Experimental Mathematics, 21(4):385–394, 2012

ISSN: 1058-6458 print / 1944-950X online
DOI: 10.1080/10586458.2012.688376

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Decomposition of Semigroup Algebras

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CONTENTS

Let $A \subseteq B$ be cancellative abelian semigroups, and let R be an integral domain. We show that the semigroup ring $R[B]$ can be decomposed, as an $R[A]$-module, into a direct sum of $R[A]$-submodules of the quotient ring of $R[A]$. In the case of a finite extension of positive affine semigroup rings, we obtain an algorithm computing the decomposition. When $R[A]$ is a polynomial ring over a field, we explain how to compute many ring-theoretic properties of $R[B]$ in terms of this decomposition. In particular, we obtain a fast algorithm to compute the Castelnuovo–Mumford regularity of homogeneous semigroup rings. As an application we confirm the Eisenbud–Goto conjecture in a range of new cases. Our algorithms are implemented in the Macaulay2 package MonomialAlgebras.

1. INTRODUCTION

Let $A \subseteq B$ be cancellative abelian semigroups, and let R be an integral domain. Denote by $G(B)$ the group generated by B, and by $R[B]$ the semigroup ring associated to B, that is, the free R-module with basis formed by the symbols t^a for $a \in B$, and multiplication given by the R-bilinear extension of $t^a \cdot t^b = t^{a+b}$. Extending a result of [Hoa and Stückrad 03], we show that the semigroup ring $R[B]$ can be decomposed, as an $R[A]$-module, into a direct sum of $R[A]$-submodules of $R[G(A)]$ indexed by the elements of the factor group $G(B)/G(A)$.

By a *positive affine semigroup* we mean a finitely generated subsemigroup $B \subseteq \mathbb{N}^m$, for some m. If $A \subseteq B \subseteq \mathbb{N}^m$ are positive affine semigroups, K is a field, and the positive rational cones $C(A) \subseteq C(B)$ spanned by A and B are equal, then $K[B]$ is a finitely generated $K[A]$-module, and we can make the decomposition above effective. In this case, the number of submodules I_g in the decomposition is finite, and we can choose them to be ideals of $K[A]$. We give an algorithm for computing the decomposition, implemented in our Macaulay2 [Grayson and Stillman 10] package MonomialAlgebras [Böhm et al. 12].

By a *simplicial semigroup*, we mean a positive affine semigroup B such that $C(B)$ is a simplicial cone. If B is

2000 AMS Subject Classification: Primary 13D45; Secondary 13P99, 13H10
Keywords: Semigroup rings, Castelnuovo–Mumford regularity, Eisenbud–Goto conjecture, computational commutative algebra

simplicial and A is a subsemigroup generated by elements on the extremal rays of B, many ring-theoretic properties of $K[B]$ such as being Gorenstein, Cohen–Macaulay, Buchsbaum, normal, or seminormal can be characterized in terms of the decomposition; see Proposition 3.1. Using this, we can provide functions to test those properties efficiently.

Recall that every positive affine semigroup B has a unique minimal generating set $\mathrm{Hilb}(B)$ called its *Hilbert basis*. By a *homogeneous semigroup* we mean a positive affine semigroup that admits an $\mathbb{N}$-grading in which all the elements of $\mathrm{Hilb}(B)$ have degree 1.

One motivation for developing the decomposition algorithm was to have a more efficient algorithm to compute the Castelnuovo–Mumford regularity (see Section 4 for the definition) of a homogeneous semigroup ring $K[B]$. This invariant is often computed from a minimal graded free resolution of $K[B]$ as a module over a polynomial ring in n variables, where n is the cardinality of $\mathrm{Hilb}(B)$. The free resolution could have length $n-1$, and if n is large (say $n \geq 15$), this computation becomes very time-consuming. But in fact, the Castelnuovo–Mumford regularity of $K[B]$ can be computed from a minimal graded free resolution of $K[B]$ as a module over any polynomial ring, so long as $K[B]$ is finitely generated.

For example, if A is the subsemigroup generated by elements of $\mathrm{Hilb}(B)$ that lie on the extremal rays of B, and $K[B] \cong \oplus_g I_g$ is a decomposition as graded $K[A]$-modules, then the regularity of $K[B]$ is the maximum of the regularities of the I_g as $K[A]$-modules (Proposition 4.1). Since the minimal graded free resolution of I_g has length at most the cardinality of $\mathrm{Hilb}(A)$ (equal to the dimension of $K[B]$ in the simplicial case), and the decomposition can be obtained very efficiently, this method of computing the regularity is typically much faster. See Section 4 for timings.

The Eisenbud–Goto conjecture gives a bound on the Castelnuovo–Mumford regularity; see [Eisenbud and Goto 84]. It is known to hold in relatively few cases. The efficiency of our algorithm allows us to test many new cases of the conjecture for homogeneous semigroup rings of dimension 3, 4, and 5 (Proposition 4.3).

2. DECOMPOSITION

If $X \subseteq G(B)$ we write $t^X := \{t^x \mid x \in X\}$.

Theorem 2.1. *Let $A \subseteq B$ be cancellative abelian semigroups, and let R be an integral domain. The $R[A]$-module $R[B]$ is isomorphic to the direct sum of submodules $I_g \subseteq R[G(A)]$ indexed by elements $g \in G := G(B)/G(A)$.*

Proof. We think of an element $g \in G$ as a subset of $G(B)$. For $g \in G$, let

$$\Gamma'_g := \{b \in B \mid b \in g\}.$$

By construction, we have

$$R[B] = \bigoplus_{g \in G} R \cdot t^{\Gamma'_g}.$$

For each $g \in G$, choose a representative $h_g \in g \subseteq G(B)$. The module $R \cdot t^{\Gamma'_g}$ is an $R[A]$-submodule of $R[B]$, and as such, it is isomorphic to

$$I_g := R \cdot \{t^{b-h_g} \mid b \in \Gamma'_g\} \subseteq R[G(A)].$$

□

We note that such a decomposition was considered in [Bruns and Gubeladze 03] for polynomial rings $R[B]$ over a field R and certain normal affine subsemigroups A of B.

With notation as in the proof, we have

$$R[B] \cong_{R[A]} \bigoplus_{g \in G} I_g \cdot t^{h_g}.$$

This decomposition, together with the ring structure of $R[A]$ and the group structure of G, actually determines the ring structure of $R[B]$: if $x \in I_{g_1}$ and $y \in I_{g_2}$ and $xy = z$ as elements of $R[G(A)]$, then as elements in the decomposition of $R[B]$,

$$x \cdot_{R[B]} y = \frac{t^{h_{g_1}} t^{h_{g_2}}}{t^{h_{g_1+g_2}}} z \in I_{g_1+g_2}.$$

Henceforward, we assume that $A \subseteq B \subseteq \mathbb{N}^m$ are positive affine semigroups, and we work with monomial algebras over a field K.

The set $B_A = \{x \in B \mid x \notin B + (A \setminus \{0\})\}$ is the unique minimal subset of B such that t^{B_A} generates $K[B]$ as a $K[A]$-module. We define $\Gamma_g := \{b \in B_A \mid b \in g\}$. Then $\Gamma_g + A = \Gamma'_g$.

We can compute the decomposition of Theorem 2.1 if $K[B]$ is a finitely generated $K[A]$-module, or equivalently, if B_A is a finite set. This finiteness (for positive affine semigroups $A \subseteq B$) is equivalent to the property $C(A) = C(B)$, where $C(X)$ denotes the positive rational cone spanned by X in $\mathbb{Q}^m$. (Proof: If $C(A) \subsetneq C(B)$, we can choose an element $x \in B$ on a ray of $C(B)$ not in $C(A)$, so $nx \in B_A$ for all $n \in \mathbb{N}^+$. Thus B_A is not finite. Conversely, if $C(A) = C(B)$, then for all $b \in B$, there exists $n_b \in \mathbb{N}^+$ such that $n_b b \in A$. To generate $K[B]$ as a $K[A]$-module, it suffices to take all possible sums of the

Algorithm 1 Decompose a monomial algebra.

Input: A homogeneous ring homomorphism

$$\psi : K[y_1,\ldots,y_d] \to K[x_1,\ldots,x_n]$$

of $\mathbb{N}^m$-graded polynomial rings over a field K with $\deg y_i = e_i$ and $\deg x_j = b_j$ such that $\psi(y_i)$ is a monomial for all i and the gradings specify positive affine semigroups $A = \langle e_1,\ldots,e_d\rangle \subseteq B = \langle b_1,\ldots,b_n\rangle \subseteq \mathbb{N}^m$ with $C(A) = C(B)$.

Output: An ideal $I_g \subseteq K[A]$ and a shift $h_g \in G(B)$ for each $g \in G := G(B)/G(A)$ with

$$K[B] \cong \bigoplus\nolimits_{g\in G} I_g(-h_g)$$

as $\mathbb{Z}^m$-graded $K[A]$-modules (with $\deg t^b = b$).

1: Compute the set $B_A = \{b \in B \mid b \notin B + (A \setminus \{0\})\}$, and let $\{v_1,\ldots,v_r\}$ be the monomials in $K[B]$ corresponding to elements of B_A. For example, this can be done by computing the toric ideal $I_B := \ker\varphi$ associated to B, where

$$\varphi : K[x_1,\ldots,x_n] \to K[B], \quad x_i \mapsto t^{b_i},$$

and then computing a monomial K-basis $v_1,\ldots,v_r$ of

$$K[x_1,\ldots,x_n]/(I_B + \psi(\langle y_1,\ldots,y_d\rangle)).$$

2: Partition the elements v_i by their class modulo $G(A)$, forming the decomposition

$$B_A = \dot{\bigcup}_{g\in G}\Gamma_g.$$

3: For each $g \in G$, choose a representative $\bar{g} \in \Gamma_g$.

4: For each $v \in \Gamma_g$, choose $c_{v,j} \in \mathbb{Z}$ such that

$$v = \bar{g} + \sum\nolimits_{j=1}^{d} c_{v,j}e_j.$$

5: Let $\bar{c}_{g,j} := \min\{c_{v,j} \mid v \in \Gamma_g\}$.

$$\left\{h_g := \bar{g} + \sum\nolimits_{j=1}^{d} \bar{c}_{g,j}e_j, I_g := K[A]\{t^{v-h_g} \mid v \in \Gamma_g\} \mid g \in G\right\}$$

multiples mb such that $m < n_b$ for all b in a (finite) generating set for the semigroup B.) Note that if B_A is finite, then $G(B)/G(A)$ is also finite.

From these observations we obtain Algorithm 1, computing the set B_A and the decomposition of $K[B]$.

For $v \in \Gamma_g$, the element t^{v-h_g} is in $K[A]$, because

$$v - h_g = \sum\nolimits_{j=1}^{d} (c_{v,j} - \bar{c}_{g,j})\, e_j$$

is an expression with nonnegative integer coefficients. Thus, I_g is a monomial ideal of $K[A]$, and $h_g \in G(B)$ for each $g \in G$, as required.

Example 2.2. Consider

$$B = \langle(2,0,3),(4,0,1),(0,2,3),(1,3,1),(1,2,2)\rangle \subset \mathbb{N}^3$$

and the subsemigroup

$$A = \langle(2,0,3),(4,0,1),(0,2,3),(1,3,1)\rangle.$$

We get the decomposition of B_A into equivalence classes

$$B_A = \{0,(2,4,4)\} \cup \{(1,2,2),(3,6,6)\}.$$

Choosing shifts $h_1 = (-2,0,-3)$ and $h_2 = (-1,2,-1)$ in $G(B)$, we have

$$\begin{aligned} K[B] &\cong K[A]\{t^{(2,0,3)}, t^{(4,4,7)}\}(-h_1) \\ &\quad \oplus K[A]\{t^{(2,0,3)}, t^{(4,4,7)}\}(-h_2) \\ &\cong \langle x_0, x_1x_2^2\rangle(-h_1) \oplus \langle x_0, x_1x_2^2\rangle(-h_2), \end{aligned}$$

where $K[A] \cong K[x_0,x_1,x_2,x_3]/\langle x_1^2x_2^3 - x_0^3x_3^2\rangle$.

Example 2.3. Using our implementation of Algorithm 1 in the Macaulay2 package MonomialAlgebras, we compute the decomposition of $\mathbb{Q}[B]$ over $\mathbb{Q}[A]$ in the case given in Example 2.2:

```
i1: loadPackage "MonomialAlgebras";
i2: A = {{2,0,3},{4,0,1},{0,2,3},{1,3,1}};
i3: B = {{2,0,3},{4,0,1},{0,2,3},{1,3,1},{1,2,2}};
i4: S = QQ[x_0 .. x_4, Degrees=>B];
i5: P = QQ[x_0 .. x_3, Degrees=>A];
i6: f = map(S,P);
i7: dc = decomposeMonomialAlgebra f
o7: HashTable{ {0,0,0} => { ideal ( x0, x1x2^2 ), {-2,0,-3} }
               {5,0,0} => { ideal ( x0, x1x2^2 ), {-1,2,-1} }}
i8: ring first first values dc
o8:        P
     ---------------
     x1^2x2^3-x0^3x3^2
```

The keys of the hash table represent the elements of G.

3. RING-THEORETIC PROPERTIES

In this section, we will always consider simplicial semigroups. Recall that a positive affine semigroup B is simplicial if it spans a simplicial cone, or equivalently, if there are linearly independent elements $e_1,\ldots,e_d \in B$ with $C(B) = C(\{e_1,\ldots,e_d\})$. Many ring-theoretic properties of semigroup algebras can be determined from the combinatorics of the semigroup; see [García-Sánchez and Rosales 02, Hochster 72, Hochster and Roberts 76, Li 04, Stanley 78]. Here we give characterizations in terms of the decomposition of Theorem 2.1.

Proposition 3.1. *Let K be a field, $B \subseteq \mathbb{N}^m$ a simplicial semigroup, and let A be the submonoid of B that is generated by linearly independent elements $e_1,\ldots,e_d$ of B with $C(A) = C(B)$. Let B_A be as above, and let $K[B] \cong \bigoplus_{g\in G} I_g(-h_g)$ be the output of Algorithm 1 with respect to $A \subseteq B$ using minimal generators of A. We have:*

1. *The depth of $K[B]$ is the minimum of the depths of the ideals I_g.*
2. *$K[B]$ is Cohen–Macaulay if and only if every ideal I_g is equal to $K[A]$.*
3. *$K[B]$ is Gorenstein if and only if $K[B]$ is Cohen–Macaulay and the set of shifts $\{h_g\}_{g\in G}$ has exactly one maximal element with respect to $\leq$ given by $x \leq y$ if there is an element $z \in B$ such that $x + z = y$.*
4. *$K[B]$ is Buchsbaum if and only if each ideal I_g either is equal to $K[A]$ or is equal to the homogeneous maximal ideal of $K[A]$, and $h_g + b \in B$ for all $b \in \operatorname{Hilb}(B)$.*
5. *$K[B]$ is normal if and only if for every element x in B_A, there exist $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}$ with $0 \leq \lambda_i < 1$ for all i such that $x = \sum_{i=1}^d \lambda_i e_i$.*
6. *$K[B]$ is seminormal if and only if for every element x in B_A there exist $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}$ with $0 \leq \lambda_i \leq 1$ for all i such that $x = \sum_{i=1}^d \lambda_i e_i$.*

Proof. For every $x \in G(B)$ there are uniquely determined elements $\lambda_1^x, \ldots, \lambda_d^x \in \mathbb{Q}$ such that $x = \sum_{j=1}^d \lambda_j^x e_j$. Then by construction,

$$h_g = \sum_{j=1}^{d} \min\left\{\lambda_j^v \mid v \in \Gamma_g\right\} e_j.$$

Assertions 1 and 2 follow immediately; assertion 2 was already mentioned in [Stanley 78, Theorem 6.4]. Assertion 3 can be found in [Stanley 78, Corollary 6.5].

To prove assertion 4, let I_g be a proper ideal, equivalently, $\#\Gamma_g \geq 2$. The ideal I_g is equal to the homogeneous maximal ideal of $K[A]$ and $h_g + b \in B$ for all $b \in \operatorname{Hilb}(B)$ if and only if $\Gamma_g = \{m + e_1, \ldots, m + e_d\}$ for some m with $m + b \in B$ for all $b \in \operatorname{Hilb}(B)$. Now the assertion follows from [García-Sánchez and Rosales 02, Theorem 9].

For assertion 5, we set

$$D_A = \Big\{x \in G(B) \mid x = \sum_{i=1}^{d} \lambda_i e_i, \lambda_i \in \mathbb{Q} \text{ and } 0 \leq \lambda_i < 1 \ \forall i\Big\}.$$

The ring $K[B]$ is normal if and only if $B = C(B) \cap G(B)$ by [Hochster 72, Proposition 1]. We need to show that $C(B) \cap G(B) \subseteq B$ if and only if $B_A \subseteq D_A$. We have $B_A \subseteq D_A$ if and only if $D_A \subseteq B_A$, since B_A has $\#G = \#D_A$ equivalence classes and by definition of B_A. Note that $D_A \subseteq C(B) \cap G(B)$ and $D_A \cap B \subseteq B_A$. The assertion follows from the fact that every element $x \in C(B) \cap G(B)$ can be written as $x = x' + \sum_{i=1}^d n_i e_i$ for some $x' \in D_A$ and $n_i \in \mathbb{N}$.

To prove assertion 6, we set

$$\bar{D}_A := \Big\{x \in B \mid x = \sum_{i=1}^{d} \lambda_i e_i, \lambda_i \in \mathbb{Q} \text{ and } 0 \leq \lambda_i \leq 1 \ \forall i\Big\}.$$

By [Hochster and Roberts 76, Proposition 5.32] and [Li 04, Theorem 4.1.1], $K[B]$ is seminormal if and only if $B_A \subseteq \bar{D}_A$, provided that $e_1, \ldots, e_d \in \operatorname{Hilb}(B)$. Otherwise, there is $k \in \{1, \ldots, d\}$ with $e_k = e_k' + e_k''$ and $e_k', e_k'' \in B \setminus \{0\}$. We set $A' = \langle e_1, \ldots, e_k', \ldots, e_d\rangle$ and $A'' = \langle e_1, \ldots, e_k'', \ldots, e_d\rangle$. Clearly, $C(A) = C(A') = C(A'')$. We need to show that $B_A \subseteq \bar{D}_A$ if and only if $B_{A'} \subseteq \bar{D}_{A'}$. Let $x \in B_A \setminus \bar{D}_A$. If $x - e_k' \notin B$, then $x \in B_{A'} \setminus \bar{D}_{A'}$. If $x - e_k' \in B$, then $x - e_k' \in B_{A''} \setminus \bar{D}_{A''}$. Let $x \in B_{A'} \setminus \bar{D}_{A'}$, say $x = \sum_{j\neq k} \lambda_j e_j + \lambda_k e_k'$ and $\lambda_j > 1$ for some j. If $j \neq k$, then $x \in B_A \setminus \bar{D}_A$. Let $j = k$; consider the element $y = x + e_k'' - \sum_{j\neq k} n_j e_j \in B$ for some $n_j \in \mathbb{N}$ such that $\sum_{j\neq k} n_j$ is maximal. It follows that $y \in B_A \setminus \bar{D}_A$, and we are done. $\square$

Note that normality of positive affine semigroup rings can also be tested using the implementation of normalization in the program NORMALIZ [Bruns et al. 12]. We remark that from Proposition 3.1, it follows that every simplicial affine semigroup ring $K[B]$ that is seminormal and Buchsbaum is also Cohen–Macaulay. This holds more generally for arbitrary positive affine semigroups by [Bruns et al. 06, Proposition 4.15].

Example 3.2. (Smooth Rational Monomial Curves in $\mathbb{P}^3$.) Consider the simplicial semigroup

$$B = \langle(\alpha, 0), (\alpha - 1, 1), (1, \alpha - 1), (0, \alpha)\rangle \subseteq \mathbb{N}^2$$

and set $A = \langle(\alpha, 0), (0, \alpha)\rangle$, say $K[A] = K[x, y]$. Note that we have α equivalence classes. We get

$$K[B] \cong K[x, y]^3 \oplus \langle x^{\alpha-3}, y\rangle \oplus \langle x^{\alpha-4}, y^2\rangle \oplus \cdots \oplus \langle x, y^{\alpha-3}\rangle$$

as $K[x, y]$-modules, where the shifts are omitted. In the decomposition, each ideal of the form $\langle x^i, y^j\rangle$, $1 \leq i, j \leq \alpha - 3$, with $i + j = \alpha - 2$, appears exactly once. Hence $K[B]$ is not Buchsbaum for $\alpha > 4$, since $\langle x^{\alpha-3}, y\rangle$ is a direct summand. In case $\alpha = 4$, there is only one proper ideal $I_4 = \langle x, y\rangle$ and $h_4 = (2, 2)$; in fact, $(2, 2) + \operatorname{Hilb}(B) \subseteq B$, and therefore $K[B]$ is Buchsbaum. It follows immediately that $K[B]$ is Cohen–Macaulay for $\alpha \leq 3$, Gorenstein for $\alpha \leq 2$, seminormal for $\alpha \leq 3$, and normal for $\alpha \leq 3$. Note that we could also decompose $K[B]$ over the subring $K[A]$, where $A = \langle(2\alpha, 0), (0, 2\alpha)\rangle = K[x', y']$. For $\alpha = 4$, we would get

$$K[B] \cong K[x', y']^{15} \oplus \langle x', y'\rangle,$$

and the corresponding shift of $\langle x', y'\rangle$ is again $(2, 2)$.

Example 3.3. Let

$$B = \langle (1,0,0), (0,1,0), (0,0,2), (1,0,1), (0,1,1) \rangle \subset \mathbb{N}^3.$$

Moreover, let $A = \langle (1,0,0), (0,1,0), (0,0,2) \rangle$, say $K[A] = K[x,y,z]$. This example was given in [Li 04, Example 6.0.2] to study the relation between seminormality and the Buchsbaum property. We have

$$K[B] \cong K[A] \oplus \langle x, y \rangle(-(0,0,1)),$$

as $\mathbb{Z}^3$-graded $K[A]$-modules. Hence $K[B]$ is not Buchsbaum, since $\langle x, y \rangle$ is not maximal; moreover, $K[B]$ is seminormal, but not normal.

Example 3.4. Consider the semigroup

$$B = \langle (1,0,0), (0,2,0), (0,0,2), (1,0,1), (0,1,1) \rangle \subset \mathbb{N}^3,$$

and set $A = \langle (1,0,0), (0,2,0), (0,0,2) \rangle$. We get

$$K[B] \cong K[A] \oplus K[A](-(1,0,1)) \oplus K[A](-(0,1,1)) \oplus K[A](-(1,1,2)).$$

Hence $K[B]$ is Gorenstein, since $(1,0,1) + (0,1,1) = (1,1,2)$. Moreover, $K[B]$ is not normal, since $(1,0,1) = (1,0,0) + \frac{1}{2}(0,0,2)$, but seminormal.

Example 3.5. We illustrate our implementation of the characterizations given in Proposition 3.1 in the case of Example 3.4:

```
i1: B = {{1,0,0},{0,2,0},{0,0,2},{1,0,1},{0,1,1}};
i2: isGorensteinMA B
o2: true
i3: isNormalMA B
o3: false
i4: isSeminormalMA B
o4: true
```

Note that there are also commands `isCohenMacaulayMA` and `isBuchsbaumMA` available for testing the Cohen–Macaulay and the Buchsbaum properties, respectively.

4. REGULARITY

Let K be a field and let $R = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring, that is, $\deg x_i = 1$ for all $i = 1, \ldots, n$. Let R_+ be the homogeneous maximal ideal of R, and let M be a finitely generated graded R-module. We define the *Castelnuovo–Mumford regularity* $\operatorname{reg} M$ of M by

$$\operatorname{reg} M := \max \left\{ a(H^i_{R_+}(M)) + i \mid i \geq 0 \right\},$$

where $a(H^i_{R_+}(M)) := \max \left\{ n \mid [H^i_{R_+}(M)]_n \neq 0 \right\}$ and $a(0) = -\infty$; $H^i_{R_+}(M)$ denotes the ith local cohomology module of M with respect to R_+. Note that $\operatorname{reg} M$ can also be computed in terms of the shifts in a minimal graded free resolution of M. An important application of the regularity is that it bounds the degrees in certain minimal Gröbner bases by [Bayer and Stillman 87]. Thus, it is of interest to compute or bound the regularity of a homogeneous ideal. The following conjecture (Eisenbud–Goto) was made in [Eisenbud and Goto 84]: If K is algebraically closed and I is a homogeneous prime ideal of R, then for $S = R/I$,

$$\operatorname{reg} S \leq \deg S - \operatorname{codim} S.$$

Here $\deg S$ denotes the degree of S and $\operatorname{codim} S := \dim_K S_1 - \dim S$ the codimension. The conjecture has been proved for dimension 2 by Gruson, Lazarsfeld, and Peskine (see [Gruson et al. 83]); for the Buchsbaum case by [Stückrad and Vogel 88] (see also [Treger 82] and [Stückrad and Vogel 87]); for $\deg S \leq \operatorname{codim} S + 2$ by Hoa, Stückrad, and Vogel, see [Hoa et al. 91]; and in characteristic zero for smooth surfaces and certain smooth threefolds by [Lazarsfeld 87] and [Ran 90]. There is also a stronger version in which S is only required to be reduced and connected in codimension 1; this version has been proved in dimension 2 by [Giaimo 06]. For homogeneous semigroup rings of codimension 2, the conjecture was proved by [Peeva and Sturmfels 98]. Even in the simplicial setting, the conjecture is largely open, though it was proved for the isolated singularity case by [Herzog and Hibi 03], for the seminormal case by [Nitsche 12], and for a few other cases by [Hoa and Stückrad 03, Nitsche 11].

We now focus on computing the regularity of a homogeneous semigroup ring $K[B]$. Note that a positive affine semigroup B is homogeneous if and only if there is a group homomorphism $\deg : G(B) \to \mathbb{Z}$ with $\deg b = 1$ for all $b \in \operatorname{Hilb}(B)$. We always consider the R-module structure on $K[B]$ given by the homogeneous surjective K-algebra homomorphism $R \twoheadrightarrow K[B], x_i \mapsto t^{b_i}$, where $\operatorname{Hilb}(B) = \{b_1, \ldots, b_n\}$. Generalizing the results from [Hoa and Stückrad 03], the regularity can be computed in terms of the decomposition of Theorem 2.1 as follows:

Proposition 4.1. *Let K be an arbitrary field, and let $B \subseteq \mathbb{N}^m$ be a homogeneous semigroup. Fix a group homomorphism $\deg : G(B) \to \mathbb{Z}$ with $\deg b = 1$ for all $b \in \operatorname{Hilb}(B)$. Moreover, let A be a submonoid of B with $\operatorname{Hilb}(A) = \{e_1, \ldots, e_d\}$, $\deg e_i = 1$ for all i, and $C(A) = C(B)$. Let $K[B] \cong \bigoplus_{g \in G} I_g(-h_g)$ be the output of Algorithm 1 with respect to $A \subseteq B$. Then:*

Algorithm 2 The regularity algorithm.

Input: The Hilbert basis $\mathrm{Hilb}(B)$ of a homogeneous semigroup $B \subseteq \mathbb{N}^m$ and a field K.
Output: The Castelnuovo–Mumford regularity $\operatorname{reg} K[B]$.
1: Choose a minimal subset $\{e_1, \ldots, e_d\}$ of $\mathrm{Hilb}(B)$ with $C(\{e_1, \ldots, e_d\}) = C(B)$, and set $A = \langle e_1, \ldots, e_d \rangle$.
2: Compute the decomposition $K[B] \cong \bigoplus_{g \in G} I_g(-h_g)$ over $K[A]$ by Algorithm 1.
3: Compute a hyperplane $H = \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid \sum_{j=1}^m a_j t_j = c\}$ with $c \neq 0$ such that $\mathrm{Hilb}(B) \subseteq H$. Define $\deg : \mathbb{R}^m \to \mathbb{R}$ by $\deg(t_1, \ldots, t_m) = (\sum_{j=1}^m a_j t_j)/c$. $\operatorname{reg} K[B] = \max\{\operatorname{reg} I_g + \deg h_g \mid g \in G\}$.

1. $\operatorname{reg} K[B] = \max\{\operatorname{reg} I_g + \deg h_g \mid g \in G\}$, *where* $\operatorname{reg} I_g$ *denotes the regularity of the ideal* $I_g \subseteq K[A]$ *with respect to the canonical* $K[x_1, \ldots, x_d]$*-module structure.*
2. $\deg K[B] = \#G \cdot \deg K[A]$.

Proof. To prove the first assertion, consider the $T = K[x_1, \ldots, x_d]$-module structure on $K[B]$, which is given by $T \twoheadrightarrow K[A] \subseteq K[B]$, $x_i \mapsto t^{e_i}$. Since $C(A) = C(B)$, we get by [Brodmann and Sharp 98, Theorem 13.1.6],

$$H^i_{K[B]_+}(K[B]) \cong H^i_{T_+}(K[B]),$$

as $\mathbb{Z}$-graded T-modules (where $K[B]_+$ is the homogeneous maximal ideal of $K[B]$). By the same theorem, we obtain $H^i_{K[B]_+}(K[B]) \cong H^i_{R_+}(K[B])$. Then the assertion follows from $K[B] \cong \bigoplus_{g \in G} I_g(-\deg h_g)$ as $\mathbb{Z}$-graded T-modules.

Assertion 2 follows from $\deg I_g = \deg K[A]$ for all $g \in G$. □

Using Proposition 4.1, we obtain Algorithm 2, by which the computation of $\operatorname{reg} K[B]$ reduces to computing minimal graded free resolutions of the monomial ideals I_g in $K[A]$ as $K[x_1, \ldots, x_d]$-modules.

Example 4.2. We apply Algorithm 2 using the decomposition computed in Example 2.3. A resolution of $I = \langle x_0, x_1 x_2^2 \rangle$ as a $T = \mathbb{Q}[x_0, x_1, x_2, x_3]$-module is

$$0 \longrightarrow T(-4) \oplus T(-5) \xrightarrow{d} T(-1) \oplus T(-3) \longrightarrow I \longrightarrow 0$$

with

$$d = \begin{pmatrix} x_1 x_2^2 & x_0^2 x_3^2 \\ -x_0 & -x_1 x_2 \end{pmatrix},$$

whence $\operatorname{reg} I = 4$. The group homomorphism is given by $\deg b = (b_1 + b_2 + b_3)/5$, and therefore, $\operatorname{reg} \mathbb{Q}[B] = \max\{4 - 1, 4 - 0\} = 4$.

With respect to timings, we first focus on dimension 3, comparing our implementation of Algorithm 2 in the Macaulay2 package MonomialAlgebras (marked in the tables by MA) with other methods. Here we consider the computation of the regularity via a minimal graded free resolution both in Macaulay2 (M2) and Singular [Decker et al. 12] (S). Furthermore, we compare our algorithm with the algorithm of [Bermejo and Gimenez 06]. This method does not require the computation of a free resolution, and is implemented in the Singular package mregular.lib [Bermejo et al. 11] (BG-S) and the Macaulay2 package Regularity [Seceleanu and Stapleton 10] (BG-M2). For comparability we obtain the toric ideal I_B always

	Codimension c								
Algorithm	1	2	3	4	5	6	7	8	9
MA	.073	.089	.095	.10	.13	.14	.14	.19	.16
M2	.0084	.0089	.011	.017	.043	.10	.45	2.8	21
S	.0099	.0089	.011	.013	.020	.046	.18	1.1	6.8
BG-S	.016	.030	.19	1.2	15	24	59	44	77
BG-M2	.036	.053	.47	1.8	9.0	19	34	39	43

	Codimension c								
Algorithm	10	11	12	13	14	15	16	17	18
MA	.21	.26	.22	.26	.29	.30	.31	.36	.47
M2	180	*	*	*	*	*	*	*	*
S	30	*	*	*	*	*	*	*	*
BG-S	170	520	*	*	*	*	360	460	350
BG-M2	85	150	140	250	310	290	300	410	320

TABLE 1. Algorithm timing comparisons for $K = \mathbb{Q}$, $d = 3$, $\alpha = 5$, and $n = 15$ examples.

	Codimension c								
Algorithm	1	2	3	4	5	6	7	8	9
MA	.072	.088	.093	.10	.12	.13	.13	.19	.16
M2	.0075	.0095	.010	.013	.020	.032	.090	.40	2.8
S	.0067	.010	.011	.015	.023	.041	.16	.99	6.3
BG-S	.017	.020	.031	.052	.094	.12	.18	.34	.42
BG-M2	.030	.037	.064	.14	.34	.48	.80	1.5	2.0

	Codimension c								
Algorithm	10	11	12	13	14	15	16	17	18
MA	.21	.25	.22	.25	.29	.29	.31	.35	.39
M2	26	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
S	28	250	$*$	$*$	$*$	$*$	$*$	$*$	$*$
BG-S	.57	.88	.88	1.1	1.4	1.5	1.7	2.5	2.4
BG-M2	3.3	4.4	4.4	6.4	7.9	7.8	9.2	12	13

TABLE 2. Algorithm timing comparisons for $K = \mathbb{Z}/101$, $d = 3, \alpha = 5$, and $n = 15$ examples.

through the program 4TI2 [Hemmecke et al. 08], which can be called optionally in our implementation (using [Petrovic et al. 10]). We give the average computation times over n examples generated by the function `randomSemigroup(`α`,d,c,num=>n,setSeed=>true)`. Starting with the standard random seed, this function generates n random semigroups $B \subseteq \mathbb{N}^d$ such that

- $\dim K[B] = d$.
- $\operatorname{codim} K[B] = c$; that is, the number of generators of B is $d + c$.
- Each generator of B has coordinate sum equal to α.

All timings are in seconds on a single 2.7-GHz core with 4 GB of RAM. In the cases marked by a star, at least one of the computations ran out of memory or did not finish within 1200 seconds. Note that the computation of $\operatorname{reg} I_g$ in step 4 of Algorithm 2 could easily be parallelized. This is not available in our MACAULAY2 implementation so far.

Table 1 shows the comparison for $K = \mathbb{Q}$, $d = 3$, $\alpha = 5$, and $n = 15$ examples.

For small codimension c, the decomposition approach has slightly higher overhead than the traditional algorithms. For larger codimensions, however, both the resolution approach in MACAULAY2 and SINGULAR and the Bermejo–Gimenez implementation in SINGULAR fail. The average computation times of the REGULARITY package increase significantly, whereas those for Algorithm 2 stay under one second. The traditional approaches become more competitive when the same setup over the finite field $K = \mathbb{Z}/101$ is considered, but are still much slower than Algorithm 2. See Table 2.

Note that over a finite field, there may not exist a homogeneous linear transformation such that the initial ideal is of nested type; see, for example, [Bermejo and Gimenez 06, Remark 4.9]. This case is not covered and hence does not terminate in the implementation of the Bermejo–Gimenez algorithm in the REGULARITY package. In the standard configuration, the package MREGULAR.LIB can handle this case, but then does not perform well over a finite field in our setup. Hence we use its alternative option, which takes the same approach as the REGULARITY package and applies a random homogeneous linear transformation.

Increasing the dimension to $d = 4$, we compare our implementation with the most competitive one, that is, MREGULAR.LIB ($K = \mathbb{Z}/101$, $\alpha = 5$, $n = 1$). Here also the SINGULAR implementation of the Bermejo–Gimenez algorithm fails. See Table 3.

To illustrate the performance of Algorithm 2, we present the computation times ($K = \mathbb{Z}/101$, $n = 1$) of our implementation for $d = 3$ and various values of α and c in Table 4. Table 5 is based on a similar setup for $d = 4$.

	Codimension c												
Algorithm	4	8	12	16	20	24	28	32	36	40	44	48	52
MA	.13	.31	3.8	13	.69	2.2	1.7	1.9	1.5	4.4	6.0	8.9	13
BG-S	.61	2.2	46	150	380	840	940	$*$	$*$	$*$	$*$	$*$	$*$

TABLE 3. Algorithm timing comparisons for $K = \mathbb{Z}/101$, $d = 4, \alpha = 5$, and $n = 1$ example.

	Codimension c												
Coordinate Sum α	4	8	12	16	20	24	28	32	36	40	44	48	52
3	.083												
4	.073	.10	.24										
5	.11	.13	.15	.22									
6	.11	.31	.21	.22	.27	.75							
7	.10	.16	.18	.24	.29	.86	1.0	1.4					
8	.11	.22	.26	.31	.35	.54	.67	.85	1.2	3.6			
9	.13	.25	.31	.38	.56	.64	.77	.98	1.4	3.8	5.7	8.6	13

TABLE 4. Computation times of Algorithm MA for $K = \mathbb{Z}/101$, $d = 3$, and $n = 1$ example.

Obtaining the regularity via Algorithm 2 involves two main computations: decomposing $K[B]$ into a direct sum of monomial ideals $I_g \subseteq K[A]$ via Algorithm 1 and computing a minimal graded free resolution for each I_g. The computation time for the first task is increasing with the codimension. On the other hand, the complexity of the second task grows with the cardinality of $\mathrm{Hilb}(A)$, which tends to be small for big codimension. This explains the good performance of the algorithm for large codimension observed in Table 5. In particular, the simplicial case shows an impressive performance, as illustrated in Table 6 for simplicial semigroups with $d = 5$ and $\alpha = 5$ (same setup as before). The examples are generated by the function `randomSemigroup` using the option `simplicial=>true`.

In case of a homogeneous semigroup ring of dimension 2, the ideals I_g are monomial ideals in two variables. Hence we can read off $\operatorname{reg} I_g$ by ordering the monomials with respect to the lexicographic order (see, for example, [Nitsche 11, Proposition 4.1]). This further improves the performance of the algorithm.

Due to the good performance of Algorithm 2, we can actually do the regularity computation for all possible semigroups B in $\mathbb{N}^d$ such that the generators have coordinate sum α for some α and d. This confirms the Eisenbud–Goto conjecture for some cases.

Proposition 4.3. *Let B be a homogeneous semigroup. The regularity of $\mathbb{Q}[B]$ is bounded by $\deg \mathbb{Q}[B] - \operatorname{codim} \mathbb{Q}[B]$, provided that the minimal generators of B in $\mathbb{N}^d$ have fixed coordinate sum α for $d = 3$ and $\alpha \leq 5$, for $d = 4$ and $\alpha \leq 3$, as well as for $d = 5$ and $\alpha = 2$.*

Proof. The list of all minimal generating sets $\mathrm{Hilb}(B)$ together with $\operatorname{reg} \mathbb{Q}[B]$, $\deg \mathbb{Q}[B]$, and $\operatorname{codim} \mathbb{Q}[B]$ can be found under the link given in [Böhm et al. 12]. □

Figure 1 depicts the values of $\deg \mathbb{Q}[B] - \operatorname{codim} \mathbb{Q}[B]$ plotted against $\operatorname{reg} \mathbb{Q}[B]$ for all semigroups with $\alpha = 3$ and $d = 4$. For the same setup, Figure 2 shows $\operatorname{reg} \mathbb{Q}[B]$ on top of $\operatorname{codim} \mathbb{Q}[B]$ plotted against $\deg \mathbb{Q}[B]$. The line corresponds to the projection of the plane

$$\operatorname{reg} \mathbb{Q}[B] - \deg \mathbb{Q}[B] + \operatorname{codim} \mathbb{Q}[B] = 0.$$

Figures for the remaining cases can be found at [Böhm et al. 12].

	Codimension c									
Coordinate Sum α	8	16	24	32	40	48	56	64	72	80
3	.18	.51								
4	.26	.32	.54							
5	.31	13	2.2	1.9	4.4	8.9				
6	9.6	120	$*$	$*$	3.4	7.8	15	36	66	120

TABLE 5. Computation times of Algorithm MA for $K = \mathbb{Z}/101$, $d = 4$, and $n = 1$ example.

	Codimension c											
Coordinate Sum α	10	20	30	40	50	60	70	80	90	100	110	120
5	13	13	17	32	69	86	110	170	250	400	650	1000

TABLE 6. Computation times of Algorithm MA for $K = \mathbb{Z}/101$, $d = 5$, and $n = 1$ simplicial example.

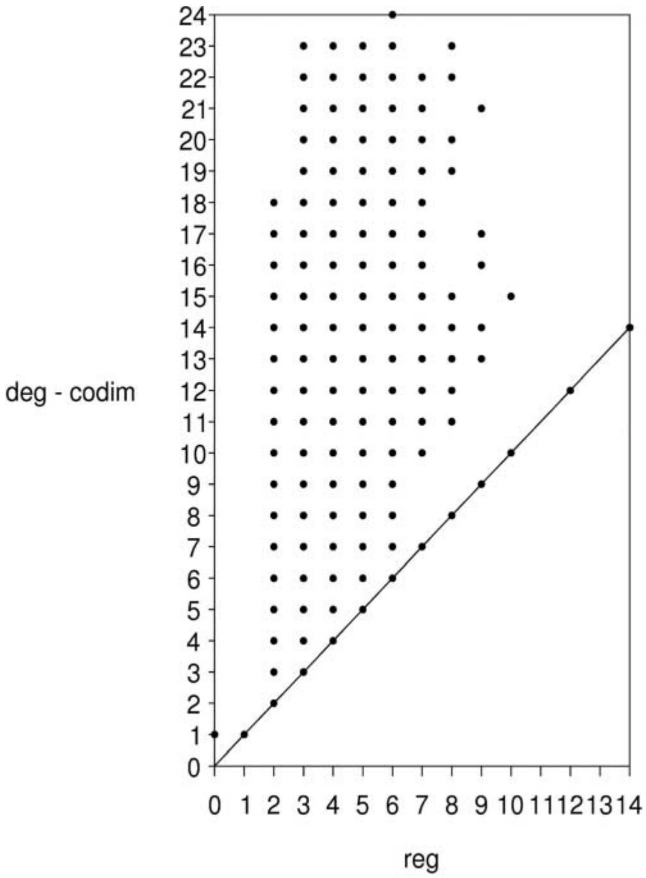

FIGURE 1. $\deg \mathbb{Q}[B] - \operatorname{codim} \mathbb{Q}[B]$ against $\operatorname{reg} \mathbb{Q}[B]$ for $\alpha = 3$ and $d = 4$.

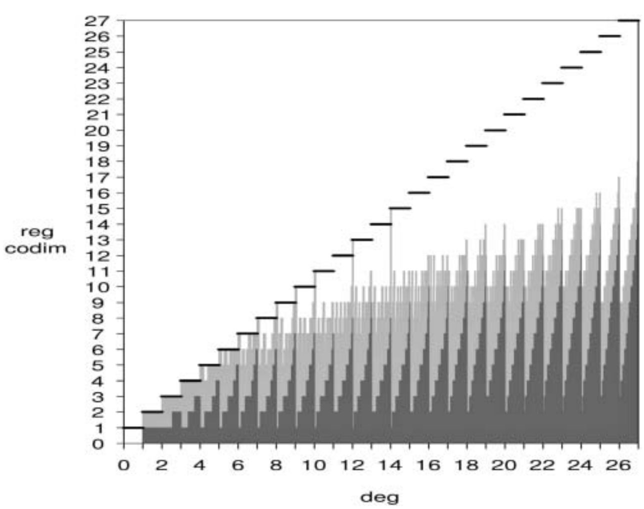

FIGURE 2. $\operatorname{reg} \mathbb{Q}[B] + \operatorname{codim} \mathbb{Q}[B]$ against $\deg \mathbb{Q}[B]$ for $\alpha = 3$ and $d = 4$.

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