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An Empirical Approach to the Normality of π

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CONTENTS

Using the results of several extremely large recent computations [Yee and Kondo 11], we tested positively the normality of a prefix of roughly four trillion hexadecimal digits of π. This result was used by a Poisson process model of normality of π: in this model, it is extraordinarily unlikely that π is not asymptotically normal base 16, given the normality of its initial segment.

1. INTRODUCTION

The question whether (and why) the digits of well-known constants of mathematics are statistically random in some sense has long fascinated mathematicians. Indeed, one prime motivation in computing and analyzing digits of π is to explore the age-old question whether and why these digits appear "random." The first computation on ENIAC in 1949 of π to 2037 decimal places was proposed by John von Neumann to shed some light on the distribution of π (and of e) [Berggren et al. 04, pp. 277–281].

Since then, numerous computer-based statistical checks of the digits of π have failed to disclose any deviation from reasonable statistical norms. See, for instance, Table 1, which presents the counts of individual hexadecimal digits among the first trillion hex digits, as obtained by Yasumasa Kanada. By contrast, early computations revealed provable abnormalities in the behavior of e [Borwein and Borwein 98, Section 11.2]. Figure 1 shows π as a random walk drawn as we describe below.

We use the normality for strings introduced and studied in [Calude 94]: a sequence whose prefixes are normal is normal, but the converse is not true. Using the results of several extremely large recent computations [Yee and Kondo 11], we tested positively the normality of a prefix of roughly four trillion hexadecimal digits of π. This result was used by a Poisson process model of normality of π: in this model, it is extraordinarily unlikely that π is not asymptotically normal base 16, given the normality of its initial segment.
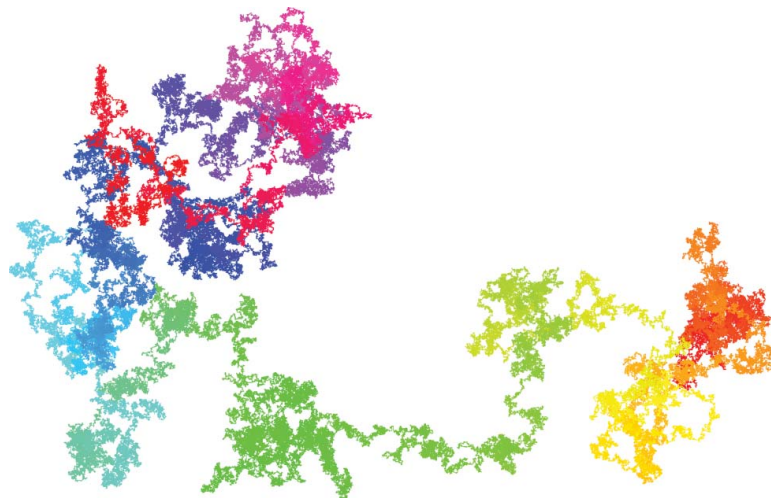
FIGURE 1. A random walk on the first two billion bits of π (normal?) (color figure available online).

2. NORMALITY OF REAL NUMBERS

In Figures 1 through 4, a digit string for a given number is used to determine the angle of unit steps (multiples of 120 degrees base 3, 90 degrees base four, etc.), while the color is shifted up the spectrum after a fixed number of steps (light and dark, and red-orange-yellow-green-cyan-blue-purple-red in the online version). In Figure 1, we
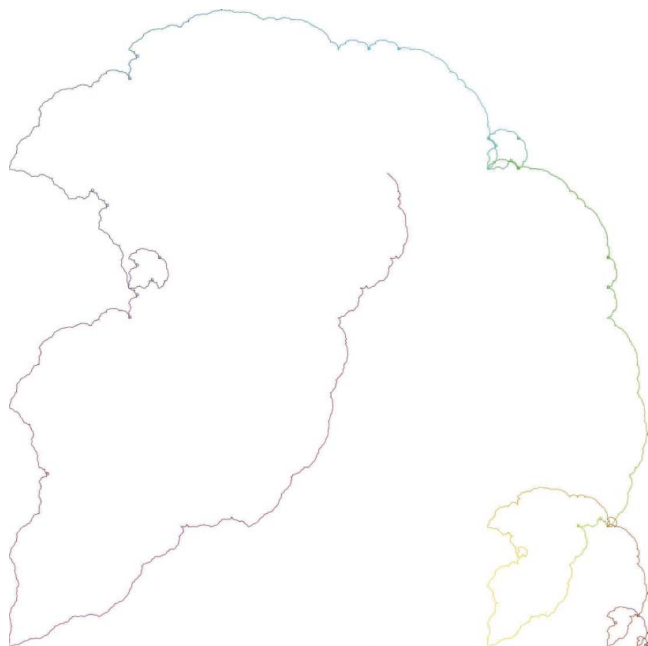

FIGURE 2. A 600 000-step walk on Champernowne's number base 4 (normal) (color figure available online).

show a walk on the first billion base-4 digits of π.[1] We note that the random walks in Figures 2 and 4 look entirely different from the expected behavior of a genuine pseudorandom walk as in Figure 5, which is similar to the random walk in Figure 1.

In the following, given some positive integer base b, we will say that a real number α is b-*normal* if every string of base-b digits of length m appears in the base-b expansion of α with precisely the expected limiting frequency $1/b^m$. It follows, from basic measure theory, that almost all real numbers are b-normal for any specific base b and even for all bases simultaneously. But proving normality for specific constants of interest in mathematics has proven remarkably difficult.

Borel was the first to conjecture that all irrational algebraic numbers are b-normal for *every* integer $b \geq 2$. Yet not a single instance of this conjecture has ever been proven. We do not even know for certain whether the limiting frequency of zeros in the binary expansion of $\sqrt{2}$ is one-half, although numerous large statistical analyses have failed to show any significant deviation from statistical normals.

Recently, two of the present authors, together with Richard Crandall and Carl Pomerance, proved the following: If a real number y has algebraic degree $D > 1$, then the number $\#(|y|, N)$ of 1-bits in the binary

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[1]This may be viewed in more detail online at http://carma. newcastle.edu.au/walks/.
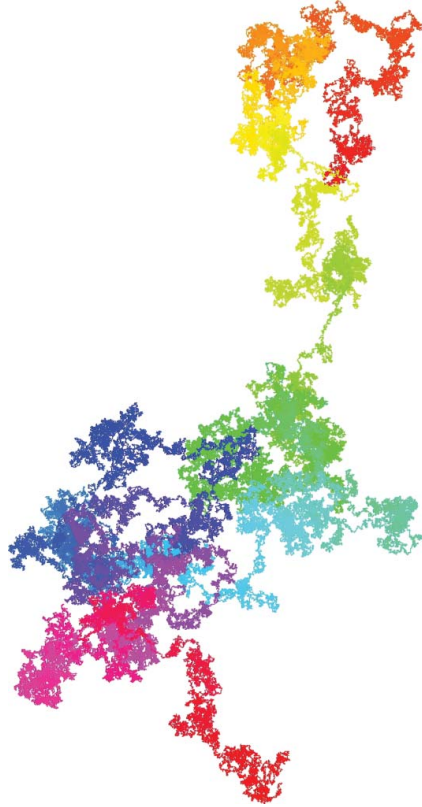
FIGURE 3. A million-step walk on $0.23571113\ldots$ base 2 (normal?) (color figure available online).

expansion of $|y|$ through bit position N satisfies

$$\#(|y|, N) > C N^{1/D}$$

for a positive number C (depending on y) and all sufficiently large N [Bailey et al. 04]. For example, there must be at least $\sqrt{N}$ 1-bits in the first N bits in the binary expansion of $\sqrt{2}$, in the limit. A related and more refined result has been obtained by Hajime Kaneko, of Kyoto University, in Japan. He obtained the bound in

$$\frac{C(\log N)^{3/2}}{(\log(6D))^{1/2}(\log\log N)^{1/2}}$$

and extended his results to a very general class of bases and algebraic irrationals [Kaneko 10]. However, each of these results falls far short of establishing b-normality for any irrational algebraic number in any base b, even in the single-digit sense.

The same can be said for π and other basic constants, such as e, $\log 2$, and $\zeta(3)$. Clearly, any result (one way or the other) for one of these constants would be a mathematical development of the first magnitude.

We record the following known stability result [Borwein and Bailey 08, pp. 165–166].

Theorem 2.1. *If α is normal in base b, and r, s are positive rational numbers, then $r\alpha + s$ is also normal in base b.*

3.   THE CHAMPERNOWNE NUMBER AND RELATIVES

The first mathematical constant proven to be 10-normal is the *Champernowne number*, which is defined as the concatenation of the decimal values of the positive integers, i.e., $C_{10} = 0.12345678910111213141516\ldots$, which can also be written as

$$C_{10} = \sum_{n=1}^{\infty} \sum_{k=10^{n-1}}^{10^n - 1} \frac{k}{10^{kn - 9\sum_{k=0}^{n-1} 10^k (n-k)}}.$$

Champernowne proved in 1933 that C_{10} is 10-normal [Champernowne 33]. This was later extended to base-b normality (for base-b versions of the Champernowne constant).

In 1946, Copeland and Erdős established that the corresponding concatenation of primes $0.23571113171923\ldots$ and also the concatenation of composites $0.46891012141516\ldots$, among others, are 10-normal [Copeland and Erdős 46]. In general, they proved the following theorem.

Theorem 3.1. [Copeland and Erdős 46] *If $a_1, a_2, \ldots$ is an increasing sequence of integers such that for every $\theta < 1$, the number of a_i's up to N exceeds N^θ, for N sufficiently large, then the infinite decimal*

$$0.a_1 a_2 a_3 \ldots$$

is normal with respect to the base β in which these integers are expressed.

Hex Digits	Occurrences	Hex Digits	Occurrences
0	62499881108	8	62500216752
1	62500212206	9	62500120671
2	62499924780	A	62500266095
3	62500188844	B	62499955595
4	62499807368	C	62500188610
5	62500007205	D	62499613666
6	62499925426	E	62499875079
7	62499878794	F	62499937801
Total:			1000000000000

TABLE 1. Digit counts in the first trillion hexadecimal (base-16) digits of π. Note that deviations from the average value $62\,500\,000\,000$ occur only after the first six digits, as expected from the central limit theorem.

FIGURE 4. A random walk on the first 100 000 bits of the primes base two (normal) (color figure available online).

This clearly applies to the Champernowne numbers (Figure 2), to the primes of the form $ak + c$ with a and c relatively prime in any given base, and to the integers that are the sum of two squares (since every prime of the form $4k + 1$ is included).

In further illustration, using the primes in binary leads to normality in base two of the number

$$0.1011101111101111011000110011101111111011111$$
$$110010110100110101110111\ldots,$$
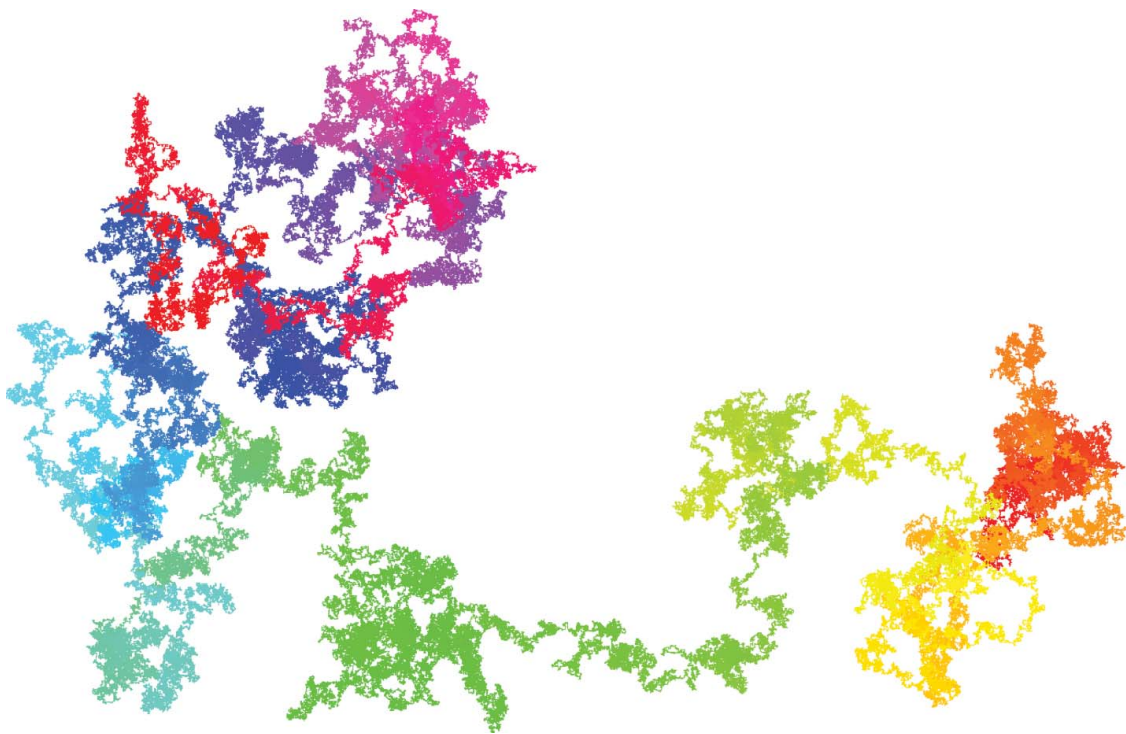
as shown as a random walk in Figure 4.

FIGURE 5. A uniform pseudorandom walk (color figure available online).

Some related results were established by Schmidt, including the following.

Theorem 3.2. [Schmidt 60] *Write $p \sim q$ if there are positive integers r and s such that $p^r = q^s$. If $p \sim q$, then any real number that is p-normal is also q-normal. However, if $p \not\sim q$, then there are uncountably many p-normal real numbers that are not q-normal.*

The above result is described in the recent survey [Queffelec 06], which also presents the following theorem.

Theorem 3.3. (Korobov.) *Numbers of the form $\sum_k p^{-2^k} q^{-p^{2^k}}$, where $p > 1$ and $q > 1$ are relatively prime, are q-normal.*

We are still completely in the dark as to the *b*-normality of "natural" constants of mathematics.

4. NORMALITY FOR STRINGS

Let x be a (finite) binary string. We denote by $N_i^m(x)$ the number of occurrences of the ith string of length m $(1 \leq i \leq 2^m)$, ordered lexicographically, where $|x|_m = \lfloor |x|/m \rfloor$ is the number of (contiguous, nonoverlapping) strings in x of length m. The *prefix* of length n of the infinite (binary) sequence $\mathbf{x} = x_1 x_2 \ldots x_m \ldots$ is denoted by $\mathbf{x} \upharpoonright n = x_1 x_2 \ldots x_n$.

Definition 4.1. [Calude 94, Calude 02] Let $\varepsilon > 0$ and let m be a positive integer. We say that

1. x is (ε, m)-*normal* if for every $1 \leq i \leq 2^m$,

$$\left| \frac{N_i^m(x)}{|x|_m} - \frac{1}{2^m} \right| \leq \varepsilon;$$

2. x is m-*normal* if for every $1 \leq i \leq 2^m$,

$$\left| \frac{N_i^m(x)}{|x|_m} - \frac{1}{2^m} \right| \leq \sqrt{\frac{\log_2 |x|}{|x|}}; \qquad (4\text{--}1)$$

3. x is *normal* if it is m-normal for every $1 \leq m \leq \log_2(\log_2 |x|)$.

If for every positive integer n, the string $\mathbf{x} \upharpoonright n$ is normal, then $\mathbf{x}$ is normal, but the converse is not necessarily true (because $\mathbf{x}$ can be normal but with a different "speed").

5. TESTING NORMALITY OF PREFIXES OF π

In 1996, one of the present authors (Bailey), together with Peter Borwein (brother of Jonathan Borwein) and Simon Plouffe, published what is now known as the BBP formula for π [Bailey et al. 97], [Borwein and Bailey 08, Chapter 3]:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$
$$(5\text{--}1)$$

We had access to an extremely large dataset, thanks to recent record computations, by Kondo and Yee, of π initially to five trillion hexadecimal (base-16) places in August 2010 and then to ten trillion in October 2011 [Yee and Kondo 11]. We first converted these bits—which Kondo and Yee had confirmed by a computation with (5–1)—to a true binary string of bits using the Python module `binascii`.

All input lines contained an even number of characters, so it was easy to convert pairs of hexadecimal digits to bytes:

```
import sys, binascii
for line in sys.stdin.readlines():
    sys.stdout.write(binascii.unhexlify(line.strip()))
```

For our normality test we needed to split a *big* binary string of length n into $\lfloor n/k \rfloor$ pieces (nonoverlapping strings) of length $k = 1, 2, \ldots, \log \log n$. We use the term *string* to denote a binary string of length k. We then proceeded to calculate the minimum and maximum frequencies of occurrences of such strings.

This calculation is done by running Algorithm 5.1 once for each different value of k.

It is essential to do an efficient streaming implementation of Algorithm 5.1 so that the actual bits of input X are read into main memory only as needed.

Algorithm 5.1: Frequency range of strings of a given length.

Input: Binary string X, string length k
Output: Minimum and maximum counts over all possible 2^k strings of length k in string X
integer array $counts[0, \ldots, 2^k - 1] = [0, 0, \ldots, 0]$;
for $i = 0$ **to** $|X| - k$ **step** k **do**
    $w = integer(X[i, \ldots, i + k - 1])$;
    increment $counts[w]$;

return $\min(counts)$, $\max(counts)$;

---

m	Minimum Frequency	Maximum Frequency	Expected Range
1	**7 962 9**33 149 184	**79 629**35 392 216	7 962 907 842 460, ..., 7 962 960 698 940
2	**1 990 7**32 495 242	**1 990 7**35 357 049	1 990 720 353 555, ..., 1 990 746 781 795
3	**663 5**76 589 836	**663 5**79 050 172	663 569 046 478, ..., 663 586 665 305
4	**248 84**1 171 873	**248 84**2 651 924	248 835 088 899, ..., 248 848 303 020
5	**99 535** 989 611	**99 537** 473 460	99 531 392 735, ..., 99 541 964 032

TABLE 2. Frequency summary for $N = 15\,925\,868\,541\,400$ bits of π.

Finally, to check that these minimum and maximum frequencies satisfy the expected range for the normality test, we used the following Python code snippet to generate a table using our earlier formula (4–1):

```
import math, sys
n=int(sys.argv[1]) # n = |X|
r = int(math.floor(math.log(math.log(n,2),2))) # r = lg lg n
m1,m2=[0]*(r+1),[0]*(r+1)
sqrtV = math.sqrt(math.log(n,2)/n)
for k in range(1,r+1):
    floorNk = math.floor(n/k)
    m1[k] = int(math.floor(((1.0/2.0**i)-sqrtV)*floorNk))
    m2[k] = int(math.ceil((sqrtV+1.0/2.0**k))*floorNk))
    print "expected range k=",k, "[",m1[k],"...",m2[k],"]"
```

We tested normality for the prefix of $N = 15\,925\,868\,541\,400$ bits of π—nearly 16 trillion bits—calculated with the *y-cruncher-multi-threaded pi* program [Yee 10], and we have found it to be within the normality range as described above. The frequency counts passed our `expectedCheck.py` test script as shown in Table 2.

6. NORMALITY OF π

We have tested the prefix of $N = 15\,925\,868\,541\,400$ bits of π—nearly 16 trillion bits—and we have found it to be normal as described above.

Does this "information" tell us anything about the classical normality of π? In the next subsection, we will use a Poisson process model to provide an affirmative answer to this question.

6.1. A Poisson Process Model

We denote by

$$\mathbf{b} = b(1)\, b(2) \ldots b(n) \ldots$$

the (infinite) binary expansion of π (b is a computable function) and by

$$\mathbf{b} \upharpoonright n = b(1)\, b(2) \ldots b(n)$$

the finite prefix of $\mathbf{b}$ of length n.

We base our model on the distribution of 1's and 0's only, i.e., we work with $N_1^1 (\mathbf{b} \upharpoonright n)$, the number of occurrences of 1 in $\mathbf{b} \upharpoonright n$, so $N_0^1 (\mathbf{b} \upharpoonright n) = n - N_1^1 (\mathbf{b} \upharpoonright n)$. A

similar, slightly more elaborate, model can be developed for strings of any length.

The number $N_1^1 (\mathbf{b} \upharpoonright n)$ can be connected with π by means of a *counting (Poisson) process* [Ross 83]:

$$Y_n = \# \{j \mid 1 \le j \le n,\ b(j) = 1\}, \quad n = 1, 2, \ldots,$$
$$Y_0 = 0,$$

where $Y_n = N_1^1 (\mathbf{b} \upharpoonright n),\ n = 1, 2, \ldots$.

Theorem 6.1. *If π is normal, then $\{Y_n,\ n = 0, 1, 2, \ldots\}$ can be approximated by a homogeneous Poisson process with intensity $\lambda = 0.5$.*

Proof. By construction, $\{Y_n,\ n = 0, 1, 2, \ldots\}$ is a Poisson process with an unspecified parameter λ. Hence Y_n is a random variable with parameter $n\lambda$ with the following properties: $E(Y_n) = V(Y_n) = n\lambda$, $\lim_{n \to \infty} Y_n = \infty$ almost surely.

We apply Chebyshev's inequality, so for every $c > 0$,

$$P\big(|Y_n - E(Y_n)| < c\big) \ge 1 - \frac{V(Y_n)}{c^2},$$

we have

$$P\big(|Y_n - n\lambda| < c\big) \ge 1 - \frac{n\lambda}{c^2}.$$

Hence

$$P\left(\left|\frac{Y_n}{n} - \lambda\right| < \frac{c}{n}\right) \ge 1 - \frac{n\lambda}{c^2}.$$

In view of (4–1), we take

$$\frac{c}{n} = \varepsilon = \sqrt{\frac{\log_2 n}{n}},$$

so we obtain

$$P\left(\left|\frac{Y_n}{n} - \lambda\right| < \varepsilon\right) \ge 1 - \frac{n\lambda}{(n\varepsilon)^2} = 1 - \frac{\lambda}{\log_2 n}. \quad (6\text{–}1)$$

If π is normal, then

$$\left|\frac{N_1^1 (x_{(n)})}{n} - \frac{1}{2}\right| \le \varepsilon = \sqrt{\frac{\log_2 n}{n}},$$

or

$$\left| \frac{Y_n}{n} - \frac{1}{2} \right| \le \varepsilon = \sqrt{\frac{\log_2 n}{n}}. \qquad (6\text{--}2)$$

If we identify the random event in relation (6–1) and the certain event in relation (6–2), we get $\lambda = 1/2$ and

$$P\left(\left| \frac{Y_n}{n} - \frac{1}{2} \right| < \sqrt{\frac{\log_2 n}{n}} \right) \ge 1 - \frac{1}{2\log_2 n}.$$

$\square$

A Poisson process with intensity λ has the following properties [Stoneham 73]:

- The Poisson process $\{Y_n,\ n = 0, 1, 2, \dots\}$ has independent increments.

- For $n > r$, $Y_n - Y_r$ has a Poisson distribution with parameter $\lambda(n - r)$, and $Y_n - Y_r$ is independent of $\{Y_t,\ t \le r\}$.

Let us denote the positions at which 1's occur (*jump moments*) by

$$\tau_r = \inf\{n \mid Y_n = r\}, \quad r = 1, 2, \dots.$$

Then

$$Y_n = \begin{cases} 0, & \text{if } n < \tau_1, \\ r, & \text{if } \tau_r \le n < \tau_{r+1}. \end{cases}$$

With the convention $\tau_0 = 0$, we can introduce the *sojourn times*, or *interarrival times*,

$$T_r = \tau_r - \tau_{r-1}, \quad r = 1, 2, \dots.$$

Note that the sojourn times represent the distances between two successive 1's. Thus, for the string 10^s1, the sojourn time is $s + 1$. Furthermore, the sequence $\{T_r,\ r = 1, 2, \dots\}$ consists of independent identically distributed random variables, with the exponential distribution Expo(λ). Then

$$E(T_r) = \frac{1}{\lambda}, \quad V(T_r) = \frac{1}{\lambda^2}.$$

Note that the jump moments $\tau_r = T_1 + \cdots + T_r$ have an Erlang distribution with parameters $(r; \lambda)$. Hence

$$E(T_r) = \frac{r}{\lambda}, \quad V(T_r) = \frac{r}{\lambda^2}.$$

Corollary 6.2. *If π is normal, then the sojourn times $\{T_r,\ r = 1, 2, \dots\}$ form a sequence of independent identically distributed random variables with the exponential*

distribution Expo$(1/2)$. *Hence*

$$P(T_r > t_r,\ r = 1, \dots, k)$$
$$= \prod_{r=1}^{k} \left(\exp\left(-\frac{t_r}{2}\right) \right) = \exp\left(-\frac{1}{2} \sum_{r=1}^{k} t_r\right).$$

6.2. Testing the Hypothesis That π Is Normal

We test the hypothesis H: "π is normal" against the alternative H_A: "π is not normal." If H is true, then for every d, there exists K_d such that the sojourn time exceeds the value d if we wait long enough, up to the rank $(K_d + 1)$:

$$P(T_1 \le d, \dots, T_{K_d} \le d, T_{K_d+1} > d \mid H \text{ true})$$
$$= \prod_{r=1}^{K_d} \left(1 - \exp\left(-\frac{d}{2}\right)\right) \cdot \exp\left(-\frac{d}{2}\right)$$
$$= \exp\left(-\frac{d}{2}\right) \left(1 - \exp\left(-\frac{d}{2}\right)\right)^{K_d} > 0.$$

We can base our decision of accepting/rejecting normality (hypothesis H) on the following implication: "π is a normal sequence" *implies* "for every d, there exists K_d such that $P(T_1 \le d, \dots, T_{K_d} \le d, T_{K_d+1} > d) > 0$."

Since we cannot explore the whole sequence π, we deal with an *evidence body* represented by a prefix of π, of length N. In this evidence body, we look for the largest value $d_{\max}$ for which a rank $K_{d\max}$ can be identified, or equivalently, we look for the first value $(d + 1)$ that is not reached by the sojourn time T. Accordingly, the decision of accepting/rejecting the hypothesis H : "π is normal" is taken according to the following algorithm:

1. If there is no such $d_{\max}$ in the evidence body, we conclude that the sequence π is normal.

2. If $d_{\max}$ and the corresponding $K_{d_{\max}}$ exist, we can decide that the sequence π is not normal. The decision is based on the event

$$\left\{T_1 \le d_{\max}, \dots, T_{K_{d_{\max}}} \le d_{\max}, T_{K_{d_{\max}}+1} > d_{\max}\right\},$$

whose probability is

$$P\left(T_1 \le d_{\max}, \dots, T_{K_{d_{\max}}} \le d_{\max}, T_{K_{d_{\max}}+1} > d_{\max}\right)$$
$$= \exp\left(-\frac{d_{\max}}{2}\right) \left(1 - \exp\left(-\frac{d_{\max}}{2}\right)\right)^{K_{d_{\max}}}.$$

We interpret the above probability to mean that the decision "π is normal" has credibility equal to

$$1 - \exp\left(-\frac{d_{\max}}{2}\right) \left(1 - \exp\left(-\frac{d_{\max}}{2}\right)\right)^{K_{d_{\max}}}.$$

d	1	2	3	4	5	6	7
K_d	9	1	14	3	46	56	41
d	8	9	10	11	12	13	14
K_d	78	1276	446	2090	18082	8633	4175
d	15	16	17	18	19	20	21
K_d	239183	5856	56453	218007	643030	363117	2787207
d	22	23	24	25	26	27	28
K_d	13733056	1003213	21127913	100317701	not found	85745944	not found
d	29						
K_d	not found						

TABLE 3. Values of d and K_d for 400 million bits of π.

6.3. Results

Suppose first that the evidence body is represented by a prefix of 400 million bits of π. The d-values and their corresponding ranks K_d are given in Table 3; $\max K_d = 100\,317\,701$.

The value $d = 28$ has the property that for every K, the event

$$\{T_1 \leq 28, \ldots, T_K \leq 28, T_{K+1} > 28\}$$

has not been identified in the evidence body; so, based on the algorithm in Section 6.2, the decision "π is not normal" has credibility

$$P\left(T_s \leq 27, \ s = 1, \ldots, 100317701, T_{100317702} > 27\right)$$
$$= \left(1 - \exp\left(-\frac{27}{2}\right)\right)^{100317701} \exp\left(-\frac{27}{2}\right)$$
$$= 2.5576 \times 10^{-66}.$$

Suppose now that the evidence body has increased to the prefix of π of $N = 15\,925\,868\,541\,400$ bits. The d-values and their corresponding ranks K_d are given in Table 4; $\max K_d = 9\,274\,770\,297\,096$.

The value $d = 43$ has the property that for every K, the event

$$\{T_1 \leq 43, \ldots, T_K \leq 43, T_{K+1} > 43\}$$

has not been identified in the evidence body, so based on the algorithm in Section 6.2, the decision "π is not normal" has credibility

$$P\big(T_s \leq 42, \ s = 1, \ldots, 9274770297096,$$
$$T_{9274770297097} > 42\big)$$
$$= \left(1 - \exp\left(-\frac{42}{2}\right)\right)^{9274770297096} \exp\left(-\frac{42}{2}\right)$$
$$= 4.3497 \times 10^{-3064}.$$

d	1	2	3	4	5
K_d	9	1	14	3	46
d	6	7	8	9	10
K_d	56	41	78	1276	446
d	11	12	13	14	15
K_d	2090	18082	8633	4175	239183
d	16	17	18	19	20
K_d	5856	56453	218007	643030	363117
d	21	22	23	24	25
K_d	2787207	13733056	1003213	21127913	100317701
d	26	27	28	29	30
K_d	273575848	85745944	234725219	611367301	1075713943
d	31	32	33	34	35
K_d	703644000	10621041176	27019219636	15063287853	10887127703
d	36	37	38	39	40
K_d	48115888750	19128531469	1218723032299	1334087352175	792460189481
d	41	42	43	44	45
K_d	9274770297096	4368224447710	not found	not found	not found

TABLE 4. Values of d and K_d for $15\,925\,868\,541\,400$ bits of π.

7.   CONCLUSION

A prime motivation in computing and analyzing digits of π is to explore the age-old question whether and why these digits appear "random." Numerous computer-based statistical checks of the digits of π have failed to disclose any deviation from reasonable statistical norms. A new avenue for studying the normality of π was explored: we proved that the prefix of $15\,925\,868\,541\,400$ bits of π is normal when viewed as a binary string [Calude 94].

This result was used in a Poisson process model to show that the probability that π is not normal is extraordinarily small, reinforcing the empirical evidence we have presented for the normality of π. In future work we intend to look methodically at other numerical constants.

ACKNOWLEDGMENTS

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