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Computation of Harmonic Weak Maass Forms

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Harmonic weak Maass forms of half-integral weight have been the subject of much recent work. They are closely related to Ramanujan's mock theta functions, and their theta lifts give rise to Arakelov Green functions, and their coefficients are often related to central values and derivatives of Hecke L-functions. We present an algorithm to compute harmonic weak Maass forms numerically, based on the automorphy method due to Hejhal and Stark. As explicit examples we consider harmonic weak Maass forms of weight 1/2 associated to the elliptic curves 11a1, 37a1, 37b1. We have made extensive numerical computations, and the data we obtained are presented in this paper. We expect that experiments based on our data will lead to a better understanding of the arithmetic properties of the Fourier coefficients of harmonic weak Maass forms of half-integral weight.

1. INTRODUCTION

Half-integral weight modular forms play an important role in arithmetic geometry and number theory. Their coefficients serve as generating functions for various interesting number-theoretic functions, such as representation numbers of quadratic forms in an odd number of variables and class numbers of imaginary quadratic fields. Moreover, employing the Shimura correspondence [Shimura 73], it has been shown in [Waldspurger 81] and [Kohnen and Zagier 81, Kohnen 85] that the coefficients of half-integral weight cusp forms essentially are square roots of central values of quadratic twists of modular L-functions. In analogy with these works, a Shimura correspondence is used in [Katok and Sarnak] to relate coefficients of weight $1/2$ Maass forms to sums of values and sums of line integrals of Maass cusp forms.

In more recent work, Zagier discovered that the generating function for the traces of singular moduli (the CM values of the classical j-function) is a weakly holomorphic modular form of weight $3/2$ [Zagier 02]. This result, which has been generalized in various directions (see, for example, [Bringmann and Ono 07, Bruinier and Funke 06, Duke and Jenkins 08, Kim 04]), demonstrates that the coefficients of automorphic forms with singularities at the cusps also carry interesting arithmetic information.

In a similar spirit, it was proved in [Bruinier and Ono 10] that the coefficients of harmonic weak Maass forms of weight 1/2 are related to both the values and central derivatives of quadratic twists of weight 2 modular L-functions. Harmonic weak Maass forms are also closely related to mock modular forms and to Ramanujan's mock theta functions, which have been the subject of recent work (see, for example, [Bringmann and Ono 06, Bringmann and Ono 10, Ono 08, Zagier 07, Zwegers 01, Zwegers 02]).

In view of these connections, it is desirable to develop tools for the computation of such automorphic forms. In the present paper, we propose an approach to this problem that yields an efficient algorithm.[1] Moreover, we compute some harmonic weak Maass forms that are related to rational elliptic curves as in [Bruinier and Ono 10].

The nonholomorphic nature of harmonic weak Maass forms prevents the use of existing well-developed algorithms for (weakly) holomorphic modular forms, such as modular symbols. The use of Poincaré series does not work well either in small weights due to the poor convergence of the infinite series that appear in the explicit formulas for the coefficients. Instead, we adapt the automorphy method originally developed by Hejhal for the computation of Maass cusp forms on Hecke triangle groups (see, for example, [Hejhal 99]) to the setting of harmonic weak Maass forms.

We now describe the content of this paper in more detail. Let $k \in \frac{1}{2}\mathbb{Z}$, and let N be a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). A *harmonic weak Maass form* of weight k on $\Gamma_0(N)$ is a smooth function on $\mathbb{H}$, the upper half of the complex plane, that satisfies the following:

(i) $f \mid_k \gamma = f$ for all $\gamma \in \Gamma_0(N)$.

(ii) $\Delta_k f = 0$, where Δ_k is the weight k hyperbolic Laplacian on $\mathbb{H}$ (see (2–3)).

(iii) There is a polynomial $P_f = \sum_{n \le 0} c^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

Throughout, for $\tau \in \mathbb{H}$, we let $\tau = u + iv$, where $u, v \in \mathbb{R}$, and we let $q := e^{2\pi i \tau}$. The polynomial P_f is called the *principal part* of f at ∞.

[1] An implementation of this algorithm can be found online at http://code.google.com/r/fredrik314-psage/.

Such a harmonic weak Maass form f has a Fourier expansion at infinity of the form

$$f(\tau) = \sum_{n \gg -\infty} c^+(n) q^n + \sum_{n<0} c^-(n) \Gamma\left(1-k, 4\pi |n| v\right) q^n, \tag{1–1}$$

where $\Gamma(a, x)$ denotes the incomplete gamma function. The series $\sum_{n \gg -\infty} c^+(n) q^n$ is called the *holomorphic part* of f, and its complement is called the *nonholomorphic part*. Naturally, f has similar expansions at the other cusps. There is an antilinear differential operator taking f to the cusp form

$$\xi_k(f) := 2iv^k \overline{\frac{\partial f}{\partial \bar{\tau}}}$$

of weight $2-k$; see (2-5). The kernel of ξ_k consists of the space of *weakly holomorphic* modular forms, those meromorphic modular forms whose poles (if any) are supported at cusps.

Every weight $(2-k)$ cusp form is the image under ξ_k of a weight k harmonic weak Maass form. Ramanujan's mock theta functions correspond to those forms whose images under $\xi_{1/2}$ are weight 3/2 unary theta functions. Here we mainly consider those weight 1/2 harmonic weak Maass forms whose images under $\xi_{1/2}$ are orthogonal to the unary theta series. According to [Bruinier and Ono 10], their coefficients are related to both the values and central derivatives of quadratic twists of weight 2 modular L-functions.

We now briefly describe this result in the special case that the level is a prime p. Let $G \in S_2(\Gamma_0(p))$ be a normalized Hecke eigenform whose Hecke L-function $L(G, s)$ satisfies an odd functional equation. That is, the completed L-function

$$\Lambda(G, s) = p^{s/2} (2\pi)^{-s} \Gamma(s) L(G, s)$$

satisfies $\Lambda(G, 2-s) = \varepsilon_G \Lambda(G, s)$ with root number $\varepsilon_G = -1$. Therefore, the central critical value $L(G, 1)$ vanishes. By Kohnen's theory of plus spaces [Kohnen 85], there is a half-integral weight newform $g \in S^+_{3/2}(\Gamma_0(4p))$, unique up to a multiplicative constant, that lifts to G under the Shimura correspondence. We choose g so that its coefficients are in F_G, the totally real number field generated by the Hecke eigenvalues of G. There exists a weight 1/2 harmonic weak Maass form f on $\Gamma_0(4p)$ in the plus space whose principal part P_f has coefficients in F_G such that

$$\xi_{1/2}(f) = \|g\|^{-2} g,$$

where $\|g\|$ denotes the usual Petersson norm.

For a fundamental discriminant Δ, let χ_Δ be the Kronecker character for $\mathbb{Q}(\sqrt{\Delta})$, and let $L(G, \chi_\Delta, s)$ be the quadratic twist of $L(G,s)$ by χ_Δ. One can show that the root number of $L(G,\chi_\Delta,s)$ is equal to $\operatorname{sign}(\Delta)\cdot\chi_\Delta(p)\varepsilon_G$.

Theorem 1.1. [Bruinier and Ono 10] *Assume that G, g, and f are as above, and let $c^\pm(n)$ denote the Fourier coefficients as in (1–1). Then the following hold:*

(i) *If $\Delta<0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then*

$$L(G,\chi_\Delta,1)=8\pi^2\|G\|^2\|g\|^2\sqrt{\frac{|\Delta|}{N}}\cdot c^-(\Delta)^2.$$

(ii) *If $\Delta>0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then $L'(G,\chi_\Delta,1)=0$ if and only if $c^+(\Delta)$ is algebraic.*

Note that the harmonic weak Maass form f is uniquely determined up to the addition of a weight $1/2$ weakly holomorphic modular form with coefficients in F_G. Furthermore, the absolute values of the nonvanishing coefficients $c^+(\Delta)$ are typically asymptotic to subexponential functions in n. For these reasons, the connection between $L'(G,\chi_\Delta,1)$ and the coefficients $c^+(\Delta)$ in Theorem 1.1(ii) cannot be modified in a simple way to obtain a formula as in the first part of the Theorem. In fact, the proof of Theorem 1.1(ii) is rather indirect. It relies on the Gross–Zagier formula and on transcendence results of Waldschmidt and Scholl on periods of differentials on algebraic curves.

The above result is one of the main motivations for the present paper. Our goal is to carry out numerical computations for the involved harmonic weak Maass forms. In that way, we hope to find more-direct connections of the coefficients $c^+(\Delta)$ to periods or L-functions. When $L'(G,\chi_\Delta,1)$ vanishes, meaning that $c^+(\Delta)$ is algebraic (actually contained in F_G), it would be interesting to see whether $c^+(\Delta)$ carries any arithmetic information related to G. In a forthcoming paper [Bruinier 11], the coefficients $c^+(n)$ will be linked to periods of certain algebraic differentials of the third kind on modular curves. It leads to a conjecture on differentials of the third kind on elliptic curves, which is based on the numerical data presented in Section 4 of the present paper.

Our computations make use of an adaption of the so-called automorphy method. The key point of this method is to view an automorphic form on a noncocompact (but cofinite) Fuchsian group Γ as a function on the upper half-plane with certain transformation properties under the group Γ as well as convergent Fourier series expansions at all cusps. This classical point of view, in terms of functions on the upper half-plane, stands in contrast to the more algebraic point of view, in terms of Hecke modules, usually taken in computing holomorphic modular forms.

By *computing* an automorphic form ϕ in this setting we mean that for any given (small) $\epsilon>0$ we compute a sufficient number of Fourier coefficients, each to high enough precision, so that we are able to evaluate the function ϕ at any point in the upper half-plane with an error at most ϵ.

To calculate these Fourier coefficients, we truncate the Fourier series representing ϕ and view the resulting trigonometric sum as a finite Fourier series. Using the Fourier inversion theorem together with the automorphic properties of ϕ (which will additionally intertwine the Fourier series at various cusps or components), we are able to obtain a set of linear equations satisfied approximately by the coefficients. See, for example, [Hejhal 99, Strömberg 05, Avelin 07]. The (surprising) effectiveness of this algorithm is closely related to the equidistribution properties of closed horocycles (see, for example, [Hejhal 96, Strömbergsson 04]). We describe the main algorithm in detail in Section 3. The implementation of the software package is briefly described in Section 3.3.

In Section 4, we describe our computational results in three cases of particular interest. We consider the elliptic curves 11a1, 37a1, and 37b1 and their corresponding weight 2 newforms. For instance, the elliptic curve 37a1 is the curve of smallest conductor with rank 1. It corresponds to the unique weight 2 normalized newform G on $\Gamma_0(37)$ whose L-function has an odd functional equation. We verified the statement of Theorem 1.1 for all fundamental discriminants Δ that are squares modulo 148 in the range $0<\Delta<15000$. For eight of these fundamental discriminants, the quantity $L'(G,\chi_\Delta,1)$ vanishes. In all these cases we found a stronger statement than that of Theorem 1.1 to be true, namely, that the associated coefficient $c^+(\Delta)$ is an integer. For the corresponding data, see Tables 4 and 5. We conclude Section 4 by describing some analogous experiments for newforms G of weight 4, where g is of weight $5/2$ and f of weight $-1/2$.

The present paper is organized as follows. In Section 2, we recall some facts on (half-integral weight) harmonic weak Maass forms. In working with an arbitrary (not necessarily prime) level, it is convenient to use vector-valued modular forms. In Section 2.3, we therefore recall from [Bruinier and Ono 10] the vector-valued version of Theorem 1.1. In Section 3, we describe the automorphy method in the context of harmonic weak Maass forms. In Section 4, we collect our computational results. In particular, we present results for the elliptic curves 11a1, 37a1, and 37b1; see, for example, Tables 2, 5, and 8. More extensive tables can be obtained from the authors on request.

2. PRELIMINARIES

In order to be able to work with newforms of arbitrary level, it is convenient to work with vector-valued modular forms of half-integral weight for the metaplectic extension of $\mathrm{SL}_2(\mathbb{Z})$. We describe the necessary background in this section.

2.1. A Weil Representation

Let $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im(\tau) > 0\}$ be the complex upper half-plane. We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic twofold cover of $\mathrm{SL}_2(\mathbb{R})$, realized as the group of pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi(\tau)^2 = c\tau + d$. The multiplication is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

We denote the inverse image of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ under the covering map by $\widetilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z})$. It is well known that $\widetilde{\Gamma}$ is generated by

$$T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

Let N be a positive integer. There is a certain representation $\widetilde{\rho}$ of $\widetilde{\Gamma}$ on $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$, the group ring of the finite cyclic group of order $2N$. For a coset $h \in \mathbb{Z}/2N\mathbb{Z}$, we denote by $\mathfrak{e}_h$ the corresponding standard basis vector of $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$. We write $\langle \cdot, \cdot \rangle$ for the standard scalar product (antilinear in the second entry) such that $\langle \mathfrak{e}_h, \mathfrak{e}_{h'} \rangle = \delta_{h,h'}$. In terms of the generators T and S of $\widetilde{\Gamma}$, the representation $\widetilde{\rho}$ is given by

$$\widetilde{\rho}(T)\mathfrak{e}_h = e\left(\frac{h^2}{4N}\right)\mathfrak{e}_h, \tag{2–1}$$

$$\widetilde{\rho}(S)\mathfrak{e}_h = \frac{1}{\sqrt{2iN}} \sum_{h'\ (2N)} e\left(-\frac{hh'}{2N}\right)\mathfrak{e}_{h'}. \tag{2–2}$$

Here the sum runs through the elements of $\mathbb{Z}/2N\mathbb{Z}$, and we have put $e(a) = e^{2\pi i a}$. Note that $\widetilde{\rho}$ is the Weil representation associated to the one-dimensional positive definite lattice $K = (\mathbb{Z}, Nx^2)$ in the sense of [Borcherds 98, Bruinier 02, Bruinier and Ono 10]. It is unitary with respect to the standard scalar product. Using (2–2), a simple computation shows that $\widetilde{\rho}(Z)(\mathfrak{e}_h) = -i\mathfrak{e}_{-h}$, where $Z := S^2 = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i \right)$ is the generator of the center of $\widetilde{\Gamma}$.

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define $j_M(\tau) = \sqrt{c\tau + d}$ by the principal branch of the argument and set

$$\rho(M) := \widetilde{\rho}\big((M, j_M(\tau))\big).$$

Since $\widetilde{\rho}(Z^2)\mathfrak{e}_h = -\mathfrak{e}_h$, it is easy to see that if $A, B \in \Gamma$, then $\rho(A)\rho(B)\rho(AB)^{-1} = \sigma(A, B)$, where $\sigma(A, B)$ is the two-cocycle with values in ± 1 defined by

$$\sigma(A, B) = j_A(B\tau) j_B(\tau) j_{AB}(\tau)^{-1}.$$

It follows that ρ is a projective representation of Γ, and it can be viewed as a *matrix-valued multiplier system* on Γ. That is, it is possible to write $\rho(M) = v(M)R(M)$, where $R : \Gamma \to \mathrm{U}(\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}])$ is a unitary representation and $v : \Gamma \to S^1$ is a half-integral weight multiplier system in the classical sense satisfying $v(AB) = \sigma(A, B)v(A)v(B)$. Compare, for example, [Petersson 38] and [Hejhal 83, Chapters 9.2, 9.3].

From this point on, we will deal only with the Weil representation in the guise of the induced multiplier system ρ. Abusing notation slightly, we continue to use the name Weil representation also for ρ.

If $k \in \frac{1}{2}\mathbb{Z}$, we write $M^!_{k,\rho}$ for the space of $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$-valued weakly holomorphic modular forms of weight k for Γ with multiplier system ρ, that is, holomorphic functions $f : \mathbb{H} \to \mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$ that satisfy $f(M\tau) = j_M(\tau)^{2k}\rho(M)f(\tau)$ for all $M \in \Gamma$ and have possible poles at the cusps of Γ. The subspaces of holomorphic modular forms and cusp forms are denoted by $M_{k,\rho}$ and $S_{k,\rho}$, respectively.

2.2. Harmonic Weak Maass Forms

In this subsection, we assume that $k \le 1$. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$ is called a *harmonic weak Maass form* (of weight k with respect to Γ and ρ) if it satisfies the following:

(i) $f(M\tau) = j_M(\tau)^{2k}\rho(M)f(\tau)$ for all $M \in \Gamma$;

(ii) $\Delta_k f = 0$;

(iii) there is a $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$-valued Fourier polynomial

$$P_f(\tau) = \sum_{h\ (2N)} \sum_{n \in \mathbb{Z}_{\le 0}} c^+(n, h) q^{\frac{n}{4N}} \mathfrak{e}_h$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$.

Here we have that

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+ikv\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right) \quad (2\text{–}3)$$

is the usual weight k hyperbolic Laplace operator (see [Bruinier and Funke 04]). The Fourier polynomial P_f is called the *principal part* of f. We denote the vector space of these harmonic weak Maass forms by $H_{k,\rho}$ (it was called $H^+_{k,\rho}$ in [Bruinier and Funke 04]). Any weakly holomorphic modular form is a harmonic weak Maass form. The Fourier expansion of any $f \in H_{k,\rho}$ gives a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{h\ (2N)}\sum_{\substack{n\in\mathbb{Z}\\ n\gg-\infty}} c^+(n,h)q^{n/4N}\,\mathfrak{e}_h, \quad (2\text{–}4a)$$

$$f^-(\tau) = \sum_{h\ (2N)}\sum_{\substack{n\in\mathbb{Z}\\ n<0}} c^-(n,h)\Gamma\left(1-k, 4\pi\left|\frac{n}{4N}\right|v\right)q^{n/4N}\,\mathfrak{e}_h. \quad (2\text{–}4b)$$

We refer to f^+ as the *holomorphic part* and to f^- as the *nonholomorphic part* of f. Note that $c^\pm(n,h) = 0$ unless $n \equiv h^2\ (4N)$.

Recall that there is an antilinear differential operator $\xi = \xi_k : H_{k,\rho} \to S_{2-k,\bar\rho}$, defined by

$$f(\tau) \mapsto \xi(f)(\tau) := 2iv^k\,\overline{\frac{\partial f}{\partial\bar\tau}}. \quad (2\text{–}5)$$

Here $\bar\rho$ denotes the complex conjugate of the representation ρ, which can be identified with the dual representation. The map ξ is surjective, and its kernel is the space $M^!_{k,\rho}$. There is a bilinear pairing between $M_{2-k,\bar\rho}$ and $H_{k,\rho}$ defined by the Petersson scalar product

$$\{g, f\} = \big(g, \xi(f)\big) := \int_{\Gamma\backslash\mathbb{H}} \langle g, \xi(f)\rangle v^{2-k}\,\frac{du\,dv}{v^2}, \quad (2\text{–}6)$$

for $g \in M_{2-k,\bar\rho}$ and $f \in H_{k,\rho}$. If g has the Fourier expansion $g = \sum_{h,n} b(n,h)q^{n/4N}\mathfrak{e}_h$, and if we denote the Fourier expansion of f as in (2–4), then by [Bruinier and Funke 04, Proposition 3.5], we have

$$\{g, f\} = \sum_{h\ (2N)}\sum_{n\le 0} c^+(n,h)b(-n,h). \quad (2\text{–}7)$$

Hence $\{g, f\}$ depends only on the principal part of f.

2.3. The Shimura Lift

Let $k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$. According to [Eichler and Zagier 85, Chapter 5], the space $M_{k,\bar\rho}$ is isomorphic to $J_{k+1/2,N}$, the space of holomorphic Jacobi forms of weight $k+1/2$ and index N. According to [Skoruppa 90a] and [Skoruppa and Zagier 88], $M_{k,\rho}$ is isomorphic to $J^{\text{skew}}_{k+1/2,N}$, the space of skew-holomorphic Jacobi forms of weight $k+1/2$ and index N. There is an extensive Hecke theory for Jacobi forms (see [Eichler and Zagier 85, Skoruppa 90a, Skoruppa and Zagier 88]), which gives rise to a Hecke theory on $M_{k,\rho}$ and $M_{k,\bar\rho}$, and which is compatible with the Hecke theory on vector-valued modular forms considered in [Bruinier and Stein 10]. In particular, there is an Atkin–Lehner theory for these spaces.

The subspace $S^{\text{new}}_{k,\rho}$ of newforms of $S_{k,\rho}$ is isomorphic as a module over the Hecke algebra to the space of newforms $S^{\text{new},+}_{2k-1}(N)$ of weight $2k-1$ for $\Gamma_0(N)$ on which the Fricke involution acts by multiplication by $(-1)^{k-1/2}$. The isomorphism is given by the Shimura correspondence. Similarly, the subspace $S^{\text{new}}_{k,\bar\rho}$ of newforms of $S_{k,\bar\rho}$ is isomorphic as a module over the Hecke algebra to the space of newforms $S^{\text{new},-}_{2k-1}(N)$ of weight $2k-1$ for $\Gamma_0(N)$ on which the Fricke involution acts by multiplication by $(-1)^{k+1/2}$ (see [Skoruppa and Zagier 88, Gross et al. 87, Skoruppa 90a]). Observe that the Hecke L-series of any $G \in S^{\text{new},\pm}_{2k-1}(N)$ satisfies a functional equation under $s \mapsto 2k-1-s$ with root number $\varepsilon_G = \pm 1$.

We now state the vector-valued version of Theorem 1.1. Let $G \in S^{\text{new}}_2(N)$ be a normalized newform (in particular, a common eigenform of all Hecke operators) of weight 2 and write F_G for the number field generated by the eigenvalues of G. If $\varepsilon_G = -1$, we put $\rho' = \rho$, and if $\varepsilon_G = +1$, we put $\rho' = \bar\rho$. There is a newform $g \in S^{\text{new}}_{3/2,\bar\rho'}$ mapping to G under the Shimura correspondence. It is well known that we may normalize g in such a way that all its coefficients are contained in F_G. According to [Bruinier and Ono 10, Lemma 7.3], there is a harmonic weak Maass form $f \in H_{1/2,\rho'}$ whose principal part has coefficients in F_G with the property that

$$\xi_{1/2}(f) = \|g\|^{-2}g.$$

This form is unique up to addition of a weakly holomorphic form in $M^!_{1/2,\rho'}$ whose principal part has coefficients in F_G.

In practice, the principal part of such an f can be computed as follows: We may complete the weight 3/2 form g to an orthogonal basis $g, g_2, \ldots, g_d$ of $S_{3/2,\bar\rho'}$ consisting of cusp forms with Fourier coefficients in F_G. Let $f \in H_{1/2,\rho'}$ be such that

$$\{g, f\} = 1, \quad \text{and} \quad \{g_i, f\} = 0 \text{ for } i = 2, \ldots, d. \quad (2\text{–}8)$$

Then f has the required properties. In view of (2–7), the conditions of (2–8) translate into an inhomogeneous system of linear equations for the principal part of f.

Theorem 2.1. *Let $G \in S^{\text{new}}_2(N)$ be a normalized newform. Let $g \in S^{\text{new}}_{3/2,\bar\rho'}$, and $f \in H_{1/2,\rho'}$ be as above. Denote the*

Fourier coefficients of f *by* $c^{\pm}(n,h)$ *for* $n \in \mathbb{Z}$ *and* $h \in \mathbb{Z}/2N\mathbb{Z}$*. Then the following are true:*

(i) *If* $\Delta \neq 1$ *is a fundamental discriminant and* $r \in \mathbb{Z}$ *is such that* $\Delta \equiv r^2 \pmod{4N}$ *and* $\varepsilon_G \Delta > 0$*, then*

$$L(G,\chi_\Delta,1) = 8\pi^2 \|G\|^2 \|g\|^2 \sqrt{\frac{|\Delta|}{N}} \cdot c^-(\Delta)^2.$$

(ii) *If* $\Delta \neq 1$ *is a fundamental discriminant and* $r \in \mathbb{Z}$ *is such that* $\Delta \equiv r^2 \pmod{4N}$ *and* $\varepsilon_G \Delta < 0$*, then*

$$L'(G,\chi_\Delta,1) = 0 \iff \quad c^+(-\varepsilon_G\Delta, r) \in \bar{\mathbb{Q}}$$
$$\iff c^+(-\varepsilon_G\Delta, r) \in F_G.$$

When $S_{1/2,\rho'} = \{0\}$, the above result also holds for $\Delta = 1$; see also [Bruinier and Ono 10, Remark 18]. This is, for instance, the case when N is a prime. If N is a prime and $\varepsilon_G = -1$, then the space $H_{1/2,\rho'}$ can be identified with a space of scalar-valued modular forms satisfying a Kohnen-plus-space condition. In that way, one obtains Theorem 1.1 stated in the introduction.

3. COMPUTATIONAL ASPECTS

3.1. The Automorphy Method for Vector-Valued Weak Maass Forms

To compute the Fourier coefficients of the harmonic weak Maass forms, we use the so-called automorphy method, sometimes called Hejhal's method. This is a general method that has been used successfully to compute various kinds of automorphic functions and forms on $\mathrm{GL}_2(\mathbb{R})$. It was originally developed by Hejhal in order to compute Maass cusp forms for the modular group and other Hecke triangle groups (see, for example, [Hejhal 99]). The method was later generalized in [Strömberg 05] to computations of Maass waveforms with nontrivial multiplier systems and arbitrary real weights, as well as to general subgroups of the modular group (see also [Strömberg 08]). Another generalization to automorphic forms with singularities (Eisenstein series, Poincaré series, and Green's functions) was made in [Avelin 10a, Avelin 10b].

In the current paper, we show how to adapt the algorithm to the case of vector-valued harmonic weak Maass forms for the Weil representation. To treat this case, we need to deal with a combination of difficulties: singularities, nontrivial weights, and matrix-valued multiplier systems. Additionally, we also give an explicit (a posteriori) error bound (3-5), which can be used to determine the accuracy for the computed Fourier coefficients.

For simplicity, we consider the representation ρ (the case of $\overline{\rho}$ is analogous) and $k \in \mathbb{Z} + \frac{1}{2}$. Furthermore, in order to avoid questions of uniqueness, we assume that either $k < 0$ or that $k = \frac{1}{2}$ and that N is prime. In these cases, a harmonic weak Maass form is uniquely determined by its principal part.

It should be emphasized that the algorithm described in this paper, as well as the implementation noted above, works for any integral or half-integral weight, although in Section 4 we give examples only of weights $\frac{1}{2}$ and $-\frac{1}{2}$. The first reason for focusing on these weights is that they are connected to holomorphic modular forms of weights 2 and 4. The weight $\frac{1}{2}$ is special in that the nonholomorphic Poincaré series (as in [Bruinier 02] or [Hejhal 83], for example) cannot be used to compute harmonic weak Maass forms. This is because of a lack of absolute convergence of certain sums of twisted Kloosterman sums. For weight $-\frac{1}{2}$, these sums converge absolutely, albeit very slowly.

For computational purposes it is not feasible to use the definition of ρ in terms of the action on the generators of the metaplectic group. Such an approach would involve a large number of matrix multiplications with $2N \times 2N$ complex matrices. Instead, we use explicit formulas from [Strömberg 11] to evaluate the matrix coefficients of $\rho(M)$ for any $M \in \Gamma$ in terms of p-adic invariants of the associated lattice $(\mathbb{Z}, Nx^2)$.

Although we are mostly interested in those harmonic weak Maass forms that are not related to mock theta functions (this is the setting of [Bruinier and Ono 10]), we would like to stress that it is also possible to use the method described in this paper to compute these. They can be obtained as the nonholomorphic parts of the appropriate harmonic weak Maass forms. The challenge is to find the principal part of the Maass form in its entirety from the form in which the mock theta function is given. It should also be noted that it is usually more efficient to use a combinatorial interpretation (if it exists) to compute the coefficients of a mock theta function then to use our method.

3.1.1. The Algorithm: Phase 1.

Let $f \in H_{k,\rho}$ with a given (fixed) principal part $P_f(\tau) = \sum_h P_{f,h}(\tau)\,\mathfrak{e}_h$, where

$$P_{f,h}(\tau) = \sum_{n=-K}^{0} a(n,h)\, q^{n/4N}$$

(for some finite $K \geq 0$), and write $f = f^+ + f^-$ (as in 2.4a and 2.4b) with $f^+ = \sum_{h(2N)} f_h^+ \mathfrak{e}_h$ and $f^- =$

$\sum_{h(2N)} f_h^- \mathfrak{e}_h$, where

$$f_h^+(\tau) = \sum_{n=-K}^{0} a(n,h)\, q^{n/4N} + \sum_{n>0} c^+(n,h)\, q^{n/4N},$$
$$f_h^-(\tau) = \sum_{n<0} c^-(n,h)\, \Gamma(1-k, 4\pi\,|n/4N|\, v)\, q^{n/4N},$$

for $\tau = u + iv \in \mathbb{H}$. Our goal is to obtain numerical approximations to the coefficients $c^{\pm}(n,h)$.

To formulate our algorithm, we prefer to separate the u- and the v-dependence in f and therefore introduce the function W defined by

$$W(v) = \begin{cases} e^{-2\pi v} & \text{if } v > 0, \\ e^{-2\pi v}\Gamma(1-k, 4\pi|v|) & \text{if } v < 0. \end{cases}$$

We also set $c(n,h) = c^+(n,h)$ for $n > 0$ and $c^-(n,h)$ for $n < 0$ and write $e_{4N}(u) = e^{2\pi i u/4N}$. With this notation,

$$f_h(\tau) = \sum_{n=-K}^{0} a(n,h)\, q^{n/4N} + \sum_{n\neq 0} c(n,h)\, W\left(\frac{nv}{4N}\right) e_{4N}(nu).$$

By standard inequalities for the incomplete gamma function, one can show that

$$|W(v)| < c_k\, e^{-2\pi|v|} \begin{cases} 1, & v > 0, \\ (4\pi\,|v|)^{-k}, & v < 0, \end{cases}$$

where c_k is an explicit constant depending only on k. To be able to determine a truncation point of the Fourier series above, we also need bounds on the coefficients $c(n,h)$. Using [Bruinier and Funke 04, Lemma 3.4], it follows that there exists an explicit constant $C > 0$ such that

$$c(n,h) = \begin{cases} O\left(\exp\left(4\pi C\sqrt{n}\right)\right), & n \to +\infty, \\ O(|n|^{k/2}), & n \to -\infty. \end{cases}$$

For $k < 0$, we are able to make the implied constants explicit using nonholomorphic Poincaré series as in, for instance, [Bruinier 02] or [Hejhal 83]. For $k = 1/2$, we rely on numerical a posteriori tests to assure ourselves that the truncation point was chosen correctly. See, for example, Section 3.2.

Let $\epsilon > 0$ and fix $Y < Y_0 = \sqrt{3}/2$. By the estimates above, we can find an $M_0 = M(Y, \epsilon)$ such that the function $\hat{f} = \sum_{h(2N)} \hat{f}_h \mathfrak{e}_h$ given by the truncated Fourier series

$$\hat{f}_h(\tau) = P_{f,h}(\tau) + \sum_{0<|n|\leq M_0} c(n,h)\, W\left(\frac{nv}{4N}\right) e_{4N}(nu)$$

satisfies

$$\left\|\hat{f}(\tau) - f(\tau)\right\|^2 < \epsilon$$

for any $\tau \in \mathcal{H}_Y = \{\tau \in \mathcal{H} \mid \Im\tau \geq Y\}$. Here

$$\|z\|^2 = \sum_{h=1}^{2N} |z_h|^2$$

for $z \in \mathbb{C}^{2N}$. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and set $z = x + iy = A\tau$. Then

$$y = \Im A\tau = \frac{v}{|c\tau + d|^2} \leq \frac{v}{c^2 v^2} \leq \frac{1}{v},$$

and hence

$$|j_A(\tau)|^4 = |c\tau + d|^2 = \frac{v}{y} \leq \frac{1}{y^2}.$$

Using the fact that ρ is unitary, it is now easy to see that if $\tau, A\tau \in \mathcal{H}_Y$, then

$$\left\|\hat{f}(A\tau) - j_A(\tau)^{2k}\, \rho(A)\, \hat{f}(\tau)\right\|^2 < \epsilon\left(1 + Y^{-2k}\right) \quad (3\text{–}1)$$
$$< 2\epsilon \cdot \max(1, Y^{-2k}).$$

Consider now a horocycle at height Y and a set of $2Q$ (with $Q > M_0$) equally spaced points

$$z_m = x_m + iY, \quad x_m = \frac{1-2m}{4Q}, \quad 1 - Q \leq m \leq Q.$$

If we view the series $\hat{f}_h$ as a finite Fourier series, we can invert it over this horocycle, and it is easy to see that if n is an integer with $0 < |n| \leq M_0$ and $n \equiv h^2\ (4N)$, then

$$\frac{1}{2Q} \sum_{m=1-Q}^{Q} \hat{f}_h(z_m)\, e_{4N}(-n x_m) \quad (3\text{–}2)$$
$$= W\left(\frac{n}{4N} Y\right) c(n,h) + a(n,h)\; e^{-\frac{2\pi n}{4N} Y}.$$

One can also interpret the left-hand side as a Riemann-sum approximation to the integral

$$\int_{-1/2}^{1/2} f_h(z)\, e_{4N}(-nx)\, dx.$$

Let $z_m^* = x_m^* + i y_m^* = T_m^{-1} z_m$ $(T_m \in \mathrm{PSL}_2(\mathbb{Z}))$ denote the pullback of z_m to the standard (closed) fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$,

$$\mathcal{F} = \left\{ z = x + iy \mid |x| \leq \frac{1}{2},\ |z| \geq 1 \right\}.$$

Using (3–1), we obtain

$$\hat{f}_h(z_m) = j_{T_m}(z_m^*) \sum_{h'(2N)} \rho_{hh'}(T_m)\, \hat{f}_{h'}(z_m^*) + [\![2\epsilon Y_k]\!],$$

where $\rho_{hh'}(T_m)$ is the (h,h')-element of the matrix $\rho(T_m)$, and we use $[\![2\epsilon Y_k]\!]$ to denote a quantity bounded in absolute value by $2\epsilon Y_k$ and $Y_k = \max(1, Y^{-2k})$.

Inserting this into (3–2), we see that the left-hand side can be written as

$$\begin{aligned}\frac{1}{2Q}\sum_{m=1-Q}^{Q} j_{T_m}(z_m^*)\sum_{h'(2N)}\rho_{hh'}(T_m) \\ \times\Bigg[\sum_{l=-K}^{0} a(l,h')\exp\left(-\frac{2\pi l}{4N}y_m^*\right)e_{4N}(lx_m^*) \\ +\sum_{0<|l|\le M_0} c(l,h')W\left(\frac{l}{4N}y_m^*\right)e_{4N}(lx_m^*)\Bigg]e_{4N}(-nx_m) \\ =\sum_{h'(2N)}\sum_{0<|l|\le M_0} c(l,h')\widetilde{V}_{nl}^{hh'}+\widetilde{W}_n^h+[\![2\epsilon Y_k]\!],\end{aligned} \tag{3–3}$$

where

$$\begin{aligned}\widetilde{V}_{nl}^{hh'} &= \frac{1}{2Q}\sum_{m=1-Q}^{Q} j_{T_m}(z_m^*)\rho_{hh'}(T_m)W\left(\frac{l}{4N}y_m^*\right) \\ &\quad\times e_{4N}(lx_m^*-nx_m), \\ \widetilde{W}_n^h &= \frac{1}{2Q}\sum_{h'(2N)}\sum_{l=-K}^{0} a(l,h') \\ &\quad\times\sum_{m=1-Q}^{Q} j_{T_m}(z_m^*)\rho_{hh'}(T_m)\exp\left(-\frac{2\pi l}{4N}y_m^*\right) \\ &\quad\times e_{4N}(lx_m^*-nx_m).\end{aligned}$$

We thus have an inhomogeneous system of linear equations that is (approximately) satisfied by the coefficients $c(n,h)$. Let

$$\mathcal{D} = \{(n,h) \mid 0<|n|\le M_0,\ 0\le h<2N\}$$

(with a fixed ordering) and note that $|\mathcal{D}| = 4M_0N$. If we set

$$\begin{aligned}\vec{D} &= (d(n,h))_{(n,h)\in\mathcal{D}}, \\ V &= V(Y) = \left(V_{nl}^{hh'}\right)_{(h,n),(h',l)\in\mathcal{D}}, \\ V_{nl}^{hh'} &= \widetilde{V}_{nl}^{hh'} - \delta_{nl}\delta_{hh'}W\left(\frac{n}{4N}Y\right), \\ \vec{W} &= \vec{W}(Y) = \left(W_n^h\right)_{(h,n)\in\mathcal{D}}, \\ W_n^h &= \widetilde{W}_n^h - a(n,h)e^{-\frac{2\pi n}{4N}Y},\end{aligned}$$

we can write this linear system as $|\mathcal{D}|$ linear equations in $|\mathcal{D}|$ variables:

$$V\vec{D}+\vec{W}=\vec{0}. \tag{3–4}$$

In practice, it turns out that the the matrix V is nonsingular as soon as the subspace of $H_{k,\rho}$ consisting of functions with a given singular part is one-dimensional. In these cases, we can immediately obtain the solution as

$$\vec{D} = -V^{-1}\vec{W},$$

and since we know that the vector of the "true" coefficients, $\vec{C} = (c(n,h))_{(n,h)\in\mathcal{D}}$, satisfies

$$\left\|V\vec{C}+\vec{W}\right\|_\infty \le 2\epsilon Y_k,$$

we see that

$$\begin{aligned}\left\|\vec{C}-\vec{D}\right\|_\infty &= \left\|\vec{C}+V^{-1}\vec{W}\right\|_\infty \le \left\|V^{-1}\right\|_\infty\cdot\left\|V\vec{C}+\vec{W}\right\|_\infty \\ &\le 2\epsilon Y_k\left\|V^{-1}\right\|_\infty.\end{aligned} \tag{3–5}$$

To obtain a theoretical error estimate, we thus need to estimate $\left\|V^{-1}\right\|_\infty$ from below. Unfortunately, this does not seem to be possible from the formulas above, and we have to use numerical methods to estimate this norm. Hence, to obtain the Fourier coefficients up to a (proven) desired precision, we might have to go back and decrease the original ϵ or increase either M_0 or Q.

At this point, one should also remark that the error bound $\left\|V^{-1}\right\|_\infty$ is in general much worse than the actual apparent error, as verified by studying coefficients known to be integers. The reason for this is that the sums $\widetilde{V}_{nl}^{hh'}$ exhibit massive cancellation and are therefore overpowered by the terms $W\left(\frac{n}{4N}Y\right)$ on the diagonal.

3.1.2. The Algorithm: Phase 2.

Returning to (3–3) and solving for $c(n,h)$, we see that

$$\begin{aligned}c(n,h) &= W\left(\frac{n}{4N}Y\right)^{-1} \\ &\quad\times\left[\sum_{h'(2N)}\sum_{|l|\le M_0} c(l,h')\widetilde{V}_{nl}^{hh'} + W_n^h + [\![2\epsilon Y_k]\!]\right]\end{aligned} \tag{3–6}$$

for *every* n, i.e., also when $|n|>M_0$, provided that $Q>M(Y)$. If we first choose Y such that $W\left(\frac{n}{4N}Y\right)$ is not too small, then we can in fact use this equation to compute $c(n,h)$ with an error of size $\epsilon W\left(\frac{n}{4N}Y\right)^{-1}$. In this manner, we may produce long stretches of coefficients (before we need to decrease Y again) at arbitrary intervals $N_A\le n\le N_B$ without the need of computing intermediate coefficients above the initial set up to $n=M_0$.

Remark 3.1. The exact same algorithm, with the nonholomorphic parts set to zero, also lets one compute holomorphic vector-valued modular forms for the Weil representation. This has been exploited in [Ryan et al. 12] to verify computations of holomorphic Poincaré series.

3.2. Heuristic Error Estimates

For $k < 0$, all implied constants and therefore all error estimates can be made explicit. In the remaining case that interests us, $k = 1/2$, the known bounds for the twisted Kloosterman sums are not enough to prove the necessary explicit bounds for the Fourier coefficients of the associated Poincaré series. We are therefore not able to give effective theoretical error estimates in this case. However, this is not a serious problem, since there are several tests that we may perform on the resulting coefficients to assure ourselves of their accuracy. We mention a few tests that we have used.

First of all, one can simply use two different values of Y and verify that the resulting vectors $\vec{D} = \vec{D}(Y)$ are independent of Y. This test is completely general and can be used for all instances in which the algorithm can be applied. Suppose now that we have a harmonic weak Maass form $f \in H_{k,\rho}$ of half-integral weight k such that $\xi_k(f) = \|g\|^{-2}g$, with $g \in S_{2-k,\bar{\rho}}$. We then know the following:

The coefficients $\sqrt{|\Delta|}c^-(-\varepsilon_G \cdot \Delta)$ are proportional to the coefficients $b(\varepsilon_G \cdot \Delta)$ of g (see, for example, [Bruinier and Ono 10, p. 3]). If additionally, the Shimura lift of g is a newform $G \in S^{\text{new}}_{3-2k}(\Gamma_0(N))$, then we can predict that certain coefficients $c^+(\Delta)$ are algebraic (see, for example, [Bruinier and Ono 10, Section 7]), and if we are able to identify these coefficients as algebraic numbers to a certain precision, this can be used as another measure of accuracy.

3.3. Implementation

The first implementation of the above-described algorithm was made in Fortran 90, using the package ARPREC [Bailey et al. 09] for arbitrary-precision (fixed-point) arithmetic.

The second and current implementation uses Python and Sage, and the included package `mpmath` for arbitrary-precision (fixed-point) arithmetic. The core parts of the algorithm also use Cython [Bradshaw et al. 11] for efficiency.[2] The goal is for it to be included as a standard package in Sage or Purple Sage.

4. RESULTS

4.1. Harmonic Maass Forms Corresponding to Elliptic Curves

In this section, we present the numerical results we have obtained for harmonic weak Maass forms corresponding to weight 2 holomorphic forms associated to elliptic curves. We have concentrated on three particular examples. In Cremona's notation, these correspond to the curve 11a1 of level 11 and the two curves 37a1 and 37b1 of level 37.

Recall that if the holomorphic weight 2 newform G of level N has Atkin–Lehner eigenvalue ± 1, then the L-function $L(G, s)$ has root number $\varepsilon_G = \mp 1$. Furthermore, since the root number of the twisted L-function $L(G, \chi_\Delta, s)$ is $\operatorname{sign}(\Delta)\chi_\Delta(N)\varepsilon_G$ and we always consider fundamental discriminants for which $\chi_\Delta(N) = 1$, we see that the central value $L(G, \chi_\Delta, 1)$ vanishes if $\operatorname{sign}(\Delta)\varepsilon_G = -1$; that is, if $L(G, s)$ has an even functional equation, we consider $\Delta < 0$, and otherwise $\Delta > 0$.

For each of these examples, we computed a large set of central derivatives of the twisted L-functions with the appropriate Δ using Sage with the standard algorithms developed by Dokchitser. We then fixed a harmonic weak Maass form with nonzero principal part P_f such that $\xi_{1/2}(f)$ maps to G under the Shimura lift. In all cases, we took a Poincaré series P_{-d} having principal part $q^{-d/4N}$ and computed an initial set of Fourier coefficients for this function using the methods described in the previous section. We then used the second phase of the algorithm and computed more Fourier coefficients.

Note that for the results in this section, all initial "phase 1" computations were performed using the new Sage package, and all further, "phase 2," computations were done in Fortran 90.

We would like to give a flavor of the CPU times involved. The initial computations, using our Sage code, took in all cases approximately two hours on a 2.66-GHz Xeon processor. On the same processor, the CPU time for a single stretch of phase-2 calculations ranged between less than an hour for the smallest discriminant up to several days for the largest discriminant.

As a measure of the accuracy of our computations, one can consider the difference between the coefficients in Tables 1, 4, and 7 and the nearest integer (the third column). To further support the correctness, we also list, in Tables 3, 6, and 9, normalized coefficients of the nonholomorphic parts, i.e., $\sqrt{|\Delta|}c^-(\Delta)/\sqrt{|\Delta_0|}c^-(\Delta_0)$ where $c^-(\Delta_0)$ is some fixed nonzero coefficient of index Δ_0.

4.1.1. The Elliptic Curve 11a1.

Here the unique newform of weight 2 and level 11 is given by

$$G = \eta(\tau)^2\eta(11\tau)^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots \in S_2^{\text{new}}(\Gamma_0(11)),$$

[2] Current versions of the algorithm can be obtained at http://code.google.com/r/fredrik314-psage/.

Δ	$c^+(\Delta)$	$\lvert c^+(\Delta) - [c^+(\Delta)]\rvert$
−824	−5798520	$3.0 \cdot 10^{-76}$
−1799	−2708450784	$2.7 \cdot 10^{-46}$
−4399	−68135748249936640	$2.3 \cdot 10^{-21}$
−8483	21444576071639138821 6704	$9.1 \cdot 10^{-28}$
−11567	−1241226714909991920509289945 6	$1.6 \cdot 10^{-25}$
−14791	66850179291021019012709832099520	$3.1 \cdot 10^{-30}$
−15487	−47873223940518244876241503088128 0	$5.6 \cdot 10^{-30}$
−15659	−80448981445459761864806477015941 5	$6.8 \cdot 10^{-30}$
−15839	−11621224950043446417995241161356 80	$7.0 \cdot 10^{-30}$
−16463	45425759225337282286439348622301 44	$1.5 \cdot 10^{-30}$
−17023	−23302350713109514450879400185948 800	$2.0 \cdot 10^{-29}$
−17927	11013323818195929170363415880837 4784	$1.2 \cdot 10^{-29}$
−18543	46472679186428248933410405816448 2624	$1.9 \cdot 10^{-29}$

TABLE 1. $E = 11\text{a}1,\ P_{-5} \in H_{1/2,\bar{\rho}}$.

and the corresponding L-function $L(G, s)$ has an even functional equation. Using Sage, we computed all values of $L'(G, \chi_\Delta, 1)$ for fundamental discriminants $\Delta < 0$ such that $\left(\frac{\Delta}{11}\right) = 1$ and $|\Delta| \leq 19703$. This set consists of 2749 fundamental discriminants, and among these, we found 14 discriminants for which $L'(G, \chi_\Delta, 1)$ vanishes up to our numerical precision (see Table 1).

As a representative for the harmonic weak Maass form in the space $H_{1/2,\bar{\rho}}$ corresponding to G, we chose the Poincaré series P_{-5} with principal part $q^{-5/44}(\mathfrak{e}_7 - \mathfrak{e}_{-7})$.

Δ	$c^+(\Delta)$	$L'(G, \chi_\Delta, 1)$
−7	$2.846337019028598018665157651971175139394807386198 8 \cdot 10^{00}$	$1.22556687406888 \cdot 10^{00}$
−8	$2.513848200257572916589271112443577446003001234176 2 \cdot 10^{00}$	$1.88791354720393 \cdot 10^{00}$
−19	$4.819242814896325586187092443704311900067351951904 2 \cdot 10^{00}$	$7.51391655667451 \cdot 10^{00}$
−24	$5.101849408833970318750717774759732759823241447640 2 \cdot 10^{00}$	$4.02744559024300 \cdot 10^{00}$
−35	$4.751589263610172376964907963967516200401724536239 9 \cdot 10^{00}$	$7.64786334637073 \cdot 10^{00}$
−39	$-1.646669069701048116627209102821932767735644291480 4 \cdot 10^{01}$	$2.97721567216550 \cdot 10^{00}$
−40	$1.147094138813807468374776831472368929286053996290 0 \cdot 10^{01}$	$5.58789208952436 \cdot 10^{00}$
−43	$-1.762243963850332772273778036004642323756836780504 8 \cdot 10^{01}$	$1.18814465355690 \cdot 10^{01}$
−51	$2.073622299987874171862971843268299555258288078606 5 \cdot 10^{01}$	$1.30416363302768 \cdot 10^{01}$
−52	$1.572352868391499038710321670014631756241143849761 5 \cdot 10^{01}$	$5.14853759817659 \cdot 10^{00}$
−68	$9.688967332293849399200604340412746997924737006792 6 \cdot 10^{00}$	$3.80344864881298 \cdot 10^{00}$
−79	$1.755735175543616073938856434002776029131708922925 4 \cdot 10^{01}$	$4.75620653690677 \cdot 10^{00}$
−83	$-7.176766438342767560986124290741795054461168316285 9 \cdot 10^{01}$	$6.43843846621214 \cdot 10^{00}$
−84	$6.166620062658731515996812679960365052558653936560 1 \cdot 10^{01}$	$6.53746327159376 \cdot 10^{00}$
−87	$-7.723003642443333454169748405033843902397964748328 0 \cdot 10^{01}$	$2.35584785481347 \cdot 10^{00}$
−95	$7.846757208406415155666183904650414442619922789799 4 \cdot 10^{01}$	$3.03660486030085 \cdot 10^{00}$
−811	$3.004624798306728533655343117548976584738204290710 5 \cdot 10^{06}$	$1.25949136911120 \cdot 10^{01}$
−820	$-6.049375425038730426209114733215857804651074901931 5 \cdot 10^{06}$	$1.19119437485937 \cdot 10^{01}$
−824	$-5.798519999999999999999999999999999999999999999999 9 \cdot 10^{06}$	$-6.6 \cdot 10^{-24}$
−827	$1.853548940787122285952805942373606773652122252855 4 \cdot 10^{06}$	$1.60961273159300 \cdot 10^{01}$
−831	$-6.791139222583541613108302669941131060842015199498 6 \cdot 10^{06}$	$3.36744068632019 \cdot 10^{00}$
−996	$-3.551629450568582040021104504706342212916808289294 1 \cdot 10^{07}$	$1.15828152096335 \cdot 10^{01}$
−1003	$1.081193474207930307380276640618147643760866892840 9 \cdot 10^{07}$	$2.53076681967579 \cdot 10^{01}$
−1007	$-3.946924800000000000000000000000000000000000000000 0 \cdot 10^{07}$	$-1.1 \cdot 10^{-22}$
−1011	$-3.768514082442963648893477501006010654734127510105 4 \cdot 10^{07}$	$1.84592490209627 \cdot 10^{01}$
−1019	$3.379031595754974944221876965059381799788862881833 8 \cdot 10^{07}$	$1.68145450009782 \cdot 10^{01}$

TABLE 2. $E = 11\text{a}1,\ P_{-5} \in H_{1/2,\bar{\rho}}$.

Δ	$\sqrt{\Delta}\, c^-(\Delta)$	$\lvert c^-(\Delta) - [c^-(\Delta)]\rvert$
4	-3	$2.0 \cdot 10^{-100}$
5	5	$2.1 \cdot 10^{-99}$
9	-2	$1.7 \cdot 10^{-100}$
12	5	$8.0 \cdot 10^{-100}$
16	4	$1.5 \cdot 10^{-99}$
20	5	$1.1 \cdot 10^{-100}$
25	0	$1.0 \cdot 10^{-100}$
36	6	$1.0 \cdot 10^{-99}$
37	5	$4.2 \cdot 10^{-99}$
45	0	$6.4 \cdot 10^{-99}$

TABLE 3. $E = 11\text{a}1$, $P_{-5} \in H_{1/2,\bar{\rho}}$. Coefficients are scaled by $c^-(1)$.

To compute the Fourier coefficients of P_{-5}, we used the method described in the previous section with initial values $\epsilon = 10^{-40}$ and $Y = 0.5$, which gave us a truncation point of $M_0 = 42$, corresponding to Δ between -1847 and 1885. For a short selection of computed values of $c^+(\Delta)$, see Table 2, and for a table of coefficients corresponding to all vanishing $L'(G, \chi_\Delta, 1)$, see Table 1. The first few normalized "negative" coefficients are displayed in Table 3. These values should be compared to the list in [Skoruppa 90b, p. 505].

4.1.2. The Elliptic Curve 37a1.

Consider the newform of weight 2 and level 37 that has an odd functional equation. The q-expansion is given by

$$G = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} + \cdots \in S_2^{\text{new}}(\Gamma_0(37)).$$

Using Sage, we computed all values of $L'(G, \chi_\Delta, 1)$ for fundamental discriminants $\Delta > 0$ such that $\left(\frac{\Delta}{37}\right) = 1$ and $|\Delta| \leq 15000$. This set consists of 2217 fundamental discriminants, and among these, we found 8 discriminants for which $L'(G, \chi_\Delta, 1)$ vanishes up to our numerical precision (see Table 4). For the corresponding harmonic weak Maass form in $H_{1/2,\rho}$, we took P_{-3}, which has principal part $q^{-3/148}(\mathfrak{e}_{21} + \mathfrak{e}_{-21})$. The initial computation was done in Sage, using $\epsilon = 1 \cdot 10^{-35}$, which gave a value of $M_0 = 30$, corresponding to discriminants in the range $-4440 \leq \Delta \leq 4585$. For examples of the coefficients $c^+(\Delta)$, see Tables 4 and 5. The first few normalized "negative" coefficients are displayed in Table 6.

Δ	$c^+(\Delta)$	$\lvert c^+(\Delta) - [c^+(\Delta)]\rvert$
1489	9	$1.6 \cdot 10^{-72}$
4393	66	$1.5 \cdot 10^{-45}$
5116	-746	$8.5 \cdot 10^{-23}$
5281	153	$8.2 \cdot 10^{-23}$
5560	-1124	$1.2 \cdot 10^{-22}$
5761	-974	$1.1 \cdot 10^{-22}$
6040	-1404	$4.2 \cdot 10^{-23}$
6169	336	$1.1 \cdot 10^{-22}$

TABLE 4. $\text{E} = 37\text{a}1$, $P_{-3} \in H_{1/2,\rho}$.

4.1.3. The Elliptic Curve 37b1.

In this case we consider the newform of weight 2 and level 37 that has an even functional equation. The q-expansion is given by

$$G = q + q^3 - 2q^4 - q^7 - 2q^9 + 3q^{11} + \cdots \in S_2^{\text{new}}(\Gamma_0(37)).$$

Using Sage, we computed all values of $L'(G, \chi_\Delta, 1)$ for fundamental discriminants $\Delta < 0$ such that $\left(\frac{\Delta}{37}\right) = 1$ and $|\Delta| \leq 12000$. This set consists of 1631 fundamental discriminants, and among these, we found 15 discriminants for which $L'(G, \chi_\Delta, 1)$ vanishes up to our numerical precision (see Table 7). For the corresponding harmonic weak Maass form in $H_{1/2,\bar{\rho}}$, we took P_{-12}, which has principal part $q^{-12/148}(\mathfrak{e}_{30} - \mathfrak{e}_{-30})$. The initial computation was done in Sage, using $\epsilon = 1 \cdot 10^{-30}$, which gave a value of $M_0 = 33$, corresponding to discriminants in the range $-4883 \leq \Delta \leq 5029$. For examples of the coefficients $c^+(\Delta)$, see Tables 7 and 8. The first few normalized "negative" coefficients are displayed in Table 9.

4.2. Further Computations

To investigate whether a result analogous to Theorem 1.1 also holds for newforms of weight 4, we computed $L'(2, G, \chi_\Delta)$ for all newforms G of weight 4 on $\Gamma_0(N)$ with $5 \leq N \leq 150$ and fundamental discriminants Δ with $|\Delta| \leq 300$ and the property that the twisted L-function $L(s, G, \chi_\Delta)$ has an odd functional equation. For $5 \leq N \leq 10$, we additionally computed these values for fundamental discriminants Δ with $|\Delta| \leq 5000$. Among all these values, we did not find a single example of a vanishing derivative. Even though we did not get any positive case in which we could test the theorem, we still wanted to make sure that there was no easily accessible counterexample.

We therefore computed the Fourier coefficients, with up to 40-digit precision, of the associated weight $(-1/2)$ harmonic Maass form corresponding to all weight 4 newforms defined over $\mathbb{Q}$ for N up to 100. To test the accuracy of our computations (and to make sure that the implementation was correct), we did not only rely on the

Δ	$c^+(\Delta)$	$L'(G, \chi_\Delta, 1)$
1	$-2.8176178498959956879756075537515493438922975370716 \cdot 10^{-01}$	$3.05999773834052 \cdot 10^{-01}$
12	$-4.8852723826201225228227029607337071669095284814788 \cdot 10^{-01}$	$4.29861479867736 \cdot 10^{00}$
21	$-1.7273925723265275652082007397068992218426924398791 \cdot 10^{-01}$	$9.00238680032537 \cdot 10^{00}$
28	$6.7819399530394779828450578400669420938246859928076 \cdot 10^{-01}$	$4.32726024966011 \cdot 10^{00}$
33	$5.6630232015906998168220545669245622604190884430064 \cdot 10^{-01}$	$3.62195679113882 \cdot 10^{00}$
37	$-9.1326561374611652958506448407204050631184401026129 \cdot 10^{-01}$	$3.47328771649229 \cdot 10^{00}$
40	$4.0098509269543637915254766073122850557290259963615 \cdot 10^{-01}$	$3.70588717878444 \cdot 10^{00}$
41	$6.5637495744757231959699415722023547525400239084778 \cdot 10^{-01}$	$5.93680171871573 \cdot 10^{00}$
44	$9.6886404434506397321859573425794267139455920322171 \cdot 10^{-01}$	$1.01334656625280 \cdot 10^{01}$
53	$-5.6688852568232517859984506645723944339238503996096 \cdot 10^{-01}$	$2.61746665637296 \cdot 10^{01}$
65	$-6.0328072889521477971996798071175156059595671497733 \cdot 10^{-01}$	$7.67818286326206 \cdot 10^{00}$
73	$3.4874711835362408853804154923777452565842552803223 \cdot 10^{-01}$	$2.92507284795068 \cdot 10^{00}$
77	$2.2699132373705254600799448087564660809534768699467 \cdot 10^{-01}$	$3.42067600398534 \cdot 10^{10}$
85	$-7.6894617048676272061865441758881289699552927122551 \cdot 10^{-01}$	$9.90133670369251 \cdot 10^{00}$
1481	$-3.2715595098273932057423414526408419506801164996884 \cdot 10^{00}$	$5.26994449124823 \cdot 10^{00}$
1484	$-1.3432792297590353562651264178555980321660674399890 \cdot 10^{01}$	$3.86746474997364 \cdot 10^{01}$
1489	$8.9999999999999999999999999999999999999999999999999 \cdot 10^{00}$	$-3.7 \cdot 10^{-23}$
1496	$1.1199440423162819213593329208218112792285448658029 \cdot 10^{01}$	$2.27616829409607 \cdot 10^{01}$
1501	$-5.8188238119388864901078937905792951783427273426771 \cdot 10^{02}$	$6.06007663972706 \cdot 10^{00}$
4376	$-3.6731327299348159991042234350611468700145535059868 \cdot 10^{02}$	$2.03155740209437 \cdot 10^{01}$
4377	$-5.0062522276143084997015960658866819832113068397294 \cdot 10^{02}$	$2.27150950159608 \cdot 10^{00}$
4393	$6.6000000000000000000000000000000000000000000001468 \cdot 10^{01}$	$5.8 \cdot 10^{-23}$
4396	$-2.3023069110811173762943326075771836710063196221488 \cdot 10^{02}$	$2.00437958330233 \cdot 10^{00}$
4412	$-3.1500483730098996665306117169085504925545562420809 \cdot 10^{02}$	$3.73011222569745 \cdot 10^{01}$

TABLE 5. $E = 37\mathrm{a}1,\ P_{-3} \in H_{1/2,\rho}$.

provable error bounds, but also checked algebraicity of certain coefficients corresponding to nonfundamental discriminants. These coefficients were indeed all found to be integers or rational with fairly small denominators. In contrast to this, the Fourier coefficients corresponding to fundamental discriminants were found not to be similarly "simple" rational numbers.

Another test case that we have used to check the implementation of the algorithm is the generating function for the partition function. This can be given as a weight $(-1/2)$ weakly holomorphic modular form for the dual of the Weil representation corresponding to the lattice $(\mathbb{Z}, 6x^2)$, and it is uniquely determined by

Δ	$\sqrt{\lvert\Delta\rvert}\, c^-(\Delta)$	$\lvert c^-(\Delta) - [c^-(\Delta)]\rvert$
-4	1	$4.0 \cdot 10^{-84}$
-7	-1	$5.0 \cdot 10^{-84}$
-11	1	$4.5 \cdot 10^{-84}$
-12	-1	$2.0 \cdot 10^{-84}$
-16	-2	$1.1 \cdot 10^{-83}$
-27	-3	$1.3 \cdot 10^{-83}$
-28	3	$1.4 \cdot 10^{-83}$
-36	-2	$1.0 \cdot 10^{-83}$
-40	2	$3.6 \cdot 10^{-85}$
-44	-1	$1.1 \cdot 10^{-83}$

TABLE 6. $E = 37\mathrm{a}1,\ P_{-3} \in H_{1/2,\rho}$. Coefficients are scaled by $\sqrt{3}\, c^-(-3)$.

Δ	$c^+(\Delta)$	$\lvert c^+(\Delta) - [c^+(\Delta)]\rvert$
-139	-6	$1.5 \cdot 10^{-85}$
-823	312	$9.1 \cdot 10^{-80}$
-2051	-26724	$1.0 \cdot 10^{-67}$
-2599	122048	$3.4 \cdot 10^{-63}$
-3223	-472416	$3.2 \cdot 10^{-57}$
-3371	-674712	$7.4 \cdot 10^{-56}$
-5227	5816	$5.5 \cdot 10^{-31}$
-5307	-5192	$4.6 \cdot 10^{-31}$
-6583	-13320	$4.6 \cdot 10^{-31}$
-7892	-79552	$1.2 \cdot 10^{-30}$
-7951	28152	$4.0 \cdot 10^{-31}$
-9112	-224548	$1.6 \cdot 10^{-30}$
-9715	236934	$2.8 \cdot 10^{-32}$
-11444	-1437956	$2.0 \cdot 10^{-33}$
-11651	563716	$7.0 \cdot 10^{-34}$

TABLE 7. $E = 37\mathrm{b}1,\ P_{-12} \in H_{1/2,\bar{\rho}}$.

Δ	$c^+(\Delta)$	$L'(G, \chi_\Delta, 1)$
-3	$1.0267149116920354474451980654263083626994977508118 \cdot 10^{00}$	$1.47929949208 \cdot 10^{00}$
-4	$1.2205364009670316625279102409757685190938519711297 \cdot 10^{00}$	$1.81299789722 \cdot 10^{00}$
-7	$1.6900297463200076214148752932012403965170838158011 \cdot 10^{00}$	$2.11071898018 \cdot 10^{00}$
-11	$5.8849982354849175483779961900586424239744874288522 \cdot 10^{-01}$	$3.65679089534 \cdot 10^{00}$
-40	$1.2669706585839831188366862729215921230412462308976 \cdot 10^{00}$	$4.16362898338 \cdot 10^{00}$
-47	$3.0756790552662277517712909874001702657447701023621 \cdot 10^{00}$	$5.26739088546 \cdot 10^{00}$
-67	$2.1608356105538234382282266707128748893591830597455 \cdot 10^{00}$	$4.98143961845 \cdot 10^{00}$
-71	$-1.5945418432752378367351454423028372659103804842831 \cdot 10^{00}$	$5.33295381308 \cdot 10^{00}$
-83	$2.9631171578917930530100644900469583789690213329975 \cdot 10^{00}$	$7.30522465208 \cdot 10^{00}$
-84	$-3.8773494709413749500399075799371212202544017987791 \cdot 10^{00}$	$1.00026475317 \cdot 10^{01}$
-95	$-2.6554862688645143792016861758519887392731139540185 \cdot 10^{00}$	$5.83606039003 \cdot 10^{00}$
-132	$4.1944733541115532186541330550136737538249181082859 \cdot 10^{00}$	$9.99216716471 \cdot 10^{00}$
-136	$-4.8392675993443437829864850814885823635034770966657 \cdot 10^{00}$	$5.73824076491 \cdot 10^{00}$
-139	$-5.9999999999999999999999999999999999999999999999991 \cdot 10^{00}$	$-8.5 \cdot 10^{-23}$
-151	$-8.3135688179267692046624844818371994826339638923811 \cdot 10^{-01}$	$6.69750855159 \cdot 10^{00}$
-152	$4.3274351625459058613812696410805017025617476195953 \cdot 10^{00}$	$7.95190347996 \cdot 10^{00}$
-811	$-1.4731293182498551151700589944493338505308298027148 \cdot 10^{02}$	$5.32436617837 \cdot 10^{00}$
-815	$1.2194410312093092058885476868302234805268383148338 \cdot 10^{02}$	$4.74925836935 \cdot 10^{00}$
-823	$3.1200000000000000000000000000000000000000000000000 \cdot 10^{02}$	$-1.5 \cdot 10^{-23}$
-824	$-3.2299860660409750567356931348586086493552010570382 \cdot 10^{02}$	$1.75028741141 \cdot 10^{01}$
-835	$-2.4035736526655124690110045874885626910322384359422 \cdot 10^{02}$	$8.64359690730 \cdot 10^{00}$

TABLE 8. $E = 37\text{b}1,\ P_{-12} \in H_{1/2,\bar{\rho}}$.

Δ	$\sqrt{\Delta}\, c^-(\Delta)$	$\lvert c^-(\Delta) - [c^-(\Delta)]\rvert$
4	-1	$1.6 \cdot 10^{-85}$
9	0	$3.2 \cdot 10^{-85}$
12	3	$9.6 \cdot 10^{-85}$
16	-2	$1.6 \cdot 10^{-85}$
21	3	$2.7 \cdot 10^{-85}$
25	-1	$1.7 \cdot 10^{-85}$
28	3	$3.9 \cdot 10^{-85}$
33	3	$3.6 \cdot 10^{-85}$
36	0	$3.9 \cdot 10^{-85}$
40	0	$5.6 \cdot 10^{-85}$

TABLE 9. $E = 37\text{b}1,\ P_{-12} \in H_{1/2,\bar{\rho}}$. Coefficients are scaled by $c^-(1)$.

its principal part $P(q) = q^{-1/24}(\mathfrak{e}_1 - \mathfrak{e}_5 - \mathfrak{e}_7 + \mathfrak{e}_{11})$.[3] See also [Bruinier and Ono ??].

The L-value computations were performed in Sage, using the included version of Rubinstein's `lcalc` library.[4]

[3] The interested reader is encouraged to use http://code.google.com/r/fredrik314-psage/ to compute this function.

[4] `lcalc`, a library for computing zeros and values of L-functions. Available at www.math.uwaterloo.ca/~mrubinst.

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