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Small Subgroups of SL(3, $\mathbb{Z}$)

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CONTENTS

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Motivated in part by various questions of Serre, Labourie and Lubotzky, we consider the question of representing the fundamental group of the figure eight knot complement into SL(3, $\mathbb{Z}$). We explore questions of faithfulness and finite index for such representations.

1. INTRODUCTION

Representations of the fundamental groups of finite-volume hyperbolic surfaces and finite-volume hyperbolic 3-manifolds into Lie groups have long been studied. Classical cases such as the case of the Lie groups $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbf{C})$, and $\mathrm{SU}(2)$ have provided powerful tools for the study of these groups, and the geometry and topology of the manifolds. More recently, this has been pursued in other Lie groups (see [Cooper et al. 06], [Goldman 90], and [Schwartz 07], to name a few).

In particular, [Cooper et al. 06] provides a powerful method for the construction of representations into the groups $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$. This paper was motivated by an examination of the integral points of such representations with a view to addressing questions about the subgroup structure of $\mathrm{SL}(3,\mathbb{Z})$. In particular, we give a partial answer to a question of Lubotzky, which we now describe.

The group $\mathrm{SL}(3,\mathbb{Z})$ has the congruence subgroup property, and in this sense its finite-index subgroup structure is much simpler than that of a lattice in $\mathrm{SL}(2,\mathbf{C})$. However, some interesting questions about the structure of subgroups of finite index remain. For example, the following question is asked in [Lubotzky 86, Section 4, Problem 1]:

Question 1.1. For $n \geq 3$, does $\mathrm{SL}(n,\mathbb{Z})$ contain arbitrarily small two-generator finite-index subgroups?

By "arbitrarily small," Lubotzky means that every finite-index subgroup of $\mathrm{SL}(3,\mathbb{Z})$ contains a two-generator subgroup of finite index.

Some progress on Question 1.1 is given in [Sharma and Venkataramana 05], where it was shown that any noncocompact irreducible lattice in a higher-rank real semisimple Lie group contains a subgroup of finite index that is generated by three elements. In addition, it is known that Question 1.1 has an affirmative answer for $SL(n,\mathbb{Z}_p)$ (see [Lubotzky and Mann 87]). In this paper we provide evidence for an affirmative answer for the case $n = 3$:

Theorem 1.2. *The group* $SL(3,\mathbb{Z})$ *contains a family* $\{N_j\}$ *of two-generator subgroups of finite index with the property that* $\cap N_j = 1$.

The nature of the subgroups used to resolve this question is perhaps as interesting as the resolution itself: Using the method developed in [Cooper et al. 06], we produce two one-parameter families of representations of $\pi_1(S^3 \setminus K)$ into $SL(3,\mathbb{R})$, where K is the figure-eight knot. These families have the property that integral specializations of subgroups of this image group, in particular the group itself and the image of the fiber group, give some potentially very interesting subgroups of $SL(3,\mathbb{Z})$. A sketch of this construction is described in Section 7.1.

We now give an overview of the content and some further results in this note. In Section 2 we introduce two families of representations, $\mathcal{F}_k$ and $\mathcal{F}_T$, of the figure-eight-knot group into $SL(3,\mathbb{R})$ that are irreducible with a small number of exceptions. Integral specializations of the parameters k and T give representations into $SL(3,\mathbb{Z})$. In order to prove Theorem 1.2, we first prove the following result.

Theorem 1.3. *Fix* $k \in \mathbb{Z}$ *(respectively nonzero* $T \in \mathbb{Z}$*). Then the image of the fiber groups* $\rho_k(F)$ *(respectively* $\beta_T(F)$*) are Zariski-dense subgroups of* $SL(3,\mathbb{R})$.

This has the interesting consequence that the figure-eight-knot group surjects all but finitely many of the finite simple groups $PSL(3,p)$.

In Section 3, we examine in greater detail the family β_T, which is used to prove Theorem 1.2.

Theorem 1.4. *Fix a nonzero integer value of* T*. Then the group* $\beta_T(F)$ *(and therefore* $\beta_T(\Gamma)$*) has finite index in* $SL(3,\mathbb{Z})$*. Furthermore,* $\cap_{T>0}\beta_T(F) = 1$.

The fact that each $\beta_T(F)$ has finite index rests on a result of Venkataramana (see [Venkataramana 87, Theorem 3.7]), which requires Zariski denseness and the construction of certain unipotent elements. The fact that the family is cofinal in T exploits a reducible specialization. We remark that while it follows from the statement that the index $[SL(3,\mathbb{Z}) : \beta_T(F)]$ approaches ∞ as T approaches ∞, the proof gives little idea what these indices actually are. They can be estimated, however, and a method for this is described at the end of Section 2; the indices are typically enormous. For example, $[SL(3,\mathbb{Z}) : \beta_7(F)]$ must be divisible by $1064332260 = 2^2 \cdot 3^2 \cdot 5 \cdot 17 \cdot 347821$.

In Section 4, we do some similar analysis for the family ρ_k. The situation for these representations is a good deal more delicate, and there is apparently none of the uniform behavior that made the family β_T tractable. We are able to prove finite index only for $k = 0, 2, 3, 4, 5$, and our method fails for other values. It appears to be very difficult to decide whether the subgroups $\rho_k(F)$ have finite index for $k \geq 6$. However, as in the previous paragraph, we are able to estimate the indices, and if they are finite, they are gigantic, which seems to be independently interesting.

In Section 5, we indulge in some speculation and potential applications directed toward the nature of finitely generated infinite-index subgroups; these remain very mysterious. Some work has been done on this (see [Venkataramana 87] and Section 3). However, some very basic questions remain unanswered. For example, an old question from [Serre 74] asks whether $SL(3,\mathbb{Z})$ is *coherent* (i.e., whether finitely generated subgroups of $SL(3,\mathbb{Z})$ are finitely presented). A question of a similar flavor is whether $SL(3,\mathbb{Z})$ has the *finitely generated intersection property* (i.e., the intersection of finitely generated subgroups of $SL(3,\mathbb{Z})$ is finitely generated).

One of the reasons that such questions have remained mysterious is the extraordinary difficulty of producing subgroups inside $SL(3,\mathbb{Z})$ that are interesting. If the representations ρ_k (for $k \geq 6$) have infinite index, they seem to be potentially useful in this regard, since one could then conjecture that the image of the stable letter does not power into the image of the fiber group, which suffices to disprove the finitely generated intersection property. This is explained in Theorem 5.2. With a little more, one can address the coherence question (see Section 5.2). A natural question raised by this work is whether there are any injections of finite-volume hyperbolic 3-manifold groups into $\mathrm{SL}(3,\mathbb{Z})$. This and some related issues are also touched upon in Sections 5 and 6 (in which we also collect some assorted final comments). The appendix contains some hints about calculations.

2. TWO REPRESENTATIONS OF THE FIGURE-EIGHT-KNOT GROUP

Let K denote the figure-eight knot and let $\Gamma = \pi_1(S^3 \setminus K)$. As is well known, Γ admits a presentation coming from the fact that $S^3 \setminus K$ is a once-punctured torus bundle over S^1. If we choose generators x and y for the fiber group (which we shall denote by F) and z as the stable letter, then Γ is presented as

$$\langle x, y, z \mid z \cdot x \cdot z^{-1} = x \cdot y,\ z \cdot y \cdot z^{-1} = y \cdot x \cdot y \rangle .$$

Given this presentation, the following proposition can be checked directly by matrix multiplication.

Proposition 2.1. *Define a map* $\rho_k : \Gamma \longrightarrow \mathrm{SL}(3,\mathbb{Z}[k])$ *by*

$$\rho_k(x) = X_k = \begin{pmatrix} 1 & -2 & 3 \\ 0 & k & -1-2k \\ 0 & 1 & -2 \end{pmatrix},$$

$$\rho_k(y) = Y_k = \begin{pmatrix} -2-k & -1 & 1 \\ -2-k & -2 & 3 \\ -1 & -1 & 2 \end{pmatrix},$$

$$\rho_k(z) = Z_k = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -k \\ 0 & 1 & -1-k \end{pmatrix}.$$

Then ρ_k *is a homomorphism.*

While these matrices appear fairly innocuous, we will show that they generate rather interesting subgroups. For example, we shall show that if $k = 5$, then $\langle X_5, Y_5 \rangle$ has finite index in $\mathrm{SL}(3,\mathbb{Z})$. While we are unable to say exactly what this index is, we can prove that it must be divisible by $2^2 \cdot 3^3 \cdot 5 \cdot 31^2 \cdot 127 \cdot 331$.

The second family of representations is described as follows.

Proposition 2.2. *Define a map* $\beta_T : \Gamma \longrightarrow \mathrm{SL}(3,\mathbb{Z}[T])$ *by*

$$\beta_T(x) = X_T = \begin{pmatrix} -1+T^3 & -T & T^2 \\ 0 & -1 & 2T \\ -T & 0 & 1 \end{pmatrix},$$

$$\beta_T(y) = Y_T = \begin{pmatrix} -1 & 0 & 0 \\ -T^2 & 1 & -T \\ T & 0 & -1 \end{pmatrix},$$

$$\beta_T(z) = Z_T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & T^2 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then β_T *is a homomorphism.*

In either case, an integral specialization gives the following corollary.

Corollary 2.3. *For integral* k *or* T, $\rho_k(\Gamma), \beta_T(\Gamma) \leq \mathrm{SL}(3,\mathbb{Z})$.

Henceforth we shall refer to these families of representations as $\mathcal{F}_k$ and $\mathcal{F}_T$. We begin with some basic analysis of this pair of families of representations, beginning with the issue of irreducibility.

Lemma 2.4. *The representations* ρ_k *and* β_T *are each irreducible except possibly for four exceptional values of their parameter.*

In particular, ρ_k *is irreducible for all* $k \in \mathbb{Z}$, *and* β_T *is irreducible for all nonzero* $T \in \mathbb{Z}$.

Proof. Suppose that the representation ρ_k is reducible. Then since $n = 3$, there must be an invariant one-dimensional subspace either for ρ_k or for the associated contragredient representation (i.e., the representation obtained by composing the given representation with the inverse transpose).

It follows that if the representation is reducible, then $\rho_k([x,y])$ must have eigenvalue 1. A computation shows that the characteristic polynomial of this element is

$$\begin{aligned} p_k(Q) = 1 &+ (-17 - 2k - 2k^2)Q \\ &+ (6 - 8k - 7k^2 + 2k^3 + k^4)Q^2 - Q^3, \end{aligned}$$

which when evaluated at $Q = 1$ gives

$$p_k(1) = (-11 + k + k^2)(1 + k + k^2).$$

Thus ρ_k is irreducible (even when restricted to F) except possibly for the four values of k that are roots of this equation. In particular, it is irreducible for any specialization $k \in \mathbb{Z}$.

We argue similarly for β_T. In this case, the characteristic polynomial of the image of the commutator evaluated at 1 is $-T^3(-8+3T^3)$, and the result follows as above. □

Remark 2.5. The case of the factor $(-8+3T^3)$ that arises in the analysis of the family $\mathcal{F}_T$ indeed determines a reducible representation, and we will make use of this later (see Section 3 for an explicit discussion of this).

Our next general observation concerns the Zariski denseness of these representations. This will be needed for the proof of Theorem 1.2.

Theorem 2.6. *Fix $k \in \mathbb{Z}$ (respectively nonzero $T \in \mathbb{Z}$). Then the image of the fiber groups $\rho_k(F)$ (respectively $\beta_T(F)$) are Zariski-dense subgroups of* $\mathrm{SL}(3,\mathbb{R})$.

Of course, this implies that the groups $\rho_k(\Gamma)$ and $\beta_T(\Gamma)$ are Zariski dense. The case $T=0$ is rather different, since $\beta_0(\Gamma)$ is finite. An easy computation shows that the image of the group F is a $\mathbb{Z}/2 \times \mathbb{Z}/2$ group on which $\beta_0(z)$ acts as the obvious element of order three.

Notation 2.7. Throughout, we will denote the finite groups $(\mathrm{P})\,\mathrm{SL}(n,\mathbb{F}_p)$ and $(\mathrm{P})\,\mathrm{GL}(n,\mathbb{F}_p)$ by $(\mathrm{P})\,\mathrm{SL}(n,p)$ and $(\mathrm{P})\,\mathrm{GL}(n,p)$ respectively.

The proof of Theorem 2.6 is structured in the following way. A key ingredient is the following result [Lubotzky 97, Proposition 1 with $n=3$].

Proposition 2.8. *Let $\Gamma < \mathrm{SL}(3,\mathbb{Z})$ and assume that for some odd prime $p \geq 3$, Γ surjects onto $\mathrm{SL}(3,p)$, under the reduction homomorphism modulo p. Then Γ is a Zariski-dense subgroup of* $\mathrm{SL}(3,\mathbb{R})$.

We then combine Proposition 2.8 with the following theorem.

Theorem 2.9. *Let G be a finitely generated nonsolvable subgroup of $\mathrm{SL}(3,\mathbb{Z})$. Suppose that there is an element $g \in G$ whose characteristic polynomial is $\mathbb{Z}$-irreducible and noncyclotomic.*

Then for infinitely many primes p, reduction modulo p surjects G onto $\mathrm{SL}(3,p)$.

The result of Theorem 2.6 will then follow by exhibiting some explicit elements of the type required by Theorem 2.9.

Proof of Theorem 2.9. Our strategy will be to apply results about the structure of subgroups of $\mathrm{SL}(3,p)$ due to Bloom. In fact, [Bloom 67] deals with subgroups of $\mathrm{PSL}(3,p)$, but we will simply blur this distinction here. Indeed, it is easy to see that G surjects $\mathrm{SL}(3,p)$ if and only if it surjects $\mathrm{PSL}(3,p)$, so there is no loss in considering only $\mathrm{SL}(3,p)$. We give the argument here.

One way is clear, and so if now G surjects $\mathrm{PSL}(3,p)$ and not $\mathrm{SL}(3,p)$, then the image of G in $\mathrm{SL}(3,p)$ is some proper subgroup $G_0 < \mathrm{SL}(3,p)$. Denoting the center of $\mathrm{SL}(3,p)$ by Z, it follows that $\mathrm{SL}(3,p) = \langle G_0, Z\rangle$. Now, Z is either the trivial group or a cyclic group of order 3. Thus, we will now assume that Z is cyclic of order 3. It follows from this that G_0 is a normal subgroup of $\mathrm{SL}(3,p)$ of index 3. However, this is impossible, since $\mathrm{SL}(3,p)$ is a perfect group.

We state only what will be needed from [Bloom 67] for us. This statement follows directly from [Bloom 67, Theorems 1.1 and 7.1]. Note that in the notation of [Bloom 67], $\alpha = 1$.

Theorem 2.10. [Bloom 67] *Suppose that p is a prime and H is a proper subgroup of $\mathrm{PSL}(3,p)$. Then H has one of the following forms:*

(1) *If H has no nontrivial normal elementary abelian subgroup, then H is isomorphic to one of $\mathrm{PSL}(2,p)$, $\mathrm{PSL}(2,7)$, A_5, A_6, and A_7.*

(2) *If H contains a nontrivial normal elementary subgroup, then H has a normal subgroup N that is one of the following: cyclic of index ≤ 3, a diagonal subgroup with H/N isomorphic to a subgroup S_3, a normal elementary abelian p-subgroup with H/N isomorphic to a subgroup of* $\mathrm{GL}(2,p)$.

We will use this result to show that for infinitely many p, the modulo-p reduction of G cannot be any of the exceptional groups provided by Theorem 2.10. This proves that G must surject $\mathrm{SL}(3,p)$ for any such p.

We begin by noting the following. The first two exceptional types in clause (2) are solvable groups of class at most three. Since G is nonsolvable, there is a nontrivial element in any term of the derived series, so that if we fix an element in the third term of the derived series, then except for possibly finitely many primes, the mod-p reduction of this element will be nontrivial.

It follows that by restricting to sufficiently large primes, we can assume that the mod-p reduction of G is of neither of those two types.

Let $g \in G$ be an element with irreducible noncyclotomic characteristic polynomial provided by the hypothesis. In particular, g has infinite order. Let n be the least common multiple of the orders of all elements of all of the finite groups $\mathrm{PSL}(2,7)$, A_5, A_6, and A_7 coming from the list given in Theorem 2.10(1).

The element g^n is not the identity, and its entries are bounded above by M, say, so that as long as we consider primes $p > M$, the reduction modulo p of g^n will not be trivial, since g has order too large for the image group to be on that list.

Henceforth we consider only primes that are sufficiently large for the considerations of the previous two paragraphs to apply. We next make the following claim.

Claim 2.11. *Let $p(Q)$ be the characteristic polynomial of the element g. Then there are infinitely many primes p for which $p(Q)$ is irreducible over $\mathbb{F}_p$.*

Proof. This is a standard consequence of the Čebotarev density theorem (see [Narkiewicz 04, Section 7.3]). We sketch the details.

Let K denote the number field generated over $\mathbb{Q}$ by a root of p, and R the ring of integers of K. By assumption, p has degree 3 and is $\mathbb{Z}$-irreducible. Hence $[K : \mathbb{Q}] = 3$.

The claim will follow once we establish that there are infinitely many rational primes p that remain totally inert to K; i.e., the ideal pR has norm p^3.

Let M denote the Galois closure of $K/\mathbb{Q}$. The possibilities for the Galois group of $M/\mathbb{Q}$ are the cyclic group of order 3 and the symmetric group S_3. In the former case, $M = K$, and the conclusion follows from the statement of the Čebotarev density theorem applied to the generator of the Galois group.

For the case that the Galois group is S_3, we argue as follows. The possible splitting types for rational primes p that are unramified to M are for p to split completely, or split as a product of prime ideals of M of the form P_1P_2 with $NP_1 = NP_2 = p^3$ or $P_1P_2P_3$ with $NP_1 = NP_2 = NP_3 = p^2$. The Čebotarev density theorem implies that there are infinitely many such rational primes p of each type. By considering the factorization of p in K and then in M/K, it follows that the case in which p splits with $NP_1 = NP_2 = p^3$ gives infinitely primes $\mathcal{P}$ in K with $N\mathcal{P} = p^3$ as required. This concludes the proof of Claim 2.11.

We further restrict attention to those primes p for which Claim 2.11 holds. The argument is now completed by showing that for these primes, we may simultaneously rule out both the remaining case from clause (1) (i.e., $\mathrm{PSL}(2,p)$) and the nonsolvable possibility of clause (2).

Let p be a prime that leaves $p(Q)$ irreducible over $\mathbb{F}_p$. This polynomial defines a unique cubic extension $L = \mathbb{F}_p(\lambda)$ of degree 3 over $\mathbb{F}_p$. Associated to the field extension $L/\mathbb{F}_p$ there is a norm map $N : L \to \mathbb{F}_p$ (see [Morandi 96, Section II.8], for example) that in our setting can be described as follows (see [Morandi 96, Proposition II.8.6]): If $\alpha \in L$ has $f(x) = x^m + \cdots + a_1x + a_0$ as its irreducible polynomial over $\mathbb{F}_p$, then $N(\alpha) = (-1)a_0^{3/m}$.

Restricting the norm map to the nonzero elements, we obtain a multiplicative homomorphism $\mu : L^* \longrightarrow \mathbb{F}_p^*$. Note that our given λ lies in the kernel of μ, since $N(\lambda) = (-1)\cdot(-1) = 1$.

We claim that $\ker(\mu)$ has order $p^2 + p + 1 = (p^3 - 1)/(p-1)$. The reason is this: Note that any extension of finite fields $L/\mathbb{F}_p$ is always Galois with cyclic Galois group. Thus, we may apply Hilbert's Theorem 90 (See [Morandi 96, Section II.10]). Here this says that if one fixes a generator σ of $\mathrm{Gal}(L/\mathbb{F}_p)$, then every element of norm 1 may be written as $a/\sigma(a)$ for some element $a \in L^*$.

Now consider the homomorphism $L^* \longrightarrow \ker(\mu)$ defined by $a \to a/\sigma(a)$. Hilbert's result implies that this is surjective, and the kernel is those elements of the field fixed by the Galois group, i.e., $\mathbb{F}_p^*$. Thus $|\ker(\mu)| = \mathrm{p}^2 + \mathrm{p} + 1$, as required.

Hence λ has multiplicative order dividing $p^2 + p + 1$. It follows that for the primes under consideration, the order of g divides $p^2 + p + 1$.

Now observe that $p^2 + p + 1$ is prime to both p and $p + 1$. Furthermore, an easy argument shows that the only prime that could divide both $p^2 + p + 1$ and $p - 1$ is 3.

Moreover, if p is congruent to 2 modulo 3, then $p^2 + p + 1$ is not divisible by 3, and if p is congruent to 1 modulo 3, then writing $p = 3r + 1$, we see that $p^2 + p + 1 = 3(1 + 3r + 3r^2)$. In particular, 3 divides $p^2 + p + 1$ with multiplicity at most one.

The upshot of this simple discussion is that the order of g modulo p is a divisor of 3τ, where τ divides $p^2 + p + 1$ and is prime to 3. Therefore, the element g^3 modulo p has order τ, where τ is prime to p, $p - 1$, and $p + 1$ and therefore prime to the orders of both $\mathrm{PSL}(2,p)$

and GL$(2,p)$ (see [Newman 72], for example). In either of these cases we deduce easily that the mod-p reduction of g^3 must be trivial; however, this was ruled out by the use of large primes. □

Proof of Theorem 2.6. The proof of Theorem 2.6 is concluded by exhibiting elements of the type required by Theorem 2.9; it is easily seen that for the given integral specializations, the image of the fiber group contains a free group of rank 2, which rules out the possibility of a solvable image.

We work with the representations ρ_k; the computation for β_T is entirely analogous. Fix some integral value of k and focus attention on the commutator element $[X_k, Y_k]$; we claim that this satisfies the conditions of Theorem 2.9.

Recall from the proof of Lemma 2.4 that the characteristic polynomial of this element is

$$p_k(Q) = 1 + (-17 - 2k - 2k^2)Q + (6 - 8k - 7k^2 + 2k^3 + k^4)Q^2 - Q^3.$$

One sees easily from this that the commutator has infinite order for any value of k. □

Claim 2.12. *$p_k(Q)$ is irreducible over $\mathbb{Z}$ for all $k \in \mathbb{Z}$.*

Proof. It suffices to prove the claim by reducing $p_k(Q)$ modulo 2. Since $p_k(Q)$ is cubic, one need only check that $p_k(Q)$ cannot have a linear factor.

Thus assume first that k is even. Then $p_k(Q)$ modulo 2 becomes $Q^3 + Q + 1$, which has no linear factor. When k is odd, notice that the coefficient of Q^2 becomes $k^4 + k^2$, which is even, and so once again the reduction of $p_k(Q)$ modulo 2 is $Q^3 + Q + 1$.

This concludes the proof of the Zariski denseness for the representations ρ_k (and β_T). □

As in [Lubotzky 97], Strong Approximation can be applied to prove the following corollary (using [Weisfeiler 84]).

Corollary 2.13. *For all but a finite number of primes $p \in \mathbb{Z}$, Γ surjects the finite simple group* PSL$(3,p)$.

Remark 2.14. The figure-eight knot admits a Seifert-fibered space surgery with base orbifold group the $(2,3,7)$-triangle group. Finite quotients of this group (so-called Hurwitz groups) have been widely studied, and using this, it can be shown that the figure-eight-knot group surjects many infinite families of nonabelian finite simple groups. However, it was shown in [Cohen 81] that the only Hurwitz group of the form PSL$(3,p)$ for p a prime has $p = 2$. Thus our construction gives more information.

We close this section with some comparisons between the families $\mathcal{F}_k$ and $\mathcal{F}_T$.

Remark 2.15. For $T \neq 0$, the image group $\beta_T(\Gamma)$ contains many "obvious" unipotent elements, whereas for $k \geq 6$, the groups $\rho_k(\Gamma)$ do not. This is easily seen by checking that y^2, $(yxy)^2$, and $(x^{-1}y)^2$ are all mapped to unipotent elements by β_T. Although this does not directly account for the finite-index results proven below, it is perhaps suggestive. For example, despite extensive searching, we have been unable to find a rank-one unipotent element in $\rho_6(F)$.

Remark 2.16. Somewhat amazingly, the following relation holds in $\beta_T(\Gamma)$ for every value of the parameter T:

$$X^{-1}YX^{-1}YX^{-1}X^{-1}YYYXYYXY^{-1}X = XY^{-1}XYYXYYYX^{-1}X^{-1}YX^{-1}YX^{-1}.$$

This appears to be the *shortest* relation for all but very small values of T. We have been unable to find an analogous universal relation for the family $\mathcal{F}_k$, although we have found some relations in these groups for $k \leq 5$.

3. THE IMAGE OF β_T

We now discuss each family of representations separately in greater detail, beginning with the image groups $\beta_T(F)$ (and $\beta_T(\Gamma)$).

We will prove the following result, from which Theorem 1.2 follows.

Theorem 3.1. *Fix a nonzero integer value of T. Then the group $\beta_T(F)$ (and therefore $\beta_T(\Gamma)$) has finite index in* SL$(3,\mathbb{Z})$. *Furthermore, $\cap_{T>0}\beta_T(F) = 1$.*

Note that by Margulis's normal subgroup theorem [Margulis 89], it follows that $\beta_T(\Gamma)$ is of finite index in SL$(3,\mathbb{Z})$ if and only if $\beta_T(F)$ has finite index in SL$(3,\mathbb{Z})$.

As indicated in Remark 2.15, the groups $\beta_T(\Gamma)$ contain unipotent elements. To prove finite index, we make use of the following result of Venkataramana (see [Venkataramana 87, Theorem 3.7]):

Theorem 3.2. *Suppose that $n \geq 3$ and $x \in$ SL$(n,\mathbb{Z})$ is a unipotent matrix such that $x - 1$ has matrix rank 1. Suppose that $y \in$ SL$(n,\mathbb{Z})$ is another unipotent matrix such that x and y generate a free abelian group N of rank 2.*

Then any Zariski-dense subgroup of $\mathrm{SL}(n,\mathbb{Z})$ *containing* N *virtually is of finite index in* $\mathrm{SL}(n,\mathbb{Z})$.

Proof of Theorem 3.1. We shall exhibit unipotent matrices b_1 and b_2 in $\beta_T(F)$ such that $b_1 - 1$ has rank 1 and $\langle b_1, b_2\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$. That $\beta_T(F)$ has finite index will then follow from Theorem 3.2 together with Theorem 2.6.

Taking

$$b_1 = X_T^{-1}\cdot Y_T\cdot Y_T\cdot Y_T\cdot X_T\cdot Y_T\cdot Y_T\cdot X_T\cdot Y_T^{-1}\cdot X_T$$

and

$$b_2 = X_T\cdot Y_T^{-1}\cdot X_T\cdot Y_T\cdot Y_T\cdot X_T\cdot Y_T\cdot Y_T\cdot Y_T\cdot X_T^{-1},$$

elementary linear algebra calculations show that both b_1 and b_2 are unipotent elements (having characteristic polynomials $-(-1+x)^3$), and $b_1 - 1$ and $b_2 - 1$ have rank 1. Conjugating by the matrix P,

$$\begin{pmatrix} 0 & 1 & 1\\ 2T & 0 & 1\\ 1 & 0 & 1\end{pmatrix},$$

shows that

$$c_1 = P^{-1}b_1P = \begin{pmatrix} 1 & 0 & -T^2(-1+2T)(-5+3T^3)\\ 0 & 1 & -T(-1+2T)(-2+3T^3)\\ 0 & 0 & 1\end{pmatrix}$$

and

$$c_2 = P^{-1}b_2P = \begin{pmatrix} 1 & 0 & -3T^2(-1+2T)\\ 0 & 1 & -T(-1+2T)(-2+3T^3)\\ 0 & 0 & 1\end{pmatrix}.$$

This exhibits the group $\langle c_1, c_2\rangle$ as acting affinely on the plane as two translations, so that the group is clearly free abelian, and it will be isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, provided the translations are linearly independent. Since the second components of the translation vectors are equal, this will be so if and only if their first components are equal, which is to say

$$T^2(-1+2T)(-5+3T^3) = 3T^2(-1+2T),$$

i.e., when $T^2(-1+2T)(-8+3T^3) = 0$. There are never any nonzero integral solutions, and the proof that $\beta_T(F)$ has finite index is complete.

To prove that these groups intersect in the identity as T varies over positive integers, we argue as follows.

Suppose that there is a nontrivial element $g \in \cap_{T>0}\beta_T(F)$. Notice that for any prime divisor p of T, reducing the coefficients of $\beta_T(F)$ modulo p coincides with the image of the group F under the representation β_0, that is, $\mathbb{Z}/2\oplus\mathbb{Z}/2$, with matrix image

$$\beta_0(F) = \left\langle \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\right\rangle.$$

By abuse of notation we will not distinguish $\beta_0(F)$ from its images in $\mathrm{SL}(3,p)$.

It follows that for any prime p, the image of g on reduction modulo p is one of four possible matrices, so that one of the matrices of $\beta_0(F)$ must occur infinitely often. Denoting this matrix by A, we see that $A\cdot g$ lies in infinitely many different principal congruence subgroups, so that $A\cdot g = \mathrm{Id}$, and therefore $g = A \in \beta_0(F)$.

So far, we have shown that $\cap_{T>0}\beta_T(F)$ contains at most the four elements of $\beta_0(F)$.

To rule out the three nontrivial elements, we need to delve somewhat more deeply into the reducible representations alluded to earlier. Taking

$$P = \begin{pmatrix} 4/T^2 & 0 & 0\\ -2/T & 1 & 1\\ 1 & 1 & 0\end{pmatrix},$$

we can conjugate the contragredient representation so that reducibility at $(-8+3T^3) = 0$ becomes obvious: $P^{-1}\beta_T(x)^*P =$

$$\begin{pmatrix} \frac{-4+3T^3}{4} & \frac{-T^3(1+2T)}{4} & \frac{-T^4}{2}\\ \frac{3(-8+3T^3)}{4} & \frac{-(1+2T)(-4+3T^3)}{4} & \frac{-T(-4+3T^3)}{2}\\ \frac{-(2+3T)(-8+3T^3)}{4T} & \frac{-8-8T+2T^2+7T^3+6T^4}{4} & \frac{-2-4T+2T^3+3T^4}{2}\end{pmatrix}$$

and $P^{-1}\beta_T(y)^*P =$

$$\begin{pmatrix} \frac{-4+3T^3}{4} & \frac{-T^3(1+2T)}{4} & \frac{-T^4}{2}\\ \frac{8-3T^3}{4} & \frac{-4-4T+T^3+2T^4}{4} & \frac{T(-2+T^3)}{2}\\ \frac{(2+T)(-8+3T^3)}{4T} & \frac{-(2+T)(-4+T^2+2T^3)}{4} & \frac{2+2T-2T^3-T^4}{2}\end{pmatrix}.$$

In particular, any fixed matrix $g\in\mathrm{SL}(3,\mathbb{Z})$ lying in $\cap_{T>0}\beta_T(F)$ must have the property that $P^{-1}g^*P$ has first column with $(2,1)$ and $(3,1)$ entries both divisible by $(-8+3T^3)$. However, this does not happen for the three nontrivial elements of $\beta_0(F)$. For example, one can compute that

$$P^{-1}\begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}P = \begin{pmatrix} -1 & 0 & 0\\ 2 & 1 & 0\\ -2 & -2 & -1\end{pmatrix}.$$

This completes the proof of Theorem 3.1 □

A more detailed examination of the last aspect of this proof gives some estimates for the index of these subgroups. For example, consider the case $T = 7$. Then

$3 \cdot 7^3 - 8 = 1021$, a prime. Reduction of the group $\beta_7(F)$ modulo 1021 is reducible, and the above computation shows that the image group Δ fits into a short exact sequence

$$1 \longrightarrow \mathbb{Z}/1021 \oplus \mathbb{Z}/1021 \longrightarrow \Delta \longrightarrow \mathrm{SL}(2, \mathbb{Z}/1021) \longrightarrow 1$$

and therefore has order

$$1021^2 \cdot 1021(1021-1)(1021+1) = 1109502522156840.$$

Of course, the group $\mathrm{SL}(3,\mathbb{Z})$ will surject $\mathrm{SL}(3,\mathbb{Z}/1021)$, which has size

$$1021^8 \left(1 - \frac{1}{1021^2}\right)\left(1 - \frac{1}{1021^3}\right) = 1180879326882889591658400$$

(see [Newman 72]), so that the index $[\mathrm{SL}(3,\mathbb{Z}) : \beta_7(F)]$ must be divisible by the ratio of these two group sizes, i.e.,

$$1064332260 = 2^2 \cdot 3^2 \cdot 5 \cdot 17 \cdot 347821.$$

4. THE IMAGE OF ρ_k

In this section we consider the family $\mathcal{F}_k$. Despite a certain uniformity linking the two constructions (see the appendix), the families of representations β_T and ρ_k appear to behave very differently.

We first prove that for some small values of k, suitable unipotent elements can be found.

Theorem 4.1. *The group $\rho_k(F)$ (and therefore $\rho_k(\Gamma)$) has finite index in* $\mathrm{SL}(3,\mathbb{Z})$ *for $k = 0, 2, 3, 4, 5$.*

Proof. The strategy is the same as that for the proof of Theorem 3.1, namely for these values of k we are able to locate inside $\rho_k(F)$ rank-one unipotents and unipotent elements that commute with them. The result will then follow as before.

Unlike the β_T representations, there seems to be no uniform way to construct the elements in question as k varies over the values above.

The following are the shortest words $\langle u_1, u_2\rangle$ known to the authors for which one can apply this method. To avoid unnecessarily cluttering the notation, we give the words as words in the fiber group F; their ρ_k images are the required unipotents:

$k = 0$: Then $u_1 = a_1 b_1$, $u_2 = c_1 \cdot d_1$, where

$$\begin{aligned} a_1 &= x^2 y^{-3} x y x^{-1}, \\ b_1 &= x^{-1} y x y^{-3} x^2, \\ c_1 &= x^2 y^{-3} x^3 y^{-1} x, \\ d_1 &= x y^{-1} x^3 y^{-3} x^2. \end{aligned}$$

$k = 2$: Then $u_1 = a_1 b_1 a_1^{-1} b_1^{-1}$, $u_2 = a_1 c_1 a_1^{-1} c_1^{-1}$, where

$$\begin{aligned} a_1 &= x^3 (yx)^3 y, \\ b_1 &= y^{-1} x^{-1} y x y^{-1} x y x^{-1} y^{-1}, \\ c_1 &= x^{-1} y x y^{-1} x^{-1} y x^{-1} y^{-1} x y. \end{aligned}$$

$k = 3$: Then $u_1 = a_1 b_1$, $u_2 = c_1 a_1 b_1 c_1^{-1}$, where

$$\begin{aligned} a_1 &= y x^{-2} y x^3, \\ b_1 &= x^{-3} y x^4 y, \\ c_1 &= x^2 y^{-1} x^{-1} y^{-1} x (xy)^{-2}. \end{aligned}$$

$k = 4$: Then $u_1 = a_1 \cdot b_1$, $u_2 = c_1 \cdot d_1$, where

$$\begin{aligned} a_1 &= (xy)^2 (yx)^{-2} x^2 y^{-1} x^{-2} y, \\ b_1 &= y x^{-2} y^{-1} x^2 (xy)^{-2} (yx)^2, \\ c_1 &= y^{-1} x^2 y x^{-2} (yx)^2 (xy)^{-2}, \\ d_1 &= (yx)^{-2} (xy)^2 x^{-2} y x^2 y^{-1}. \end{aligned}$$

$k = 5$: Then $u_1 = a_1 \cdot b_1$, $u_2 = b_1 \cdot c_1$, where

$$\begin{aligned} a_1 &= y x^{-3} y x^{-1} y^{-1} x y^{-1} x^{-1}, \\ b_1 &= x^{-1} y^{-1} x y^{-1} x^{-1} y x^{-3} y, \\ c_1 &= y^{-1} x^3 y^{-1} x^{-1} y^{-1} x y^{-1} x^{-1}. \end{aligned}$$

This completes the proof of Theorem 4.1. □

Remark 4.2. (i) We do not know whether $\rho_k(F)$ has finite index for the values not on this list; the approach outlined above seems to encounter difficulties, since for these other values, we have been unable to locate the required unipotents. As a byproduct of the proof, we generate the relation $[u_1, u_2] = 1$ in the image of the free group.

The case $k = 1$ seems different, and this is discussed in greater detail below.

(ii) As in the case for β_T, this method does not find the index of the subgroup. However, as we shall show below, one can estimate the index. For example, $[\mathrm{SL}(3,\mathbb{Z}) : \rho_5(F)]$ is divisible by $2^2 \cdot 3^3 \cdot 5 \cdot 31^2 \cdot 127 \cdot 331$.

A possible alternative approach to the finite-index question is the following: One of our motivations for consideration of these representations was to try to construct a representation of the figure-eight-knot group for which the stable letter does not power into the image of the fiber group. (See Section 5.1 for why this is of interest.) However, as we have observed above, [Margulis 89] implies that $\rho_k(\Gamma)$ is of finite index in $\mathrm{SL}(3,\mathbb{Z})$ if and only if $\rho_k(F)$ has finite index in $\mathrm{SL}(3,\mathbb{Z})$. Since $\rho_k(F)$ has finite index in $\mathrm{SL}(3,\mathbb{Z})$ implies that $Z_k^N \in F_k = \langle X_k, Y_k\rangle$ for some integer N, we can ask the following question.

Question 4.3. Is there a value of k for which Z_k does not power into $\langle X_k, Y_k\rangle$?

We note that for the values $k = 0, 1, 2$, the element Z_k must power into $\langle X_k, Y_k\rangle$. This follows from Theorem 4.1 for $k = 0, 2$, but this can be shown to be true for elementary reasons, as we now explain.

The reason is that for these values, the characteristic polynomial of the matrix Z_k has exactly one real root. It is well known that Dirichlet's unit theorem (see [Narkiewicz 04, Section 3.3]) implies that the free part of the unit group of the ring of integers for a field generated by a root of such a characteristic polynomial must be cyclic. Thus, some power of Z_k must be equal to some power of $[X_k, Y_k]$. A simple computation shows that

$$Z_0^{10} = [X_0, Y_0], \quad Z_1^4 = [X_1, Y_1], \quad Z_2^3 = [X_2, Y_2].$$

The case $k = 1$ seems particularly interesting. Note from the discussion of the previous paragraph that ρ_1 is a representation of Γ that factors through the fundamental group of the -4-surgery on the figure-eight knot. Since -4 is a boundary slope of the figure-eight knot, the result of this surgery is a Haken manifold that can be described as the union of the trefoil-knot exterior and the twisted I-bundle over the Klein bottle. Some degree of collapsing of this representation must occur (see Theorem 6.4). As such, its behavior might indeed be different from that of other values of k. Some experimentation suggests that $\rho_1(\Gamma)$ is virtually free.

It is easily shown that for $k \geq 3$ (for such k, the characteristic polynomial has three distinct real roots), a power of Z_k can never be a power of $[X_k, Y_k]$.

One way in which a positive answer to Question 4.3 could obtain is if for *generic* k, Z_k powered into F_k. However, this we can rule out, as the following theorem shows.

Theorem 4.4. *For generic k, the element Z_k does not power into the subgroup F_k.*

Proof. Let M be the matrix

$$M = \begin{pmatrix} (7-3\sqrt{5})/2 & 1 & 1 \\ (-3+\sqrt{5})/2 & 0 & -1 \\ (-3+\sqrt{5})/2 & 0 & -1 \end{pmatrix}.$$

Form a new representation $r : \Gamma \to \mathrm{SL}(3,\mathbb{R})$, by setting

$$r(g) = M^{-1} \cdot \rho_k(g)^* \cdot M,$$

where as above, $*$ denotes the contragredient. Now setting $k = (-1+3\sqrt{5})/2$, this representation becomes reducible. Notice that this k is a root of $(-11 + k + k^2)$; this is as it must be, given our observations about reducibility and the commutator.

At this value for k, the matrices for $r(x)$ and $r(y)$ have a common eigenvector, both with eigenvalue one, and this is an eigenvector for $r(z)$ with eigenvalue $(-3+\sqrt{5})/2$. It follows that $r(z)$ cannot power into $\langle r(x), r(y)\rangle$. □

While it differs in detail, one can use the kind of method that was described in Section 3 and exploit the exceptional representations coming from the roots of $(-11 + k + k^2)(k^2 + k + 1)$ to give estimates on the index of the subgroup $\rho_k(F)$. For example, if $k = 5$, then $5^2 + 5 - 11 = 19$, and as above, this gives that the index must be divisible by 6858.

In fact, one can go further for the ρ_k case. One can compute that for integral k, those primes p that divide $k^2 + k + 1$ do not give rise to reducible representations, but correspond to representations for which the image group is abstractly isomorphic to $\mathrm{PSL}(2, p)$, where the 3-dimensional integral representation arises via $\mathrm{SO}(\tau, \mathbb{F}_p)$ for a suitable and easily computed form τ. For example, taking $k = 5$, so that $5^2 + 5 + 1 = 31$, an analogous computation shows that the index must be divisible by 57256380. Putting these two computations together gives that the index of the finite-index subgroup $\rho_5(F)$ is divisible by $21814680780 = 2^2 \cdot 3^3 \cdot 5 \cdot 31^2 \cdot 127 \cdot 331$.

Using the same analysis, one can estimate indices for other values of k that are not known to be of finite index, for example, if $\rho_6(F)$ is of finite index, then this index must be divisible by $486591826140 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 43^2 \cdot 331 \cdot 631$.

5. FINITE OR INFINITE INDEX FOR $k \geq 6$? SOME SPECULATION AND APPLICATIONS

It is an intriguing and apparently difficult problem to understand the situation regarding the finite-index question for $k \geq 6$. In this section we indulge in some speculation and offer some applications. These are centered on an old question of Serre's concerning coherence and the finitely generated intersection property.

For the applications, the most useful information is whether z powers into the image of F, but one can quite naturally ask for a strengthening of this:

Question 5.1. Is there any value of k for which ρ_k is faithful?

We note that nontrivial normal subgroups inside a hyperbolic 3-manifold group must intersect. The fiber group F is normal in Γ and therefore must meet $\ker(\rho_{\mathrm{k}})$ in the event that this kernel is nontrivial. Since the fiber group is free, and free groups are well known to be Hopfian, it follows that ρ_k is faithful if and only if it is faithful when restricted to F. It is therefore a reasonable (and convenient) measure of the complication of the representation to check how much collapsing of F there is for a given integral specialization of k.

One way to proceed to quantify this collapsing is as follows. Fixing a (small) value for k, we can compare the number of reduced words in the group $\langle X_k, Y_k\rangle$ of length at most n with the number of reduced words in the free group of rank two of length at most n.

One can check that in fact, there is not too much collapsing for most values of k. Moreover, although we know by Theorem 4.1 that $\rho_k(\Gamma)$ has finite index in $\mathrm{SL}(3,\mathbb{Z})$ for $k = 0, 2, 3, 4, 5$, and so there must be collapsing, the analysis outlined above still gives some information.

For $k = 0, 1, 2, 3$ one finds that these sets are strictly smaller than that of a free group for rather small values of n. For example, for $k = 3$, there are 52 elements of length at most 3, and in the free group there are 53. It follows that $\langle X_3, Y_3\rangle$ has a relation of length at most six coming from the fact that there are two different reduced words with the same matrix; it is easily computed that X has order six.

However, the situation changes dramatically for larger values of k. For example, at $k = 4$, the number of words of length 16 or less is the same as that of the free group. So there are no relations of length 32 or less in $\langle X_4, Y_4\rangle$ despite the fact that this subgroup has finite index. The shortest relation we know (coming from the proof of Theorem 4.1) has length 112.

We now discuss why one might use 3-manifold groups as an approach to the questions of coherence and the finitely generated intersection property.

Note that whenever $n \geq 4$, $\mathrm{SL}(n,\mathbb{Z})$ is easily seen not to be coherent nor to have the finitely generated intersection property. This is because one can inject $F \times F$, where F is a free group of rank 2.

5.1. Finitely Generated Intersection Property

Our strategy to violate the finitely generated intersection property for $\mathrm{SL}(3,\mathbb{Z})$ using 3-manifolds is based on the well-known fact that if M is a finite-volume hyperbolic 3-manifold that fibers over the circle, then $\pi_1(M)$ does not have the finitely generated intersection property. In fact, in our context, in order to disprove the finitely generated intersection property for $\mathrm{SL}(3,\mathbb{Z})$, one needs less than the faithfulness of ρ_k.

Theorem 5.2. *Suppose that for some integral value k (ℓ, say), Z_ℓ does not power into F_ℓ. Then $\mathrm{SL}(3,\mathbb{Z})$ does not have the finitely generated intersection property.*

Proof. Take a power of X_ℓ such that the subgroup $H = \langle Z_\ell, X_\ell^R\rangle$ is free of rank two. Then since $F_\ell = \langle X_\ell, Y_\ell\rangle$ is normal in $\rho_\ell(\Gamma)$, we have that $H \cap F_\ell$ is normal in the free group H, but the hypothesis implies that it contains no powers of Z_ℓ and therefore has infinite index. It follows that $H \cap F_\ell$ is infinitely generated. □

Remark 5.3. The virtual cohomological dimension of $\mathrm{SL}(3,\mathbb{Z})$ is 3 (see [Brown 89, Chapter VII], for instance). Thus there is no cohomological obstruction for $\mathrm{SL}(3,\mathbb{Z})$ to contain the fundamental group of a finite-volume hyperbolic 3-manifold. Indeed, $\mathrm{SL}(3,\mathbb{Z})$ contains some 3-manifold groups, for example the integral Heisenberg group.

5.2. Coherence

With regard to coherence, a strategy to exploit the family $\mathcal{F}_k$ is summarized in the following proposition.

Proposition 5.4. *Suppose that for some $k \in \mathbb{Z}$ we can arrange that $\rho_k(F)$ is of infinite index in $\rho_k(\Gamma)$ and is not free. Further suppose that the virtual cohomological dimension of $\rho_k(\Gamma)$ is 2. Then $\mathrm{SL}(3,\mathbb{Z})$ is not coherent.*

Proof. By a theorem in [Bieri 76], in a group of cohomological dimension 2, any finitely presented normal subgroup is free or is of finite index. Thus applying this to $\rho_k(\Gamma)$, we argue as follows.

We are assuming that $\rho_k(F)$ is not free and that it has infinite index in $\rho_k(\Gamma)$, that is, we have exhibited an infinite-index normal subgroup of $\rho_k(\Gamma)$ that is finitely generated but not free.

By passing to a torsion-free subgroup of finite index Δ_k in $\rho_k(\Gamma)$, it follows from standard properties of cohomological dimension that Δ_k has cohomological dimension 2. The only possibility from Bieri's result is that $F_k \cap \Delta_k$ is not finitely presented. This completes the proof. □

6. FINAL COMMENTS

We raise a natural question motivated by this note.

Question 6.1. Does there exist an orientable finite-volume hyperbolic 3-manifold M for which $\pi_1(M)$ admits a faithful representation into $\mathrm{SL}(3,\mathbb{Z})$?

It is not hard to see that if Σ_g is a closed orientable surface of genus g, then $\pi_1(\Sigma_g)$ admits a faithful representation into $\mathrm{SL}(3,\mathbb{Z})$.

Briefly, the case $g = 0, 1$ is obvious, and so we can assume that $g \geq 2$. Consider the ternary quadratic form $f = x^2 - 3y^2 - 3z^2$. The group $\mathrm{SO}(f,\mathbb{Z})$ is a subgroup of $\mathrm{SL}(3,\mathbb{Z})$ and contains as a subgroup of finite index the $(2, 4, 6)$ triangle group (see [Mennicke 67]).

By [Edmonds et al. 92], the minimal index of a torsion-free subgroup in this triangle group is 24, and this has to be a genus-2 surface group. Since these representations lie in $\mathrm{SO}(2,1)$, they are not Zariski dense in $\mathrm{SL}(3,\mathbb{R})$. However, we have been informed by Bill Goldman (private communication) that Kac-Vinberg have constructed faithful Zariski-dense representations of some Fuchsian triangle groups into $\mathrm{SL}(3,\mathbb{Z})$. He has kindly allowed us to include the matrices for one such example.

Example 6.2. [Kac and Vinberg 67] Kac-Vinberg have shown that the following matrices determine a faithful Zariski-dense representation of the $(3, 3, 4)$ triangle group into $\mathrm{SL}(3,\mathbb{Z})$:

$$a = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easily checked that $b^3 = c^3 = 1$ and $a = c \cdot b$ with $a^4 = 1$. Note that Zariski denseness can easily be checked using Theorem 2.9.

Given this, we formulate another version of Question 6.1:

Question 6.3. Does there exist a compact orientable hyperbolizable 3-manifold M that is not an I-bundle over a surface and for which $\pi_1(M)$ admits a faithful representation into $\mathrm{SL}(3,\mathbb{Z})$?

As remarked in the previous section, it is well known that $\mathrm{SL}(3,\mathbb{Z})$ contains subgroups isomorphic to the fundamental group of some closed orientable 3-manifolds. Indeed, the fundamental groups of the torus bundles modeled on NIL and SOLV geometries all are subgroups.

We now show why NIL geometry gives rise to the only interesting class of Seifert-fibered spaces with infinite fundamental group that admit a faithful representation into $\mathrm{SL}(3,\mathbb{Z})$. We exclude as uninteresting the case that the manifold is covered by $S^2 \times \mathbb{R}$.

Theorem 6.4. *Let M be a compact orientable Seifert-fibered space with infinite fundamental group, not covered by $S^2 \times \mathbb{R}$ or admitting a geometric structure modeled on* NIL. *Then $\pi_1(M)$ does not admit a faithful representation into* $\mathrm{SL}(3,\mathbb{Z})$.

Proof. Firstly, $\mathrm{SL}(3,\mathbb{Z})$ does not contain $\mathbb{Z}^3$. This will automatically exclude those M admitting a Euclidean geometry, for in that case, M is covered by the 3-torus.

This follows from an analysis of centralizers of elements in $\mathrm{SL}(3,\mathbb{Z})$. Briefly, let $\gamma \in \mathrm{SL}(3,\mathbb{Z})$ be an element of infinite order. Then the eigenvalues of γ are either ± 1, three distinct real numbers, or one real number and one pair of complex conjugate numbers. If γ is an element of a subgroup $V = \mathbb{Z}^3 < \mathrm{SL}(3,\mathbb{Z})$, then it follows that V must consist of virtually unipotent elements. Otherwise, the centralizer of γ is (virtually) $\mathbb{Z} \oplus \mathbb{Z}$ or (virtually) $\mathbb{Z}$ by Dirichlet's unit theorem.

Now if $x \in V$, then x also has all eigenvalues ± 1. Now every such element has a square that is unipotent, and so we deduce from this that V contains a subgroup of finite index consisting entirely of unipotent elements (consider the subgroup generated by $\{g^2 : g \in V\}$. We can then deduce the existence of a $\mathbb{Z}^3$ subgroup inside a Borel subgroup of $\mathrm{SL}(3,\mathbb{Z})$, and this is false.

The proof of the theorem is now easily completed. For let M admit a geometric structure based on $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{PSL}}_2$, with $Z = \langle c \rangle$ the center of $\pi_1(M)$ and $\rho : \pi_1(M) \to \mathrm{SL}(3,\mathbb{Z})$ a faithful representation.

The discussion above on centralizers applied to $\rho(c)$ shows that $\rho(c)$ cannot have three real distinct eigenvalues or one real eigenvalue and one pair of complex conjugate eigenvalues. Moreover, if $\rho(c)$ has all eigenvalues ± 1, it follows from the above discussion that M admits a geometric structure modeled on NIL. □

Remark 6.5. In Section 4 we noted that for $k = 2$, we have $Z_2^3 = [X_2, Y_2]$. This shows that the representation ρ_2 factors through $(-3/1)$-Dehn surgery on the figure-eight-knot complement. This manifold is a Seifert-fibered space whose base orbifold is a quotient of $\mathbb{H}^2$ by the $(3, 3, 4)$ triangle group. Thus Theorem 6.4 shows that in fact, ρ_2 factors through the $(3, 3, 4)$ triangle group. Notice that this triangle group is the triangle group in Kac-Vinberg's example. However, Theorem 4.1 shows that the image of ρ_2 is of finite index in $\mathrm{SL}(3, \mathbb{Z})$, and so these representations are very different.

7. APPENDIX

7.1. Construction of ρ_k and β_T

We briefly outline our method for producing the representations ρ_k and β_T. It is based on [Cooper et al. 06], which takes a representation of a group into some higher-rank Lie group (for our purposes here, $\mathrm{SL}(3, \mathbb{R})$) and attempts to deform it. In this way, we may produce an exact expression for the representation variety through that point. Of course, this is not always possible, since there are obstructions to deformation, but the method is usually rather effective if the representation is deformable.

In [Cooper et al. 06], this was applied in the context of closed hyperbolic 3-manifolds, where one has a canonical representation into $\mathrm{SO}(3, 1)$ that one tries to deform into $\mathrm{SL}(4, \mathbb{R})$.

In the setting of the figure-eight knot, one does have a small supply of 3-dimensional real representations (for example coming from the reduced Burau representation). This idea was exploited in [Mangum and Shanahan 97]. However, the representations ρ_k and β_T have a rather more number-theoretic flavor, which we now describe.

We started with a surjection $h : \Gamma \to \mathrm{SL}(3, 3)$ given by

$$h(z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix}, \quad h(x) = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix},$$

which was promoted (basically using Hensel's lemma) to a representation $h_3 : \Gamma \to \mathrm{SL}(3, \mathbb{Z}_3)$ (where $\mathbb{Z}_3$ denotes the 3-adic integers) with the property that it has $\mathbb{Z}$-integral character. This representation can be conjugated into $\mathrm{SL}(3, \mathbb{Z})$, and one can compute that it has a 2-dimensional character variety of $\mathrm{SL}(3, \mathbb{R})$ deformations. This variety was then computed exactly using the method of [Cooper et al. 06]. (The authors thank Morwen Thistlethwaite for doing much of the heavy lifting involved in implementing [Cooper et al. 06] in this last computation.) The two families $\mathcal{F}_k$ and $\mathcal{F}_T$ correspond to certain specializations of the parameters.

Remark 7.1. It is easily checked that Γ has no irreducible representation with infinite image in $\mathrm{SL}(2, \mathbb{Z})$ (see [Long and Reid 03]). Indeed, it is shown in [Long and Reid 03] that there are no infinite representations of Γ into $\mathrm{SL}(2, \mathbf{C})$ with $\mathbb{Z}$-characters.

On the other hand, Γ admits a faithful representation into $\mathrm{SL}(4, \mathbb{Z})$. This is seen as follows: Γ has a representation as an arithmetic Kleinian group appearing as a subgroup of index 12 in the Bianchi group $\mathrm{PSL}(2, O_3)$. Moreover, this group admits a faithful representation as a subgroup of $\mathrm{SO}(p; \mathbb{Z}) < \mathrm{SL}(4, \mathbb{Z})$, where p is the quaternary quadratic form $x^2 + y^2 + z^2 - 3t^2$ (see, for example, [Elstrodt et al. 99, Chapter 10.2, Example 7]).

7.2. Computations

We have used Magma to compute some indices. Of course, this is possible only in the very simplest cases, since as we have outlined above, usually the index must be too gigantic for current technology.

For example, for $T = -2$, the index $[\mathrm{SL}(3, \mathbb{Z}) : \beta_{-2}(\Gamma)]$ was computed to be 3670016, and $[\mathrm{SL}(3, \mathbb{Z}) : \beta_{-2}(F)] = 48 \cdot 3670016 = 2^{21} \cdot 7$. We now give a brief discussion of the computation and the Magma routine that is used.

The basic idea is that using the presentation for $\mathrm{SL}(3, \mathbb{Z})$ that is given in [Steinberg 85], we write the matrix elements generating $\rho_0(\Gamma)$ and $\beta_2(\Gamma)$ in terms of these generators. The generators are the six elementary matrices xij. Computations were done in Mathematica to arrive at these expressions.

For example, for $k = 0$, we have

$$X_0 = x12 * x23^{-1} * x32 * x23^{-1} * x12^{-1} * x12^{-1} * x23^{-1} * x23^{-1}$$

and

$$Y_0 = x12 * x31^{-1} * x31^{-1} * x13 * x31^{-1} * x21^{-1} * x21^{-1} * x23^{-1} * x32^{-1} * x23 * x32^{-1} * x23 * x23 * x32^{-1}.$$

The following routine was run in MAGMA (following a suggestion of Eamonn O'Brien):

$G\langle x12, x13, x21, x23, x31, x32\rangle :=$ Group$\langle x12, x13, x21, x23, x31, x32 \mid (x12, x13), (x21, x23), (x31, x32), (x12, x32), (x21, x31), (x21, x13) * x23^{-1}, (x12, x23) * x13^{-1}, (x13, x23), (x13, x32) * x12^{-1}, (x31, x12) * x32^{-1}, (x23, x31) * x21^{-1}, (x32, x21) * x31^{-1}, (x12 * x21^{-1} * x12)^4\rangle;$

$S :=$ sub$\langle G \mid x12 * x23^{-1} * x32 * x23^{-1} * x12^{-1} * x12^{-1} * x23^{-1} * x23^{-1}, x12 * x31^{-1} * x31^{-1} * x13 * x31^{-1} * x21^{-1} * x21^{-1} * x23^{-1} * x32^{-1} * x23 * x32^{-1} * x23 * x23 * x32^{-1}\rangle;$

ToddCoxeter $(G, S :$ Hard, Workspace $:= 10^8$, Print $:= 10^6);$

In the case $T = -2$, the elements X_{-2} and Z_{-2} in terms of the generators are

$$X_{-2} = x21 * x31 * x32 * x23^{-1} * x13 * x31^{-1} * (x12 * x13^{-5} * x23)^{-1}$$

and

$$Z_{-2} = x31^2 * x23^{-2} * x12 * (x13 * x31^{-1} * x13)^{-2} * x12 * x21^{-4}.$$

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