_j$, for $u : i \to j$ in I. Assume that $\alpha : \mathtt{X} \to \lim\ \mathcal{D}$ is given and let $p_i : \lim\ \mathcal{D} \to D_i,\ i \in I$, be the projections of the limit. By 3.5(4), for every $i \in I$, there is a promap $\beta_i : \mathtt{Y} \to D_i$, such that $\beta_i \circ \mathtt{f} = p_i \circ \alpha$. If $I(i, j) = \emptyset$, for all $i \in I,\ i \neq j$, put $\mathtt{h}_j = \beta_j$. If $u \in I(i, j)$, define $\mathtt{h}_i = D_u \circ \mathtt{j}$. In this way one obtains a natural cone from $\mathtt{Y}$ to the vertices of the diagram, which induces a unique promap $\mathtt{h} : \mathtt{Y} \to \lim\ \mathcal{D}$, with $\mathtt{h} \circ \mathtt{f} = \alpha$. □

The proof of the following theorem is partially inspired by Thm. 4.4 of [**8**].

Theorem 3.9. *Let* $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ *be a strong shape equivalence in* pro$\mathcal{T}op$*. Then* $\mathtt{f}$ *induces a bijection* $\mathtt{f}^* : [\mathtt{Y}, \mathtt{Z}] \to [\mathtt{X}, \mathtt{Z}]$*, for every strongly fibrant prospace* $\mathtt{Z} \in$ *pro*$\mathcal{ANR}$*.*

Proof. First of all we may assume, as usual, that $\mathtt{f}$ is a level promap with cofinite index set. Moreover, using the mapping cylinder decomposition of $\mathtt{f}$, we may also assume that $\mathtt{f}$ is a level cofibration. Let $\mathtt{g} : \mathtt{X} \to \mathtt{Z}$ be a given promap, with $\mathtt{Z} = (Z_\mu, z_{\mu\mu'}, M) \in \mathrm{pro}\mathcal{ANR}$ strongly fibrant. The fact that $\mathtt{f}$ is a shape equivalence, by 3.5(4), implies that, for every $\mu \in M$, there is a $\mathtt{k}_\mu : \mathtt{Y} \to Z_\mu$ such that $\mathtt{k}_\mu \circ \mathtt{f} = \mathtt{g}_\mu$. By induction on the number $\#(\mu)$ of the predecessors of μ, let us define $\mathtt{h}_\mu = \mathtt{k}_\mu$ if $\#(\mu) = 0$, and assume to have defined $\mathtt{h}_\mu$, for every $\mu \in M$ with $1 \leqslant \#(\mu) < n$, in such a way that $z_{\mu\mu'} \circ \mathtt{h}_{\mu'} = \mathtt{h}_\mu$, for $\mu \leqslant \mu'$. Let $\mu^* \in M$ having $\#(\mu^*) = n$. The promaps $\mathtt{h}_\mu$, for $\mu < \mu^*$, define a map $z_{\mu^*} : Z_{\mu^*} \to \lim_{\mu \leqslant \mu^*} Z_{\mu^*}$, and one has $z_{\mu^*} \circ \mathtt{k}_{\mu^*} \circ \mathtt{f} = z_{\mu^*} \circ \mathtt{g}_{\mu^*}$. By Lemma 3.8, there is a promap $\gamma : \mathtt{Y} \to \lim_{\mu \leqslant \mu^*} Z_{\mu^*}$, with the property that $\gamma \circ \mathtt{f} = z_{\mu^*} \circ \mathtt{k}_{\mu^*}$. In diagram

$$\begin{tikzcd}
\mathtt{X} \arrow[rr, "\mathtt{f}"] \arrow[dr, "\mathtt{g}_{\mu^*}"'] & & \mathtt{Y} \arrow[dl, "\mathtt{k}_{\mu^*}"] \arrow[ddl, "\gamma"] \\
& Z_{\mu^*} \arrow[d, "z_{\mu^*}"'] & \\
& \lim_{\mu \leqslant \mu^*} Z_{\mu^*} &
\end{tikzcd}$$

□

Then, $z_{\mu^*} \circ \mathtt{k}_{\mu^*} \circ \mathtt{f} \simeq \gamma \circ \mathtt{f}$. Again by Lemma 3.8, since $\mathtt{f}$ is a strong shape equivalence, there is a homotopy $\mathtt{H} : \mathtt{Y} \times I \to \lim_{\mu \leqslant \mu^*} Z_{\mu^*}$, with $\mathtt{H} : \gamma \simeq z_{\mu^*} \circ \mathtt{k}_{\mu^*}$. Since z_{μ^*} is a fibration, there is a homotopy $\mathtt{H}^* : \mathtt{Y} \times I \to Z_{\mu^*}$, such that $\mathtt{H}^* \circ \mathtt{e}_0 = \mathtt{k}_{\mu^*}$ and $z_{\mu^*} \circ \mathtt{H}^* = \mathtt{H}$. If we put $\mathtt{h}_{\mu^*} = \mathtt{H}^* \circ \mathtt{e}_1$, the the definition of $\mathtt{h} : \mathtt{Y} \to \mathtt{Z}$ is complete and one has $\mathtt{h} \circ \mathtt{f} = \mathtt{g}$. Let now $\mathtt{h}, \mathtt{h}' : \mathtt{Y} \to \mathtt{Z}$ be such that $\mathtt{h} \circ \mathtt{f} \simeq \mathtt{h}' \circ \mathtt{f}$, by means of a homotopy $\mathtt{F} : \mathtt{X} \times I \to \mathtt{Z}$. For every $\lambda \in \Lambda$, $DM(f_\lambda) = X_\lambda \times I \cup Y_\lambda \times \{0,1\}$ and the inclusion $V_\lambda : DM(f_\lambda) \to Y_\lambda \times I$ is a cofibration. If $\tilde{F}_\lambda : X_\lambda \times I \cup Y_\lambda \times \{0,1\} \to Z_\lambda$ is defined by

$$\tilde{F}_\lambda(x,t) = \begin{cases} F_\lambda(x,y), & \text{for } \ (x,t) \in X_\lambda \times I \\ h_\lambda(x), & t = 0 \\ h'_\lambda(x), & t = 1 \end{cases}$$

Then $\tilde{\mathtt{F}} : DM(\mathtt{f}) \to \mathtt{Z}$, $\quad \tilde{\mathtt{F}} = \{\tilde{F}_\lambda \mid \lambda \in \Lambda\}$, is a level promap. Since the promap $\mathtt{V} : DM(\mathtt{f}) \to \mathtt{Y} \times I$ is a shape equivalence and a level cofibration, it follows that $\tilde{\mathtt{F}}$ has an extension $\mathtt{G} : \mathtt{Y} \times I \to \mathtt{Z}$, which turns out to be a homotopy connecting $\mathtt{h}$ to $\mathtt{h}'$.

Theorem 3.10. *A continuous map $f : X \to Y$ is a strong shape equivalence iff it induces a bijection $f^* : [Y, Z] \to [X, Z]$, for every strongly fibered space Z.*

Proof. Let f be a strong shape equivalence, then by Thm 3.9 it induces a bijection $f^* : [Y, \mathtt{Z}] \to [X, \mathtt{Z}]$, for every strongly fibrant prospace $\mathtt{Z} \in \mathrm{pro}\mathcal{ANR}$. Let $Z = \mathit{lim}\ \mathtt{Z}$. Since the projection of the limit $\mathtt{p} : Z \to \mathtt{Z}$ induces bijections $[X, Z] \to [X, \mathtt{Z}]$ and

$[Y, Z] \to [Y, \mathtt{Z}]$, the first part of the theorem easily follows. Conversely, let f induce bijections $f^* : [Y, Z] \to [X, Z]$, for every strongly fibered space Z. Since every ANR-space is strongly fibered, it follows at once that f is a shape equivalence. Taking $\mathtt{Z} = \Phi(X)$, there is a $\mathtt{g} : Y \to \Phi(X)$ such that $[\mathtt{g} \circ f] = [\mu]$. Since $\mu : X \to \Phi(X)$ is a strong shape equivalence, it follows that f is such. □

Recently, Prasolov [**17**] has defined the strong shape category of prospaces $sSh(\mathrm{pro}\mathcal{T}op)$ by localizing $\mathrm{pro}\mathcal{T}op$ at the class of strong shape equivalences as defined by the properties (SSE1) and (SSE2) above. The two categories coincide. In fact, from the construction of $sSh(\mathrm{pro}\mathcal{T}op)$, since $\mathrm{ho}(\Phi)$ is reflective, it follows that

$$\begin{aligned} sSh(\mathrm{pro}\mathcal{T}op) &\cong (\mathrm{ho}(\mathrm{pro}\mathcal{T}op))_{\mathrm{ho}(\Phi)} \cong \mathrm{ho}(\mathrm{pro}\mathcal{T}op)[\Sigma^{-1}_{\mathrm{ho}(\Phi)}] \cong \\ &\cong \mathrm{ho}(\mathrm{pro}\mathcal{T}op)[\Sigma^{-1}_{sS}] \cong \mathrm{pro}\mathcal{T}op[\mathcal{SSE}^{-1}], \end{aligned}$$

where $\mathcal{SSE}$ is the class of strong shape equivalences in $\mathrm{pro}\mathcal{T}op$, that is those promaps $\mathtt{f} \in \mathrm{pro}\mathcal{T}op$ such that $L(\mathtt{f}) \in \Sigma_{sS}$.

4. SSDR-promaps.

In this section we discuss some points from [**9**] in connection with the results obtained in the previous section. We need some preliminary results before to go on.

Let us recall the following definition from [**9**] :

4.1. *a promap $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ is called an SSDR-promap provided that any commutative diagram in* $\mathrm{pro}\mathcal{T}op$

$$\begin{tikzcd} \mathtt{X} \arrow[r, "a"] \arrow[d, "\mathtt{f}"'] & Map(K, A) \arrow[d, "i^*"] \\ \mathtt{Y} \arrow[r, "b"] & Map(L, A) \end{tikzcd}$$

*has a filler $\mathtt{Y} \to Map(K, A)$, whenever K is a finite CW complex, L is a finite subcomplex, $i : L \to K$ is the inclusion and $A \in \mathcal{ANR}$. This notion is a generalization of that of SSDR-map introduced in [**4**]. $Map(K, A)$ denotes the space of mappings with the compact-open topology.*

In the sequel we shall denote by SSDR the class of SSDR-promaps while $\mathrm{SSDR}_{\mathcal{T}op}$ will be the subclass of SSDR whose elements are of the form $\mathtt{f} : \mathtt{X} \to Y$, $Y \in \mathcal{T}op$.

Thm. 3.5 of [**9**] states that $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ is an SSDR promap iff it satisfies the following two conditions :

(SSDR1) *for every $A \in \mathcal{ANR}$ and for every $\mathtt{h} : \mathtt{X} \to A$, there is a $\mathtt{g} : \mathtt{Y} \to A$, such that $\mathtt{g} \circ \mathtt{f} = \mathtt{h}$,*

(SSDR2) *given morphisms $\mathtt{g}$, $\mathtt{h} : \mathtt{Y} \to A$, $A \in \mathcal{ANR}$, and a global homotopy $\mathtt{F} : \mathtt{X} \times I \to A$ joining $\mathtt{f} \circ \mathtt{g}$ and $\mathtt{f} \circ \mathtt{h}$, there exists a global homotopy $\mathtt{G} : \mathtt{Y} \times I \to A$ joining $\mathtt{g}$ and $\mathtt{h}$, such that $\mathtt{F} = \mathtt{G} \circ (\mathtt{f} \times 1)$.*

Since (SSDR2) says exactly that $\mathtt{f}$ has the SHEP w.r.t. $\mathcal{ANR}$, from theorems 3.5(4) and 3.6, one obtains the

Theorem 4.2. THEOREM 4.2 *Let* $\mathtt{f} : \mathtt{X} \to \mathtt{Y}$ *be a level cofibration in* pro$\mathcal{T}op$. $\mathtt{f}$ *is an SSDR promap iff it is a strong shape equivalence.*

Remark 4.3. As a consequence of the theorem, it follows that every trivial cofibration is an SSDR-promap. In fact, by ([**10**], 3.3.36), one may assume, up to isomorphisms, that $\mathtt{f}$ is a level trivial cofibration. Then, it is clear that every (strongly) SSDR-fibrant prospace is also (strongly) fibrant. Moreover, if $\mathtt{Z}$ is a (strongly) SSDR-fibered prospace, then its inverse limit lim $\mathtt{Z}$ is SSDR$_{\mathcal{T}op}$-fibrant: let $\mathtt{f} : \mathtt{X} \to Y \in \mathrm{SSDR}_{\mathcal{T}op}$ and $\mathtt{a} : \mathtt{X} \to \lim \mathtt{Z}$, be given. If $\mathtt{p} : \lim \mathtt{Z} \to \mathtt{Z}$ is the limiting cone, there is a $\mathtt{y} : Y \to \mathtt{Z}$, such that $\mathtt{y} \circ \mathtt{f} = \mathtt{p} \circ \mathtt{a}$ and, by the universal property of the limit, there is also a $\mathtt{t} : Y \to \lim \mathtt{Z}$, with $\mathtt{p} \circ \mathtt{t} = \mathtt{y}$. It follows that $\mathtt{t} \circ \mathtt{f} = \mathtt{a}$.

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L. Stramaccia `stra@dipmat.unipg.it`

Dipartimento di Matematica e Informatica
Universitá di Perugia
via Vanvitelli, I-06123 Perugia