Experimental Mathematics, 20(4):457–466, 2011

ISSN: 1058-6458 print / 1944-950X online
DOI: 10.1080/10586458.2011.565260

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Equivalence Classes for the μ-Coefficient of Kazhdan–Lusztig Polynomials in S_n

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CONTENTS

We study equivalence classes relating to the Kazhdan–Lusztig $\mu(x, w)$ coefficients in order to help explain the scarcity of distinct values. Each class is conjectured to contain a "crosshatch" pair. We also compute the values attained by $\mu(x, w)$ for the permutation groups S_{10} and S_{11}.

1. INTRODUCTION

The Kazhdan–Lusztig polynomials, introduced in [Kazhdan and Lusztig 79], arose in the context of constructing representations of the Hecke algebra associated to a Weyl group. It was soon apparent that these polynomials encode important information relating to geometry and representation theory. For example, they encode the singularities of Schubert varieties and the multiplicities of irreducibles in Verma modules [Beilinson and Bernstein 81, Brylinski and Kashiwara 81, Kazhdan and Lusztig 79]. They are also of interest from a purely combinatorial viewpoint (see [Björner and Brenti 05]).

We restrict our attention to the type-A case, in which there is one Kazhdan–Lusztig polynomial $P_{x,w}(q)$ associated to every pair of permutations $x, w \in S_n$. Kazhdan and Lusztig give a simple recurrence for these polynomials in their original paper (see Section 2.2 below). However, our combinatorial understanding of these polynomials is still far from complete. For example, there is neither a *combinatorial* proof that the coefficients of $P_{x,w}(q)$ are nonnegative nor a closed formula for the degree of a given polynomial. It is important to note that a noncombinatorial proof of nonnegativity arises from the interpretation of the coefficients of Kazhdan–Lusztig polynomials in terms of intersection cohomology [Kazhdan and Lusztig 80].

The μ-coefficient, $\mu(x, w)$, is defined to be the coefficient of $q^{(\ell(w)-\ell(x)-1)/2}$ in $P_{x,w}(q)$ (where $\ell(w)$ denotes the *length* of w; see Section 2.1). The reason the above problems are still open is that $\mu(x, w)$ controls

2000 AMS Subject Classification: 05E10, 20F55
Keywords: Kazhdan–Lusztig polynomials, equivalence classes, mu-coefficient

a correction term in Kazhdan and Lusztig's original recurrence; $\mu(x, w)$ is a number we know little about. While there are known to be a few simple combinatorial *necessary* conditions for $\mu(x, w)$ to be nonzero, these conditions are by no means sufficient. In fact, there are no nontrivial sufficient conditions known for arbitrary x and w (however, see [Shi 08, Xi 05]). A combinatorial rule for the value $\mu(x, w)$ would likely lead to insights wherever Kazhdan–Lusztig polynomials arise.

A major difficulty in the study of these μ-coefficients is that (as shown in [McLarnan and Warrington 03]) S_{10} is the smallest symmetric group for which $\mu(x, w)$ can be anything other than 0 or 1. There is little overlap between what is computationally feasible and what is computationally illuminating. Nonetheless, there are a number of important combinatorial results regarding these polynomials. See the book [Björner and Brenti 05] for an overview and the papers of Brenti (such as [Brenti 98] and [Brenti 04]) in particular.

The organization of the paper is as follows. Section 2 provides the necessary definitions, while Section 3 outlines the properties of $\mu(x, w)$ from the literature that we will be using. The results of this paper are of two types. First, we present new data regarding the values $\mu(x, w)$ takes; how we do this is outlined in Section 4.2. Set $M(n) = \{\mu(x, w) : x, w \in S_n\} \setminus \{0\}$.

Theorem 1.1. *We have*

- $M(10) = \{1, 4, 5\}$;
- $M(11) = \{1, 3, 4, 5, 18, 24, 28\}$;
- $M(12) \supseteq \{1, 2, 3, 4, 5, 6, 7, 8, 18, 23, 24, 25, 26, 27, 28, 158, 163\}$.

Particular pairs x, w realizing each of these values are given in Table 2. The only μ-values that have already appeared in the literature for S_n are $\{0, 1, 2, 3, 4, 5\}$.

We also offer computer code [Warrington 11] that can quickly produce a database of all Kazhdan–Lusztig polynomials in S_{10}; this code is discussed in Section 4.1. There are over one billion "extremal pairs" (x, w) in S_{10} for which one might hope that $\mu(x, w) > 0$. More than 100 million of these pairs cannot be reduced to equivalent pairs in smaller symmetric groups. Altogether, approximately one million different polynomials appear. Even stored efficiently, this yields a gigabyte of data. The comparable database for S_{11} would be on the order of 50 times larger.

Second, we consider the question why there are so few different values of $\mu(x, w)$. For example, in S_{10} there are 664 752 noncovering pairs $x < w$ for which $\mu(x, w) > 0$. Yet the only nonzero values taken are 1, 4, and 5. We explain this in Section 4.3 by showing that for S_{10} and S_{11}, the μ-positive pairs fall into a handful of equivalence classes. The μ-coefficient is constant on each class by construction. The equivalence relation, $\sim$, is defined in Section 4.3; the corresponding class of a pair (x, w) is denoted by $[[x, w]]$.

A class is *n-minimal* if it does not intersect S_m for $m < n$. Pairs in n-minimal classes are also themselves referred to as n-minimal. As a consequence of Theorem 1.1, the numbers of 10- and 11-minimal classes are at least 2 and 4, respectively.

Theorem 1.2. *The* 2*-minimal class* $[[01, 10]]$ *is the only class intersecting* S_m *for some* $m < 10$*. The numbers of* 10*- and* 11*-minimal classes are at most* 4 *and* 7*, respectively.*

Finally, in Section 5 we speculate that each $\sim$-equivalence class contains a "crosshatch" pair.

2. DEFINITIONS

2.1. The Symmetric Group

The symmetric group S_n has the following presentation as a Coxeter group:

$$S_n = \langle s_1, \ldots, s_{n-1}, : s_i^2 = 1,\ s_i s_{i\pm1} s_i = s_{i\pm1} s_i s_{i\pm1}, \\ s_i s_j = s_j s_i,\ \text{for } |i - j| > 1 \rangle.$$

We write $\mathcal{S}$ for the set of generators $\{s_1, \ldots, s_{n-1}\}$. The group S_n is often described as the group of bijections from $\{0, 1, \ldots, n-1\}$ to itself (i.e., permutations) under the usual composition of functions. From this perspective, it is most convenient to identify the generator s_i with the adjacent transposition that switches $i - 1$ and i. For clarity in examples, we will write a for 10, b for 11, etc. The *one-line notation* for $\sigma \in S_n$ lists the elements $[\sigma(0), \sigma(1), \ldots, \sigma(n-1)]$ in order. We often omit commas and brackets. For example, the permutation $\sigma \in S_6$ that sends i to $5 - i$ would be written $[5, 4, 3, 2, 1, 0]$ or simply 543210.

The group S_n has the structure of a ranked poset as follows. An *inversion* of a permutation $w = [w(0), w(1), \ldots, w(n-1)]$ is a pair $i < j$ for which $w(i) > w(j)$. The *length* $\ell(w)$ of w is the total number of inversions. The rank of an element is then given by its length.

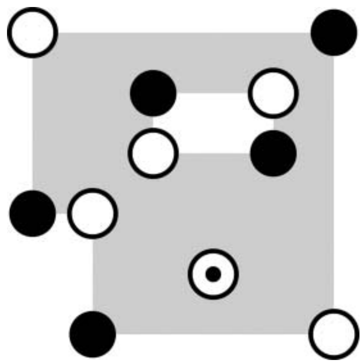

FIGURE 1. Bruhat picture for $x = [2, 0, 4, 1, 3, 5]$, $w = [5, 2, 3, 1, 4, 0]$.

To define the partial order under which we will be relating our elements, we first make two auxiliary definitions. Let $x, w \in S_n$ and $p, q \in \mathbb{Z}$. Define $r_w(p, q) = |\{i \le p : w(i) \ge q\}|$ and the *difference function* $d_{x,w}(p, q) = r_w(p, q) - r_x(p, q)$. Then the *Bruhat partial order* $\le$ is determined by setting $x \le w$ if $d_{x,w}(p, q) \ge 0$ for all p, q. This definition is equivalent to more common ones such as the tableau criterion (cf. [Billey and Lakshmibai 00, Fulton 97, Humphreys 90]).

For a permutation w, let $\mathcal{D}_w$ denote the permutation matrix oriented such that for each i there is a 1 in the ith column from the left and $w(i)$th row from the bottom. We will frequently display a pair of permutations x and w graphically using *Bruhat pictures*: Such a picture consists of $\mathcal{D}_w$ and $\mathcal{D}_x$ overlaid along with shading given by the difference function. An example is given in Figure 1. Entries of $\mathcal{D}_x$ and $\mathcal{D}_w$ are denoted by black disks and circles, respectively. Positions corresponding to 1's of both $\mathcal{D}_x$ and $\mathcal{D}_w$ (termed *capitols*) are denoted by a black disk and a larger concentric circle. Shading denotes regions in which $d_{x,w} \ge 1$. Successively darker shading denotes successively higher values of $d_{x,w}$.

Finally, there are two sets we associate to any permutation w. We define the *right descent set rds(w)* of w as $\{s \in \mathcal{S} : ws < w\}$. Similarly, the *left descent set* is $\text{lds}(w) = \{s \in \mathcal{S} : sw < w\}$.

2.2. Kazhdan–Lusztig Polynomials

We now define the Kazhdan–Lusztig polynomials $P_{x,w}(q)$ associated to pairs of elements $x, w \in S_n$. For motivation and more general definitions applicable to any Coxeter group, we refer the reader to [Humphreys 90, Kazhdan and Lusztig 79]. Set

$$\mu(x, w) = \text{ coefficient of } q^{(\ell(w)-\ell(x)-1)/2} \text{ in } P_{x,w}(q)$$

and define

$$c_s(x) = \begin{cases} 1 & \text{if } xs < x, \\ 0 & \text{if } xs > x. \end{cases}$$

We have the following paraphrased theorem of Kazhdan and Lusztig.

Theorem 2.1. [Kazhdan and Lusztig 79] *There is a unique set of polynomials $\{P_{x,w}(q)\}_{x,w \in S_n}$ such that for $x, w \in S_n$:*

- $P_{w,w}(q) = 1$*;*
- $P_{x,w}(q) = 0$ *when* $x \not\le w$*;*
- *for* $s \in \text{rds}(w)$*,*

$$P_{x,w}(q) = q^{c_s(x)} P_{x,ws}(q) + q^{1-c_s(x)} P_{xs,ws}(q) - \sum_{\substack{z \le ws \\ zs < z}} \mu(z, ws) q^{\frac{\ell(w)-\ell(z)}{2}} P_{x,z}(q). \quad (2\text{–}1)$$

When $x < w$, we have an upper bound on the degrees:

$$\deg(P_{x,w}(q)) \le \frac{\ell(w) - \ell(x) - 1}{2}.$$

Note that $\mu(x, w)$ is the coefficient of the highest possible power of q in $P_{x,w}(q)$.

3. PROPERTIES SATISFIED BY $\mu(x, w)$

We now proceed to describe various well-known properties satisfied by the μ-coefficient. If $x \not\le w$, then $\mu(x, w)$ is automatically zero. So assume $x \le w$. There are two easily recognized instances in which the μ-coefficient is zero. The first follows directly from the definitions, since $P_{x,w}$ is a polynomial in q rather than $q^{1/2}$.

Fact 3.1. *If $\ell(w) - \ell(x)$ is even, then $\mu(x, w) = 0$.*

We will refer to a pair x, w for which $\ell(w) - \ell(x)$ is odd as an *odd pair*.

The second follows from an important set of equalities satisfied by the Kazhdan–Lusztig polynomials (see [Humphreys 90, Corollary 7.14] for a proof):

$$P_{x,w}(q) = \begin{cases} P_{xs,w}(q) & \text{if } s \in \text{rds}(w), \\ P_{sx,w}(q) & \text{if } s \in \text{lds}(w). \end{cases} \quad (3\text{–}1)$$

Define the set of *extremal pairs*

$$\text{EP}(n) = \{x \le w \in S_n \times S_n : \text{lds}(x) \supseteq \text{lds}(w) \text{ and } \text{rds}(x) \supseteq \text{rds}(w)\}.$$

Fact 3.2. *If $\ell(x) < \ell(w) - 1$ and $(x, w) \notin \mathrm{EP}(n)$, then $\mu(x, w) = 0$.*

To see why Fact 3.2 is true, suppose we have a noncovering pair $x < w$ along with some $s \in \mathcal{S}$ such that $xs > x$ and $ws < w$. The equality $P_{x,w}(q) = P_{xs,w}(q)$ combined with the degree bound of Theorem 2.1 implies, since $\ell(w) - \ell(xs) = \ell(w) - \ell(x) - 1$, that the coefficient of $q^{(\ell(w)-\ell(x)-1)/2}$ in $P_{x,w}(q)$ must be zero.

According to computations in [Hammett and Pittel 08] , there are approximately 800 billion comparable pairs x, w in S_{10}. It turns out that whenever w covers x, $P_{x,w}(q) = \mu(x, w) = 1$; ignore these pairs for the moment. Then, considering only pairs for which $\mu(x, w) > 0$, Facts 3.1 and 3.2 allow us to restrict our attention to the odd extremal pairs. The number of such pairs in S_{10} is a modest 626 145 374, yet still much larger than $|M(10)| = 3$.

The idea of considering equivalence classes to explain the redundancy of μ-values is not new. Lascoux and Schützenberger, and probably others, entertained the possibility that any pair x, w with $\mu(x, w) > 0$ could be generated from a cover by applying certain operators (see the L-S operators below). By construction, all pairs generated in this way would have the same μ-value. Our main contribution in this paper in this regard is to consider "compression" (and "decompression") *in conjunction with* the L-S operators and symmetry. Our hope is that these classes are large enough to explain fully the scarcity of distinct values of μ. The three relations from which we build these classes exist already in the literature. We now describe them.

The simplest relations (of various symmetries) can be derived from the definitions in [Kazhdan and Lusztig 79].

Fact 3.3. *Let w_0 denote the* long word $[n-1, n-2, \ldots, 1, 0]$ *in S_n. Then for $x, w \in S_n$,*

$$\mu(x, w) = \mu(x^{-1}, w^{-1}) = \mu(w_0 w, w_0 x) = \mu(w w_0, x w_0).$$

Our second relation arises from the *Lascoux–Schützenberger (L-S) operators* (which, their name notwithstanding, were known to Kazhdan and Lusztig [Kazhdan and Lusztig 79]). Define $\mathcal{R}_k$ as the set of permutations w for which $ws_k < w$ or $ws_{k+1} < w$, but not both. In other words, $\mathcal{R}_k$ consists of all permutations in which $w(k)$, $w(k+1)$, $w(k+2)$ do *not* appear in increasing or decreasing order. Then wR_k is defined to be the unique element in the intersection $\mathcal{R}_k \cap \{ws_k, ws_{k+1}\}$. The operators R_k act "on the right" in the sense that they act on positions. Operators L_k that act "on the left" can be defined analogously by having them act on values. More precisely, we set $\mathcal{L}_k = \{w : w^{-1} \in \mathcal{R}_k\}$ and $L_k w = (w^{-1} R_k)^{-1}$. (These operators, elementary Knuth transformations and their duals, are closely connected to the Robinson–Schensted correspondence; for details, see [Fulton 97, Knuth 70].) For $x, w \in S_n$, set

$$\mu[x, w] = \begin{cases} \mu(x, w), & \text{if } x \le w, \\ \mu(w, x), & \text{if } w \le x, \\ 0, & \text{if } x \text{ and } w \text{ are not comparable.} \end{cases}$$

Fact 3.4. [Kazhdan and Lusztig 79] *If $x, w \in \mathcal{L}_k$, then $\mu[x, w] = \mu[L_k x, L_k w]$. If $x, w \in \mathcal{R}_k$, then $\mu[x, w] = \mu[xR_k, wR_k]$.*

Note that the L-S operators *do not* preserve the lower-order coefficients of Kazhdan–Lusztig polynomials. Also note that $\mu(\cdot, \cdot)$ is not invariant under the L-S operators (consider L_0 acting on the pair $(021, 201)$). In the rest of this paper, when we refer to μ being constant on an equivalence class, we are referring to $\mu[\cdot, \cdot]$ rather than $\mu(\cdot, \cdot)$.

Our third relation, unlike the L-S operators, has the potential to take a pair in one symmetric group into a pair in a *different* symmetric group.

We say that a capitol for a pair $x, w \in S_n$ is *naked* if it lies within an unshaded region of the corresponding Bruhat picture. The *compression* $(x^{\hat{\imath}}, w^{\hat{\imath}})$ of (x, w) at the naked capitol $(i, x(i)) = (i, w(i))$ corresponds to deleting the ith columns and $w(i)$th rows of $\mathcal{D}_x$ and $\mathcal{D}_w$. Running the process in reverse is termed a *decompression*. The pair (x, w) is *incompressible* if its Bruhat picture has no naked capitols. Note that compressing a pair $x, w \in S_n$ produces a pair in S_{n-1}, while decompression produces one in S_{n+1}. In the figures, compression(s) will be denoted by a C and decompressions by a D. A proof of the following can be found in [Billey and Warrington 03, Lemma 39].

Fact 3.5. *For any naked capitol $(i, x(i)) = (i, w(i))$, we have both $P_{x,w}(q) = P_{x^{\hat{\imath}}, w^{\hat{\imath}}}$ and $\ell(w) - \ell(x) = \ell(w^{\hat{\imath}}) - \ell(x^{\hat{\imath}})$. Hence, $\mu(x, w) = \mu(x^{\hat{\imath}}, w^{\hat{\imath}})$.*

4. RESULTS

4.1. Computation of Kazhdan–Lusztig Polynomials

Construction of the database encoding all Kazhdan–Lusztig polynomials for pairs $x, w \in S_m$ with $m \le$

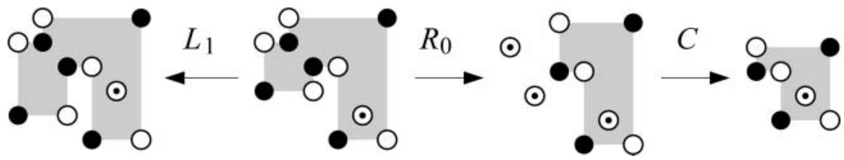

FIGURE 2. Example actions of L_1 and R_0 on the pair $x = 243015$, $w = 452310$. The simultaneous compression of (xR_0, wR_0) at two capitols is displayed in the rightmost picture.

10 proceeded by a direct application of (2–1). Our algorithm is basically that of the original recurrence in [Kazhdan and Lusztig 79] as described in [Humphreys 90]. However, two aspects of our algorithm merit note.

First, equation (3–1) allows us to focus on extremal pairs. As in the program [du Cloux 11], when required to compute $P_{x,w}(q)$ for any pair $(x, w) \notin \mathrm{EP}(n)$, we simply move x up in the Bruhat order through the action of elements of $\mathrm{rds}(w)$ and $\mathrm{lds}(w)$. Second, Fact 3.5 allows us to focus on incompressible pairs. When required to compute the Kazhdan–Lusztig polynomial for a compressible pair, we take the novel approach of first compressing a pair (x', w') as much as possible. Often, the resulting pair is not extremal. Moving x' up in the Bruhat order can then lead to additional naked capitols. The process can repeat, as illustrated in Figure 3.

A great deal of redundancy is avoided by keeping track of only the incompressible extremal pairs. In S_{10}, for example, 90% of the extremal pairs are compressible.

Table 1 collects various data regarding Kazhdan–Lusztig polynomials and their computation. The first five rows list the numbers of extremal pairs, incompressible extremal pairs, extremal pairs with positive μ-value, irreducible pairs, and $(n, 0)$-minimal pairs (these last two terms are defined in Sections 4.2 and 4.3). The final two rows reflect (among *all* $P_{x,w}(q)$ with $x, w \in S_n$) the maximum coefficient encountered and the number of distinct nonconstant polynomials appearing. Due to memory constraints, we have only partial results for S_{11}.

Remark 4.1. It is not clear how to take full advantage of parallel computation in computing collections of Kazhdan–Lusztig polynomials via (2–1). The computation of $P_{x,w}(q)$ is not local, in the sense that it is not clear which $P_{u,v}(q)$ will be required during the recursive steps. In fact, due to the structure of the recursive branching, any given $P_{u,v}(q)$ may be required *many* times. As such, the most efficient approach appears to store the intermediate $P_{u,v}(q)$ whenever possible. For S_{11}, however, such a database (useful in this way only if kept in RAM) would run to roughly 50 gigabytes.

4.2. Computing Possible μ-Values

For $n \leq 10$, the possible μ-values can be extracted directly from the database. For $n = 11$, the memory constraints discussed in Remark 4.1 prevented us from computing the Kazhdan–Lusztig polynomials for all incompressible extremal pairs. Fortunately, the identities of Section 3 provide a simple way to filter out pairs x, w for which $\mu(x, w) \notin M(n) \setminus M(n-1)$.

Define two pairs in S_n to be $\sim_{\mathrm{LS}}$-*equivalent* if they can be connected by a finite chain of L-S operators. Denote the corresponding equivalence classes by $[[x, w]]_{\mathrm{LS}}$. Let x, w be an odd pair. Suppose $[[x, w]]_{\mathrm{LS}}$ contains a pair u, v that is (1) compressible, (2) not extremal and with $\ell(u) < \ell(v) - 1$, or (3) not related in the Bruhat order. In the first case, $\mu(x, w) \in M(m)$ for some $m < n$. But the following lemma already tells us that such values are contained in $M(n)$.

Lemma 4.2. *For $n \geq 2$, $M(n-1) \subseteq M(n)$.*

Proof. Any pair $x, w \in S_{n-1}$ can be decompressed by adding a capitol in the nth row and nth column. The lemma then follows by Fact 3.5. □

In the second and third cases, $\mu(x, w)$ must be 0. So in looking for elements of $M(n) \setminus M(n-1)$, we can restrict our attention to odd extremal pairs in S_n for which none

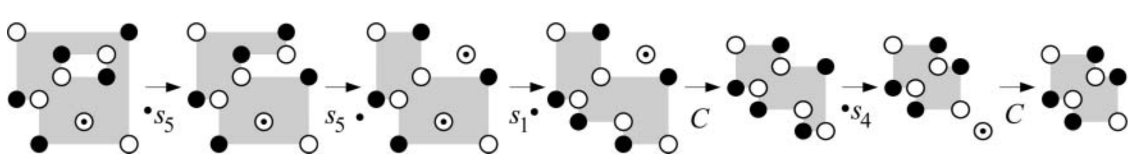

FIGURE 3. Example of how compression can lead to an extremal pair no longer being extremal.

n	4	5	6	7	8	9	10	11
$\lvert \mathrm{EP}(n)\rvert$	6	122	2 220	45 184	1 107 636	33 487 176	1 248 544 230	56 786 656 838
$\lvert \mathrm{EP}_{\mathrm{unc}}(n)\rvert$	2	10	152	3 114	84 624	2 896 168	122 345 174	6 252 533 464
$\lvert \mathrm{EP}_{\mu>0}(n)\rvert$	2	2	30	176	2 312	33 550	664 752	
$\lvert \mathrm{Irr}(n)\rvert$	0	0	0	0	0	16	2 663	54 214
$\lvert(n,0)$-minimal$\rvert$	0	0	0	0	0	12	2 512	51 060
max coeff.	1	2	4	15	73	460	4 176	≥18 915
$\lvert\{P_{x,w}(q)\}\rvert$	1	4	16	97	1 118	24 361	981 174	

TABLE 1. Kazhdan–Lusztig data for various S_n.

of the above three cases apply. Such pairs will be termed *irreducible*. It is significantly faster to compute whether a pair is irreducible than to compute the corresponding Kazhdan–Lusztig polynomial.

Even though there are over half a million μ-positive pairs in S_{10}, there are only 2663 irreducible pairs. The computation of the Kazhdan–Lusztig polynomials for the 54 214 irreducible pairs in S_{11} can be done in a few thousand hours of CPU time.

This completes the description of the work required for the first two parts of Theorem 1.1. The elements of $M(12)$ given there stem from individual Kazhdan–Lusztig polynomials we chose to compute guided by Conjecture 5.1. See Table 2 for representative pairs yielding these μ-values. (In the table, the polynomial $a_0 + a_1 q + a_2 q^2 + \cdots$ is described by its coefficient list: $a_0, a_1, a_2, \ldots$.)

4.3. Equivalence Classes of Pairs

Let $\mathrm{EP}'_{\mu>0}(n) = \mathrm{EP}_{\mu>0}(n) \cup \{(x,w) : w \text{ covers } x\}$ denote the set of pairs $(x,w) \in S_n \times S_n$ for which $\mu(x,w) > 0$. Write $\mathrm{EP}'_{\mu>0}$ for the union of $\mathrm{EP}'_{\mu>0}(n)$ as n runs over the positive integers. The identities in Facts 3.3, 3.4, and 3.5 allow us to define the following equivalence relation on the elements of $\mathrm{EP}'_{\mu>0}$: Two pairs in $\mathrm{EP}'_{\mu>0}$ are $\sim$-equivalent if they can be connected by a finite chain consisting of LS-moves, compressions/decompressions, and symmetries. (That is, $\sim$ is the transitive closure of the union of the relations arising from Facts 3.3, 3.4, and 3.5.)

By construction, $\mu[\cdot,\cdot]$ is constant on $\sim$-equivalence classes. Hence, the number of classes intersecting S_m for $m \le n$ gives an upper bound on the size of $M(n)$. Unfortunately, we have no algorithm (in the precise sense of the word) for computing the equivalence classes: To show that (x,w) and (y,v) are equivalent, we must provide a chain $(x,w) \sim (x',w') \sim \cdots \sim (y,v)$ in which each successive pair is connected by an L-S operator, a compression, a decompression, or a symmetry. However, we have no bound on how large a symmetric group we might have to pass through in order to construct such a chain; we can *always* decompress. In other words, given pairs with the same μ-value, we have no effective method for showing that they are *not* in the same $\sim$-equivalence class. In light of this problem, we define μ-positive pairs $(x,w) \in S_m$ and (x',w') in S_n to be $\overset{k}{\sim}$-*equivalent* if they can be connected by a chain that does not pass through $S_{\max(m,n)+k+1}$. An (n,k)-*minimal* pair is one whose $\overset{k}{\sim}$-equivalence class does not intersect S_m with $m < n$. The irreducible pairs in S_n with positive μ-value are the $(n,0)$-minimal pairs.

Let A be the $(\lvert \mathrm{EP}'_{\mu>0}(n)\rvert + 1) \times (\lvert \mathrm{EP}'_{\mu>0}(n)\rvert + 1)$ zero–one matrix with the first row and column indexed by a "sink" and all other rows/columns indexed by the elements of $\mathrm{EP}'_{\mu>0}(n)$. The sink will identify all pairs in $\mathrm{EP}'_{\mu>0}(n)$ that are not (n,k)-minimal. There is a straightforward algorithm for determining the (n,k)-minimal equivalence classes:

1. Pick k. Initialize all entries of A to 0.
2. For each pair $(x,w) \in \mathrm{EP}'_{\mu>0}(n)$ (indexing row/column i), perform a breadth-first search of the members of its $\overset{k}{\sim}$-equivalence class by considering L-S moves, symmetries, compressions, and decompressions. (Allow decompressions only in the case that the resulting pair lies in S_m for some $m \le n+k$.)
3. For each pair (y,v) (indexing row/column j) encountered in Step 2, set $A(i,j) = 1$.
4. If (x,w) is related to a pair in some S_m, $m < n$, then set $A(i,1) = 1$.

n	μ	x	w	$P_{x,w}(q)$
1	1	01	10	1
10	4	0432187659	4678091235	1,14,60,96,43,4
	5	2106543987	5678901234	1, 10, 43, 86, 84, 37, 5
11	3	108765432a9	789a4560123	1,14,82,247,420,420,235,60,3
	18	21076543a98	792a4560813	1, 16, 112, 442, 1038, 1485, 1309, 698, 200, 18
	24	1065432a987	689a1345702	1, 17, 129, 556, 1416, 2143, 1919, 993, 269, 24
	28	21076543a98	6789a123450	1, 18, 145, 646, 1654, 2516, 2283, 1197, 325, 28
12	6	107654328ba9	b6789a123450	1, 24, 267, 1772, 7554, 21518, 41845, 55849, 50705, 30547, 11637, 2552, 259, 6
	7	21076543ba98	b6789a501234	1, 4, 18, 83, 233, 514, 1045, 1571, 1648, 1373, 869, 341, 73, 7
	8	054321ba9876	9ab834567012	1, 11, 59, 213, 579, 1216, 1920, 2216, 1823, 1034, 386, 89, 8
	23	543210ba9876	9ab345678012	1,13,71,207,337,311,153,23
	25	10765432ba98	9ab345678012	1, 24, 253, 1527, 5662, 13109, 18983, 16997, 9166, 2836, 453, 25
	26	10765432ba98	789ab1234560	1, 21, 191, 933, 2561, 4008, 3573, 1735, 387, 26
	27	10765432ba98	b6789a012345	1, 21, 191, 933, 2554, 3994, 3583, 1772, 415, 27
	158	210876543ba9	b6789a123450	1, 24, 266, 1752, 7380, 20722, 39703, 52400, 47388, 28667, 10969, 2301, 158
	163	21076543ba98	b6789a123450	1, 23, 250, 1682, 7564, 23555, 51779, 80733, 88768, 67850, 35154, 11769, 2280, 163
13	796	321087654cba9	c789ab1234560	1, 27, 347, 2808, 15615, 62330, 183306, 401999, 658761, 802957, 721035, 469418, 215528, 66010, 12044, 796

TABLE 2. Known values of $\mu(x, w)$ and pairs that achieve them.

5. We then compute the connected components using Matlab's `graphconncomp` command. (Since A may be missing edges originating at the sink, we use the `weak` option.)

Table 3 illustrates how the various equivalence classes coalesce for $9 \le n \le 11$ as k ranges from 0 to 2. An s entry (for "sink") indicates that some of the pairs are not (n, k)-minimal. Theorem 1.2 is immediate. We computed the corresponding $(n, 3)$-minimal classes for all cases except the $\mu = 1$, $n = 11$ class, for which we ran out of memory. For the computed cases, the $(n, 3)$-minimal classes equaled the $(n, 2)$-minimal classes. Figure 4 gives the Bruhat pictures for (noncanonical) representatives of each $(n, 2)$-minimal class.

We suspect that some of these classes may coalesce further as k is increased. However, already at $k = 3$, com-

			k		
n	μ	No.	0	1	2
9	1	12	3	s	s
10	1	586	31	s+1	s+1
	4	428	10	3	2
	5	1498	27	2	1
11	1	26336	419	s+1	s+1
	3	2466	36	2	1
	4	5166	59	s+3	s+1
	5	17052	170	s	s
	18	16	1	1	1
	24	16	1	1	1
	28	8	2	2	2

TABLE 3. Coalescence of $\overset{k}{\sim}$-equivalence classes.

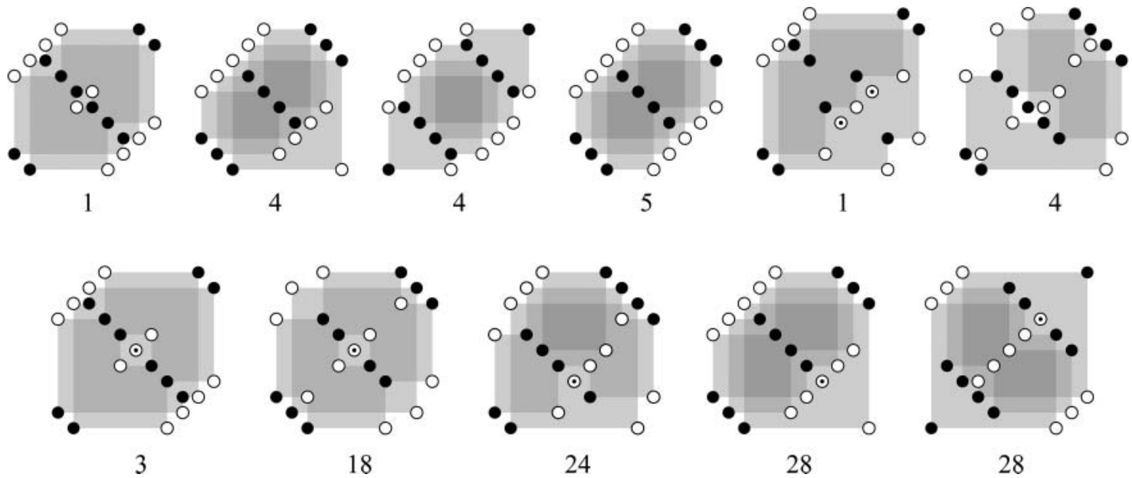

FIGURE 4. Representatives of $(n, 2)$-minimal classes.

putations become demanding. For example, consider the $(11, 0)$-minimal pair $x = 21076543a98$, $w = 6789a123450$. The size of its $\overset{k}{\sim}$-equivalence class grows from 1032 to 879 316 to 331 361 376 as k goes from 1 to 2 to 3.

As an example of coalescence, we consider one of the twelve $(9, 0)$-minimal pairs in S_9. Figure 5 demonstrates the equality $[[216540873, 567812340]] = [[01, 10]]$. Any chain connecting these two pairs must pass through S_{10}. This example also serves to illustrate that the Kazhdan–Lusztig polynomials are not preserved by the L-S operators: $P_{01,10}(q) = 1$, while

$$P_{216540873,567812340}(q) = 1 + 8q + 16q^2 + 11q^3 + q^4.$$

5. REPRESENTATIVES OF EQUIVALENCE CLASSES

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$, let x_α be the permutation

$$[n - \alpha_1, n - \alpha_1 + 1, \ldots, n - 1, n - \alpha_1 - \alpha_2, \\ n - \alpha_1 - \alpha_2 + 1, \ldots, n - \alpha_1 - 1, \ldots, 0, 1, \ldots, \alpha_k - 1].$$

Let $X_n = \{x_\alpha : \alpha \models n\}$. We define a *crosshatch pair* to be a pair $x \le w$ for which xw_0, $w \in X_n$.

Conjecture 5.1. *Every $\sim$-equivalence class contains a crosshatch pair.*

In particular, while we conjecture that each n-minimal class has a crosshatch pair, there may be such pairs in

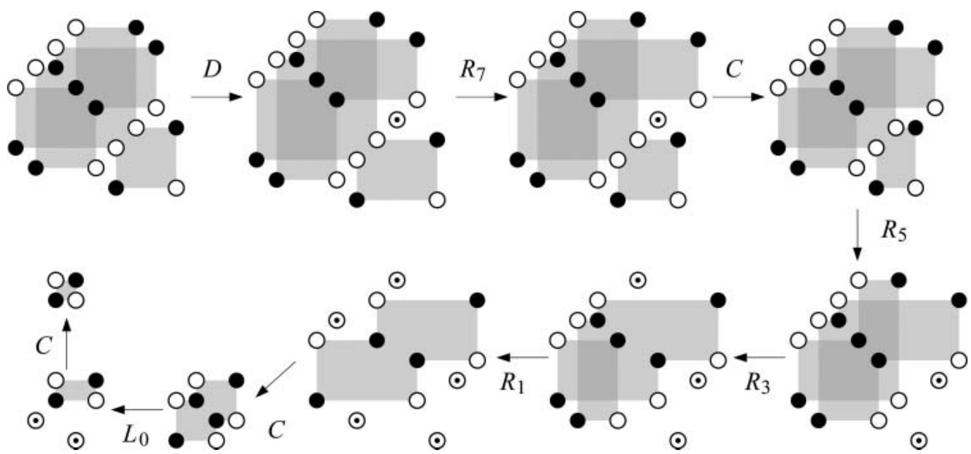

FIGURE 5. Reduction of $(x, w) = (216540873, 567812340)$.

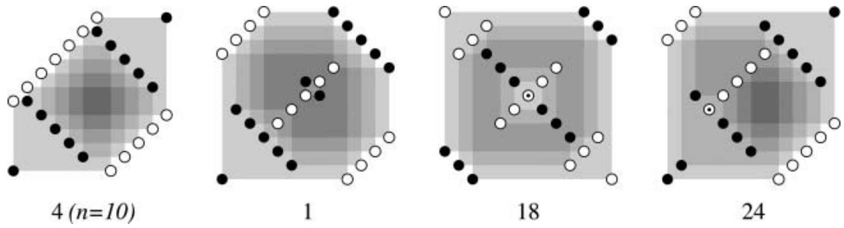

FIGURE 6. Crosshatch representatives.

S_m only with $m > n$. Even after factoring out symmetry, such putative representatives are not unique. Recall that Figure 4 gives representatives for the various $(n, 2)$-minimal equivalence classes that we have been able to compute. For five of these classes (one $n = 10$, $\mu = 4$ class and the $\mu = 1, 4, 18, 24$ classes for $n = 11$), the representative given in that figure is not a crosshatch pair. Figure 6 remedies this for four of the classes by giving crosshatch representatives lying in S_m with m equal to 12 or 13. The class for which we were unable to find a crosshatch representative is the $n = 11$, $\mu = 4$ class. However, given our above remark about the sizes of $\stackrel{k}{\sim}$-equivalence classes, we do not feel that this is a significant mark against Conjecture 5.1. The three possibilities are that this class is not $(11, k)$-minimal for some $k > 3$, that its smallest crosshatch pair lies in S_m for some $m \geq 15$, and that it does not contain a crosshatch pair at all.

In light of Conjecture 5.1, it is reasonable to ask whether there are simple criteria for the μ-value of a crosshatch pair to be nonzero, or even more ambitiously, to ask for a simple closed formula for the value of μ on such an interval. We note here that Brenti (along with various coauthors; see [Brenti et al. 06, Brenti and Incitti 06]) has closed formulas for Kazhdan–Lusztig polynomials based on alternating sums of paths that might be specialized for this purpose.

It would also be interesting to understand geometrically why such intervals appear so prevalent among pairs with μ-values greater than 1; the crosshatch intervals are minimal coset representatives for certain Richardson varieties with respect to independent partial flag manifolds [Knutson 70]. Of course, everything in this section may be attributable to working with values of n that are too small. On the other hand, crosshatch pairs are relatively rare even for these small values of n. Of the 1.2 billion extremal pairs in S_{10}, only 4708 are crosshatch pairs.

ACKNOWLEDGMENTS

I would like to thank Allen Knutson for helpful discussions. The hash-generation code released by Bob Jenkins [Jenkins 11] and the hash table code released by Troy D. Hanson [Hanson 11] are much more robust and efficient than anything I could have written. I am also grateful to the University of Vermont and its Vermont Advanced Computing Center for generously providing access to its cluster. The author was supported in part by National Security Agency grant H98230-09-1-0023.

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Received July 21, 2010; accepted November 18, 2010.