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On the Periodized Square of L^2 Cardinal Splines

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CONTENTS

We establish properties of and propose a conjecture concerning $\sum_m (S(x+m))^2$, where S is a piecewise polynomial cardinal spline in $L^2(\mathbb{R})$.

1. INTRODUCTION

Let $S(x)$ be a piecewise polynomial cardinal spline of order n in $L^2(\mathbb{R})$ as considered in [Schoenberg 73]. Assume that all scalars are real.

Then $S(x)$ enjoys the representation

$$S(x) = \sum_{k=-\infty}^{\infty} b_k B(x+k),$$

where

$$B(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)_+^{n-1}$$

is the B-spline of order n, and

$$\|\{b_k\}\|^2 = \sum_{k=-\infty}^{\infty} |b_k|^2 < \infty.$$

We are interested in

$$\Phi(x) = \sum_{k=-\infty}^{\infty} (S(x+k))^2.$$

In the case $n=1$, the function $\Phi(x)$ is equal to a positive constant c_0. More generally, $\Phi(x)$ is a piecewise polynomial of degree $2(n-1)$, periodic with

$$\Phi(x+1) = \Phi(x),$$

and in $C^{n-2}(\mathbb{R})$.

Proposition 1.1. *The function $\Phi(x)$ has the development*

$$\Phi(x) = c_0 + 2\sum_{k=1}^{\infty} c_k \cos 2\pi k x, \tag{1–1}$$

where

$$c_k = (n-1)!^2 \int_{-\infty}^{\infty} \left(\frac{\sin \pi\xi}{\pi}\right)^{2n} \left(\frac{1}{(\xi-k)\xi}\right)^{n} |\hat{b}(\xi)|^2 \, d\xi \tag{1–2}$$

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with $\hat{b}(\xi) = \sum_{k=-\infty}^{\infty} b_k e^{i2\pi k\xi}$. *The series converges pointwise, and if* $n \geq 4$, *it converges absolutely.*

Corollary 1.2. *If* $n \geq 2$, *then on the interval* $[0, 1)$ *the maximum of* $\Phi(x)$ *is taken on at only one point, which is* 0 *when* n *is even and* $1/2$ *when* n *is odd.*

Further considerations of symmetry lead to the following result.

Proposition 1.3. *The function* $\Phi(x)$ *can be expressed as*

$$\Phi(x) = \sum_{k=0}^{n-1} a_k x^k (1-x)^k \tag{1–3}$$

whenever $0 \leq x \leq 1$.

Note that

$$a_0 = \sum_{k=-\infty}^{\infty} (S(k))^2 = \sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^{n-1} b_{k+j} B(j) \right)^2,$$

and it is clear that $a_0 > 0$.

Considerations of smoothness imply the following.

Proposition 1.4. *For* $n \geq 2$, *the coefficients* a_k *satisfy*

$$\sum_{j=0}^{(m-1)/2} (-1)^j \binom{m-j}{j} a_{m-j} = 0 \tag{1–4}$$

for m *odd and less than or equal to* $n-2$.

Roughly speaking, the coefficients a_k are certain combinations of l^2 norms squared of various differences of the coefficients $\{b_k\}$.

Direct computations give us the following.

Proposition 1.5. *If* $n \geq 2$, *then*

$$a_{n-1} = (-1)^{n-1} \left\| \{\Delta^{n-1} b_m\} \right\|^2 .$$

If $n \geq 4$, *then*

$$a_{n-2} = (-1)^{n-1} (n-1)(n-3) \left\| \{\Delta^{n-2} b_m\} \right\|^2 .$$

If $n \geq 6$, *then*

$$\begin{aligned} a_{n-3} &= (-1)^{n-1}(n-1)(n-2) \\ &\quad \times \left\{ \frac{(n-4)(n-5)}{4} \left\| \{\Delta^{n-3} b_m + \Delta^{n-3} b_{m+1}\} \right\|^2 \right. \\ &\qquad \left. + \frac{(n-3)(n-4)}{6} \left\| \{\Delta^{n-2} b_m\} \right\|^2 \right\}, \end{aligned}$$

and if $n \geq 8$,

$$\begin{aligned} a_{n-4} &= (-1)^{n-1}(n-1)(n-2)(n-3)(n-5) \\ &\quad \times \left\{ \frac{(n-6)(n-7)}{36} \right. \\ &\qquad \times \left\| \{\Delta^{n-4} b_m + 4\Delta^{n-4} b_{m+1} + \Delta^{n-4} b_{m+2}\} \right\|^2 \\ &\qquad + \frac{(n-1)(n-4)}{60} \left\| \{\Delta^{n-2} b_m\} \right\|^2 \\ &\qquad \left. + \frac{(n-4)(n-6)}{30} \left\| \{\Delta^{n-4} b_m - \Delta^{n-4} b_{m+2}\} \right\|^2 \right\}. \end{aligned}$$

Here Δ denotes the standard forward difference operator. For its definition and that of its higher powers, see the discussion immediately following identity (5–2).

Proposition 1.5 together with Proposition 1.4 allows us to determine all the coefficients in the cases $n \leq 9$. These coefficients are all listed in Section 6. As a consequence we have the following.

Proposition 1.6. *If* $2 \leq n \leq 9$, *then*

$$(-1)^{n-1} a_k \geq 0 \quad \textit{for } k = 1, \ldots, n-1.$$

Presumably, an explicit formula for a_{n-5} if $n \geq 10$ will allow us to come to the same conclusion in the cases $n = 10$ and $n = 11$.

Conjecture 1.7. *If* $n \geq 2$, *then*

$$(-1)^{n-1} a_k \geq 0 \quad \textit{for } k = 1, \ldots, n-1.$$

The properties of splines exploited here can all be found in [de Boor 78, Schoenberg 73]. We bring attention to the fact that for the sake of convenience we use variants of the classical B-splines that are not normalized.

Our interest in $\Phi(x)$ arises from a question related to statistics in which the independent variable, here x, is usually denoted by t. If $\{\phi(t+k)\}_{k\in\mathbb{Z}}$ is an orthogonal basis for a subspace V of $L^2(\mathbb{R})$, namely if $\phi(t)$ satisfies

$$\int_{-\infty}^{\infty} \phi(t)\phi(t+k)dt = \delta_{0,k},$$

then $\Phi(t) = \sum_{k\in\mathbb{Z}} (\phi(t+k))^2$ is the variance of the Gaussian process

$$X(t) = \int_{-\infty}^{\infty} \sum_{k\in\mathbb{Z}} \phi(t+k)\phi(s+k)dW(s),$$

where W is Brownian motion and

$$\sum_{k\in\mathbb{Z}} \phi(t+k)\phi(s+k)$$

is the kernel of the orthogonal projection onto the subspace V.

This process plays a role in density estimation by projections of wavelet type. More concretely, under conditions on the smoothness of the density and on the resolution levels of the estimates, the sup norm over a fixed interval of the discrepancy between the density and its estimate has the same distributional behavior in the limit as the sup norm over increasing intervals of the Gaussian process $X(t)$, and the limiting distribution of its sup norm is determined by its variance $\Phi(t)$. See [Giné and Nickl 10, Proposition 5 and Section 4.2.3] for more details.

If $\Phi(t)$ has a unique maximum on $[0,1)$, then [Piterbarg and Seleznjev 94, Theorem 1] yields the limiting distribution of $\sup_{0\le t\le T}|X(t)|$. See also [Konstant and Piterbarg 93, Theorem 3.1] and [Giné and Nickl 10, Theorem 2]. That $\Phi(t)$ indeed does have a unique maximum on $[0,1)$ was established in [Giné and Nickl 10, Lemma 1] in the cases in which $\phi(t)=\sum_m b_m B(t+m)$ is such that $\{\phi(t+k)\}_{k\in\mathbb{Z}}$ is an orthogonal basis for the subspace $V=V_n$ consisting of piecewise polynomial cardinal splines of order $n=2,3,4$. In this article we note that when the subspace V consists of piecewise polynomial cardinal splines of order n, the above-mentioned unique maximum property of $\Phi(t)$ remains valid even if ϕ is replaced by any member S of V. We extend this result to all n, record several additional properties of Φ, and propose a conjecture concerning its nature.

2. PROOF OF PROPOSITION 1.1 AND ITS COROLLARY

To prove Proposition 1.1, we use the normalization

$$\hat{f}(\xi)=\int_{-\infty}^{\infty} f(x)\,e^{-i2\pi\xi x}\,dx$$

for the Fourier transform $\hat{f}(\xi)$ of the integrable function f. The smoothness and periodicity properties of Φ allow us to express it as

$$\Phi(x)=\sum_{k=-\infty}^{\infty} c_k e^{i2\pi kx}. \tag{2–1}$$

The above series converges pointwise, and if $n\ge 4$, then it converges absolutely.

The fact that Φ is an even function of x, which follows, for instance, from (3–5) below, implies that $c_k=c_{-k}$ and allows us to rewrite (2–1) as the cosine series (1–1). In view of the Poisson summation formula, the coefficients c_k may be expressed as

$$c_k=\hat{S}*\hat{S}(k)=\int_{-\infty}^{\infty}\hat{S}(k-\xi)\hat{S}(\xi)d\xi. \tag{2–2}$$

To see the explicit formula for c_k, write

$$\hat{S}(\xi)=\hat{b}(\xi)\hat{B}(\xi), \tag{2–3}$$

where

$$\hat{b}(\xi)=\sum_{k=-\infty}^{\infty} b_k e^{i2\pi k\xi}.$$

The periodic function $\hat{b}(\xi)$ enjoys

$$\hat{b}(k-\xi)=\hat{b}(-\xi)=\overline{\hat{b}(\xi)},$$

because the coefficients $\{b_k\}$ are real. Since

$$\hat{B}(\xi)=ae^{-i\pi n\xi}\left(\frac{\sin\pi\xi}{\pi\xi}\right)^n$$

with $a=(n-1)!$ (see, for example, [Schoenberg 73, (1.4), (1.5), and (1.7) in Lecture 2]), it follows that

$$\hat{S}(k-\xi)\hat{S}(\xi)=a^2\left(\frac{\sin\pi\xi}{\pi}\right)^{2n}\left(\frac{1}{(\xi-k)\xi}\right)^n|b(\xi)|^2,$$

which yields (1–2) and completes the proof of Proposition 1.1.

To prove the corollary, observe that if n is even, then in view of (1–2), the coefficients in (1–1) satisfy $c_k>0$ for all k. From this it follows that in this case, $\Phi(0)>\Phi(x)$ for all x, $0<x<1$.

The case of odd n is a bit more intricate. Combine (2–1), (2–2), and (2–3) to write

$$\Phi(x)=\sum_{k=\infty}^{\infty}\left(\int_{-\infty}^{\infty}\hat{B}(k-\xi)\hat{B}(\xi)|\hat{b}(\xi)|^2d\xi\right)e^{i2\pi kx}.$$

Express each term in the sum as

$$\begin{aligned}
&\int_{-\infty}^{\infty}\hat{B}(k-\xi)\hat{B}(\xi)e^{i2\pi kx}|\hat{b}(\xi)|^2d\xi\\
&\quad=\int_{-1/2}^{1/2}\left(\sum_{m=-\infty}^{\infty}\hat{B}(k-(\xi+m))\hat{B}(\xi+m)\right)e^{i2\pi kx}\\
&\qquad\times\left|\hat{b}(\xi)\right|^2d\xi\\
&\quad=\int_{-1/2}^{1/2}\left(\sum_{m=-\infty}^{\infty}\hat{B}(k-m-\xi)e^{i2\pi(k-m)x}\hat{B}(\xi+m)e^{i2\pi mx}\right)\\
&\qquad\times\left|\hat{b}(\xi)\right|^2d\xi,
\end{aligned}$$

so that summing over k and interchanging the order of summation results in

$$\Phi(x) = \int_{-1/2}^{1/2} \left(\sum_{k=-\infty}^{\infty} \hat{B}(k-\xi) e^{i2\pi kx} \right) \times \left(\sum_{m=-\infty}^{\infty} \hat{B}(\xi+m) e^{i2\pi mx} \right) |\hat{b}(\xi)|^2 d\xi.$$

Note that

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} \hat{B}(\xi+m) e^{i2\pi mx} \\ &= \left(\sum_{m=-\infty}^{\infty} \hat{B}(\xi+m) e^{i2\pi(\xi+m)x} \right) e^{-i2\pi\xi x} \\ &= \left(\sum_{m=-\infty}^{\infty} B(x+m) e^{-i2\pi m\xi} \right) e^{-i2\pi\xi x}, \end{aligned}$$

where the second equality follows from the Poisson summation formula and fact that $\hat{B}(\xi+m)e^{i2\pi(\xi+m)x}$ is the Fourier transform evaluated at $\xi+m$ of $B(y+x)$ as a function of y. Similar reasoning shows that

$$\sum_{k=-\infty}^{\infty} \hat{B}(k-\xi) e^{i2\pi kx} = \overline{\left(\sum_{k=-\infty}^{\infty} B(x+k) e^{-i2\pi k\xi} \right) e^{i2\pi\xi x}}.$$

The last three displayed expressions imply that Φ can be expressed as

$$\Phi(x) = \int_{-1/2}^{1/2} \left| \sum_{k=-\infty}^{\infty} B(x+k) e^{-i2\pi k\xi} \right|^2 |\hat{b}(\xi)|^2 \, d\xi. \quad (2\text{–}4)$$

Now, $\sum_{m=-\infty}^{\infty} B(x+k)e^{-i2\pi k\xi}$ is simply a constant multiple of $S_{n-1,2\pi\xi}(x)$, the so-called Euler exponential spline [Schoenberg 73, Schoenberg 83] expressed in de Boor's notation [de Boor 76].

In view of [de Boor 76, item (15), p. 934], for odd n we know that for all x, $0 \le x < 1$, we have $|S_{n-1,2\pi\xi}(x)| \le 1$, with equality if $x = 1/2$ and strict inequality otherwise. This, together with (2–4), implies the conclusion of the corollary in the case of odd n.

3. PROOF OF PROPOSITION 1.3

Recall that $B(x)$ has support in $[0, n]$, which we may write as

$$\operatorname{supp}(B(x)) \in [0, n],$$

so that

$$\operatorname{supp}\big(B(x)B(x+k)\big) \in \begin{cases} [0, n-k] & \text{if } 0 \le k \le n-1, \\ [-k, n] & \text{if } 1-n \le k \le 0, \end{cases}$$

and is identically zero otherwise. Also recall that

$$B(n-x) = B(x). \quad (3\text{–}1)$$

With these properties of B in mind, consider the function

$$\Psi_m(x) = \sum_{k=-\infty}^{\infty} B(x+k)B(x+k+m)$$

for every integer m. Then $\Psi_m(x)$ is a piecewise polynomial of degree $2(n-1)$, is periodic with

$$\Psi_m(x+1) = \Psi_m(x),$$

is in $C^{n-2}(\mathbb{R})$, and satisfies

$$\Psi_{-m}(x) = \Psi_m(x).$$

It is identically zero when the integer m is outside the range $[1-n, n-1]$.

Furthermore, the function $\Psi_m(x)$ has the following properties:

$$\Psi_m(-x) = \Psi_m(x), \quad (3\text{–}2)$$
$$\Psi_m(1-x) = \Psi_m(x), \quad (3\text{–}3)$$

and can be expressed as

$$\Psi_m(x) = \sum_{k=0}^{n-1} c_k x^k (1-x)^k \quad (3\text{–}4)$$

whenever $0 \le x \le 1$.

To establish (3–2), write

$$\begin{aligned} \Psi_m(x) &= \sum_{k=-\infty}^{\infty} B(x+k)B(x+k+m) \\ &= \sum_{k=-\infty}^{\infty} B(n-k-x)B(n-k-m-x) \quad \text{by (3–1)} \\ &= \sum_{k=-\infty}^{\infty} B(k-x)B(k-m-x) \quad \text{by periodicity} \\ &= \sum_{k=-\infty}^{\infty} B(k+m-x)B(k-x) \quad \text{by periodicity} \\ &= \Psi_m(-x). \end{aligned}$$

Identity (3–3) follows from (3–2) and periodicity.

To prove (3–4), observe that the fact that $\Psi_m(x)$ is a polynomial of degree $2(n-1)$ on $[0, 1]$ allows us to write, with $\gamma_0 = 0$,

$$\Psi_m(x) = \sum_{k=0}^{n-1} \left\{ c_k x^k (1-x)^k + \gamma_k x^{2k-1} \right\}$$

whenever $0 \le x \le 1$. In view of (3–3), it follows that

$$\sum_{k=1}^{n-1} \gamma_k (1-x)^{2k-1} = \sum_{k=1}^{n-1} \gamma_k x^{2k-1}$$

for all $x \in [0,1]$, from which we may conclude that $\gamma_1 = \cdots = \gamma_{n-1} = 0$.

Note that

$$\Psi_{n-1}(x) = x^{n-1}(1-x)^{n-1}.$$

For other values of m, that is, for $m = 0, \ldots, n-2$, $\Psi_m(x)$ is not so easy to determine.

Proposition 1.3 is a consequence of relation (3–4). Take x in $[0,1]$ and write

$$\begin{aligned}
\Phi(x) &= \sum_{j=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} b_k B(x+j+k) \right\}^2 \\
&= \sum_j \sum_{k,l} b_k b_l B(x+j+k) B(x+j+l) \\
&= \sum_{k,l} b_k b_l \sum_j B(x+j+k) B(x+j+l) \\
&= \sum_{k,l} b_k b_l \sum_j B(x+j) B(x+j+l-k) \\
&= \sum_{k,l} b_k b_l \Psi_{l-k}(x) \\
&= \sum_{m=0}^{n-1} \epsilon_m \left\{ \sum_{k=-\infty}^{\infty} b_k b_{k+m} \right\} \Psi_m(x),
\end{aligned}$$

where

$$\epsilon_m = \begin{cases} 1 & \text{if } m = 0, \\ 2 & \text{otherwise.} \end{cases}$$

If we use the abbreviation

$$\beta_m = \sum_{k=-\infty}^{\infty} b_k b_{k+m},$$

then the last identity for $\Phi(x)$ can be expressed more succinctly as

$$\Phi(x) = \sum_{m=0}^{n-1} \epsilon_m \beta_m \Psi_m(x). \tag{3–5}$$

Identities (3–4) and (3–5) imply (1–3).

4. PROOF OF PROPOSITION 1.4

If $-1 \le x < 0$, then $0 \le 1+x < 1$, and for such x we may write

$$\Phi(x) = \Phi(1+x) = \sum_{k=0}^{n-1} a_k (1+x)^k (-x)^k. \tag{4–1}$$

Note that

$$x^j(-x)^k = (-1)^{j+k} x^k (-x)^j,$$

so that the coefficients of odd powers of x in (4–1) are the negatives of those of the corresponding powers in (1–3). Since $\Phi(x)$ is in $C^{n-2}(\mathbb{R})$, it follows that these coefficients must be 0. (For the record, note that the coefficients of the even powers of x in (4–1) are the same as of those of the corresponding powers in (1–3).)

For the reader's convenience and future reference, we list in Table 1 the coefficients of x^k in $\Phi(x)$ when $0 \le x < 1$ for $k = 1, 2, \ldots, 2(n-1)$, in the case $n = 10$.

k	Coefficient of x^k								
1	a_1								
2	$-a_1$	$+a_2$							
3		$-2a_2$	$+a_3$						
4		$+a_2$	$-3a_3$	$+a_4$					
5			$+3a_3$	$-4a_4$	$+a_5$				
6			$-a_3$	$+6a_4$	$-5a_5$	$+a_6$			
7				$-4a_4$	$+10a_5$	$-6a_6$	$+a_7$		
8				$+a_4$	$-10a_5$	$+15a_6$	$-7a_7$	$+a_8$	
9					$+5a_5$	$-20a_6$	$+21a_7$	$-8a_8$	a_9
10					$-a_5$	$+15a_6$	$-35a_7$	$+28a_8$	$-9a_9$
11						$-6a_6$	$+35a_7$	$-56a_8$	$+36a_9$
12						$+a_6$	$-21a_7$	$+70a_8$	$-84a_9$
13							$+7a_7$	$-56a_8$	$+126a_9$
14							$-a_7$	$+28a_8$	$-126a_9$
15								$-8a_8$	$+84a_9$
16								$+a_8$	$-36a_9$
17									$+9a_9$
18									$-a_9$

TABLE 1. Coefficients of x^k in $\Phi(x)$ when $0 \le x < 1$ for $k = 1, 2, \ldots, 2(n-1)$, in the case $n = 10$.

The coefficients of all the odd powers k that are less than or equal to $n-2$ must vanish. In the cases $n = 9, 10$, this of course implies that

$$\begin{aligned} a_1 \quad & = 0,\\ -2a_2 + a_3 \quad & = 0,\\ +3a_3 - 4a_4 + a_5 \quad & = 0,\\ -4a_4 + 10a_5 - 6a_6 + a_7 & = 0. \end{aligned}$$

In the general case we have (1–4).

Note that it follows from these constraints that when $n = 2m$ or $2m + 1$, all the coefficients $a_1, \dots, a_{n-1}$ in the expression (1–3) for $\Phi(x)$ can be deduced from the top m terms $a_{n-1}, \dots, a_{n-m}$.

5. PROOF OF PROPOSITION 1.5

In what follows, we use the notation $S^{(k)}(x)$ to denote the derivative of order k of S. Also, because we will be working with B-splines of various orders simultaneously, the B-spline of order n will be denoted by $B_n(x)$, and extensive use will be made of the identity

$$B_n^{(1)}(x) = (n-1)\{B_{n-1}(x) - B_{n-1}(x-1)\}. \tag{5–1}$$

Furthermore, we will need to know some of the values $B_n(k)$. To this end, recall that the B-splines enjoy the recurrence relation

$$B_{n+1}(x) = xB_n(x) + (n+1-x)B_n(x-1)$$

with

$$B_1(x) = \begin{cases} 1 & \text{if } 0 \le x < 1,\\ 0 & \text{otherwise.}\end{cases}$$

Thus the values $B_n(k)$ can be computed directly from the definition or the recurrence formula. For easy reference, we include a table of these values for $n = 1, \dots, 7$; see Table 2. Finally, unless it makes sense otherwise, all values of x should be assumed to be in the range $0 \le x < 1$.

n	$k=0$	1	2	3	4	5	6	7
1	1	0						
2	0	1	0					
3	0	1	1	0				
4	0	1	4	1	0			
5	0	1	11	11	1	0		
6	0	1	26	66	26	1	0	
7	0	1	57	302	302	57	1	0

TABLE 2. Some values of $B_n(k)$.

The evaluation of a_{n-l}, $l = 1, 2, 3, 4$, relies on the corresponding expressions for $\Phi^{(2(n-l))}(x)$.

Because $S^{(k)}(x) = 0$ for $0 < x < 1$ whenever $k \ge n$, for $l \le n/2$ we may write

$$\Phi^{(2(n-l))}(x) = \sum_{j=0}^{l-1} \epsilon_j \binom{2(n-l)}{n-l+j} \times \sum_{m=-\infty}^{\infty} S^{(n-l+j)}(x-m) S^{(n-l-j)}(x-m), \tag{5–2}$$

where

$$\epsilon_j = \begin{cases} 1 & \text{if } j = 0,\\ 2 & \text{otherwise.}\end{cases}$$

Next, using the notation

$$\Delta b_k = b_k - b_{k+1}$$

and by induction

$$\Delta^m b_k = \sum_{j=0}^{m} (-1)^j \binom{m}{j} b_{k+j},$$

we may write

$$\begin{aligned} S^{(n-j)}(x-m) &= \sum_{k=-\infty}^{\infty} b_k B_n^{(n-j)}(x+k-m)\\ &= \frac{(n-1)!}{(j-1)!} \sum_{k=-\infty}^{\infty} \Delta^{n-j} b_k B_j(x+k-m)\\ &= \frac{(n-1)!}{(j-1)!} \sum_{k=0}^{j-1} \Delta^{n-j} b_{m+k} B_j(x+k) \end{aligned}$$

when $0 \le x \le 1$. Hence if $0 \le x \le 1$, then

$$\begin{aligned} & S^{(n-j_1)}(x-m) S^{(n-j_2)}(x-m)\\ &= \frac{((n-1)!)^2}{(j_1-1)!(j_2-1)!} \sum_{k_1=0}^{j_1-1} \sum_{k_2=0}^{j_2-1} \Delta^{n-j_1} b_{m+k_1} \Delta^{n-j_2} b_{m+k_2}\\ &\qquad \times B_{j_1}(x+k_1) B_{j_2}(x+k_2). \end{aligned}$$

Now choose $j_1 = l - j$, $j_2 = l + j$; sum over m; use the fact that

$$\sum_{m=-\infty}^{\infty} (\Delta \alpha_m) \beta_m = -\sum_{m=-\infty}^{\infty} \alpha_{m+1} \Delta \beta_m,$$

so that

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} \Delta^{n-l+j} b_{m+k_1} \Delta^{n-l-j} b_{m+k_2}\\ &= \sum_{m=-\infty}^{\infty} (-1)^j \Delta^{n-l} b_{m+k_1+j} \Delta^{n-l} b_{m+k_2}; \end{aligned}$$

and let

$$c_m = \Delta^{n-l} b_m$$

to get

$$\sum_{m=-\infty}^{\infty} S^{(n-l+j)}(x-m) S^{(n-l-j)}(x-m) = \frac{((n-1)!)^2}{(l+j-1)!(l-j-1)!} \times \sum_{m=-\infty}^{\infty} \sum_{k_1=0}^{l-j-1} \sum_{k_2=0}^{l+j-1} (-1)^j c_{m+k_1+j} c_{m+k_2} \times B_{l-j}(x+k_1) B_{l+j}(x+k_2).$$

Combining with (5–2), we have

$$\Phi^{(2(n-l))}(x) = \sum_{j=0}^{l-1} \epsilon_j \binom{2(n-l)}{n-l+j} \frac{((n-1)!)^2}{(l+j-1)!(l-j-1)!} \times \sum_{m=-\infty}^{\infty} \sum_{k_1=0}^{l-j-1} \sum_{k_2=0}^{l+j-1} (-1)^j c_{m+k_1+j} c_{m+k_2} \times B_{l-j}(x+k_1) B_{l+j}(x+k_2).$$

For $l = 1$ we have

$$B_1(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and thus

$$\Phi^{(2(n-1))}(x) = \binom{2(n-l)}{n-1} ((n-1)!)^2 \sum_{m=-\infty}^{\infty} c_m c_m = (2(n-1))! \sum_{m=-\infty}^{\infty} (\Delta^{n-1} b_m)^2.$$

Since

$$(-1)^{n-1}(2(n-1))! a_{n-1} = \Phi^{(2(n-1))}(0) = \Phi^{(2(n-1))}(x),$$

we may conclude that

$$a_{n-1} = (-1)^{n-1} \left\| \{ \Delta^{n-1} b_m \} \right\|^2. \qquad (5\text{–}3)$$

If $l > 1$ we may take

$$\Phi^{(2(n-1))}(0) = \lim_{x \to 0} \Phi^{(2(n-1))}(x) = \sum_{j=0}^{l-1} \epsilon_j \binom{2(n-l)}{n-l+j} \frac{((n-1)!)^2}{(l+j-1)!(l-j-1)!} \times \sum_{m=-\infty}^{\infty} \sum_{k_1=0}^{l-j-1} \sum_{k_2=0}^{l+j-1} (-1)^j c_{m+k_1+j} c_{m+k_2} B_{l-j}(k_1) \times B_{l+j}(k_2).$$

For $l = 2$, reading off the appropriate values of $B_j(k)$ from Table 2 results in

$$\Phi^{(2(n-2))}(0) = \binom{2(n-2)}{n-2} ((n-1)!)^2 \sum_{m=-\infty}^{\infty} c_{m+1} c_{m+1} + 2 \binom{2(n-2)}{n-1} \frac{((n-1)!)^2}{2} (-1) \times \sum_{m=-\infty}^{\infty} \{ c_{m+1} c_{m+1} + c_{m+1} c_{m+2} \},$$

which can be simplified to

$$\frac{\Phi^{(2(n-2))}(0)}{(2(n-2))!} = (n-1)^2 \sum_{m=-\infty}^{\infty} c_m c_m - (n-1)(n-2) \times \sum_{m=-\infty}^{\infty} \{ c_m c_m + c_m c_{m+1} \}.$$

This should be compared to

$$\frac{\Phi^{(2(n-2))}(0)}{(2(n-2))!} = (-1)^{n-2} a_{n-2} + (-1)^{n-3} \binom{n-1}{n-3} a_{n-1} = (-1)^{n-2} \left\{ a_{n-2} - \frac{(n-1)(n-2)}{2} (-1)^{n-1} \times \sum_{m=-\infty}^{\infty} (c_m - c_{m+1})^2 \right\}.$$

Solving for a_{n-2} results in

$$a_{n-2} = (-1)^{n-1} (n-1)(n-3) \sum_{m=-\infty}^{\infty} c_m^2,$$

which can be reexpressed as

$$a_{n-2} = (-1)^{n-1} (n-1)(n-3) \left\| \{ \Delta^{n-2} b_m \} \right\|^2. \qquad (5\text{–}4)$$

For $l = 3$, reading off the appropriate values of $B_j(k)$ from Table 2 results in

$$\begin{aligned}
&\Phi^{(2(n-3))}(0)\\
&\quad= \binom{2(n-3)}{n-3}\frac{((n-1)!)^2}{2!2!}\\
&\qquad\times \sum_{m=-\infty}^{\infty}\{2c_{m+1}c_{m+1} + 2c_{m+1}c_{m+2}\}\\
&\qquad - 2\binom{2(n-3)}{n-2}\frac{((n-1)!)^2}{3!}\\
&\qquad\quad\times \sum_{m=-\infty}^{\infty}\{c_{m+2}c_{m+1} + 4c_{m+2}c_{m+2} + c_{m+2}c_{m+3}\}\\
&\qquad + 2\binom{2(n-3)}{n-1}\frac{((n-1)!)^2}{4!}\\
&\qquad\quad\times \sum_{m=-\infty}^{\infty}\{c_{m+2}c_{m+1} + 11c_{m+2}c_{m+2}\\
&\qquad\qquad\qquad + 11c_{m+2}c_{m+3} + c_{m+2}c_{m+4}\},
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
\frac{\Phi^{(2(n-3))}(0)}{(2(n-3))!} &= (n-1)(n-2)\\
&\times\Bigg\{\frac{(n-1)(n-2)}{4}\sum_{m=-\infty}^{\infty}\left\{2c_m^2 + 2c_m c_{m+1}\right\}\\
&\quad - 2\frac{(n-1)(n-3)}{3!}\sum_{m=-\infty}^{\infty}\left\{4c_m^2 + 2c_m c_{m+1}\right\}\\
&\quad + 2\frac{(n-3)(n-4)}{4!}\\
&\qquad\times\sum_{m=-\infty}^{\infty}\{11c_m^2 + 12c_m c_{m+1} + c_m c_{m+2}\}\Bigg\}.
\end{aligned}$$

This should be compared to

$$\begin{aligned}
&\frac{\Phi^{(2(n-3))}(0)}{(2(n-3))!}\\
&\quad= (-1)^{n-3}a_{n-3} + (-1)^{n-4}\binom{n-2}{n-4}a_{n-2}\\
&\qquad + (-1)^{n-5}\binom{n-1}{n-5}a_{n-1}\\
&\quad= (-1)^{n-3}\Bigg\{a_{n-3} - \binom{n-2}{n-4}(-1)^{n-1}(n-1)(n-3)\\
&\quad\times\sum_{m=-\infty}^{\infty}(c_m - c_{m+1})^2\\
&\qquad + \binom{n-1}{n-5}(-1)^{n-1}\sum_{m=-\infty}^{\infty}(c_m - 2c_{m+1} + c_{m+2})^2\Bigg\}\\
&\quad= (-1)^{n-3}a_{n-3} - \frac{(n-1)(n-2)(n-3)^2}{2}\\
&\qquad\times\sum_{m=-\infty}^{\infty}\{2c_m^2 - 2c_m c_{m+1}\}\\
&\qquad + \frac{(n-1)(n-2)(n-3)(n-4)}{4!}\\
&\qquad\quad\times\sum_{m=-\infty}^{\infty}\{6c_m^2 - 8c_m c_{m+1} + 2c_m c_{m+2}\},
\end{aligned}$$

while paying particular attention to the term involving $c_m c_{m+2}$. Solving for a_{n-3} results in

$$\begin{aligned}
(-1)^{n-3}a_{n-3} &= (n-1)(n-2)\\
&\times\Bigg\{\frac{(n-1)(n-2)}{4}\sum_{m=-\infty}^{\infty}\{2c_m^2 + 2c_m c_{m+1}\}\\
&\quad - 2\frac{(n-1)(n-3)}{3!}\sum_{m=-\infty}^{\infty}\{4c_m^2 + 2c_m c_{m+1}\}\\
&\quad + \frac{(n-3)^2}{2}\sum_{m=-\infty}^{\infty}\{2c_m^2 - 2c_m c_{m+1}\}\\
&\quad + \frac{(n-3)(n-4)}{4!}\sum_{m=-\infty}^{\infty}\{16c_m^2 + 32c_m c_{m+1}\}\Bigg\}.
\end{aligned}$$

To make sense of this, let

$$X = \sum_{m=-\infty}^{\infty} 2c_m^2, \quad Y = \sum_{m=-\infty}^{\infty} 2c_m c_{m+1},$$

and let $\alpha, \ldots, \delta$ be the appropriate coefficients so that the expression in large braces above reduces to

$$\alpha(X+Y) + \beta(2X+Y) + \gamma(X-Y) + \delta(X+2Y).$$

We want to reexpress it as

$$A(X+Y) + B(X-Y).$$

This implies that

$$\begin{aligned}
2A &= 2\alpha + 3\beta + 3\delta\\
&= 2\frac{(n-1)(n-2)}{4} + 3\left(-\frac{(n-1)(n-3)}{3}\right)\\
&\quad + 3\frac{(n-3)(n-4)}{3}\\
&= \frac{(n-4)(n-5)}{2}
\end{aligned}$$

and

$$\begin{aligned}2B &= \beta + 2\gamma - \delta \\ &= -\frac{(n-1)(n-3)}{3} + 2\frac{(n-3)^2}{2} \\ &\quad - \frac{(n-3)(n-4)}{3} \\ &= \frac{(n-3)(n-4)}{3}.\end{aligned}$$

Thus the last expression for a_{n-3} reduces to

$$\begin{aligned}&(-1)^{n-3}a_{n-3} \\ &\quad = (n-1)(n-2)\left\{\frac{(n-4)(n-5)}{4}(X+Y)\right. \\ &\qquad\qquad \left. + \frac{(n-3)(n-4)}{6}(X-Y)\right\}.\end{aligned}$$

Since

$$\begin{aligned}X+Y &= \sum_{m=-\infty}^{\infty}\left\{2c_m^2 + 2c_m c_{m+1}\right\} \\ &= \sum_{m=-\infty}^{\infty}(c_m + c_{m+1})^2 \\ &= \|\{\Delta^{n-3}b_m + \Delta^{n-3}b_{m+1}\}\|^2, \\ X-Y &= \sum_{m=-\infty}^{\infty}\left\{2c_m^2 - 2c_m c_{m+1}\right\} \\ &= \sum_{m=-\infty}^{\infty}(c_m - c_{m+1})^2 = \|\{\Delta^{n-2}b_m\}\|^2,\end{aligned}$$

and $(-1)^{n-3} = (-1)^{n-1}$, we may express a_{n-3} succinctly as

$$\begin{aligned}a_{n-3} &= (-1)^{n-1}(n-1)(n-2) \\ &\quad\times\left\{\frac{(n-4)(n-5)}{4}\|\{\Delta^{n-3}b_m + \Delta^{n-3}b_{m+1}\}\|^2\right. \\ &\qquad \left. + \frac{(n-3)(n-4)}{6}\|\{\Delta^{n-2}b_m\}\|^2\right\}.\end{aligned} \tag{5–5}$$

For $l = 4$, reading off the appropriate values of $B_j(k)$ from Table 2 and simplifying as in the cases $l = 2, 3$ results in

$$\begin{aligned}\frac{\Phi^{(2(n-4))}(0)}{(2(n-4))!} &= (n-1)(n-2)(n-3) \\ &\times\left\{\frac{(n-1)(n-2)(n-3)}{3!3!}\right. \\ &\quad\times\sum_{m=-\infty}^{\infty}\left\{18c_m^2 + 16c_m c_{m+1} + 2c_m c_{m+1}\right\} \\ &\quad - 2\frac{(n-1)(n-2)(n-4)}{4!2!} \\ &\qquad\times\sum_{m=-\infty}^{\infty}\left\{22c_m^2 + 24c_m c_{m+1} + 2c_m c_{m+2}\right\} \\ &\qquad + 2\frac{(n-1)(n-4)(n-5)}{5!} \\ &\qquad\times\sum_{m=-\infty}^{\infty}\left\{66c_m^2 + 52c_m c_{m+1} + 2c_m c_{m+2}\right\} \\ &\qquad - 2\frac{(n-4)(n-5)(n-6)}{6!} \\ &\qquad\times\sum_{m=-\infty}^{\infty}\left\{302c_m^2 + 359c_m c_{m+1} + 58c_m c_{m+2}\right. \\ &\qquad\left.\left. + c_m c_{m+3}\right\}\right\}.\end{aligned}$$

This should be compared to

$$\begin{aligned}\frac{\Phi^{(2(n-4))}(0)}{(2(n-4))!} &= (-1)^{n-4}a_{n-4} + (-1)^{n-5}\binom{n-3}{n-5}a_{n-3} \\ &\quad + (-1)^{n-6}\binom{n-2}{n-6}a_{n-2} + (-1)^{n-7}\binom{n-1}{n-7}a_{n-1} \\ &= (-1)^{n-4}\left\{a_{n-4} - \binom{n-3}{n-5}(-1)^{n-1}(n-1)(n-2)\right. \\ &\quad\times(n-4)\left\{\frac{(n-5)}{4}\sum_{m=-\infty}^{\infty}\left\{2c_m^2 - 2c_m c_{m+2}\right\}\right. \\ &\qquad\left. + \frac{(n-3)}{6}\sum_{m=-\infty}^{\infty}\left\{6c_m^2 - 8c_m c_{m+1} + 2c_m c_{m+2}\right\}\right\} \\ &\quad + \binom{n-2}{n-6}(-1)^{n-1}(n-1)(n-3) \\ &\quad\times\sum_{m=-\infty}^{\infty}\left\{6c_m^2 - 8c_m c_{m+1} + 2c_m c_{m+2}\right\} \\ &\quad - \binom{n-1}{n-7}(-1)^{n-1}\sum_{m=-\infty}^{\infty}\left\{20c_m^2 - 30c_m c_{m+1}\right. \\ &\qquad\left.\left. + 12c_m c_{m+2} - 2c_m c_{m+3}\right\}\right\},\end{aligned}$$

while paying particular attention to the term involving $c_m c_{m+3}$.

Solving for a_{n-4} results in

$$\begin{aligned}
&(-1)^{n-4}a_{n-4}\\
&= \binom{n-1}{n-4}^2 \sum_{m=-\infty}^{\infty} \{18c_m^2 + 16c_m c_{m+1} + 2c_m c_{m+2}\}\\
&\quad - \binom{n-1}{n-5}(n-1)(n-2)\\
&\quad \times \sum_{m=-\infty}^{\infty} \{22c_m^2 + 24c_m c_{m+1} + 2c_m c_{m+2}\}\\
&\quad + \binom{n-1}{n-6}(n-1)\\
&\quad \times \sum_{m=-\infty}^{\infty} 2\{66c_m^2 + 52c_m c_{m+1} + 2c_m c_{m+2}\}\\
&\quad - \binom{n-1}{2}\binom{n-3}{2}\binom{n-4}{2} \sum_{m=-\infty}^{\infty} \{2c_m^2 - 2c_m c_{m+2}\}\\
&\quad - \binom{n-1}{n-5}(n-3)^2\\
&\quad \times \sum_{m=-\infty}^{\infty} \{6c_m^2 - 8c_m c_{m+1} + 2c_m c_{m+2}\}\\
&\quad - \binom{n-1}{n-7}\\
&\quad \times \sum_{m=-\infty}^{\infty} \{624c_m^2 + 688c_m c_{m+1} + 128c_m c_{m+2}\}.
\end{aligned}$$

To make sense of this, let

$$X = \sum_{m=-\infty}^{\infty} 2c_m^2, \quad Y = \sum_{m=-\infty}^{\infty} 2c_m c_{m+1},$$
$$Z = \sum_{m=-\infty}^{\infty} 2c_m c_{m+2},$$

and let $\alpha, \dots, \zeta$ be the appropriate coefficients so that the last expression for $(-1)^{n-4}a_{n-4}$ above reduces to

$$\begin{aligned}
&\alpha(9X + 8Y + Z) + \beta(11X + 12Y + Z)\\
&\quad + \gamma(66X + 52Y + 2Z) + \delta(X - Z) + \varepsilon(3X - 4Y + Z)\\
&\quad + \zeta(312X + 344Y + 64Z).
\end{aligned}$$

We want to reexpress it as

$$A(9X + 8Y + Z) + B(3X - 4Y + Z) + C(X - Z).$$

This implies that

$$\begin{aligned}
18A &= 18\alpha + 24\beta + 120\gamma + 720\zeta\\
&= -\frac{(n-1)(n-2)(n-3)(n-5)(n-6)(n-7)}{2},
\end{aligned}$$

while

$$\begin{aligned}
4B &= 8A - (8\alpha + 12\beta + 52\gamma - 4\varepsilon + 344\zeta)\\
&= -\frac{(n-1)^2(n-2)(n-3)(n-4)(n-5)}{15}
\end{aligned}$$

and

$$\begin{aligned}
C &= A + B - (\alpha + \beta + 2\gamma - \delta + \varepsilon + 64\zeta)\\
&= -\frac{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{30}.
\end{aligned}$$

Finally, the fact that

$$\begin{aligned}
&9X + 8Y + Z\\
&\quad = \|\{(\Delta^{n-4}b_m + 4\Delta^{n-4}b_{m+1} + \Delta^{n-4}b_{m+2}\}\|^2,\\
&3X - 4Y + Z = \|\{\Delta^{n-2}b_m\}\|^2,\\
&\qquad X - Z = \|\{\Delta^{n-4}b_m - \Delta^{n-4}b_{m+2}\}\|^2
\end{aligned}$$

together with the observation that $(-1)^{n-4} = -(-1)^{n-1}$ allows us to express a_{n-4} succinctly as

$$\begin{aligned}
a_{n-4} &= (-1)^{n-1}(n-1)(n-2)(n-3)(n-5) \qquad (5\text{–}6)\\
&\quad \times \left\{ \frac{(n-6)(n-7)}{36}\|\{\Delta^{n-4}b_m + 4\Delta^{n-4}b_{m+1} \right.\\
&\qquad + \Delta^{n-4}b_{m+2}\}\|^2 + \frac{(n-1)(n-4)}{60}\|\{\Delta^{n-2}b_m\}\|^2\\
&\qquad \left. + \frac{(n-4)(n-6)}{30}\|\{\Delta^{n-4}b_m - \Delta^{n-4}b_{m+2}\}\|^2 \right\}.
\end{aligned}$$

In view of subsequent applications, it is useful to note that

$$\|\{\Delta^{n-4}b_m - \Delta^{n-4}b_{m+2}\}\|^2 = \|\{\Delta^{n-3}b_m + \Delta^{n-3}b_{m+1}\}\|^2.$$

6. PROOF OF PROPOSITION 1.6

$n = 1$. In the case $n = 1$, $\Phi(x)$ is simply the constant

$$\Phi(x) = a_0 = \sum_{k=-\infty}^{\infty} b_k^2 = \|\{b_k\}\|^2.$$

$n = 2$. The case $n = 2$ follows from formula (5–3) for general a_{n-1}. Namely,

$$a_0 = \|\{b_k\}\|^2 \quad \text{and} \quad a_1 = -\|\{\Delta b_k\}\|^2.$$

$n = 3$. The case $n = 3$ follows from (5–3) and

$$a_1 = 0, \qquad (6\text{–}1)$$

which is valid when $n \geq 3$. Namely,

$$a_0 = \|\{b_k + b_{k+1}\}\|^2, \quad a_1 = 0, \quad \text{and} \quad a_2 = \|\{\Delta^2 b_k\}\|^2.$$

$n = 4$. The case $n = 4$ can be deduced from (5–3), (6–1), and the general formula (5–4) for a_{n-2}, which is valid when $n \geq 4$. Specifically,

$$a_0 = \|\{b_k + 4b_{k+1} + b_{k+2}\}\|^2 \quad \text{and} \quad a_1 = 0,$$

while

$$a_2 = -3\|\{\Delta^2 b_k\}\|^2 \quad \text{and} \quad a_3 = -\|\{\Delta^3 b_k\}\|^2.$$

$n = 5$. The case $n = 5$ can be deduced from (5–3), (5–4), (6–1), and the general formula

$$2a_2 = a_3, \tag{6–2}$$

which is valid when $n \geq 5$. Specifically,

$$a_0 = \|\{b_k + 11b_{k+1} + 11b_{k+2} + b_{k+3}\}\|^2 \quad \text{and} \quad a_1 = 0,$$

while

$$\begin{aligned} a_2 &= 4\|\{\Delta^3 b_k\}\|^2, \\ a_3 &= 8\|\{\Delta^3 b_k\}\|^2, \\ a_4 &= \|\{\Delta^4 b_k\}\|^2. \end{aligned}$$

$n = 6$. The case $n = 6$ can be deduced from (5–3), (5–4), (6–1), (6–2), and the general formula (5–5) for a_{n-3}, which is valid when $n \geq 6$. Specifically,

$$\begin{aligned} a_0 &= \|\{b_k + 26b_{k+1} + 66b_{k+2} + 26b_{k+3} + b_{k+4}\}\|^2, \\ a_1 &= 0, \end{aligned}$$

while

$$\begin{aligned} a_2 &= -5\|\{\Delta^3 b_k + \Delta^3 b_{k+1}\}\|^2 - 10\|\{\Delta^4 b_k\}\|^2, \\ a_3 &= 2a_2, \quad a_4 = -15\|\{\Delta^4 b_k\}\|^2, \quad a_5 = -\|\{\Delta^5 b_k\}\|^2. \end{aligned}$$

$n = 7$. The case $n = 7$ can be deduced from (5–3), (5–4), (5–5), (6–1), (6–2), and the general formula

$$3a_3 = 4a_4 - a_5, \tag{6–3}$$

which is valid when $n \geq 7$. Specifically,

$$\begin{aligned} a_0 &= \|\{b_k + 57b_{k+1} + 302b_{k+2} + 302b_{k+3} + 57b_{k+4} \\ &\qquad + b_{k+6}\}\|^2, \\ a_1 &= 0, \end{aligned}$$

while

$$\begin{aligned} a_2 &= \frac{a_3}{2}, \\ a_3 &= 60\|\{\Delta^4 b_k + \Delta^4 b_{k+1}\}\|^2 + 72\|\{\Delta^5 b_k\}\|^2, \\ a_4 &= 45\|\{\Delta^4 b_k + \Delta^4 b_{k+1}\}\|^2 + 60\|\{\Delta^5 b_k\}\|^2, \\ a_5 &= 24\|\{\Delta^5 b_k\}\|^2, \\ a_6 &= \|\{\Delta^6 b_k\}\|^2. \end{aligned}$$

$n = 8$. In the case $n = 8$, in addition to the formulas used above, we also make use of (5–6), which is valid when $n \geq 8$, to get

$$\begin{aligned} a_7 &= -\|\{\Delta^7 b_k\}\|^2, \\ a_6 &= -35\|\{\Delta^6 b_k\}\|^2, \\ a_5 &= -126\|\{\Delta^5 b_k + \Delta^5 b_{k+1}\}\|^2 - 140\|\{\Delta^6 b_k\}\|^2, \\ a_4 &= -35\|\{\Delta^4 b_k + 4\Delta^4 b_{k+1} + \Delta^4 b_{k+2}\}\|^2 \\ &\quad - 294\|\{\Delta^6 b_k\}\|^2 - 168\|\{\Delta^5 b_k + \Delta^5 b_{k+1}\}\|^2, \\ 3a_3 &= 4a_4 - a_5, \quad 2a_2 = a_3, \quad a_1 = 0, \end{aligned}$$

and it is clear that both a_3 and a_2 are negative.

$n = 9$. In the case $n = 9$, in addition to all the formulas used above, we make use of

$$4a_4 = 10a_5 - 6a_6 + a_7, \tag{6–4}$$

which is valid whenever $n \geq 9$, to get

$$\begin{aligned} a_8 &= \|\{\Delta^8 b_k\}\|^2, \quad a_7 = 48\|\{\Delta^7 b_k\}\|^2, \\ a_6 &= 280\|\{\Delta^6 b_k + \Delta^6 b_{k+1}\}\|^2 + 280\|\{\Delta^7 b_k\}\|^2, \\ a_5 &= 224\|\{\Delta^5 b_k + 4\Delta^5 b_{k+1} + \Delta^5 b_{k+2}\}\|^2 \\ &\quad + 896\|\{\Delta^7 b_k\}\|^2 + 672\|\{\Delta^6 b_k + \Delta^6 b_{k+1}\}\|^2, \end{aligned}$$

$$\begin{aligned} 4a_4 &= 10a_5 - 6a_6 + a_7, \quad 3a_3 = 4a_4 - a_5, \quad 2a_2 = a_3, \\ a_1 &= 0, \end{aligned}$$

from which it should be clear that all the coefficients a_k, $k = 2, \ldots, 8$, are positive.

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To establish Corollary 1.2 in the case of odd n, we used a property concerning the maximum of the modulus of the so-called Euler exponential splines. These splines were christened and studied by Schoenberg; see, for example, [Schoenberg 73, Schoenberg 83]. The authors would like to thank Carl de Boor, who kindly responded to a query concerning this matter and provided the reference [de Boor 76], which contains a proof of this property.

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