Experimental Mathematics, 21(3):266–279, 2012

ISSN: 1058-6458 print / 1944-950X online
DOI: 10.1080/10586458.2012.653273

Zero Cells of the Siegel–Gottschling Fundamental Domain of Degree 2

Takahiro Hayata, Takayuki Oda, and Tomoki Yatougo

CONTENTS

Let $\mathcal{F}_n$ be a fundamental domain of the Siegel upper half-space of degree n with respect to the Siegel modular group $\mathrm{Sp}(n, \mathbb{Z})$. According to Siegel himself, $\mathcal{F}_n$ is determined by only finitely many polynomial inequalities. In case of degree $n = 2$, Gottschling determined the minimal set of inequalities. The boundary of $\mathcal{F}_2$ is of great concern in the literature not only from a homological point of view but also from the geometry of numbers. In this paper we compute the vertices of $\mathcal{F}_2$ under the condition that the defining ideal is zero-dimensional ("0-cells"). We also discuss an equivalence relation among 0-cells.

1. INTRODUCTION

In the classical study of elliptic modular forms, the fundamental domain

$$\mathcal{F}_1 = \left\{ z = x + y\sqrt{-1} \in \mathbb{C} \mid |x| \leq \frac{1}{2},\ y > 0,\ |z| \geq 1 \right\}$$

is well known [Serre 73], and knowledge of the fundamental domain of an arithmetic subgroup plays an important role [Fricke and Klein 65]. But there seem to be few articles on the fundamental domains of classical symmetric domains of higher dimension. This paper is a case study for the next difficult case. Let $\mathcal{F}_2$ be the fundamental domain of the Siegel modular group $\mathrm{Sp}(2, \mathbb{Z})$ of genus 2 in the Siegel upper half-space $\mathbb{H}_2$. In his book [Siegel 64], C. L. Siegel proved that such a fundamental domain in general degree is determined by only finitely many inequalities of the form $|\det(CZ + D)| \geq 1$ and with the Minkowski condition [Klingen 90]. If the degree is 2, Gottschling determined the minimal set of inequalities [Gottschling 59]. In the following we specify the fundamental domain $\mathcal{F}_2$ determined by this minimal set as the Siegel–Gottschling fundamental domain.

In the literature, several papers are concerned with fixed points or conjugacy classes [Ueno 71, Ueno 72, Hashimoto 83]. Gottschling himself also computed fixed points and fixed-point subgroups of $\mathrm{Sp}(2, \mathbb{Z})$ [Gottschling 61b, Gottschling 61a]. In another direction, there is a paper [MacPherson and McConnell 93] on the

2000 AMS Subject Classification: Primary 11F46, Secondary 11F06
Keywords: Siegel modular group, Siegel upper half-space, fundamental domain, reduction theory

topology of modular groups of genus 2 that shows the existence of a spine. See also [Yasaki 06]. There also is an attempt to develop a non-Euclidean Voronoi theory [Watanabe 03, Watanabe 11].

To continue the investigation of the fundamental domains, we are interested in the "fine structure" of this Siegel–Gottschling $\mathcal{F}_2$. The idea is as follows. We regard $\mathbb{H}_2$ as a real affine space $V_{\mathbb{R}}$ of dimension 6. By the result of Siegel and Gottschling mentioned above, the boundary $\partial\mathcal{F}_2$ of the domain $\mathcal{F}_2$ consists of real 5-dimensional hypersurfaces ("walls"), and each of them has a description by polynomial equations. To understand the cells that form $\partial\mathcal{F}_2$ is to understand the real zeros of the system of polynomial equations. Generally speaking, however, the detailed structure of the intersections of these walls seem to be unknown. Since $\dim_{\mathbb{R}} \mathcal{F}_2 = 6$, it seems to be very hard to obtain a complete answer to this problem.

In this paper we restrict ourselves to a consideration of the "0-cells" under the condition that the ideal defined by the system of walls is 0-dimensional. Our goal is to prove several results on these 0-cells. To present the main result, we require some notation. By [Gottschling 59, Sätze 1 and 2], there are 28 walls defined by polynomial equations $f_1 = 0, \dots, f_{28} = 0$. For a label $L = \{i_1, \dots, i_p\}$ $(1 \le i_j \le 28)$, let $I_L = \langle f_{i_1}, \dots, f_{i_p} \rangle$ be the ideal in the polynomial ring $\mathbb{Q}[V]$ over the rational numbers $\mathbb{Q}$ of six variables. Let $V(I_L)$ be the zero set of I_L. Then we have the following theorem.

Theorem 1.1. *There are* 180 *points* $p \in V(I_L) \cap \partial\mathcal{F}_2$ *for some* L *such that* I_L *is zero-dimensional. They are divided into* $\mathrm{Sp}(2,\mathbb{Z})$*-equivalence classes, and the upper bound on the number of classes is* 40*, the lower bound is* 25.

The points obtained as 0-cells seem to have a greater chance to be informative in view of Voronoi theory [Martinet 03] and its extensions. For example, the point $Z_8 = \left(\begin{smallmatrix} \eta & (\eta-1)/2 \\ (\eta-1)/2 & \eta \end{smallmatrix}\right)$ with $\eta = (1+2\sqrt{2}i)/3$ in $\partial\mathcal{F}_2$ appears as a 0-cell in $\partial\mathcal{F}_2$, which also appeared in Gottschling's paper. Our index model (Section 3) describes this 0-cell as lying on the intersection of three "rank-1" equalities, four "rank-2" equalities, and two Minkowski conditions, and is strictly positive in the other 19 inequalities. There is a notion of Hermite constant in the case of the linear algebraic group [Watanabe 03]. We can prove that Z_8 attains the minimum $\det(Y)$ among 180 points. We remark that it was announced in [Kawamura 09] that Z_8 attains the minimum $\det(Y)$ on $\mathcal{F}_2$, which implies that Z_8 attains the Hermite constant of the symplectic group of degree 2.

Now we explain the contents of the paper. In Section 2, we review Gottschling's theorem and fix notation. We introduce an index model based on the numbering of the walls in Section 3. We also introduce an involution that works efficiently during the classification process (Section 4). Then we state the main theorem of the paper in Section 5. Since the results were obtained through exhaustive computer search, we explain the procedure using a computer algebra system step by step and give search results in Section 6 before the proof of the main theorem (Section 7). Lastly, in Section 8, we discuss the Γ-equivalence and inequivalence property of the set of 0-cells in detail.

The authors thank the referee for a very careful reading of our manuscript and many suggestions for corrections.

2. SIEGEL'S FUNDAMENTAL DOMAIN OF DEGREE 2

Let $\mathbb{H}_2$ be the Siegel upper half-space of degree 2, namely,

$$\mathbb{H}_2 = \{Z = X + \sqrt{-1}Y \in M_2(\mathbb{C}) \mid {}^tZ = Z,\ Y\text{: positive definite}\},$$

where $M_2(\mathbb{C})$ is the set of 2×2 complex matrices. The discrete group $\Gamma = \mathrm{Sp}(2,\mathbb{Z})$ is a set of symplectic matrices of degree 4 whose entries are integers:

$$\Gamma = \left\{ g \in M_4(\mathbb{Z}) \;\middle|\; {}^tg \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} g = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}.$$

The matrix $\gamma \in \Gamma$ acts on $\mathbb{H}_2$ discontinuously by the linear fractional transformation

$$\gamma \cdot Z = (AZ+B)(CZ+D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Siegel's fundamental domain $\mathcal{F}_2$ of $\mathbb{H}_2$ with respect to the action of Γ is given as follows [Klingen 90, Siegel 64]. Let X_{ij}, Y_{ij} be the (i,j)-entries of the 2×2 matrices X, Y respectively. By definition, $\mathcal{F}_2$ consists of the points $Z = X + \sqrt{-1}Y \in M_2(\mathbb{C})$ determined by the following three types of inequalities:

(1) $|X_{ij}| \le 1/2$.

(2) Y is Minkowski reduced [Klingen 90, Section I.2]). Specifically, $0 \le 2Y_{12} \le Y_{11} \le Y_{22}$.

(3) $|\det(CZ+D)| \ge 1$ for all $\gamma = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma$.

Siegel proved that it is enough to consider only a finite number of γ's in the condition (3). Gottschling

[Gottschling 59] determined a set of 15 inequalities out of (3) that form a minimal set.

To recall Gottschling's description, we prepare a set of square matrices of degree 2. Let E_{ij} be the identity matrix and O the zero matrix. For simplicity, we set

$$E_1 = E_{11}, \quad E_2 = E_{22}, \quad I_\pm = E_1 \pm E_2,$$
$$J = E_{12} + E_{21}, \quad J_i = J + E_i \quad (i = 1, 2).$$

Theorem 2.1. [Gottschling 59, Sätze 1, 2] *The condition* (3) *is exhausted by the following matrix pairs* (C, D)*, and no proper subset of them is sufficient to define the fundamental domain* $\mathcal{F}_2$*:*

$\operatorname{rank}(C) = 1$*:*

$$(C, D) \in \{(E_1, E_2), (E_2, E_1), (E_1 - E_{12}, I_+ + E_{21}), (E_1 - E_{12}, -I_+ - E_{21})\};$$

$\operatorname{rank}(C) = 2$*:*

$$(C, D) \text{ with; } C = I_+, \quad \pm D = O, E_1, E_2, I_\pm, J, J_1, J_2.$$

3. INDEX-ORIENTED MODEL OF THE BOUNDARY

Let $\partial\mathcal{F}_2$ be the boundary of $\mathcal{F}_2$. We want to understand $\partial\mathcal{F}_2$ from the viewpoint of real algebraic geometry as semialgebraic subsets in $\mathcal{F}_2$. From the definition of $\mathcal{F}_2$, we introduce the defining polynomials f_λ to describe $\partial\mathcal{F}_2$.

Now we define a finite set Λ indexing the conditions (1), (2), and (3). For the condition (1), we set

$$f_{X_1}(Z) = 1/2 - X_{11}, \quad f_{X_2}(Z) = 1/2 - X_{22},$$
$$f_{X_3}(Z) = 1/2 - X_{12} \quad f_{\theta X_1}(Z) = 1/2 + X_{11},$$
$$f_{\theta X_2}(Z) = 1/2 + X_{22}, \quad f_{\theta X_3}(Z) = 1/2 + X_{12}.$$

Secondly, for the condition (2), we set

$$f_{Y_1}(Z) = Y_{22} - Y_{11}, \quad f_{Y_2}(Z) = Y_{11} - 2Y_{12}, \quad f_{Y_3}(Z) = Y_{12}.$$

Thirdly, we consider the condition (3) with the help of Theorem 2.1. For simplicity we set $\mathbf{1} = (E_1, E_2) \in \Lambda$, $\mathbf{2} = (E_2, E_1) \in \Lambda$, $R = (E_1 - E_{12}, I_+ + E_{21}) \in \Lambda$, and $\theta R = (E_1 - E_{12}, -I_+ - E_{21}) \in \Lambda$ in the case of rank 1 in Theorem 2.1. For these $\lambda \in \Lambda$, we put $f_\lambda(Z) = |\det(CZ + D)|^2 - 1$. In the case of rank 2, C is always the identity. So we represent $(C, D) \in \Lambda$ by D. Then for $D \in \Lambda$, we put

$$f_D(Z) = |\det(Z + D)|^2 - 1.$$

Thus Λ consists of 28 elements: $6 + 3 + 4 + 15 = 28$.

Put $W_\lambda = \{Z \in \mathcal{F}_2 \mid f_\lambda(Z) = 0\}$. Obviously, $\partial\mathcal{F}_2 = \cup_{\lambda\in\Lambda} W_\lambda$. Moreover, define the extended notation $W_L = \bigcap_{\lambda \in L} W_\lambda$ for a subset L in Λ. Note that the labels are inclusion-reversing, i.e., $L \subset L' \Rightarrow W_L \supset W_{L'}$.

Though we do not know the substance of the labeled subset W_L of $\partial\mathcal{F}_2$, we hope that the contiguity of W_L are useful to parameterize the cells in ∂F_2. Besides, we can see the substance of W_L pretty much in the 0-dimensional case. We remark that the naive expectation $\dim W_L = 6 - |L|$ is not true for $|L| \geq 2$.

Let us introduce a notion of 0-cells, a candidate for "vertices" in this model. Let V be the affine space in which the coordinate $(X_{11}, X_{12}, X_{22}, Y_{11}, Y_{12}, Y_{22})$ lives. Put $I_L = \langle f_\lambda \mid \lambda \in L\rangle$, $L \subset \Lambda$, for the ideal generated by L and consider the zero set $V(I_L)$. Then one has $W_L = V(I_L) \cap \mathcal{F}_2$.

Definition 3.1.

1. A label L or W_L is called a 0-cell if the dimension of $V(I_L)$ is zero.
2. A label L is called trivial if the corresponding ideal I_L is trivial (i.e., $I_L \ni 1$).

4. INVOLUTIONS

We set $F_\infty(z) = -\overline{z}$ for $z \in \mathbb{C}$. We introduce involutive real-analytic diffeomorphisms θ_i, $i = 1, 2, 3$, on $\mathbb{H}_2$ by

$$\theta_1(Z) = \begin{pmatrix} F_\infty(Z_{11}) & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix},$$
$$\theta_2(Z) = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & F_\infty(Z_{22}) \end{pmatrix},$$
$$\theta_3(Z) = \begin{pmatrix} Z_{11} & F_\infty(Z_{12}) \\ F_\infty(Z_{12}) & Z_{22} \end{pmatrix}.$$

Moreover we put $\theta(Z) = \theta_1\theta_2\theta_3(Z)$, which is a normalizer of Γ in the group of diffeomorphisms of $\mathbb{H}_2$. For $\gamma = \left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$, we also define $\sigma_0(Z) = \gamma \cdot Z = \left(\begin{smallmatrix} Z_{22} & Z_{12} \\ Z_{12} & Z_{11} \end{smallmatrix}\right)$. Let Δ be the finite group of order 16 generated by θ_i, $i = 1, 2, 3$, and σ_0 with relation $\sigma_0\theta_1\sigma_0 = \theta_2$. We say that two points Z, Z' in $\mathbb{H}_2$ are Δ-equivalent if there is a $\delta \in \Delta$ such that $\delta(Z) = Z'$, and this fact is denoted by $Z \overset{\Delta}{\sim} Z'$. In each Δ-equivalence class, we may choose a representative point Z whose X_{ij}-coordinates are all nonnegative and $X_{11} \geq X_{22}$.

We define $f_{\theta\lambda}(Z) = f_\lambda(\theta Z)$. Then we have a more economical description of Λ:

$$\Lambda = \{\mathbf{1}, \mathbf{2}, Y_1, Y_2, Y_3, O\} \cup \{D, \theta D \mid D = R, I_+, I_-, J, E_i, J_i \ (i = 1, 2), X_k \ (k = 1, 2, 3)\}.$$

5. 0-CELLS AND MAXIMAL LABELS

In this section we exhaust the whole set of 0-cells in our definition. The procedure consists of a few steps; each step is carried out by a computer algebra system. We enumerate all the possible L such that $\dim(V_L) = 0$ to get a finite set $V_0 = \cup_{\dim V(I_L)=0} V(I_L)_{\mathbb{R}}$. A detailed explanation of the method and the computer computations are presented in Section 6. There are 752 370 possibilities by the results of **step 1**.

Next, for each L such that $\dim V(I_L) = 0$, we check $W_L = V(I_L)_{\mathbb{R}} \cap \partial\mathcal{F}_2 \neq \varnothing$ in the finite set V_0. We explain the details of the method by an example in Section 7. The corresponding algorithm is given in Section 6 (step 2).

Then the number of remaining L such that W_L is nonempty is 2146, and we find that the set

$$W_0 = \bigcup_{\substack{L\subset\Lambda \\ \dim V(I_L)=0 \\ W_L\neq\varnothing}} W_L$$

consists of 180 points. Lastly, we check the maximality condition (3) for L in Theorem 5.1 below.

To state the main result, we define 180 points and their associated labels a priori, which are to be 0-cells. Define e_i, $0 \le i \le 39$, as given in Table 1.

In the table, we put $\tau(x) = \sqrt{1-x^2}$ and set ω_j to be the algebraic number given by a real root of Ω_j specified uniquely by indicated additional conditions in Tables 2 and 3. Their floating-point expressions are given in Table 4. Finally, we define 180 actual points using the action of Δ. With a given point p, we associate a label L_p. They both are given in Tables 5 and 6. The points e_i are the representatives of 40 Δ-equivalence classes.

Summing up, we can state the main result of this paper.

Theorem 5.1. *Let p be a point and $L = L_p$ the label associated with p in Tables 5 and 6. Then we have*

(1) *I_L is zero-dimensional.*

(2) *$p \in W_L$.*

(3) *L is maximal, i.e., $p \notin W_{L'}$ for $L' \supsetneq L$.*

Computation of $\det(Y)$ from $p = X + \sqrt{-1}Y$ gives the following corollary.

Corollary 5.2. *The points $\theta_3 e_1$ and $\theta\theta_3 e_1$ attain the minimum $\det(Y)$ among the 180 points in Tables 5 and 6.*

Remark 5.3. Theorem 5.1 does not necessarily mean that we have a set of L's such that $\dim(W_L) = 0$. Specifically, there could be $L \subset \Lambda$ such that $\dim W_L = 0$ but $\dim V(I_L) > 0$.

6. PROCEDURE TO OBTAIN 0-CELLS

6.1. The Number of Nontrivial Labels

Logically speaking, the computation in this subsection is not necessary, but it is helpful in grasping the complexity of the computations.

Define $T^{(n)}$ to be the set of trivial labels with size n (cf. Definition 3.1). We put $T^{(n)}_{\text{new}} = T^{(n)} \setminus \bigcup_{k<n} T^{(k)}$. Obviously, $T^{(1)} = \varnothing$. It is also clear the trivial labels of size 2 are

$$T^{(2)}_{\text{new}} = T^{(2)} = \{[X_1, \theta X_1], [X_3, \theta X_3], [X_2, \theta X_2]\}.$$

By computer search, we obtain $|T^{(3)}_{\text{new}}| = 0$, $|T^{(4)}_{\text{new}}| = 16$, $|T^{(5)}_{\text{new}}| = 64$, $|T^{(6)}_{\text{new}}| = 1024$.

The nontrivial labels are obtained as

$$2^\Lambda \setminus \bigcup_{n\ge2} \bigcup_{L\in T^{(n)}} \{L' \in 2^\Lambda \mid L \subset L'\}.$$

The inclusion–exclusion principle computes the cardinality. We restrict ourselves to the case $n \le 4$. If we put $T = T^{(2)} \cup T^{(4)}_{\text{new}}$, then the cardinality of possibly nontrivial labels is given by $2^{28} - 204\,166\,144 = 64\,269\,312$.

To clarify the situation of nontrivial labels of smaller size, we employ the inclusion–exclusion principle again. If the size of the labels is 2, because $|T^{(2)}| = 3$, the number of remaining labels of size 2 is $\binom{28}{2} - 3 = 375$. If three, the number is $\binom{28}{3} - 3\binom{26}{1} = 3198$. If four, because $|T^{(4)}_{\text{new}}| = 16$, therefore $\binom{28}{4} - 3\binom{26}{2} - 16 + 3 = 19\,487$. If five and six, we have respectively $\binom{28}{5} - 8248 + 97 - 7 = 90\,122$ and $\binom{28}{6} - 51\,762 + 1699 - 197 + 2 = 326\,482$ for the numbers of nontrivial labels. In conclusion, we have the following result:

$\lvert L\rvert$	2	3	4	5	6
$\binom{28}{\lvert L\rvert}$	378	3276	20 475	98 280	376 740
# of nontrivial labels	375	3198	19 487	90 122	326 482

6.2. The 0-Dimensional Ideals

Here, we collect the steps to obtain the 0-cells.

Procedure 6.1. Consider the nontrivial labels L.

step 1. Collect the L's where I_L is zero-dimensional.

step 2. For each L such that I_L is zero-dimensional, compute all real zeros $\{p_{k,L}\}_k \in V(I_L)$.

j	X_{11}	X_{12}	X_{22}	Y_{11}	Y_{12}	Y_{22}
e_0	$1/2$	0	$1/2$	ω_0	0	ω_0
e_1	$1/3$	$1/3$	$1/3$	ω_1	$\omega_1/2$	ω_1
e_2	ω_6	$1/2$	ω_6	1	$1/2$	1
e_3	$1/2$	$1/2$	$1/2$	ω_0	0	ω_0
e_4	$1/2$	$1/2$	0	1	$1/2$	1
e_5	$1/2$	0	$1/2$	1	$1/2$	1
e_6	0	$1/2$	0	1	$1/2$	1
e_7	$1/2$	$\omega_0 - 3/4$	$1/2$	ω_0	$\omega_0/2$	$(3-\omega_0)/2$
e_8	$1/2$	$1/2$	$1/2$	ω_3	$\omega_3/2$	ω_3
e_9	$1/2$	$1/2$	$1/2$	ω_0	$\omega_0/2$	ω_4
e_{10}	$1/2$	$1/2$	0	ω_0	$\omega_0/2$	$\omega_5/16$
e_{11}	$1/2$	0	$1/2$	ω_0	$\omega_0/2$	$\omega_5/16$
e_{12}	ω_2	$(1-\omega_2)/2$	ω_2	$\tau(\omega_2)$	$\tau(\omega_2)/2$	$\tau(\omega_2)$
e_{13}	$(1-3\omega_2)/2$	$(1-\omega_2)/2$	ω_2	$\tau(\omega_2)$	$\tau(\omega_2)/2$	$\tau(\omega_2)$
e_{14}	$(1-3\omega_2)/2$	ω_2	$(1-3\omega_2)/2$	$\tau(\omega_2)$	$\tau(\omega_2)/2$	$\tau(\omega_2)$
e_{15}	ω_2	$(1-3\omega_2)/2$	ω_2	$\tau(\omega_2)$	$\tau(\omega_2)/2$	$\tau(\omega_2)$
e_{16}	$1/2$	$\omega_7/4$	$(\omega_7+1)/2 - \omega_8/8$	ω_0	$\omega_0/2$	$\frac{\omega_0(\omega_8+4-2\omega_7)}{4}$
e_{17}	$1/2$	$(2-\omega_7)/4$	$\omega_8/8$	ω_0	$\omega_0/2$	$\frac{\omega_0(\omega_8+4-2\omega_7)}{4}$
e_{18}	$1/2$	$1/2$	$1/2$	ω_0	ω_{21}	ω_0
e_{19}	$1/2$	$1/4 + \omega_0\omega_{21}$	$(\omega_0-\omega_{21})^2/2$	ω_0	$(\omega_0-\omega_{21})/2$	$\tau((\omega_0-\omega_{21})^2/2)$
e_{20}	$(\omega_0-\omega_{21})^2/2$	$(\omega_0-\omega_{21})^2/2$	$(\omega_0-\omega_{21})^2/2$	$\tau((\omega_0-\omega_{21})^2/2)$	$\frac{\omega_0}{2} + \frac{\omega_{21}}{4} - \omega_{21}^3$	$\tau((\omega_0-\omega_{21})^2/2)$
e_{21}	$2\omega_{14}$	$1/2$	ω_{14}	$\tau(\omega_{14})$	$\tau(\omega_{14})/2$	$\tau(\omega_{14})$
e_{22}	ω_{15}	$1/2$	ω_{16}	$\tau(\omega_{15})$	$\tau(\omega_{15})/2$	$\tau(\omega_{15}-\omega_{16})$
e_{23}	ω_{17}	$1/2$	ω_{17}	$\tau(\omega_{17})$	$\tau(\omega_{17})/2$	$\tau(\omega_{17})$
e_{24}	ω_{20}	$(1-\omega_{20})/2$	$1/2$	$\tau(\omega_{20})$	$\tau(\omega_{20})/2$	$\tau(\omega_{20}/2)$
e_{25}	$1/2$	$(1-\omega_{18})/4$	$1/2$	ω_{19}	$\omega_{19}/2$	ω_{19}
e_{26}	$1/2$	$(1+\omega_{18})/4$	$(1-\omega_{18})/2$	ω_{19}	$\omega_{19}/2$	ω_{19}
e_{27}	ω_{22}	$\omega_{23}/2$	$1/2$	$\tau(\omega_{22})$	$\tau(\omega_{22})/2$	$\tau(\omega_{22})\omega_{24}/2$
e_{28}	ω_{25}	$1/2$	ω_{26}	$\tau(\omega_{25})$	$\tau(\omega_{25})/2$	$\tau(\omega_{26})$
e_{29}	$1/2$	$1/2$	0	ω_0	$\omega_{27}/2$	1
e_{30}	$\frac{1-\omega_{27}}{2}$	$1/2$	$\frac{1-\omega_{27}}{2}$	$\tau(\frac{1-\omega_{27}}{2})$	$\tau(\frac{1-\omega_{27}}{2}) - \frac{1}{2}$	$\tau(\frac{1-\omega_{27}}{2})$
e_{31}	ω_{28}	$1/2$	$\omega_{28}/2$	$\tau(\omega_{28})$	$\tau(\omega_{28})/2$	$\tau(\omega_{28}/2)$
e_{32}	ω_{29}	ω_{30}	ω_{29}	$\tau(\omega_{29})$	$\tau(\omega_{29})/2$	$\tau(\omega_{29})$
e_{33}	$1-2\omega_{30}$	$1-\omega_{29}-\omega_{30}$	ω_{29}	$\tau(\omega_{29})$	$\tau(\omega_{29})/2$	$\tau(\omega_{29})$
e_{34}	$1-2\omega_{30}$	$\omega_{30}-\omega_{29}$	ω_{29}	$\tau(\omega_{29})$	$\tau(\omega_{29})/2$	$\tau(\omega_{29})$
e_{35}	ω_{31}	$(1+\omega_{31}-\omega_{32})/2$	$\omega_{32}/2$	$\tau(\omega_{31})$	$\tau(\omega_{31})/2$	$\tau(\omega_{32}/2)$
e_{36}	ω_{31}	$(1-\omega_{31}-\omega_{32})/2$	$\omega_{32}/2$	$\tau(\omega_{31})$	$\tau(\omega_{31})/2$	$\tau(\omega_{32}/2)$
e_{37}	ω_{33}	$\omega_{34}/2$	$\omega_{35}/4$	$\tau(\omega_{33})$	$\tau(\omega_{33})/2$	$\tau(\omega_{35}/4)$
e_{38}	ω_{33}	$\omega_{34}/2 - \omega_{33}$	$\omega_{33} - \omega_{34} - \frac{\omega_{35}}{4} + 1$	$\tau(\omega_{33})$	$\tau(\omega_{33})/2$	$\tau(\omega_{35}/4)$
e_{39}	$1/2$	$\omega_{38}/2$	$\omega_{36}/2$	$\tau(\omega_{36}/2)$	$\sqrt{\omega_{37}}/2$	$\tau(\omega_{36}/2)$

TABLE 1. Definition of the points $e_j = X + \sqrt{-1}Y$ $(0 \le j \le 39)$.

step 3. For each point $p_{k,L}$, take the maximal $L \subset L_{\max}$ by inclusion. Register $(p_{k,L}, L_{\max})$.

step 4. For each (p, L) registered, if $p \in W_L$, then output (p, L).

The ideal I is 0-dimensional if and only if $\dim \mathbb{Q}[V]/I < \infty$. So by computing the Gröbner basis of I and checking its leading exponents, one can decide on the zero-dimensionality. Using the computer algebra system ASIR,[1] this is achieved by the command `zero_dim` in the package `gr`. By an exhaustive search in 2^{Λ}, we obtain the following search result.

Search Result 6.2. *There are 752 370 labels L out of 2^{Λ}, so that I_L is zero-dimensional.*

6.3. The Minimal Polynomials of Ideals

For each L such that I_L is zero-dimensional, we want to compute all real zeros $\{p_k\} \in V(I_L)$.

In preparation for step 2, we compute the minimal polynomial of the ideal I, which we now review. The propositions below are taken from

[1] Available at http://www.math.kobe-u.ac.jp/asir/asir.html.

$$\Omega_0(t) = 4t^2 - 3, \qquad \omega_0 = \sqrt{3}/2 = 0.866\ldots$$
$$\Omega_1(t) = 9t^2 - 8, \qquad \omega_1 = 2\sqrt{2}/3 = 0.9428\ldots$$
$$\Omega_2(t) = 8t^3 - 13t^2 + 10t - 1, \qquad \omega_2 = 0.1163\ldots$$
$$\Omega_3(t) = 9t^4 + 4t^2 - 16, \qquad \omega_3 = \frac{1}{3}\sqrt{2\sqrt{37} - 2} = 1.0627\ldots$$
$$\Omega_4(t) = 65536t^4 - 133376t^2 + 61009, \qquad \omega_4 = \sqrt{(3\sqrt{3045} + 521)/2^9} = 1.1579\ldots$$
$$\Omega_5(t) = t^4 - 441t^2 + 42849, \qquad \omega_5 = 3\sqrt{(\sqrt{285} + 49)/2} = 17.2\ldots$$
$$\Omega_6(t) = 4t^4 - 8t^3 + 12t^2 - 8t + 1, \qquad \omega_6 = 0.1593\ldots$$
$$\Omega_7(t) = t^4 + 22t^2 + 96t - 39, \qquad \omega_7 = 0.3739\ldots$$
$$\Omega_8(t) = t^4 - 14t^3 + 11t^2 + 2250t - 3687, \qquad \omega_8 = (3 + 10\omega_7 - \omega_7^2)/4 = 1.6500\ldots$$
$$\Omega_{14}(t) = 9t^4 - 24t^3 + 68t^2 - 48t + 4, \qquad \omega_{14} = 0.0959\ldots$$
$$\Omega_{15}(t) = 9t^4 - 24t^3 + 52t^2 - 32t + 4, \qquad \omega_{15} = 0.1670\ldots$$
$$\Omega_{16}(t) = 1296t^4 - 3744t^3 + 5224t^2 - 4504t + 505, \qquad \omega_{16} = (3\omega_{15}^3 - 6\omega_{15}^2 + 16\omega_{15} - 2)/4 = 0.1299\ldots$$
$$\Omega_{17}(t) = 15t^4 - 32t^3 - 4t^2 + 32t - 4, \qquad \omega_{17} = 0.1291\ldots$$
$$\Omega_{18}(t) = t^6 - 2t^4 + 8t^3 - 259t^2 + 256t - 64, \qquad \omega_{18} = 0.5285\ldots$$
$$\Omega_{19}(t) = 2304t^{12} + 3712t^{10} - 1776t^8 - 2848t^6 - 1016t^4 - 240t^2 + 225, \qquad \omega_{19} = (1 - \omega_{18})/(2\sqrt{2\omega_{18} - 1}) = 0.9861\ldots$$
$$\Omega_{20}(t) = 4t^5 + 7t^4 - 28t^3 - 38t^2 + 48t - 9, \qquad \omega_{20} = 0.241164\ldots$$
$$\Omega_{21}(t) = 256t^8 - 256t^6 + 224t^4 - 848t^2 + 25, \qquad \omega_{21} = 0.1723559\ldots$$
$$\Omega_{22}(t) = 9t^6 - 2t^5 - 57t^4 + 107t^2 + 2t - 23, \qquad \omega_{22} = 0.4843\ldots$$
$$\Omega_{23}(t) = 9t^6 + 44t^5 - 36t^4 - 52t^3 + 124t^2 + 24t - 12, \qquad \omega_{23} = \sqrt{2 + 2\omega_{22}} - 1 - \omega_{22} = 0.2386\ldots$$
$$\Omega_{24}(t) = 12t^6 - 88t^5 + 148t^4 + 28t^3 - 120t^2 + 36t + 23, \quad \omega_{24} = \omega_{22}^5 + (37\omega_{22}^4 - 148\omega_{22}^3 - 152\omega_{22}^2 + 130\omega_{22} + 73)/36 = 2.402\ldots$$
$$\begin{aligned}\Omega_{25}(t) = {}& 7t^{12} - 16t^{11} - 44t^{10} + 368t^9 - 1804t^8 + 3840t^7 - 3744t^6 - 3200t^5\\ & + 12560t^4 - 6912t^3 - 6848t^2 + 6912t - 1344, \qquad \omega_{25} = 0.2914\ldots\end{aligned}$$
$$\begin{aligned}\Omega_{26}(t) = {}& 7168t^{12} - 34816t^{11} + 20224t^{10} + 231936t^9 - 599760t^8 + 222816t^7 + 713128t^6 + 31320t^5 - 1793601t^4\\ & + 1157252t^3 + 463714t^2 - 470548t + 4511,\end{aligned}$$
$$\begin{aligned}\omega_{26} = {}& (3997\omega_{25}^{11} - 4999\omega_{25}^{10} - 31374\omega_{25}^9 + 180128\omega_{25}^8 - 836696\omega_{25}^7 + 1270256\omega_{25}^6 - 551200\omega_{25}^5 - 2995952\omega_{25}^4\\ & + 4650320\omega_{25}^3 + 1295920\omega_{25}^2 - 4073760\omega_{25} + 989888)/572416 = 0.0096\ldots\end{aligned}$$
$$\Omega_{27}(t) = t^8 + 4t^6 - 16t^5 - 42t^4 - 32t^3 + 164t^2 - 16t - 47, \qquad \omega_{27} = 0.7113\ldots$$
$$\Omega_{28}(t) = 9t^8 - 24t^7 + 56t^6 + 16t^5 - 232t^4 - 416t^3 + 1760t^2 - 1088t + 144, \qquad \omega_{28} = 0.1851\ldots$$
$$\Omega_{29}(t) = t^8 - 4t^7 + 16t^6 - 20t^5 - 40t^4 + 64t^3 + 28t^2 - 40t + 4, \qquad \omega_{29} = 0.1105\ldots$$
$$\Omega_{30}(t) = 256t^8 - 512t^7 + 1280t^6 - 1536t^5 + 1888t^4 - 1120t^3 + 512t^2 + 96t - 87$$
$$\omega_{30} = (3\omega_{29}^7 - 11\omega_{29}^6 + 59\omega_{29}^5 - 88\omega_{29}^4 + 78\omega_{29}^3 - 2\omega_{29}^2 - 188\omega_{29} + 74)/132 = 0.4036\ldots$$
$$\begin{aligned}\Omega_{31}(t) = {}& t^{16} - 12t^{15} + 88t^{14} - 448t^{13} + 1716t^{12} - 5104t^{11} + 11896t^{10} - 21956t^9 + 32228t^8 - 37776t^7 + 35464t^6\\ & - 25968t^5 + 14736t^4 - 5744t^3 + 1220t^2 - 120t + 4, \qquad \omega_{31} = 0.18587\ldots\end{aligned}$$

TABLE 2. Minimal polynomials of algebraic numbers appearing in the coordinates. (Note that we do not use the labels Ω_i, ω_i $(9 \leq i \leq 13)$.)

[Noro and Yokoyama 03, Saito et al. 03] with terminology there and without proof. Let $P = \mathbb{Q}[V] = \mathbb{Q}[X_1, \ldots, X_n]$ be a polynomial ring of n-variables. Let I be an ideal of P. Let $f \in P$. We say that $m(I, f; t) \in \mathbb{Q}[t]$ is a minimal polynomial of I with respect to f if

$$\{g(t) \in P \mid g(f) \in I\} = \langle m(I, f; t)\rangle.$$

Proposition 6.3. *If I is zero-dimensional, then there is a unique $m(I, f; t)$ with respect to every $f \in P$.*

Proposition 6.4. *If $f(X) = X_i$ is a monomial of degree 1, then*

$$\{p \in V(\mathbb{C}) \mid m(I, X_i; p_i) = 0 \textit{ for all } i\} \supset V(I).$$

This proposition implies that we have candidates for points of $V(I)$ by combining the zeros of the minimal polynomials of the indeterminates. As for determining the actual zeros $V(I)$, we have the following result.

$$\Omega_{32}(t) = t^{16} - 8t^{15} + 64t^{14} - 328t^{13} + 1248t^{12} - 3584t^{11} + 6512t^{10} - 2048t^9 - 21472t^8 + 52224t^7 - 40448t^6 - 30848t^5 + 75008t^4 - 15360t^3 - 36352t^2 + 6144t + 256, \qquad \omega_{32} = 0.2019\ldots$$

$$\Omega_{33}(t) = 2704t^{18} + 6656t^{17} + 14600t^{16} + 40200t^{15} - 89391t^{14} - 422918t^{13} + 347143t^{12} + 483432t^{11} - 462293t^{10} + 218930t^9 + 255381t^8 - 256272t^7 + 3376t^6 + 5844t^5 + 5256t^4 - 384t^3 - 152t^2 - 16t + 4, \qquad \omega_{33} = 0.2726\ldots$$

$$\Omega_{34}(t) = 2704t^{18} - 14976t^{17} + 50584t^{16} - 181536t^{15} + 325297t^{14} - 275424t^{13} + 906700t^{12} - 3114868t^{11} + 12229602t^{10} - 26726196t^9 + 22096964t^8 + 20670736t^7 - 134656607t^6 - 103677140t^5 + 998831544t^4 - 1897054492t^3 + 2036832876t^2 - 398345048t - 481484588, \qquad \omega_{34} = 0.9447\ldots$$

$$\Omega_{35}(t) = 2704t^{18} - 58032t^{17} + 283180t^{16} + 4323156t^{15} - 80836455t^{14} + 586982356t^{13} - 517856644t^{12} - 33377993888t^{11} + 333714279920t^{10} - 1388330956224t^9 + 939768309312t^8 + 16067763419136t^7 - 73817655947264t^6 + 140376670666752t^5 - 73288713666560t^4 - 240280447483904t^3 + 655739410448384t^2 - 673143031070720t + 135070115430400, \qquad \omega_{35} = -\omega_{33}^3 + 2\omega_{34}\omega_{33}^2 - (\omega_{34}^2 - 1)\omega_{33} - 2\omega_{34} + 2 = 0.2599\ldots$$

$$\Omega_{36}(t) = 9t^{18} - 30t^{17} + 347t^{16} - 2316t^{15} + 1553t^{14} + 15838t^{13} - 69221t^{12} + 153688t^{11} + 299059t^{10} - 950530t^9 - 1504919t^8 - 3251596t^7 + 21274899t^6 + 19229826t^5 - 104885343t^4 + 22091520t^3 + 149954464t^2 - 126939648t + 28776704, \qquad \omega_{36} = 0.4794\ldots$$

$$\Omega_{37}(t) = 1296t^{18} - 29376t^{17} - 41624t^{16} + 14552056t^{15} - 62498599t^{14} - 2452996286t^{13} + 11707386901t^{12} + 194047036776t^{11} - 879334215598t^{10} - 6258212822748t^9 + 37344996202338t^8 + 25623399000072t^7 - 715703278407651t^6 + 1044863824619722t^5 + 3139512217604881t^4 - 11114753673284056t^3 + 13509026011131304t^2 - 7566283923018880t + 1647662728153744$$

$$\omega_{37} = 0.8570\ldots$$

$$\Omega_{38}(t) = 36t^{18} - 480t^{17} + 4136t^{16} - 22220t^{15} + 67533t^{14} - 35042t^{13} - 766479t^{12} + 4336224t^{11} - 13308798t^{10} + 27123976t^9 - 38935546t^8 + 40272664t^7 - 29588523t^6 + 13701578t^5 - 1642203t^4 - 2665532t^3 + 2118624t^2 - 805056t + 147044, \qquad \omega_{38} = 0.76594\ldots$$

TABLE 3. Minimal polynomials of algebraic numbers (2).

pts	coordinate expression	pts	coordinate expression
e_0	$[0.5, 0, 0.5, 0.866, 0, 0.866]$	e_{20}	$[0.240, 0.240, 0.240, 0.9706, 0.4709, 0.9706]$
e_1	$[0.333, 0.333, 0.333, 0.942, 0.471, 0.942]$	e_{21}	$[0.191, 0.5, 0.095, 0.9953, 0.4976, 0.9953]$
e_2	$[0.159, 0.5, 0.159, 1, 0.5, 1]$	e_{22}	$[0.167, 0.5, 0.1299, 0.985, 0.4929, 0.9993]$
e_3	$[0.5, 0.5, 0.5, 0.866, 0, 0.866]$	e_{23}	$[0.1291, 0.5, 0.1291, 0.9916, 0.495, 0.9916]$
e_4	$[0.5, 0.5, 0, 1, 0.5, 1]$	e_{24}	$[0.2411, 0.3794, 0.5, 0.9704, 0.4852, 0.992]$
e_5	$[0.5, 0, 0.5, 1, 0.5, 1]$	e_{25}	$[0.5, 0.117, 0.5, 0.9861, 0.4930, 0.9861]$
e_6	$[0, 0.5, 0, 1, 0.5, 1]$	e_{26}	$[0.5, 0.3821, 0.2357, 0.9861, 0.4930, 0.9861]$
e_7	$[0.5, 0.1160, 0.5, 0.866, 0.433, 1.066]$	e_{27}	$[0.4843, 0.1193, 0.5, 0.8748, 0.4374, 1.0508]$
e_8	$[0.5, 0.5, 0.5, 1.062, 0.531, 1.062]$	e_{28}	$[0.2914, 0.5, 0.0096, 0.9565, 0.4782, 0.9999]$
e_9	$[0.5, 0.5, 0.5, 0.866, 0.433, 1.157]$	e_{29}	$[0.5, 0.5, 0, 0.866, 0.3556, 1]$
e_{10}	$[0.5, 0.5, 0, 0.866, 0.433, 1.076]$	e_{30}	$[0.1443, 0.5, 0.1443, 0.9895, 0.4895, 0.9895]$
e_{11}	$[0.5, 0, 0.5, 0.866, 0.433, 1.076]$	e_{31}	$[0.1851, 0.5, 0.092, 0.9827, 0.4913, 0.9957]$
e_{12}	$[0.1163, 0.441, 0.1163, 0.9932, 0.4966, 0.9932]$	e_{32}	$[0.110, 0.4036, 0.110, 0.9938, 0.4969, 0.9938]$
e_{13}	$[0.325, 0.441, 0.1163, 0.9932, 0.4966, 0.9932]$	e_{33}	$[0.192, 0.4857, 0.110, 0.9938, 0.4969, 0.9938]$
e_{14}	$[0.325, 0.1163, 0.325, 0.9932, 0.4966, 0.9932]$	e_{34}	$[0.192, 0.293, 0.110, 0.9938, 0.4969, 0.9938]$
e_{15}	$[0.1163, 0.325, 0.1163, 0.9932, 0.4966, 0.9932]$	e_{35}	$[0.1858, 0.4919, 0.100, 0.9825, 0.4912, 0.9948]$
e_{16}	$[0.5, 0.093, 0.480, 0.866, 0.433, 1.061]$	e_{36}	$[0.1858, 0.3060, 0.100, 0.9825, 0.4912, 0.9948]$
e_{17}	$[0.5, 0.406, 0.206, 0.866, 0.433, 1.061]$	e_{37}	$[0.272, 0.472, 0.064, 0.962, 0.481, 0.997]$
e_{18}	$[0.5, 0.5, 0.5, 0.866, 0.172, 0.866]$	e_{38}	$[0.272, 0.199, 0.262, 0.962, 0.481, 0.997]$
e_{19}	$[0.5, 0.399, 0.240, 0.866, 0.346, 0.9706]$	e_{39}	$[0.5, 0.3829, 0.2397, 0.9708, 0.4628, 0.9708]$

TABLE 4. Numerical expression of points on $(V(I) \cap \partial\mathcal{F})/\!\sim \Delta$.

pt: p	maximal label: L_p
e_0	$[\mathbf{1},\mathbf{2},O,\theta E_1,\theta E_2,\theta I_+,Y_1,Y_3,X_1,X_2]$
$\theta_2 e_0$	$[\mathbf{1},\mathbf{2},O,\theta E_1,E_2,\theta I_-,Y_1,Y_3,X_1,\theta X_2]$
$\theta_1 e_0$	$[\mathbf{1},\mathbf{2},O,E_1,\theta E_2,I_-,Y_1,Y_3,X_2,\theta X_1]$
$\theta_1\theta_2 e_0$	$[\mathbf{1},\mathbf{2},O,E_1,E_2,I_+,Y_1,Y_3,\theta X_1,\theta X_2]$
$\theta_3 e_1$	$[\mathbf{1},\mathbf{2},\theta R,\theta E_1,\theta E_2,\theta I_+,J,Y_1,Y_2]$
$\theta_1\theta_2 e_1$	$[\mathbf{1},\mathbf{2},R,E_1,E_2,I_+,\theta J,Y_1,Y_2]$
$\theta_2 e_2$	$[R,E_2,\theta J_1,Y_1,Y_2,X_3]$
$\theta_3\theta_2 e_2$	$[\theta R,\theta E_1,J_2,Y_1,Y_2,\theta X_3]$
$\theta_1 e_2$	$[R,E_1,\theta J_2,Y_1,Y_2,X_3]$
$\theta_1\theta_3 e_2$	$[\theta R,\theta E_2,J_1,Y_1,Y_2,\theta X_3]$
e_3	$[\mathbf{1},\mathbf{2},Y_1,Y_3,X_1,X_3,X_2]$
$\theta_2 e_3$	$[\mathbf{1},\mathbf{2},Y_1,Y_3,X_1,X_3,\theta X_2]$
$\theta_3 e_3$	$[\mathbf{1},\mathbf{2},Y_1,Y_3,X_1,X_2,\theta X_3]$
$\theta_3\theta_2 e_3$	$[\mathbf{1},\mathbf{2},Y_1,Y_3,X_1,\theta X_3,\theta X_2]$
$\theta_1 e_3$	$[\mathbf{1},\mathbf{2},Y_1,Y_3,X_3,X_2,\theta X_1]$
$\theta_1\theta_2 e_3$	$[\mathbf{1},\mathbf{2},Y_1,Y_3,X_3,\theta X_1,\theta X_2]$
$\theta_1\theta_3 e_3$	$[\mathbf{1},\mathbf{2},Y_1,Y_3,X_2,\theta X_1,\theta X_3]$
θe_3	$[\mathbf{1},\mathbf{2},Y_1,Y_3,\theta X_1,\theta X_3,\theta X_2]$
e_4	$[\mathbf{2},O,\theta J_1,Y_1,Y_2,X_1,X_3]$
$\theta_2\sigma_0 e_4$	$[\mathbf{1},E_2,\theta J,Y_1,Y_2,X_3,\theta X_2]$
$\theta_3 e_4$	$[\mathbf{2},\theta E_1,J,Y_1,Y_2,X_1,\theta X_3]$
$\theta_3\theta_2\sigma_0 e_4$	$[\mathbf{1},O,J_2,Y_1,Y_2,\theta X_3,\theta X_2]$
$\theta_3\sigma_0 e_4$	$[\mathbf{1},\theta E_2,J,Y_1,Y_2,X_2,\theta X_3]$
$\theta_1 e_4$	$[\mathbf{2},E_1,\theta J,Y_1,Y_2,X_3,\theta X_1]$
$\theta_1\theta_3 e_4$	$[\mathbf{2},O,J_1,Y_1,Y_2,\theta X_1,\theta X_3]$
$\sigma_0 e_4$	$[\mathbf{1},O,\theta J_2,Y_1,Y_2,X_3,X_2]$
e_5	$[\theta R,\theta E_1,\theta E_2,Y_1,Y_2,X_1,X_2]$
$\theta_1\theta_2 e_5$	$[R,E_1,E_2,Y_1,Y_2,\theta X_1,\theta X_2]$
e_6	$[\mathbf{1},\mathbf{2},R,Y_1,Y_2,X_3]$
$\theta_3 e_6$	$[\mathbf{1},\mathbf{2},\theta R,Y_1,Y_2,\theta X_3]$
e_7	$[\mathbf{1},O,\theta E_2,Y_2,X_1,X_2]$
$\theta_2 e_7$	$[\mathbf{1},O,E_2,Y_2,X_1,\theta X_2]$
$\theta_3 e_7$	$[\mathbf{1},\theta E_1,\theta I_+,Y_2,X_1,X_2]$
$\theta_3\theta_2 e_7$	$[\mathbf{1},\theta E_1,\theta I_-,Y_2,X_1,\theta X_2]$
$\theta_1 e_7$	$[\mathbf{1},E_1,I_-,Y_2,X_2,\theta X_1]$
$\theta_1\theta_2 e_7$	$[\mathbf{1},E_1,I_+,Y_2,\theta X_1,\theta X_2]$
$\theta_1\theta_3 e_7$	$[\mathbf{1},O,\theta E_2,Y_2,X_2,\theta X_1]$
θe_7	$[\mathbf{1},O,E_2,Y_2,\theta X_1,\theta X_2]$
e_8	$[O,Y_1,Y_2,X_1,X_3,X_2]$
$\theta_2 e_8$	$[E_2,\theta J_1,Y_1,Y_2,X_1,X_3,\theta X_2]$
$\theta_3 e_8$	$[\theta I_+,J,Y_1,Y_2,X_1,X_2,\theta X_3]$
$\theta_3\theta_2 e_8$	$[\theta E_1,J_2,Y_1,Y_2,X_1,\theta X_3,\theta X_2]$
$\theta_1 e_8$	$[E_1,\theta J_2,Y_1,Y_2,X_3,X_2,\theta X_1]$
$\theta_1\theta_2 e_8$	$[I_+,\theta J,Y_1,Y_2,X_3,\theta X_1,\theta X_2]$
$\theta_1\theta_3 e_8$	$[\theta E_2,J_1,Y_1,Y_2,X_2,\theta X_1,\theta X_3]$
θe_8	$[O,Y_1,Y_2,\theta X_1,\theta X_3,\theta X_2]$
e_9	$[\mathbf{1},O,Y_2,X_1,X_3,X_2]$
$\theta_2 e_9$	$[\mathbf{1},E_2,\theta J_1,Y_2,X_1,X_3,\theta X_2]$
$\theta_3 e_9$	$[\mathbf{1},\theta I_+,J,Y_2,X_1,X_2,\theta X_3]$
$\theta_3\theta_2 e_9$	$[\mathbf{1},\theta E_1,J_2,Y_2,X_1,\theta X_3,\theta X_2]$
$\theta_1 e_9$	$[\mathbf{1},E_1,\theta J_2,Y_2,X_3,X_2,\theta X_1]$
$\theta_1\theta_2 e_9$	$[\mathbf{1},I_+,\theta J,Y_2,X_3,\theta X_1,\theta X_2]$
$\theta_1\theta_3 e_9$	$[\mathbf{1},\theta E_2,J_1,Y_2,X_2,\theta X_1,\theta X_3]$
θe_9	$[\mathbf{1},O,Y_2,\theta X_1,\theta X_3,\theta X_2]$
e_{10}	$[\mathbf{1},O,\theta J_1,Y_2,X_1,X_3]$
$\theta_3 e_{10}$	$[\mathbf{1},\theta E_1,J,Y_2,X_1,\theta X_3]$
$\theta_1 e_{10}$	$[\mathbf{1},E_1,\theta J,Y_2,X_3,\theta X_1]$
$\theta_1\theta_3 e_{10}$	$[\mathbf{1},O,J_1,Y_2,\theta X_1,\theta X_3]$
e_{11}	$[\mathbf{1},\theta E_1,\theta E_2,Y_2,X_1,X_2]$
$\theta_2 e_{11}$	$[\mathbf{1},O,\theta I_-,Y_2,X_1,\theta X_2]$
$\theta_1 e_{11}$	$[\mathbf{1},O,I_-,Y_2,X_2,\theta X_1]$
$\theta_1\theta_2 e_{11}$	$[\mathbf{1},E_1,E_2,Y_2,\theta X_1,\theta X_2]$
$\theta_2 e_{12}$	$[\mathbf{1},\mathbf{2},R,E_2,Y_1,Y_2]$
$\theta_3\theta_2 e_{12}$	$[\mathbf{1},\mathbf{2},\theta R,\theta E_1,Y_1,Y_2]$
$\theta_1 e_{12}$	$[\mathbf{1},\mathbf{2},R,E_1,Y_1,Y_2]$
$\theta_1\theta_3 e_{12}$	$[\mathbf{1},\mathbf{2},\theta R,\theta E_2,Y_1,Y_2]$
$\theta_2 e_{13}$	$[\mathbf{2},O,E_2,\theta J_1,Y_1,Y_2]$
$\theta_3\theta_2\sigma_0 e_{13}$	$[\mathbf{1},O,\theta E_1,J_2,Y_1,Y_2]$
$\theta_1\theta_3 e_{13}$	$[\mathbf{2},O,\theta E_2,J_1,Y_1,Y_2]$
$\theta_1\sigma_0 e_{13}$	$[\mathbf{1},O,E_1,\theta J_2,Y_1,Y_2]$
$\theta_3 e_{14}$	$[\theta R,O,\theta E_1,\theta E_2,Y_1,Y_2]$
$\theta_1\theta_2 e_{14}$	$[R,O,E_1,E_2,Y_1,Y_2]$
$\theta_3 e_{15}$	$[\mathbf{1},\mathbf{2},\theta R,O,Y_1,Y_2]$
$\theta_1\theta_2 e_{15}$	$[\mathbf{1},\mathbf{2},R,O,Y_1,Y_2]$
e_{16}	$[\mathbf{1},O,\theta E_1,\theta E_2,Y_2,X_1]$
$\theta_3\theta_2 e_{16}$	$[\mathbf{1},O,\theta E_1,\theta I_-,Y_2,X_1]$
$\theta_1 e_{16}$	$[\mathbf{1},O,E_1,I_-,Y_2,\theta X_1]$
θe_{16}	$[\mathbf{1},O,E_1,E_2,Y_2,\theta X_1]$
$\theta_2 e_{17}$	$[\mathbf{1},O,E_2,\theta J_1,Y_2,X_1]$
$\theta_3 e_{17}$	$[\mathbf{1},\theta E_1,\theta I_+,J,Y_2,X_1]$
$\theta_1\theta_2 e_{17}$	$[\mathbf{1},E_1,I_+,\theta J,Y_2,\theta X_1]$
$\theta_1\theta_3 e_{17}$	$[\mathbf{1},O,\theta E_2,J_1,Y_2,\theta X_1]$
e_{18}	$[\mathbf{1},\mathbf{2},O,Y_1,X_1,X_3,X_2]$
$\theta_2 e_{18}$	$[\mathbf{1},\mathbf{2},E_2,\theta J_1,Y_1,X_1,X_3,\theta X_2]$
$\theta_3 e_{18}$	$[\mathbf{1},\mathbf{2},\theta I_+,J,Y_1,X_1,X_2,\theta X_3]$
$\theta_3\theta_2 e_{18}$	$[\mathbf{1},\mathbf{2},\theta E_1,J_2,Y_1,X_1,\theta X_3,\theta X_2]$
$\theta_1 e_{18}$	$[\mathbf{1},\mathbf{2},E_1,\theta J_2,Y_1,X_3,X_2,\theta X_1]$
$\theta_1\theta_2 e_{18}$	$[\mathbf{1},\mathbf{2},I_+,\theta J,Y_1,X_3,\theta X_1,\theta X_2]$
$\theta_1\theta_3 e_{18}$	$[\mathbf{1},\mathbf{2},\theta E_2,J_1,Y_1,X_2,\theta X_1,\theta X_3]$
θe_{18}	$[\mathbf{1},\mathbf{2},O,Y_1,\theta X_1,\theta X_3,\theta X_2]$

TABLE 5. Points p and its associated label $L_p \subset \Lambda$.

Proposition 6.5. *Let $G \subset I$ be a Gröbner basis of I with respect to the lexicographic order $X_1 \succ X_2 \succ \cdots \succ X_n$. Then*

$$I \cap \mathbb{Q}\,[X_n] = \langle m(I, X_n; t)\rangle.$$

If G is reduced, then $G \cap \mathbb{Q}\,[X_n] = m(I, X_n; t)$.

One way to obtain the minimal polynomials is to compute Gröbner bases. Finally, we refer to the so-called shape basis.

pt: p	maximal label: L_p
$\theta_2 e_{19}$	$[\mathbf{1}, \mathbf{2}, O, E_2, \theta J_1, X_1]$
$\theta_3 e_{19}$	$[\mathbf{1}, \mathbf{2}, \theta E_1, \theta I_+, J, X_1]$
$\theta_1\theta_2 e_{19}$	$[\mathbf{1}, \mathbf{2}, E_1, I_+, \theta J, \theta X_1]$
$\theta_1\theta_3 e_{19}$	$[\mathbf{1}, \mathbf{2}, O, \theta E_2, J_1, \theta X_1]$
$\theta_3 e_{20}$	$[\mathbf{1}, \mathbf{2}, \theta R, O, \theta E_1, \theta E_2, Y_1]$
$\theta_1\theta_2 e_{20}$	$[\mathbf{1}, \mathbf{2}, R, O, E_1, E_2, Y_1]$
$\theta_2 e_{21}$	$[\mathbf{2}, R, \theta J_1, Y_1, Y_2, X_3]$
$\theta_2\sigma_0 e_{21}$	$[\mathbf{1}, R, E_2, Y_1, Y_2, X_3]$
$\theta_3\theta_2 e_{21}$	$[\mathbf{2}, \theta R, \theta E_1, Y_1, Y_2, \theta X_3]$
$\theta_3\theta_2\sigma_0 e_{21}$	$[\mathbf{1}, \theta R, J_2, Y_1, Y_2, \theta X_3]$
$\theta_1 e_{21}$	$[\mathbf{2}, R, E_1, Y_1, Y_2, X_3]$
$\theta_1\theta_3 e_{21}$	$[\mathbf{2}, \theta R, J_1, Y_1, Y_2, \theta X_3]$
$\theta_1\theta_3\sigma_0 e_{21}$	$[\mathbf{1}, \theta R, \theta E_2, Y_1, Y_2, \theta X_3]$
$\theta_1\sigma_0 e_{21}$	$[\mathbf{1}, R, \theta J_2, Y_1, Y_2, X_3]$
$\theta_2 e_{22}$	$[\mathbf{1}, R, E_2, \theta J_1, Y_2, X_3]$
$\theta_3\theta_2 e_{22}$	$[\mathbf{1}, \theta R, \theta E_1, J_2, Y_2, \theta X_3]$
$\theta_1 e_{22}$	$[\mathbf{1}, R, E_1, \theta J_2, Y_2, X_3]$
$\theta_1\theta_3 e_{22}$	$[\mathbf{1}, \theta R, \theta E_2, J_1, Y_2, \theta X_3]$
e_{23}	$[\mathbf{1}, \mathbf{2}, O, Y_1, Y_2, X_3]$
$\theta_3 e_{23}$	$[\mathbf{1}, \mathbf{2}, J, Y_1, Y_2, \theta X_3]$
$\theta_1\theta_2 e_{23}$	$[\mathbf{1}, \mathbf{2}, \theta J, Y_1, Y_2, X_3]$
θe_{23}	$[\mathbf{1}, \mathbf{2}, O, Y_1, Y_2, \theta X_3]$
$\theta_3 e_{24}$	$[\mathbf{1}, \theta E_2, \theta I_+, J, Y_2, X_2]$
$\theta_3\theta_2 e_{24}$	$[\mathbf{1}, O, \theta E_1, J_2, Y_2, \theta X_2]$
$\theta_1 e_{24}$	$[\mathbf{1}, O, E_1, \theta J_2, Y_2, X_2]$
$\theta_1\theta_2 e_{24}$	$[\mathbf{1}, E_2, I_+, \theta J, Y_2, \theta X_2]$
e_{25}	$[O, \theta E_1, \theta E_2, Y_1, Y_2, X_1, X_2]$
$\theta_2 e_{25}$	$[O, E_2, \theta I_-, Y_1, Y_2, X_1, \theta X_2]$
$\theta_3 e_{25}$	$[\theta E_1, \theta E_2, \theta I_+, Y_1, Y_2, X_1, X_2]$
$\theta_3\theta_2 e_{25}$	$[O, \theta E_1, \theta I_-, Y_1, Y_2, X_1, \theta X_2]$
$\theta_1 e_{25}$	$[O, E_1, I_-, Y_1, Y_2, X_2, \theta X_1]$
$\theta_1\theta_2 e_{25}$	$[E_1, E_2, I_+, Y_1, Y_2, \theta X_1, \theta X_2]$
$\theta_1\theta_3 e_{25}$	$[O, \theta E_2, I_-, Y_1, Y_2, X_2, \theta X_1]$
θe_{25}	$[O, E_1, E_2, Y_1, Y_2, \theta X_1, \theta X_2]$
$\theta_2 e_{26}$	$[O, E_2, \theta J_1, Y_1, Y_2, X_1]$
$\theta_3 e_{26}$	$[\theta E_1, \theta I_+, J, Y_1, Y_2, X_1]$
$\theta_3\theta_2\sigma_0 e_{26}$	$[O, \theta E_1, J_2, Y_1, Y_2, \theta X_2]$
$\theta_3\sigma_0 e_{26}$	$[\theta E_2, \theta I_+, J, Y_1, Y_2, X_2]$
$\theta_1\theta_2 e_{26}$	$[E_1, I_+, \theta J, Y_1, Y_2, \theta X_1]$
$\theta_1\theta_2\sigma_0 e_{26}$	$[E_2, I_+, \theta J, Y_1, Y_2, \theta X_2]$
$\theta_1\theta_3 e_{26}$	$[O, \theta E_2, J_1, Y_1, Y_2, \theta X_1]$
$\theta_1\sigma_0 e_{26}$	$[O, E_1, \theta J_2, Y_1, Y_2, X_2]$
$\theta_3 e_{27}$	$[\mathbf{1}, \theta E_1, \theta E_2, \theta I_+, Y_2, X_2]$
$\theta_3\theta_2 e_{27}$	$[\mathbf{1}, O, \theta E_1, \theta I_-, Y_2, \theta X_2]$
$\theta_1 e_{27}$	$[\mathbf{1}, O, E_1, I_-, Y_2, X_2]$
$\theta_1\theta_2 e_{27}$	$[\mathbf{1}, E_1, E_2, I_+, Y_2, \theta X_2]$
$\theta_2 e_{28}$	$[\mathbf{1}, \mathbf{2}, O, \theta J_1, Y_2, X_3]$
$\theta_3\theta_2 e_{28}$	$[\mathbf{1}, \mathbf{2}, \theta E_1, J, Y_2, \theta X_3]$
$\theta_1 e_{28}$	$[\mathbf{1}, \mathbf{2}, E_1, \theta J, Y_2, X_3]$
$\theta_1\theta_3 e_{28}$	$[\mathbf{1}, \mathbf{2}, O, J_1, Y_2, \theta X_3]$
e_{29}	$[\mathbf{1}, \mathbf{2}, O, \theta J_1, X_1, X_3]$
$\theta_3 e_{29}$	$[\mathbf{1}, \mathbf{2}, \theta E_1, J, X_1, \theta X_3]$
$\theta_1 e_{29}$	$[\mathbf{1}, \mathbf{2}, E_1, \theta J, X_3, \theta X_1]$
$\theta_1\theta_3 e_{29}$	$[\mathbf{1}, \mathbf{2}, O, J_1, \theta X_1, \theta X_3]$
$\theta_2 e_{30}$	$[\mathbf{1}, \mathbf{2}, R, E_2, \theta J_1, Y_1, X_3]$
$\theta_3\theta_2 e_{30}$	$[\mathbf{1}, \mathbf{2}, \theta R, \theta E_1, J_2, Y_1, \theta X_3]$
$\theta_1 e_{30}$	$[\mathbf{1}, \mathbf{2}, R, E_1, \theta J_2, Y_1, X_3]$
$\theta_1\theta_3 e_{30}$	$[\mathbf{1}, \mathbf{2}, \theta R, \theta E_2, J_1, Y_1, \theta X_3]$
$\theta_2 e_{31}$	$[\mathbf{1}, \mathbf{2}, R, \theta J_1, Y_2, X_3]$
$\theta_3\theta_2 e_{31}$	$[\mathbf{1}, \mathbf{2}, \theta R, \theta E_1, Y_2, \theta X_3]$
$\theta_1 e_{31}$	$[\mathbf{1}, \mathbf{2}, R, E_1, Y_2, X_3]$
$\theta_1\theta_3 e_{31}$	$[\mathbf{1}, \mathbf{2}, \theta R, J_1, Y_2, \theta X_3]$
$\theta_2 e_{32}$	$[\mathbf{1}, \mathbf{2}, O, E_2, Y_1, Y_2]$
$\theta_3\theta_2 e_{32}$	$[\mathbf{1}, \mathbf{2}, O, \theta E_1, Y_1, Y_2]$
$\theta_1 e_{32}$	$[\mathbf{1}, \mathbf{2}, O, E_1, Y_1, Y_2]$
$\theta_1\theta_3 e_{32}$	$[\mathbf{1}, \mathbf{2}, O, \theta E_2, Y_1, Y_2]$
$\theta_2 e_{33}$	$[\mathbf{2}, R, E_2, \theta J_1, Y_1, Y_2]$
$\theta_3\theta_2\sigma_0 e_{33}$	$[\mathbf{1}, \theta R, \theta E_1, J_2, Y_1, Y_2]$
$\theta_1\theta_3 e_{33}$	$[\mathbf{2}, \theta R, \theta E_2, J_1, Y_1, Y_2]$
$\theta_1\sigma_0 e_{33}$	$[\mathbf{1}, R, E_1, \theta J_2, Y_1, Y_2]$
$\theta_3 e_{34}$	$[\mathbf{2}, \theta R, O, \theta E_1, Y_1, Y_2]$
$\theta_3\sigma_0 e_{34}$	$[\mathbf{1}, \theta R, O, \theta E_2, Y_1, Y_2]$
$\theta_1\theta_2 e_{34}$	$[\mathbf{2}, R, O, E_1, Y_1, Y_2]$
$\theta_1\theta_2\sigma_0 e_{34}$	$[\mathbf{1}, R, O, E_2, Y_1, Y_2]$
$\theta_2 e_{35}$	$[\mathbf{1}, \mathbf{2}, R, E_2, \theta J_1, Y_2]$
$\theta_1\theta_3 e_{35}$	$[\mathbf{1}, \mathbf{2}, \theta R, \theta E_2, J_1, Y_2]$
$\theta_3 e_{36}$	$[\mathbf{1}, \mathbf{2}, \theta R, O, \theta E_1, Y_2]$
$\theta_1\theta_2 e_{36}$	$[\mathbf{1}, \mathbf{2}, R, O, E_1, Y_2]$
$\theta_2 e_{37}$	$[\mathbf{1}, \mathbf{2}, O, E_2, \theta J_1, Y_2]$
$\theta_1\theta_3 e_{37}$	$[\mathbf{1}, \mathbf{2}, O, \theta E_2, J_1, Y_2]$
$\theta_3 e_{38}$	$[\mathbf{1}, \theta R, O, \theta E_1, \theta E_2, Y_2]$
$\theta_1\theta_2 e_{38}$	$[\mathbf{1}, R, O, E_1, E_2, Y_2]$
$\theta_2 e_{39}$	$[\mathbf{2}, O, E_2, \theta J_1, Y_1, X_1]$
$\theta_3 e_{39}$	$[\mathbf{2}, \theta E_1, \theta I_+, J, Y_1, X_1]$
$\theta_3\theta_2\sigma_0 e_{39}$	$[\mathbf{1}, O, \theta E_1, J_2, Y_1, \theta X_2]$
$\theta_3\sigma_0 e_{39}$	$[\mathbf{1}, \theta E_2, \theta I_+, J, Y_1, X_2]$
$\theta_1\theta_2 e_{39}$	$[\mathbf{2}, E_1, I_+, \theta J, Y_1, \theta X_1]$
$\theta_1\theta_2\sigma_0 e_{39}$	$[\mathbf{1}, E_2, I_+, \theta J, Y_1, \theta X_2]$
$\theta_1\theta_3 e_{39}$	$[\mathbf{2}, O, \theta E_2, J_1, Y_1, \theta X_1]$
$\theta_1\sigma_0 e_{39}$	$[\mathbf{1}, O, E_1, \theta J_2, Y_1, X_2]$

TABLE 6. Points p and their associated labels $L_p \subset \Lambda$ (2).

Proposition 6.6. *Suppose that I is zero-dimensional and that $\dim P/I = \deg(m(I, X_n; t))$. Then there is a set of polynomials $\{g_i(t)\}$ such that the Gröbner basis with respect to the lexicographic order is*

$$G = \{X_1 - g_1(X_n), X_2 - g_2(X_n), \ldots, X_{n-1} - g_{n-1}(X_n), m(I, X_n; t)\}.$$

$$
\begin{aligned}
\mathcal{C}_0 &= \{e_0, \theta_2 e_0, \theta_1 e_0, \theta_1\theta_2 e_0\} \\
\mathcal{C}_1 &= \{\theta_3 e_1, \theta_1\theta_2 e_1\} \\
\mathcal{C}_2 &= \{\theta_2 e_2, \theta_3\theta_2 e_2, \theta_1 e_2, \theta_1\theta_3 e_2\} \\
\mathcal{C}_3 &= \{e_3, \theta_2 e_3, \theta_3 e_3, \theta_3\theta_2 e_3, \theta_1 e_3, \theta_1\theta_2 e_3, \theta_1\theta_3 e_3, \theta e_3\} \\
\mathcal{C}_4 &= \{e_4, \theta_2\sigma_0 e_4, \theta_3 e_4, \theta_3\theta_2\sigma_0 e_4, \theta_3\sigma_0 e_4, \theta_1 e_4, \theta_1\theta_3 e_4, \sigma_0 e_4, e_5, \theta_1\theta_2 e_5, e_6, \theta_3 e_6\} \\
\mathcal{C}_7 &= \{e_7, \theta_2 e_7, \theta_1 e_7, \theta_1\theta_2 e_7\}, \qquad \theta\mathcal{C}_7 = \{\theta_3 e_7, \theta_3\theta_2 e_7, \theta_1\theta_3 e_7, \theta e_7\} \\
\mathcal{C}_8 &= \{e_8, \theta_2 e_8, \theta_3 e_8, \theta_3\theta_2 e_8, \theta_1 e_8, \theta_1\theta_2 e_8, \theta_1\theta_3 e_8, \theta e_8\} \\
\mathcal{C}_9 &= \{e_9, \theta_2 e_9, \theta_3 e_9, \theta_3\theta_2 e_9, \theta_1 e_9, \theta_1\theta_2 e_9, \theta_1\theta_3 e_9, \theta e_9\} \\
\mathcal{C}_{10} &= \{e_{10}, \theta_3 e_{10}, \theta_1 e_{10}, \theta_1\theta_3 e_{10}, e_{11}, \theta_2 e_{11}, \theta_1 e_{11}, \theta_1\theta_2 e_{11}\} \\
\mathcal{C}_{12} &= \{\theta_2 e_{12}, \theta_1 e_{12}, \theta_3\theta_2\sigma_0 e_{13}, \theta_1\theta_3 e_{13}, \theta_1\theta_2 e_{14}, \theta_3 e_{15}\}, \\
\theta\mathcal{C}_{12} &= \{\theta_3\theta_2 e_{12}, \theta_1\theta_3 e_{12}, \theta_2 e_{13}, \theta_1\sigma_0 e_{13}, \theta_3 e_{14}, \theta_1\theta_2 e_{15}\} \\
\mathcal{C}_{16} &= \{e_{16}, \theta_1 e_{16}, \theta_2 e_{17}, \theta_1\theta_2 e_{17}\}, \qquad \theta\mathcal{C}_{16} = \{\theta_3\theta_2 e_{16}, \theta e_{16}, \theta_3 e_{17}, \theta_1\theta_3 e_{17}\} \\
\mathcal{C}_{18} &= \{e_{18}, \theta_2 e_{18}, \theta_3 e_{18}, \theta_3\theta_2 e_{18}, \theta_1 e_{18}, \theta_1\theta_2 e_{18}, \theta_1\theta_3 e_{18}, \theta e_{18}, \theta_2 e_{19}, \theta_3 e_{19}, \theta_1\theta_2 e_{19}, \theta_1\theta_3 e_{19}, \theta_3 e_{20}, \theta_1\theta_2 e_{20}\} \\
\mathcal{C}_{21} &= \{\theta_2 e_{21}, \theta_3\theta_2 e_{21}, \theta_1\theta_3\sigma_0 e_{21}, \theta_1\sigma_0 e_{21}\}, \qquad \theta\mathcal{C}_{21} = \{\theta_2\sigma_0 e_{21}, \theta_3\theta_2\sigma_0 e_{21}, \theta_1 e_{21}, \theta_1\theta_3 e_{21}\} \\
\mathcal{C}_{22} &= \{\theta_2 e_{22}, \theta_3\theta_2 e_{22}\}, \qquad \theta\mathcal{C}_{22} = \{\theta_1 e_{22}, \theta_1\theta_3 e_{22}\} \\
\mathcal{C}_{23} &= \{e_{23}, \theta_3 e_{23}\}, \qquad \theta\mathcal{C}_{23} = \{\theta_1\theta_2 e_{23}, \theta e_{23}\} \\
\mathcal{C}_{24} &= \{\theta_3 e_{24}, \theta_3\theta_2 e_{24}\}, \qquad \theta\mathcal{C}_{24} = \{\theta_1 e_{24}, \theta_1\theta_2 e_{24}\} \\
\mathcal{C}_{25} &= \{e_{25}, \theta_2 e_{25}, \theta_1 e_{25}, \theta_1\theta_2 e_{25}, \theta_2 e_{26}, \theta_1\theta_2 e_{26}, \theta_1\theta_2\sigma_0 e_{26}, \theta_1\sigma_0 e_{26}\}, \\
\theta\mathcal{C}_{25} &= \{\theta_3 e_{25}, \theta_3\theta_2 e_{25}, \theta_1\theta_3 e_{25}, \theta e_{25}, \theta_3 e_{26}, \theta_3\theta_2\sigma_0 e_{26}, \theta_3\sigma_0 e_{26}, \theta_1\theta_3 e_{26}\} \\
\mathcal{C}_{27} &= \{\theta_3 e_{27}, \theta_3\theta_2 e_{27}\}, \qquad \theta\mathcal{C}_{27} = \{\theta_1 e_{27}, \theta_1\theta_2 e_{27}\} \\
\mathcal{C}_{28} &= \{\theta_2 e_{28}, \theta_3\theta_2 e_{28}\}, \qquad \theta\mathcal{C}_{28} = \{\theta_1 e_{28}, \theta_1\theta_3 e_{28}\} \\
\mathcal{C}_{29} &= \{e_{29}, \theta_3 e_{29}, \theta_1 e_{29}, \theta_1\theta_3 e_{29}, \theta_2 e_{30}, \theta_3\theta_2 e_{30}, \theta_1 e_{30}, \theta_1\theta_3 e_{30}\} \\
\mathcal{C}_{31} &= \{\theta_2 e_{31}, \theta_3\theta_2 e_{31}\}, \qquad \theta\mathcal{C}_{31} = \{\theta_1 e_{31}, \theta_1\theta_3 e_{31}\} \\
\mathcal{C}_{32} &= \{\theta_2 e_{32}, \theta_1 e_{32}, \theta_2 e_{33}, \theta_1\sigma_0 e_{33}, \theta_3 e_{34}, \theta_3\sigma_0 e_{34}\}, \\
\theta\mathcal{C}_{32} &= \{\theta_3\theta_2 e_{32}, \theta_1\theta_3 e_{32}, \theta_3\theta_2\sigma_0 e_{33}, \theta_1\theta_3 e_{33}, \theta_1\theta_2 e_{34}, \theta_1\theta_2\sigma_0 e_{34}\} \\
\mathcal{C}_{35} &= \{\theta_2 e_{35}, \theta_3 e_{36}\}, \qquad \theta\mathcal{C}_{35} = \{\theta_1\theta_3 e_{35}, \theta_1\theta_2 e_{36}\} \\
\mathcal{C}_{37} &= \{\theta_2 e_{37}, \theta_3 e_{38}\}, \qquad \theta\mathcal{C}_{37} = \{\theta_1\theta_3 e_{37}, \theta_1\theta_2 e_{38}\} \\
\mathcal{C}_{39} &= \{\theta_2 e_{39}, \theta_1\theta_2 e_{39}, \theta_1\theta_2\sigma_0 e_{39}, \theta_1\sigma_0 e_{39}\}, \qquad \theta\mathcal{C}_{39} = \{\theta_3 e_{39}, \theta_3\theta_2\sigma_0 e_{39}, \theta_3\sigma_0 e_{39}, \theta_1\theta_3 e_{39}\}
\end{aligned}
$$

TABLE 7. Γ-equivalence sets $\mathcal{C}_k$ and $\theta\mathcal{C}_k$.

So the problem is reduced to obtaining the zeros of the univariate polynomial $m(I, X_n; t)$. The other coordinates of the zeros are obtained as certain values of the univariate polynomial.

The real roots of a one-variable polynomial are fairly easily obtained by Newton's method. So by the proposition above, we obtain all the real zero points of $V(I)$ when I is zero-dimensional.

Returning to our case, the computation of a Gröbner basis with respect to the lexicographic order is sometimes too expensive. In this unfortunate situation, we do a mixed strategy utilizing other kinds of term order. We divide step 2 into the following:

step 2-1. Compute the minimal polynomials $m(I, X_{ij}; t)$, $m(I, Y_{ij}; t)$ and their real roots.

step 2-2. Check $p \in V(I)$ using floating-point computation up to a certain accuracy for all combinations of real roots of the minimal polynomials.

In the system ASIR, this can be done by the command `minipoly`. The real roots are obtained by Newton's method.

To speed up the search, we apply the following pruning. We can skip to the next L if a root violates condition (1) or (2) in Section 2.

6.4. Step 4: Nonempty 0-Cells

We know the points in V_0 of all 0-dimensional ideals I_L by step 2. Then step 3 can be easily carried out by the inclusion-reversing property. In step 4, we decide whether $p \in V_0$ is in W_L. This can be done by a positivity check for f_λ $\lambda \in \Lambda \setminus L$ by a floating-point computation in view of step 3; if a point can attain zero on f_λ, $\lambda \notin L$, this would violate the maximality of L.

Search Result 6.7. *Among the zero-dimensional ideals I_L there are 2146 labels L, so that W_L is nonempty. In fact, one has $|W_L| = |V(I_L) \cap \partial\mathcal{F}_2| \leq 2$.*

Search Result 6.8. *These* 2146 *labels produce* 180 *points on the boundary of the fundamental domain* $V(I_L) \cap \partial\mathcal{F}$ *indicated in Table 5.*

Our search results suggest that Theorem 5.1 is best possible.

7. PROOF OF THEOREM 5.1

Once the point p is explicitly given, to show that the point satisfies the positivity $f_\lambda > 0$, $\lambda \in \Lambda \setminus L$, is straightforward. So the remaining part of of the proof of Theorem 5.1 is to show that $p \in V(I_L)_\mathbb{R}$. This is achieved through a point-by-point investigation, as the following example indicates.

With the result in Section 6.3 in mind, we revisit Procedure 6.1 for the case $p \overset{\Delta}{\sim} e_{12}$. This also clarifies the situation of maximality of labels.

Take $L = [\mathbf{1}, \mathbf{2}, \theta R, \theta E_1, Y_1, Y_2]$. Then I_L can be checked to be zero-dimensional. The minimal polynomial $m(t) = m(I, X_{22}; t)$ of X_{22} is of degree 8. By factorization, X_{22} should be 1/3, ±1/5, or $\Omega_2(-X_{22}) = 0$, so $X_{11} = 1/3$, ±1/5, or $-\omega_2$ as real zeros whose absolute values are less than 1/2.

Consider the case $X_{22} = -\omega_2$. Since variables other than X_{12} depend on X_{22} very simply, what is essential is the dependency on X_{12}. The shape basis says that

$$\begin{aligned}&-40411800X_{22}^7 + 2571225X_{22}^6 + 91990172X_{22}^5\\&\quad + 52140601X_{22}^4 - 53576848X_{22}^3 - 50659293X_{22}^2\\&\quad + 16986764X_{22} - 14988288X_{12} - 4052533 = 0.\end{aligned}$$

Taking this modulo $\Omega_2(-X_{22})$, we obtain the value of X_{12}. In fact, taking the basis corresponding to the graded reverse lexicographical order (`grrevlex`), we may see the basis would show a simpler dependency: $X_{12} = -(1-\omega_2)/2$. In any event, we can obtain $\theta_3\theta_2 e_{12}$. After checking positivity, we conclude that W_L is nonempty. If $X_{22} = \pm 1/5$, this also gives rise to the point $Z \in V(I_L)$, but it would make $f_O(Z)$ negative, so such is not the case. On the other hand, when $X_{22} = 1/3$, it also gives a valid point $\theta_3 e_1 \in W_L$. So $|W_L| = 2$. It turns out that the label L is the maximal label of $\theta_3\theta_2 e_{12}$. Consider the label $L' = [\mathbf{1}, \mathbf{2}, \theta R, \theta E_1, \theta E_2, \theta I_+, J, Y_1, Y_2]$. Then $L \subset L'$ and $W_{L'} = \{\theta_3 e_1\}$. This implies that the maximal label of $\theta_3 e_1$ is L'.

8. Γ-EQUIVALENCE CLASSES OF ZERO CELLS

In this section we consider the Γ-equivalence classes between the 180 0-cells obtained in the previous sections. Define 40 sets $\mathcal{C}_k$ and their θ-images as in Table 7. We remark that when $k = 0, 1, 2, 3, 4, 8, 9, 10, 18, 29$, then $\mathcal{C}_k$ is θ-stable, that is, $\mathcal{C}_k = \theta\mathcal{C}_k$. We discuss Γ-equivalence and Γ-inequivalence of the $\mathcal{C}_k$'s.

8.1. Equivalence Property of $\mathcal{C}_k$

We start with a few lemmas.

Lemma 8.1. *A set* $\mathcal{C}$ *is* Γ*-equivalent if and only if* $\theta\mathcal{C}$ *is.*

Proof. If we put $(\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right))^\theta = \left(\begin{smallmatrix} A & -B \\ -C & D\end{smallmatrix}\right)$, then the result follows by $\theta\gamma Z = \gamma^\theta \theta Z$. □

Lemma 8.2. *The sets* $\mathcal{C}_k$, $\theta\mathcal{C}_k$ *for* $k = 0, 2, 3, 7, 8, 9, 21, 22, 23, 24, 27, 28, 31, 39$ *are* Γ*-equivalent sets.*

Proof. The assertion is clear, because the equivalence inside $\mathcal{C}_k$ or $\theta\mathcal{C}_k$ can be found by σ_0 and the translations of the X_{ij}-coordinates by ±1. □

To show the other equivalence property, we first define

$$\sigma_1^\pm = \begin{pmatrix} I_- + E_{21} & \pm E_2 \\ 0 & I_- + E_{12}\end{pmatrix} \in \Gamma.$$

Lemma 8.3. *The matrix* $\sigma_1^\pm$ *transfers* 0*-cells to* 0*-cells as follows:*

$$\begin{aligned}&\sigma_1^+[\theta_1\theta_3 e_{13}, e_{16}, e_{25}] = [\theta_1\theta_2 e_{14}, \theta_2 e_{17}, \theta_2 e_{26}]\\&\sigma_1^-[e_4, e_{10}, \theta_2 e_{12}, \theta_2 e_{32}, \theta_2 e_{33}, \theta_2 e_{35}, \theta_2 e_{37}],\\&\qquad = [e_5, e_{11}, \theta_3 e_{15}, \theta_3\sigma_0 e_{34}, \theta_3 e_{34}, \theta_3 e_{36}, \theta_3 e_{38}].\end{aligned}$$

Proof. This is easily proved by a direct computation. □

Lemma 8.4. *The sets* $\mathcal{C}_k, \theta\mathcal{C}_k$ *for* $k = 10, 16, 25, 32, 35, 37$ *are* Γ*-equivalent sets.*

Proof. The equivalence of the sets $\mathcal{C}_k$ and $\theta\mathcal{C}_k$ in the assertion follows by Lemmas 8.1 and 8.3 with the help of σ_0 and the translations of X_{ij}-coordinates by ±1. □

Lemma 8.5. *The sets* $\mathcal{C}_k, \theta\mathcal{C}_k$ *for* $k = 1, 4, 12, 29$ *are* Γ*-equivalent sets.*

Proof. The assertion when $k=1$ follows from the fact that

$$\gamma_1 = \begin{pmatrix} E_2 & -I_+ \\ E_1 & E_2 \end{pmatrix}, \quad \gamma_1 \cdot \theta_3 e_1 = \theta\theta_3 e_1.$$

Consider the case $k=4$. We can check that

$$\gamma_4' = \begin{pmatrix} E_{21}+I_- & -E_2 \\ E_2 - E_{12} & E_1 \end{pmatrix}, \quad \gamma_4' \cdot e_4 = e_6.$$

By Lemma 8.3, we have $\sigma_1^- e_4 = e_5$, so the equivalence of this case is now clear. When $k=12$, we need another equality:

$$\gamma_{12} = \begin{pmatrix} E_{21} & E_{12} \\ -E_{12} & E_{21} \end{pmatrix}, \quad \gamma_{12}\theta_2 e_{12} = \theta\theta_1\sigma_0 e_{13}.$$

The equivalence when $k=12$ is now obtained by the help of Lemma 8.3.

To show the equivalence when $k=29$, it is enough to prove the following equality, which is obtained by direct computation:

$$\gamma_{29} = \begin{pmatrix} I_- + E_{21} & -E_2 \\ E_2 - E_{12} & E_1 \end{pmatrix}, \quad \gamma_{29} e_{29} = \theta_2 e_{30}.$$

□

Lemma 8.6. *The set $\mathcal{C}_{18}$ is a Γ-equivalent set.*

Proof. Since X_{ij} of e_{18} are all 1/2, the former eight elements in $\mathcal{C}_{18}$ are all Γ-equivalent. In particular, we have $e_{18} \overset{\Gamma}{\sim} \theta e_{18}$. Moreover, $\gamma_{18} e_{18} = \theta_1\theta_3 e_{19}$ with $\gamma_{18} = \left(\begin{smallmatrix} E_2 & E_1 \\ -E_1 & E_2 \end{smallmatrix}\right)$. Thus $e_{18} \overset{\Gamma}{\sim} \theta_1\theta_3 e_{19}$. This means that

$$\theta_2 e_{19} = \theta\theta_1\theta_3 e_{19} = \theta\gamma_{18} e_{18} = \gamma_{18}^{\theta}\theta e_{18} \overset{\Gamma}{\sim} e_{18}.$$

Since $x_{11} = 1/2$ for e_{19}, we have $e_{19} \overset{\Gamma}{\sim} \theta_1 e_{19}$. Hence

$$\theta_1\theta_2 e_{19} \overset{\Gamma}{\sim} \theta_2 e_{19} \overset{\Gamma}{\sim} e_{18}$$

and $\theta_3 e_{19} \overset{\Gamma}{\sim} \theta_1\theta_3 e_{19} \overset{\Gamma}{\sim} e_{18}$.

We have to fix the cases $\theta_3 e_{20}$ and $\theta_1\theta_2 e_{20}$. Since $\theta_2 e_{19} \overset{\Gamma}{\sim} e_{18}$, for $\theta_3 e_{20}$ we apply the equality

$$\gamma_{18}'\theta_2 e_{19} = \theta_3 e_{20} \quad \text{for } \gamma_{18}' = \begin{pmatrix} E_1 & E_2 \\ -E_2 & E_1 \end{pmatrix}.$$

Finally,

$$\begin{aligned} \theta_1\theta_2 e_{20} &= \theta(\theta_3 e_{20}) = \theta\gamma_{18}'\theta_2 e_{19} = (\gamma_{18}')^{\theta}\theta\theta_2 e_{19} \\ &= (\gamma_{18}')^{\theta}\theta_1\theta_3 e_{19} \overset{\Gamma}{\sim} e_{18}. \end{aligned}$$

□

degree/$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\mathcal{C})$	$\mathcal{C}$
1	$\mathbb{Q}(\sqrt{-3})$	$\mathcal{C}_0, \mathcal{C}_3$
2	$\mathbb{Q}(\sqrt{-3}, \sqrt{-1})$	$\mathcal{C}_7, \theta\mathcal{C}_7$
2	$\mathbb{Q}(\omega_4\sqrt{-1})$	$\mathcal{C}_9$
2	$\mathbb{Q}(\omega_5\sqrt{-1})$	$\mathcal{C}_{10}$
4	$\mathbb{Q}(\omega_6 + \sqrt{-1})$	$\mathcal{C}_2$
4	$\mathbb{Q}(\omega_7 + \sqrt{-3})$	$\mathcal{C}_{16}, \theta\mathcal{C}_{16}$
4	$\mathbb{Q}(\omega_{21}\sqrt{-1})$	$\mathcal{C}_{18}$
4	$\mathbb{Q}(\nu_{15})$	$\mathcal{C}_{22}, \theta\mathcal{C}_{22}$
8	$\mathbb{Q}(\omega_{27}\sqrt{-1})$	$\mathcal{C}_{29}$
8	$\mathbb{Q}(\nu_{29})$	$\mathcal{C}_{32}, \theta\mathcal{C}_{32}$
16	$\mathbb{Q}(\nu_{31})$	$\mathcal{C}_{35}, \theta\mathcal{C}_{35}$
18	$\mathbb{Q}(\nu_{33})$	$\mathcal{C}_{37}, \theta\mathcal{C}_{37}$

TABLE 8. The field $\mathbb{Q}(\mathcal{C})$ that contains $\sqrt{-3}$. Here ν_k is such that $|\nu_k| = 1$ and $\mathrm{Re}(\nu_k) = \omega_k$.

Summing up, we obtain the following theorem.

Theorem 8.7. *Let $\mathcal{C}_k$ be the set defined as in Table 7 and let $\theta\mathcal{C}_k$ be its θ-image. Then every $\mathcal{C}_k$ and every $\theta\mathcal{C}_k$ is a Γ-equivalence set.*

Corollary 8.8. *The upper bound of the number of Γ-equivalence classes $W_0/\overset{\Gamma}{\sim}$ is* 40.

8.2. Inequivalence Property of $\mathcal{C}_k$

For $Z \in \mathbb{H}_2$, let $\mathbb{Q}(Z)$ denote $\mathbb{Q}(Z_{11}, Z_{12}, Z_{22})$, the field generated by the point Z.

The following lemma is clear.

Lemma 8.9. *If $\gamma Z = Z'$ for a $\gamma \in \Gamma$, then $\mathbb{Q}(Z) = \mathbb{Q}(Z')$.*

By Lemma 8.9, $\mathbb{Q}(\mathcal{C})$ for the equivalence set $\mathcal{C}$ is well defined. We list these fields in Tables 8 and 9.

Proposition 8.10. *The lower bound of the number of Γ-equivalence classes $W_0/\overset{\Gamma}{\sim}$ is* 25.

Proof. Since one can directly check that the fields listed in Tables 8 and 9 are all distinct, the possible number of Γ-equivalence classes is greater than or equal to 24 by Lemma 8.9.

To show that the lower bound is 25, we prove that $\mathcal{C}_0$ and $\mathcal{C}_3$ are inequivalent by contradiction. Suppose there is an invertible integral matrix in Γ that transfers e_0 to

degree/$\mathbb{Q}$	$\mathbb{Q}(\mathcal{C})$	$\mathcal{C}$
2	$\mathbb{Q}(\sqrt{-1})$	$\mathcal{C}_4$
2	$\mathbb{Q}(\sqrt{-2})$	$\mathcal{C}_1$
4	$\mathbb{Q}(\omega_3\sqrt{-1})$	$\mathcal{C}_8$
6	$\mathbb{Q}(\nu_1)$	$\mathcal{C}_{12}, \theta\mathcal{C}_{12}$
8	$\mathbb{Q}(\nu_{14})$	$\mathcal{C}_{21}, \theta\mathcal{C}_{21}$
8	$\mathbb{Q}(\nu_{17})$	$\mathcal{C}_{23}, \theta\mathcal{C}_{23}$
10	$\mathbb{Q}(\nu_{20})$	$\mathcal{C}_{24}, \theta\mathcal{C}_{24}$
12	$\mathbb{Q}(\omega_{19}\sqrt{-1})$	$\mathcal{C}_{25}, \theta\mathcal{C}_{25}$
12	$\mathbb{Q}(\nu_{22})$	$\mathcal{C}_{27}, \theta\mathcal{C}_{27}$
16	$\mathbb{Q}(\nu_{28})$	$\mathcal{C}_{31}, \theta\mathcal{C}_{31}$
24	$\mathbb{Q}(\nu_{25})$	$\mathcal{C}_{28}, \theta\mathcal{C}_{28}$
36	$\mathbb{Q}(\nu_{36})$	$\mathcal{C}_{39}, \theta\mathcal{C}_{39}$

TABLE 9. The field $\mathbb{Q}(\mathcal{C})$ that does not contain $\sqrt{-3}$.

e_3. Then

$$(Ae_0 + B)(Ce_0 + D)^{-1} = e_3$$

and

$$\eta A + B = \begin{pmatrix} \eta - 1 & \eta/2 \\ \eta/2 & \eta - 1 \end{pmatrix} C + \begin{pmatrix} \eta & 1/2 \\ 1/2 & \eta \end{pmatrix} D$$

for $\eta = e^{\pi i/3}$ and integral 2×2 matrices A, B, C, and D. By extraction, we readily find that both matrices C and D must have all entries even integers, a contradiction. □

8.3. Automorphic Functions and the Inequivalence Property: A Conjecture

We conjecture that the all $\mathcal{C}_j$ are inequivalent, which should mean that $|W_0/\underset{\sim}{\Gamma}| = 40$. To show this inequivalence property between C_j's, we should consider the case $k = 7, 12, 26, 21, 22, 23, 24, 25, 27, 28, 31, 32, 35, 37, 39$. They are actually inequivalences between $\mathcal{C}$ and its θ-image. In this case, however, $\mathbb{Q}(\mathcal{C})$ coincides with $\mathbb{Q}(\theta\mathcal{C})$, so we need additional information. Here we indicate a possible method to solve this question.

Let J_1, J_2, J_3 be the Igusa generators of the modular function fields $\mathbb{C}(J_1, J_2, J_3)$ of the Siegel modular variety $\Gamma\backslash\mathbb{H}_2$, which are the quotients of modular forms with adequate weights such that all the Fourier coefficients belong to $\mathbb{Q}$ [Igusa 62]. Then if e_j and θe_j are Γ-equivalent, one has

$$J(e_j) = J(\theta e_j) = \overline{J(e_j)}, \quad J = J_1, J_2, J_3.$$

This means that $e_j \in \Gamma\backslash\mathbb{H}_2$ defines a real point on the canonical model of $\mathbb{Q}(J_1, J_2, J_3)$. Therefore, our conjecture is equivalent to the claim that for each j, there exists i such that $J_i(e_j) \notin \mathbb{R}$. We can check this condition if we can compute the numerical value of $J_i(e_j)$ with high-enough precision, applying the method of verified numerical computation [Rump et al. 08] if necessary.

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Takahiro Hayata, Department of Informatics, Faculty of Engineering, Yamagata University, 4-3-16, Jyonan, Yonezawashi, 992-8510, Japan (hayata@yz.yamagata-u.ac.jp)

Takayuki Oda, Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1, Komaba, Meguroku, Tokyo, 153-8914, Japan (takayuki@ms.u-tokyo.ac.jp)

Tomoki Yatougo, Department of Informatics, Faculty of Engineering, Yamagata University, 4-3-16, Jyonan, Yonezawashi, 992-8510, Japan (taw08385@st.yamagata-u.ac.jp)