Taylor & Francis
Taylor & Francis Group

Extended Torelli Map to the Igusa Blowup in Genus 6, 7, and 8

Valery Alexeev, Ryan Livingston, Joseph Tenini, Maxim Arap, Xiaoyan Hu, Lauren Huckaba, Patrick McFaddin, Stacy Musgrave, Jaeho Shin, and Catherine Ulrich

CONTENTS

It was conjectured in [Namikawa 73] that the Torelli map $M_g \to A_g$ associating to a curve its Jacobian extends to a regular map from the Deligne–Mumford moduli space of stable curves $\overline{M}_g$ to the (normalization of the) Igusa blowup $\overline{A}_g^{\mathrm{cent}}$. A counterexample in genus $g = 9$ was found in [Alexeev and Brunyate 11]. Here, we prove that the extended map is regular for all $g \le 8$, thus completely solving the problem in every genus.

1. INTRODUCTION

The Torelli map $M_g \to A_g$ associates to a smooth curve C its Jacobian JC, a principally polarized abelian variety. Does it extend to a regular map $\overline{M}_g \to \overline{A}_g$, where $\overline{M}_g$ is the Deligne–Mumford moduli space of stable curves, and $\overline{A}_g$ is a toroidal compactification of A_g?

This question was first asked in the pioneering paper [Namikawa 73] in the case that $\overline{A}_g = \overline{A}_g^{\mathrm{cent}}$ is the normalization of the Igusa blowup $\mathrm{Bl}_{\partial A_g^*} A_g^*$ of the Satake compactification along the boundary. This compactification was introduced in [Igusa 67], and is possibly the first toroidal compactification ever constructed. It corresponds to the central cone decomposition.

Namikawa conjectured that the extended map is always regular. He was able to prove it for the stable curves with a planar dual graph, and for curves of low genus; the case $g \le 6$ was stated without proof. (Note: the graphs in this paper may have multiple edges and loops.)

The question was recently revisited in [Alexeev and Brunyate 11], where the following was shown:

(1) Let C be a stable curve of genus g, and let Γ be its dual graph. Then the rational map $\overline{M}_g \to \overline{A}_g^{\mathrm{cent}}$ is regular in a neighborhood of the point $[C] \in \overline{M}_g$ if and only if there exists a positive definite integral-valued quadratic form q on the first cohomology $H^1(\Gamma, \mathbb{Z})$ such that $q(e_i^*) = 1$ for every non-bridge edge e_i of Γ. Such

quadratic forms q are called *integral edge-minimizing metrics* or $\mathbb{Z}$-*emms* for short.

Recall that for a graph, one has

$$H^1(\Gamma) = C^1(\Gamma)/dC^0(\Gamma),$$

where

$$C^1(\Gamma, \mathbb{Z}) = \oplus_{\text{edges } e_i} \mathbb{Z}e_i^*, \quad C^0(\Gamma, \mathbb{Z}) = \oplus_{\text{vertices } v_j} \mathbb{Z}v_j^*,$$

and

$$dv_j^* = \sum_{e_i \text{ begins with } v_j} e_i^* - \sum_{e_i \text{ ends with } v_j} e_i^*.$$

We denote the image of e_i^* in $H^1(\Gamma, \mathbb{Z})$ by the same letter e_i^* and call it a *coedge*.

(2) Call a graph *cohomology-irreducible* if there does not exist a partition of its edges into two groups $I_1 \sqcup I_2$ such that $H^1(\Gamma, \mathbb{Z}) = \langle e_i^*, \ i \in I_1 \rangle \oplus \langle e_i^*, \ i \in I_2 \rangle$. Then Γ is either a simple loop (a graph with one vertex and one edge), or Γ is loopless and 2-connected.

For every graph Γ, one has $H^1(\Gamma, \mathbb{Z}) = \oplus H^1(\Gamma_k, \mathbb{Z})$ for some cohomology-irreducible graphs Γ_k, and all coedges e_i^* lie in the direct summands. We call G_k *cohomology-irreducible components of* Γ. Then there exists a $\mathbb{Z}$-emm for Γ if and only if there exist $\mathbb{Z}$-emms for all Γ_k.

(3) If a graph Γ is cohomology-irreducible and q is a $\mathbb{Z}$-emm for Γ, then the lattice $(H^1(\Gamma, \mathbb{Z}), 2q)$ is a root lattice of type A_g, D_g ($g \geq 4$), or E_g ($g = 6, 7, 8$). Further, there exists a $\mathbb{Z}$-emm of type A_g if and only if Γ is planar, and for $g \geq 4$, there exists a $\mathbb{Z}$-emm of type D_g if and only if Γ is projective planar, i.e., can be embedded in the projective plane $P = \mathbb{RP}^2$.

The famous theorem of Kuratowski says that a graph is nonplanar if and only if it contains a subgraph homeomorphic either to K_5 or to $K_{3,3}$. A Kuratowski-type theorem for the projective plane P was proved in [Archdeacon 81, Archdeacon 80], where it was shown that the list of 103 minimal nonprojective planar graphs produced earlier in [Glover et al. 79] is complete; any

other nonprojective planar graph contains a subgraph homeomorphic to one of them. The smallest graph on their list has genus 6.

This implies that every graph of genus less than or equal to 5 has a $\mathbb{Z}$-emm, and consequently, the extended Torelli map $\overline{M}_g \to \overline{A}_g^{\text{cent}}$ is regular for $g \leq 5$. On the other hand, as noted in [Alexeev and Brunyate 11], there exist cohomology-irreducible nonprojective planar graphs of genus 9, so the extended Torelli map is *not* regular for every $g \geq 9$.

Here are the main results of this paper:

Theorem 1.1. *Let Γ be a cohomology-irreducible nonprojectively planar graph of genus $g = 6$, 7, or 8. Then Γ admits a $\mathbb{Z}$-emm of type E_g.*

Corollary 1.2. *The extended Torelli map $\overline{M}_g \to \overline{A}_g^{\text{cent}}$ is regular for $g \leq 8$.*

Corollary 1.3. *Let C be a stable curve of genus g, and Γ its dual graph. Then the extended Torelli map $\overline{M}_g \to \overline{A}_g^{\text{cent}}$ is regular in a neighborhood of the point $[C] \in \overline{M}_g$ if and only if every cohomology-irreducible component Γ_k has genus ≤ 8 or is a projectively planar graph of genus ≥ 9.*

The plan of the paper is as follows. In Section 2, we reduce the proof of Theorem 1.1 to checking finitely many graphs: one graph for $g = 6$, 14 graphs for $g = 7$, and 2394 graphs for $g = 8$. In Section 3, we give a finite algorithm for an arbitrary graph, and then run it for the only graph needed in genus 6. In Section 4, we give the 14 graphs in genus 7 that have to be checked, and explicitly list a $\mathbb{Z}$-emm for each of them. In Section 5, we state our computer-aided findings for genus 8.

2. REDUCTION TO FINITELY MANY GRAPHS

As noted in [Alexeev and Brunyate 11, Section 2], for the proof of Theorem 1.1 we may reduce to graphs that are trivalent. So let H be a cohomology-irreducible
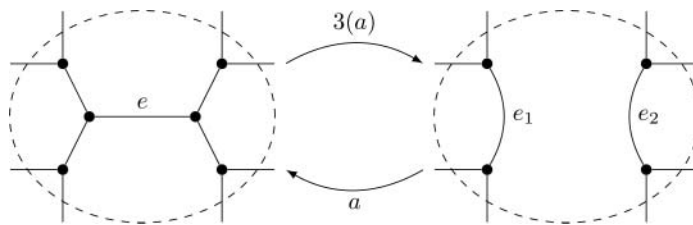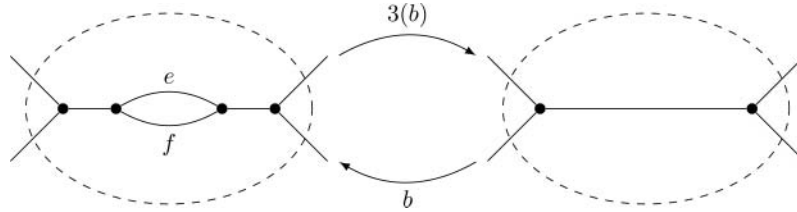

FIGURE 1. The procedures (3a) and (a).

FIGURE 2. The procedures (3b) and (b).

nonprojectively planar trivalent graph of genus $g = 6, 7$, or 8. One says that H is *irreducible with respect to P* if H does not embed into P, but for any edge e in H, $H - e$ does embed into P. We now describe a process that will reduce H to a trivalent graph irreducible with respect to P. The operations (3a), (3b), (3c) are illustrated in Figures 1, 2, 3.

1. If the graph is irreducible with respect to P, stop and call this graph H'.

2. If not, choose an edge e such that $H - e$ does not embed into P and delete e from the graph.

(3a) If e was not a loop and did not have a parallel edge, then, denoting by v_1 and v_2 the distinct vertices to which e is incident, contract an edge incident to v_1 and an edge incident to v_2.

(3b) If e was not a loop but had a parallel edge f, then, denoting by v_1 and v_2 the distinct vertices to which e and f are incident, contract the edge incident to v_1 and different from f and the edge incident to v_2 and different from f.

(3c) If e was a loop incident to v, then delete the remaining edge f incident to v and, denoting by w the other vertex to which f is incident, contract one of the other two edges incident to w and different from f.

Notice that the above operations (3a), (3b), (3c) reduce the genus of the graph by 1 except for operation (3a) when e is a bridge. Repeating this process, we get a

graph H' irreducible with respect to P that is of the form $H' = \widetilde{H} \cup \{u_1, \ldots, u_k\}$ where the u_i are isolated vertices and $\widetilde{H}$ is a trivalent graph irreducible with respect to P. By [Glover and Huneke 75, Milgram 73] (see also [Archdeacon 81, Archdeacon 80]), $\widetilde{H}$ is isomorphic to one of the following:

(i) The connected graph G of genus 6 shown in Figure 4.

(ii) The connected graphs $F_{11}, F_{12}, F_{13}, F_{14}$ of genus 7 shown in Figures 6–9.

(iii) The graph E_{42} shown in Figure 5.

Thus, we may construct H from $\widetilde{H}$ by reversing the algorithm above. We make this explicit for the relevant genera 6, 7, and 8.

2.1.   *H* has Genus 6

Since H is cohomology-irreducible, it has no bridges, and so operations (3a), (3b), and (3c) would all drop the genus. Thus H is already irreducible with respect to P, and so $H = \widetilde{H} = G$. Thus, to show the existence of $\mathbb{Z}$-emms for graphs of genus 6, it suffices to produce one for G.

2.2.   *H* has Genus 7

Either $\widetilde{H}$ equals one of $F_{11}, F_{12}, F_{13}, F_{14}$, or else $\widetilde{H} = G$. In the first case, we have that H is equal to one of $F_{11}, F_{12}, F_{13}, F_{14}$ (again since H was cohomology-irreducible, thus bridgeless). The second case is slightly more complicated. First notice that H' has at most one
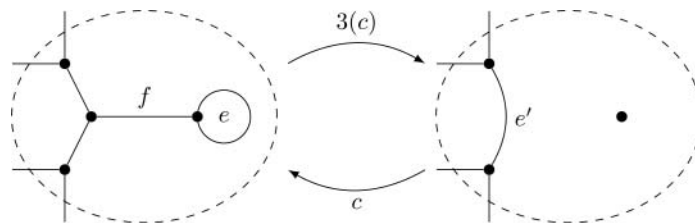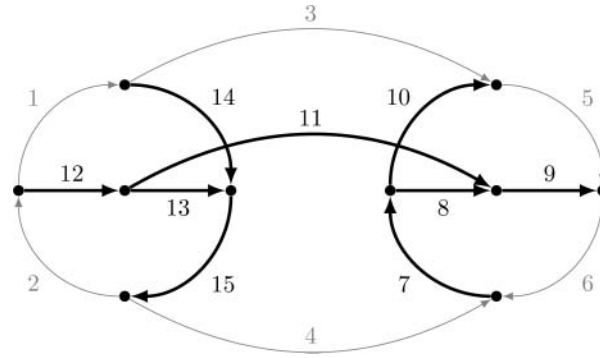

FIGURE 3. The procedures (3c) and (c).

FIGURE 4. The graph G.

isolated vertex v, because in the case of applying (3c), the genus drops by 1. Then H may be obtained from $\widetilde{H}$ by doing one of the following three operations—notice that (a), (b), and (c) are the inverse operations of (3a), (3b), and (3c) (defined above) respectively:

(a) Choose two distinct edges e_1 and e_2 and add an edge from the midpoint of e_1 to the midpoint of e_2.

(b) Choose an edge and add a handle to it.

(c) Choose an edge e' and add an edge f from the midpoint of e' to the isolated vertex v. Then add a loop e to v.

In the case (c), f is a bridge, and so we do not need to consider graphs acquired from $\widetilde{H}$ from operation (c).

A careful but elementary analysis shows that the cases (a) and (b), up to symmetries, produce ten possible graphs for H. We denote these graphs by $G_1, \ldots, G_{10}$. They appear in Figures 10–19. Thus, to show the existence of $\mathbb{Z}$-emms for graphs of genus 7, it suffices to produce one for $F_{11}, F_{12}, F_{13}, F_{14}$ and G_i for $i \in \{1, \ldots, 10\}$.

2.3.   *H* has Genus 8

Since H is cohomology-irreducible, the graphs H and $\widetilde{H}$ cannot be isomorphic to E_{42}: otherwise, H would have genus ≥ 10.

We may choose an edge e such that $H - e$ does not embed into P. Since e is not a bridge, we may construct a trivalent graph $\mathrm{Simp}(H - e)$ from $H - e$ by contracting edges that were incident to e, as in (3a) or (3b). So $\mathrm{Simp}(H - e)$ is a trivalent graph of genus 7 that does not embed into P. Hence by our above argument, $\mathrm{Simp}(H - e)$ is isomorphic to one of $F_{11}, F_{12}, F_{13}, F_{14}, G_i$ for $i \in \{1, \ldots, 10\}$, or to a graph G' obtained from G by choosing an edge e', adding an edge f from the midpoint of e' to an isolated vertex v, and then adding a loop e to v, as in (c).

In the latter case, H is obtained from the graph G by performing operation (c) and then (a). But equivalently, this can be accomplished by the operations (a) and then (b). Thus, to prove Theorem 1.1 for $g = 8$, it is sufficient to find $\mathbb{Z}$-emms for the finitely many graphs obtained from one of the graphs F_{11}–F_{14}, G_1–G_{10} by performing one operation of type (a) or (b).
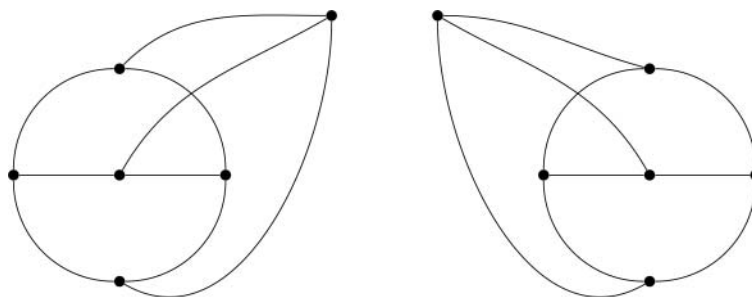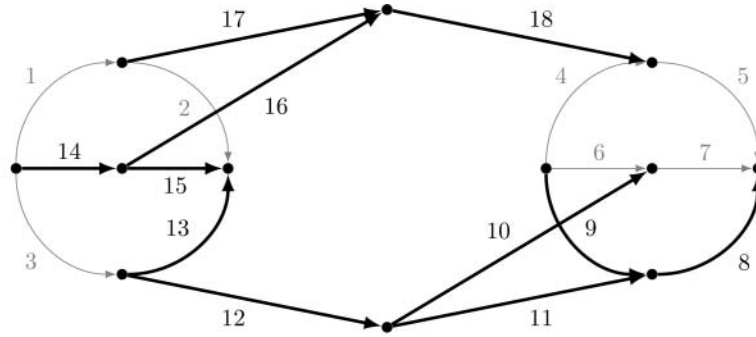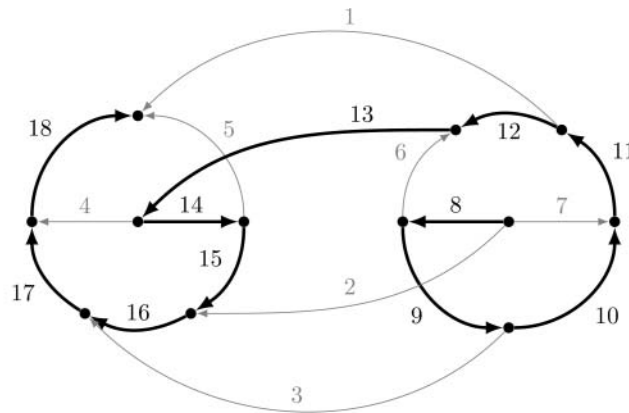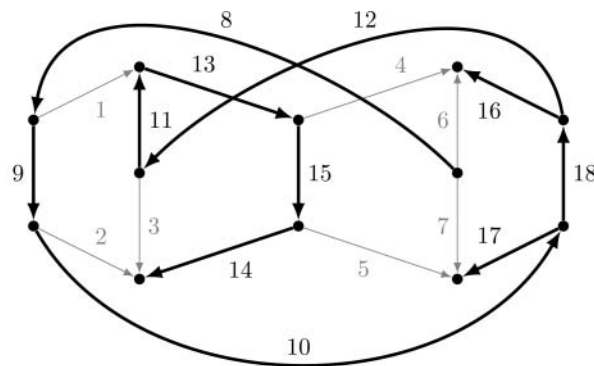

FIGURE 5. The graph E_{42}.

$$q_{F_{11}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 + x_1 x_2 - x_1 x_3 - x_1 x_6 - x_1 x_7 + x_2^2 + x_2 x_4 - x_2 x_6 + x_3^2 + x_3 x_4 + x_3 x_7 + x_4^2$$
$$+ x_4 x_5 - x_4 x_6 + x_5^2 - x_5 x_6 - x_5 x_7 + x_6^2 + x_6 x_7 + x_7^2$$
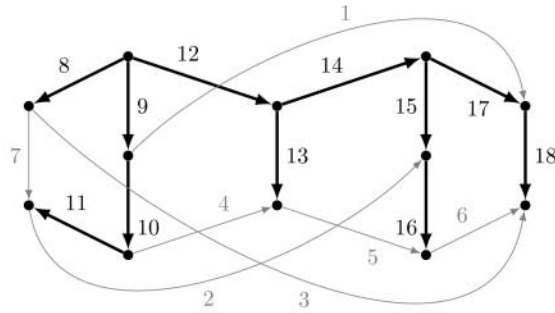
FIGURE 6. The graph F_{11}.

$$q_{F_{12}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1 x_3 - x_1 x_5 + x_2^2 - x_2 x_3 - x_2 x_4 + x_2 x_5 - x_2 x_7 + x_3^2 - x_3 x_6 + x_3 x_7 + x_4^2$$
$$- x_4 x_5 + x_5^2 + x_6^2 - x_6 x_7 + x_7^2$$
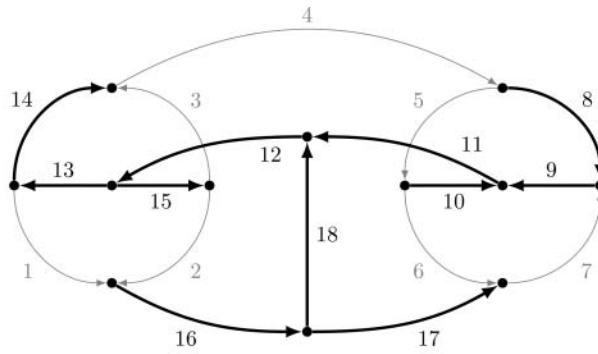
FIGURE 7. The graph F_{12}.

$$q_{F_{13}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 + x_1 x_5 - x_1 x_7 + x_2^2 - x_2 x_3 + x_2 x_5 - x_2 x_6 + x_3^2 + x_4^2 - x_4 x_5 - x_4 x_6 + x_4 x_7$$
$$+ x_5^2 - x_5 x_7 + x_6^2 - x_6 x_7 + x_7^2$$
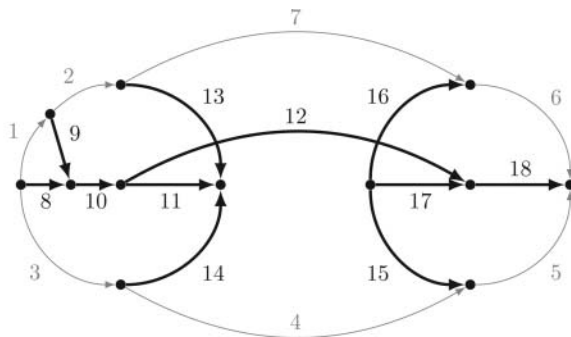
FIGURE 8. The graph F_{13}.

$$q_{F_{14}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1x_2 - x_1x_3 + x_2^2 - x_2x_4 - x_2x_5 + x_2x_7 + x_3^2 - x_3x_6 - x_3x_7 + x_4^2 + x_4x_5$$
$$+ x_5^2 + x_5x_6 - x_5x_7 + x_6^2 + x_7^2$$

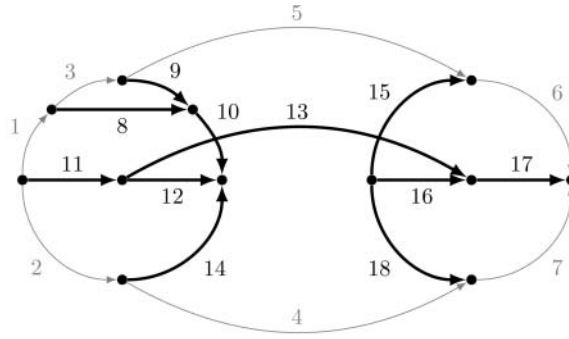**FIGURE 9.** The graph F_{14}.

$$q_{G_1}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1x_2 + x_1x_3 + x_1x_5 + x_1x_6 + x_1x_7 + x_2^2 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3^2$$
$$+ x_3x_4 + x_3x_5 + x_3x_6 + x_4^2 + x_4x_5 - x_4x_7 + x_5^2 + x_5x_6 + x_6^2 + x_6x_7 + x_7^2$$
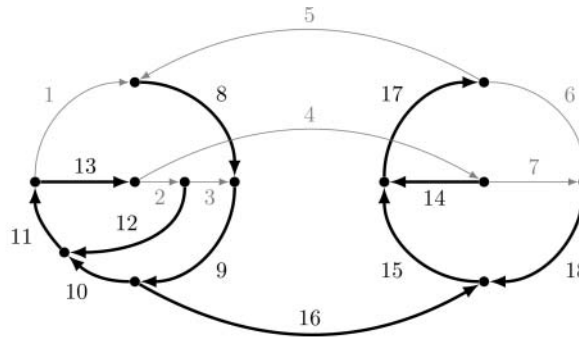
FIGURE 10. The graph G_1.

$$q_{G_2}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 + x_1x_2 - x_1x_3 + x_2^2 - x_2x_3 - x_2x_4 + x_2x_7 + x_3^2 + x_3x_4 + x_4^2 + x_4x_5 - x_4x_6$$
$$- x_4x_7 + x_5^2 - x_5x_6 + x_6^2 + x_6x_7 + x_7^2$$
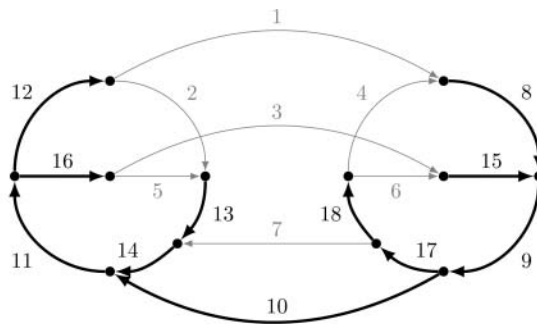
FIGURE 11. The graph G_2.

$$q_{G_3}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1 x_2 + x_1 x_3 - x_1 x_4 + x_1 x_5 + x_2^2 + x_2 x_4 + x_3^2 + x_3 x_5 + x_4^2 - x_4 x_5 - x_4 x_6$$
$$+ x_4 x_7 + x_5^2 + x_5 x_6 + x_6^2 - x_6 x_7 + x_7^2$$

FIGURE 12. The graph G_3.

$$q_{G_4}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1 x_4 - x_1 x_5 - x_1 x_7 + x_2^2 + x_2 x_3 - x_2 x_4 - x_2 x_5 + x_3^2 - x_3 x_4 - x_3 x_5 + x_4^2$$
$$+ x_4 x_5 + x_4 x_7 + x_5^2 - x_5 x_6 + x_5 x_7 + x_6^2 - x_6 x_7 + x_7^2$$

FIGURE 13. The graph G_4.

$$q_{G_5}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1 x_2 - x_1 x_4 + x_1 x_7 + x_2^2 - x_2 x_5 + x_3^2 - x_3 x_5 - x_3 x_6 + x_3 x_7 + x_4^2 - x_4 x_6 + x_5^2$$
$$+ x_5 x_6 - x_5 x_7 + x_6^2 - x_6 x_7 + x_7^2$$

FIGURE 14. The graph G_5.

$$q_{G_6}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1x_2 - x_1x_3 + x_1x_4 + x_1x_6 + x_2^2 - x_2x_4 + x_2x_5 - x_2x_6 + x_3^2 - x_3x_5 - x_3x_6$$
$$- x_3x_7 + x_4^2 - x_4x_7 + x_5^2 + x_6^2 + x_6x_7 + x_7^2$$

FIGURE 15. The graph G_6.

3. GENUS 6

In this section, we explain the general method for finding a $\mathbb{Z}$-emm for any graph, and illustrate it in the case of the trivalent genus-6 graph G.

3.1. Procedure for a General Graph

Let Γ be a directed graph of genus g with edge set $E = \{e_1, \ldots, e_n\}$. After renaming the edges, we may insist that the edges $\{e_{g+1}, \ldots, e_n\}$ induce a spanning tree T of Γ. Then for each e_i with $i \in \{1, \ldots, g\}$, we have a

corresponding basis element f_i of the homology group $H_1(\Gamma, \mathbb{Z})$, given by

$$f_i = e_i + \sum_{e_s \in T} b_{i,s} e_s, \quad b_{i,s} = 0, \pm 1, i \in \{1, \ldots, g\},$$

and the coedges $e_1^*, \ldots, e_g^*$ form a basis of the cohomology group $H^1(G, \mathbb{Z})$ (cf. [Alexeev and Brunyate 11, Lemma 2.3]).

Specifically, f_i is given by the unique simple cycle in Γ that uses only the edge e_i and edges of T. If we write the vectors f_i as the rows of a $g \times n$ matrix, then the columns

$$q_{G_7}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1x_2 - x_1x_4 - x_1x_5 + x_1x_6 + x_2^2 - x_2x_3 + x_2x_5 - x_2x_6 - x_2x_7 + x_3^2 - x_3x_5 + x_4^2$$
$$+ x_4x_5 - x_4x_6 + x_5^2 - x_5x_6 + x_6^2 + x_6x_7 + x_7^2$$

FIGURE 16. The graph G_7.
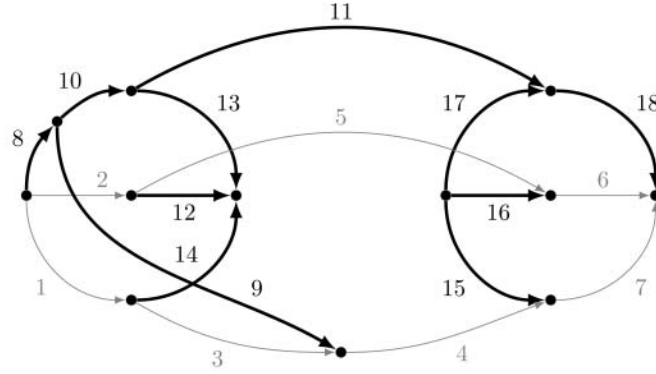
$$q_{G_8}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_5 + x_3^2 + x_3 x_4 + x_4^2 - x_4 x_5 - x_4 x_6 + x_4 x_7 + x_5^2$$
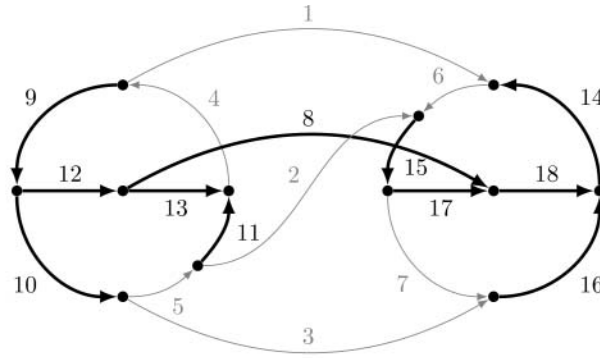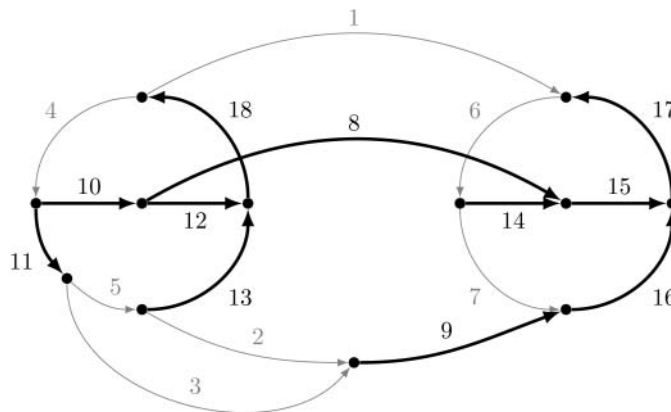$$+ x_5 x_6 + x_6^2 - x_6 x_7 + x_7^2$$

FIGURE 17. The graph G_8.

of this matrix are the coedges $e_i^* \in H^1(G, \mathbb{Z})$ written in the basis $\{e_1^*, \ldots, e_g^*\}$. In particular, the first g columns form an identity matrix.

Let q be a $\mathbb{Z}$-emm for Γ. Since q is a $\mathbb{Z}$-valued quadratic form, we may associate to q an even integral matrix $M_q = (a_{i,j})$ such that

$$q(x_1, \ldots, x_g) = (x_1, \ldots, x_g) \frac{1}{2} M_q (x_1, \ldots, x_g)^T.$$

Note here that $a_{i,j} = a_{j,i}$ is just the coefficient of the term $x_i x_j$ in $q(x_1, \ldots, x_g)$ if $i \neq j$, and $a_{i,i}$ is just twice the coefficient of the term x_i^2 in $q(x_1, \ldots, x_g)$.

We need to enforce the condition that $q(e_i^*) = 1$ for $i = 1, \ldots, n$. To ensure that $q(e_i^*) = 1$ for $i = 1, \ldots, g$, we must have $a_{i,i} = 2$. Now we must ensure that $q(e_i^*) = 1$ for $i = g+1, \ldots, n$. This is equivalent to $n - g$ linear equations on $a_{i,j}$:

$$1 = \sum_{i=1}^{g} c_i^2 + \sum_{1 \leq i < j \leq g} c_i c_j a_{i,j} \quad \text{for each column } (c_i).$$

Further, the condition that q is positive definite implies that each $a_{i,j}$ is in $\{0, \pm 1\}$. Thus, for any given graph, we have reduced the problem to a finite computation.
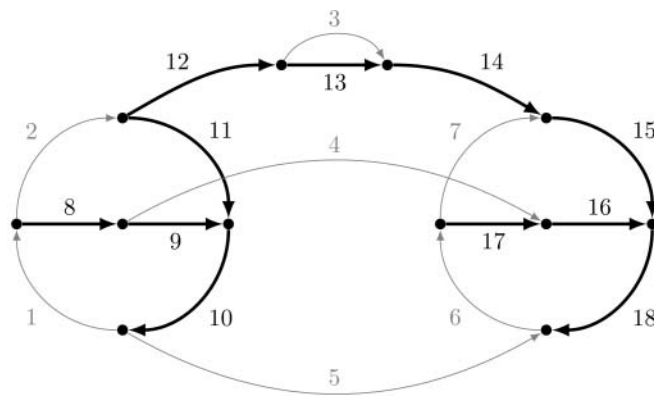
$$q_{G_9}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 + x_1 x_2 + x_1 x_4 - x_1 x_5 + x_2^2 + x_2 x_3 - x_2 x_5 + x_3^2 - x_3 x_4 + x_3 x_6 + x_4^2 - x_4 x_5$$
$$- x_4 x_6 + x_5^2 + x_5 x_6 + x_5 x_7 + x_6^2 + x_6 x_7 + x_7^2$$

FIGURE 18. The graph G_9.
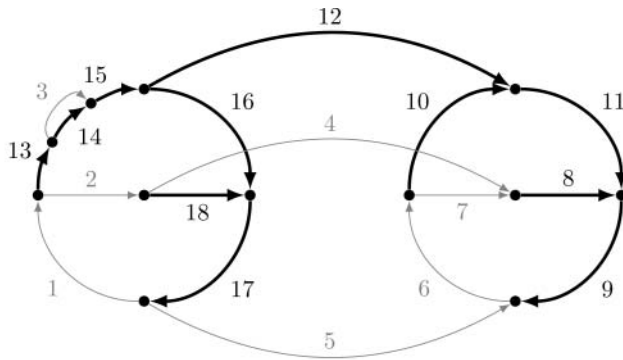
$$q_{G_{10}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 + x_1 x_2 + x_1 x_3 + x_1 x_4 - x_1 x_5 + x_2^2 + x_2 x_4 + x_3^2 + x_4^2 - x_4 x_5 - x_4 x_7 + x_5^2$$
$$+ x_5 x_6 + x_5 x_7 + x_6^2 + x_6 x_7 + x_7^2$$

FIGURE 19. The graph G_{10}.

3.2.   Computation for Graph G

We now specialize to the graph G. In Figure 4 it is shown as a labeled directed graph with a spanning tree denoted by bold edges.

Using the spanning tree drawn and the process described above, we get a basis for $H_1(G, \mathbb{Z})$, written as the rows of the matrix shown in Table 1.

The linear equations become

(1) $1 = 2 + a_{4,6}$,

(2) $1 = 4 + a_{3,4} - a_{3,5} + a_{3,6} - a_{4,5} + a_{4,6} - a_{5,6}$,

(3) $1 = 2 - a_{5,6}$,

(4) $1 = 2 - a_{3,5}$,

(5) $1 = 2 + a_{3,4}$,

(6) $1 = 2 - a_{1,2}$,

(7) $1 = 4 - a_{1,2} - a_{1,3} - a_{1,4} + a_{2,3} + a_{2,4} + a_{3,4}$,

(8) $1 = 2 - a_{1,3}$,

(9) $1 = 2 + a_{2,4}$.

So equations (1), (3), (4), (5), (6), (8), (9) immediately imply that $1 = a_{5,6} = a_{3,5} = a_{1,2} = a_{1,3}$ and $-1 = a_{4,6} =$

$a_{3,4} = a_{2,4}$. Applying this information to (2) and (7), we get $1 = a_{3,6} - a_{4,5}$ and $1 = a_{2,3} - a_{1,4}$ respectively. Let us arbitrarily choose $a_{3,6} = a_{2,3} = 1$ and $a_{4,5} = a_{1,4} = 0$. Hence, we will get a $\mathbb{Z}$-emm if we can choose the remaining terms of the following matrix in such a way that it is positive definite:

$$\begin{pmatrix} 2 & 1 & 1 & 0 & a_{1,5} & a_{1,6} \\ 1 & 2 & 1 & -1 & a_{2,5} & a_{2,6} \\ 1 & 1 & 2 & -1 & 1 & 1 \\ 0 & -1 & -1 & 2 & 0 & -1 \\ a_{5,1} & a_{5,2} & 1 & 0 & 2 & 1 \\ a_{6,1} & a_{6,2} & 1 & -1 & 1 & 2 \end{pmatrix}.$$

One such choice is to set all the unknowns to 0. Then the quadratic form corresponding to this matrix is

$$q(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3$$
$$- x_2 x_4 + x_3^2 - x_3 x_4 + x_3 x_5 + x_3 x_6 + x_4^2 - x_4 x_6 + x_5^2$$
$$+ x_5 x_6 + x_6^2.$$

One can easily check by diagonalizing that this quadratic form is indeed positive definite. Moreover, in an

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
f_1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	1	0
f_2	0	1	0	0	0	0	0	0	0	0	0	1	1	0	1
f_3	0	0	1	0	0	0	0	1	0	-1	-1	0	1	-1	0
f_4	0	0	0	1	0	0	1	1	0	0	-1	0	1	0	1
f_5	0	0	0	0	1	0	0	-1	-1	1	0	0	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0

TABLE 1. A basis for $H_1(G, \mathbb{Z})$, written as the rows of a matrix.

appropriately chosen basis, it is isomorphic to the standard quadratic form E_6.

4.   GENUS 7

We repeat the general procedure of the previous section for the graphs F_{11}–F_{14} and G_1–G_{10}. In each of Figures 6–19, we list one explicit $\mathbb{Z}$-emm for each of these graphs. The detailed computations are available in the long version of this paper.[1]

5.   GENUS 8

As we explained in Section 2.3, it is sufficient to find a $\mathbb{Z}$-emm for each of the finitely many graphs obtained from F_{11}–F_{14} and G_1-G_{10} by applying procedure (a) or (b). This gives $14 \cdot \left(\binom{18}{2} + 18 \right) = 2394$ graphs.

We have written a Mathematica program for computing the 8×13 matrices for these graphs, and a Fortran program that uses integer arithmetic for finding the $\mathbb{Z}$-emms. We have confirmed that they exist for all of these graphs. The lists of the matrices and the $\mathbb{Z}$-emms are available at http://www.math.uga.edu/~valery/vigre2010.

REFERENCES

[Alexeev and Brunyate 11] Valery Alexeev and Adrian Brunyate. "Extending Torelli Map to Toroidal Compactifications of Siegel Space." arXiv:1102.3425, 1–16, 2011.

[Archdeacon 80] Dan Steven Archdeacon. "A Kuratowski Theorem for the Projective Plane." PhD thesis, The Ohio State University, 1980.

[Archdeacon 81] Dan Archdeacon. "A Kuratowski Theorem for the Projective Plane." *J. Graph Theory* 5 (1981), 243–246.

[Glover and Huneke 75] Henry H. Glover and John P. Huneke. "Cubic Irreducible Graphs for the Projective Plane." *Discrete Math.* 13 (1975), 341–355.

[Glover et al. 79] Henry H. Glover, John P. Huneke, and Chin San Wang. "103 Graphs That Are Irreducible for the Projective Plane." *J. Combin. Theory Ser. B* 27 (1979), 332–370.

[Igusa 67] Jun-ichi Igusa. "A Desingularization Problem in the Theory of Siegel Modular Functions." *Math. Ann.* 168 (1967), 228–260.

[Milgram 73] Martin Milgram. "Irreducible Graphs. II." *J. Combinatorial Theory Ser. B* 14 (1973), 7–45.

[Namikawa 73] Yukihiko Namikawa. "On the Canonical Holomorphic Map from the Moduli Space of Stable Curves to the Igusa Monoidal Transform." *Nagoya Math. J.* 52 (1973), 197–259.

Valery Alexeev, Department of Mathematics, University of Georgia, Athens, GA 30602, USA (valery@math.uga.edu)

Ryan Livingston, Joseph Tenini, Maxim Arap, Xiaoyan Hu, Lauren Huckaba, Patrick McFaddin, Stacy Musgrave, Jaeho Shin, Catherine Ulrich, Department of Mathematics, University of Georgia, Athens, GA 30602, USA

---

[1] Available at http://arxiv.org/abs/1105.4384, download the TeX source and run with the \includecomment{longversion}option.