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Mosaic Supercongruences of Ramanujan Type

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In this article, we present analogues of supercongruences of Ramanujan type observed by L. Van Hamme and W. Zudilin. Our congruences are inspired by Ramanujan-type series that involve quadratic algebraic numbers.

1. RAMANUJAN AND RAMANUJAN–SATO SERIES

Srinivasa Ramanujan's work on elliptic integrals and modular equations led him to the discovery of 17 surprising series for $1/\pi$, which were published in [Ramanujan 14]. These series are of the following form:

$$\sum_{n=0}^{\infty} A_n (a+bn) z^n = \frac{1}{\pi},$$

where z, a, and b are algebraic numbers and

$$A_n = \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3}, \quad s = \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}.$$

The proofs of Ramanujan's 17 series, as well as many other series of the same type, are now available. For more details see [Baruah et al. 09a]. Such series are now known as Ramanujan-type series. In 2002, T. Sato surprised the mathematical community by presenting a series like Ramanujan's involving the Apéry numbers

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Inspired by this series, mathematicians discovered similar series involving the Domb numbers [Chan et al. 04]

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k},$$

the Almkvist–Zudilin numbers [Almkvist and Zudilin 06]

$$A_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{3^{n-3k}(3k)!}{(k!)^3} \binom{n}{3k} \binom{n+k}{k},$$

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and other types of numbers [Almkvist and Zudilin 06] [Chan and Verril 09, Chan et al. 11], such as

$$A_n = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2, \quad A_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^3,$$

$$A_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Series involving such types of numbers are now known as Ramanujan–Sato-type series. We will refer to the numbers A_n as Ramanujan–Sato-type numbers.

2. MOSAIC SUPERCONGRUENCES

We generalize the patterns of supercongruences of Ramanujan type noted in [Van Hamme 97] and [Zudilin 09] to series involving quadratic algebraic numbers. Let A_n be of Ramanujan–Sato type. Let z, a, b be algebraic numbers such that

$$\sum_{n=0}^{\infty} A_n (a+bn) z^n = \frac{1}{\pi}.$$

Suppose that

$$\sum_{n=0}^{p-1} A_n (a+bn) z^n = \alpha_1(p)\sqrt{d_1} + \cdots + \alpha_j(p)\sqrt{d_j},$$

where $\alpha_1(p), \ldots, \alpha_j(p)$ are rational, $a = a_1\sqrt{d_1} + a_2\sqrt{d_2} + \cdots + a_j\sqrt{d_j}$, and $d_1, \ldots, d_j$ are square-free. Then for primes $p > p_0$, where p_0 is some fixed prime, we have the supercongruences

$$\alpha_i(p) \equiv a_i \left(\frac{-d_i}{p}\right) p \pmod{p^3}, \quad i = 1, 2, \ldots, j.$$

We will refer to them as *mosaic supercongruences* because they are pieces of a single sum. We recall that for some Ramanujan–Sato-type numbers, the supercongruences hold only modulo p^2.

For the Ramanujan-like series for $1/\pi^2$ discovered by the author, we conjecture analogous mosaic supercongruences, again generalizing Zudilin's observations [Zudilin 09]. See our last two examples.

3. EXAMPLES

All congruences in the following examples are conjectures.

Example 3.1. For the Ramanujan-type series

$$\frac{\sqrt{15}}{2^7 \cdot 5^2} \cdot \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (263 + 5418n) \frac{(-1)^n}{80^{3n}} = \frac{1}{\pi},$$

we have checked that if we write

$$\sqrt{15} \sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (263 + 5418n) \frac{(-1)^n}{80^{3n}} = \alpha_p \sqrt{15},$$

then for primes $p > 5$, we have the following supercongruences:

$$\alpha_p \equiv 263 \left(\frac{-15}{p}\right) p \pmod{p^3},$$

that is,

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (263 + 5418n) \frac{(-1)^n}{80^{3n}} \equiv 263 \left(\frac{-15}{p}\right) p \pmod{p^3} \quad p > 5,$$

which is [Zudilin 09, equation 21].

Example 3.2. For the Ramanujan-type series

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} \left(\frac{7\sqrt{7}-10}{27} + \frac{13\sqrt{7}-7}{9} n\right) \times \left(\frac{13\sqrt{7}-34}{54}\right)^n = \frac{1}{\pi},$$

we have checked that if we write

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} \left[(7\sqrt{7}-10) + (39\sqrt{7}-21)n \right] \times \left(\frac{13\sqrt{7}-34}{54}\right)^n = \alpha_p + \beta_p \sqrt{7},$$

then for primes $p > 7$, we have the following supercongruences:

$$\alpha_p \equiv -10 \left(\frac{-1}{p}\right) p, \quad \beta_p \equiv 7 \left(\frac{-7}{p}\right) p \pmod{p^3}.$$

Example 3.3. Consider the Apéry numbers

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad n = 0, 1, 2, \ldots.$$

One of Sato's series is [Sato 02]

$$\sum_{n=0}^{\infty} A_n \left[(60\sqrt{15} - 134\sqrt{3}) + (72\sqrt{15} - 160\sqrt{3})n \right] \times \left(\frac{\sqrt{5}-1}{2}\right)^{12n} = \frac{1}{\pi}.$$

If we write

$$\sum_{n=0}^{p-1} A_n \left[(60\sqrt{15} - 134\sqrt{3}) + (72\sqrt{15} - 160\sqrt{3})n \right] \times \left(\frac{\sqrt{5}-1}{2} \right)^{12n} = \alpha_p \sqrt{3} + \beta_p \sqrt{15},$$

then for primes $p > 5$, we have the supercongruences

$$\alpha_p \equiv -134 \left(\frac{-3}{p} \right) p, \quad \beta_p \equiv 60 \left(\frac{-15}{p} \right) p \pmod{p^3}.$$

Example 3.4. The Ramanujan-type series in [Baruah and Berndt 09b, equation 6.1] is

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \times \big[(73 + 52\sqrt{2} - 42\sqrt{3} - 30\sqrt{6}) + (168 + 120\sqrt{2} - 96\sqrt{3} - 69\sqrt{6})n \big] \times (-18872 - 13344\sqrt{2} + 10896\sqrt{3} + 7704\sqrt{6})^n = \frac{1}{\pi}.$$

It can be derived from Ramanujan series [Baruah and Berndt 09b, equation 6.4] by Zudilin's translation method. Writing

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \times \big[(73 + 52\sqrt{2} - 42\sqrt{3} - 30\sqrt{6}) + (168 + 120\sqrt{2} - 96\sqrt{3} - 69\sqrt{6})n \big] \times (-18872 - 13344\sqrt{2} + 10896\sqrt{3} + 7704\sqrt{6})^n = \alpha_p + \beta_p\sqrt{2} + \gamma_p\sqrt{3} + \delta_p\sqrt{6},$$

we have for primes $p > 3$, the supercongruences

$$\alpha_p \equiv 73 \left(\frac{-1}{p} \right) p, \quad \beta_p \equiv 52 \left(\frac{-2}{p} \right) p,$$
$$\gamma_p \equiv -42 \left(\frac{-3}{p} \right) p, \quad \delta_p \equiv -30 \left(\frac{-6}{p} \right) p \pmod{p^3}.$$

Example 3.5. The "complex" Ramanujan series [Guillera and Zudilin 12, equation 45] is

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3} \left(\frac{7\sqrt{7} - 13\sqrt{-1}}{64} + \frac{15\sqrt{7} - 21\sqrt{-1}}{32} n \right) \times \left(\frac{47 + 45\sqrt{-7}}{128} \right)^n = \frac{1}{\pi}.$$

Writing

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^3}{(1)_n^3} \left[(7\sqrt{7} - 13\sqrt{-1}) + (30\sqrt{7} - 42\sqrt{-1})n \right] \times \left(\frac{47 + 45\sqrt{-7}}{128} \right)^n = \alpha_p\sqrt{-1} + \beta_p\sqrt{7},$$

we have for primes $p > 7$, the supercongruences

$$\alpha_p \equiv -13p, \quad \beta_p \equiv 7 \left(\frac{-7}{p} \right) p \pmod{p^3}.$$

Example 3.6. For the Ramanujan-like series [Guillera 10, equation 10]

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} (-1)^n \left(\frac{3}{4}\right)^{6n} \times (45 + 549n + 1930n^2) = \frac{384}{\pi^2},$$

we have checked that if we write

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} (-1)^n \left(\frac{3}{4}\right)^{6n} \times (45 + 549n + 1930n^2) = \alpha_p,$$

then for primes $p > 3$, we have the supercongruences

$$\alpha_p \equiv 45p^2 \pmod{p^5},$$

that is, they follow Zudilin's pattern [Zudilin 08].

Example 3.7. The only known (unproven) hypergeometric Ramanujan-like series for $1/\pi^2$ [Guillera 10, equation 9] with an irrational z is

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{15\sqrt{5} - 33}{2} \right)^{3n} \times \left[\left(56 - 25\sqrt{5}\right) + \left(303 - 135\sqrt{5}\right) n + \left(\frac{1220}{3} - 180\sqrt{5} \right) n^2 \right] = \frac{1}{\pi^2}.$$

Writing

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{15\sqrt{5} - 33}{2} \right)^{3n} \times \left[\left(56 - 25\sqrt{5}\right) + \left(303 - 135\sqrt{5}\right) n + \left(\frac{1220}{3} - 180\sqrt{5} \right) n^2 \right] = \alpha_p + \beta_p\sqrt{5},$$

we have for primes $p > 5$, the supercongruences

$$\alpha_p \equiv 56p^2, \quad \beta_p \equiv -25 \left(\frac{5}{p} \right) p^2 \pmod{p^5}.$$

This generalizes Zudilin's pattern.

Example 3.8. To provide more evidence to support our observation, we have considered other series involving only simple square roots in [Baruah and Berndt 09b, Borwein and Borwein 88, Chan and Verril 09]. The expected mosaic supercongruences appear to hold in all these cases.

4. CONCLUDING REMARKS

For an excellent survey on Ramanujan-type series and a beautiful survey on recent advances in this topic, see [Baruah et al. 09a] and [Zudilin 08] respectively. There are many examples of convergent Ramanujan-type and Ramanujan–Sato-type series in the literature. From the modular theory of Ramanujan-type series we know that there are functions $z(q)$, $b(q)$, and $a(q)$, with $q = e^{i\pi\tau}$ and $\Im(\tau) > 0$, that take algebraic values when τ is a quadratic irrational. Obviously the series converges faster when $\Im(\tau)$ increases. If $\Im(\tau)$ is small, then the series may diverge. An example of a "divergent" Ramanujan-type series is given in [Borwein and Borwein 88, p. 371], which corresponds to $\Im(\tau) = \sqrt{253}/11$. Convergent or divergent series lead to supercongruences following exactly the same patterns [Guillera and Zudilin 12].

Taking into account that the Jacobi symbols are the quadratic residues, perhaps this work can provide some clues for discovering similar congruences when the algebraic numbers involved are more complicated. Our future project would be to carry this out.

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