Critical Values of Derivatives of Twisted Elliptic L-Functions

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CONTENTS

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Let $L(E/\mathbb{Q},s)$ be the L-function of an elliptic curve E defined over the rational field $\mathbb{Q}$. We examine special values of the derivatives $L'(E,1,\chi)$ of twists by Dirichlet characters of $L(E/\mathbb{Q},s)$ when $L(E,1,\chi)=0$.

1. INTRODUCTION

Let $E/\mathbb{Q}$ be an elliptic curve defined over the field $\mathbb{Q}$. Denote by

$$L(E/\mathbb{Q},s)=\sum_{n\geq 1}a_n n^{-s}$$

its L-function.

By the proof [Breuil et al. 01, Taylor and Wiles 95] of the modularity of elliptic curves over $\mathbb{Q}$, we know that $L(E,s)$ has an analytic continuation for all $s\in\mathbb{C}$, and satisfies the functional equation

$$\Lambda(E,s)=w_E\Lambda(E,2-s),$$

where $\Lambda(E,s)=(\sqrt{N_E}/2\pi)^s\Gamma(s)L(E,s)$, N_E is the conductor of $E/\mathbb{Q}$, and $w_E=\pm 1$ is the *sign* of the functional equation.

For a primitive Dirichlet character χ of conductor $\mathfrak{f}_\chi$, the *twist* of $L(E/\mathbb{Q},s)$ by χ is

$$L(E,s,\chi)=\sum_{n\geq 1}\chi(n)a_n n^{-s}.$$

We also know that the L-function $L(E,s,\chi)$ has an analytic continuation and that if $\mathfrak{f}_\chi$ is coprime to N_E, it satisfies the functional equation

$$\Lambda(E,s,\chi)=w_E\chi(N_E)\tau(\chi)^2\mathfrak{f}_\chi^{-1}\Lambda(E,2-s,\overline{\chi}), \quad (1\text{–}1)$$

where $\Lambda(E,s,\chi)=(\mathfrak{f}_\chi\sqrt{N_E}/2\pi)^s\Gamma(s)L(E,s,\chi)$ and $\tau(\chi)$ is the Gauss sum

$$\tau(\chi)=\sum_{a=0}^{\mathfrak{f}_\chi-1}\chi(a)\exp(2\pi ia/\mathfrak{f}_\chi).$$

The Birch–Swinnerton-Dyer conjecture asserts that for any finite extension $K/\mathbb{Q}$,

$$\mathrm{ord}_{s=1} L(E/K,s) = \mathrm{rank}_{\mathbb{Z}}\, E(K),$$

where $E(K)$ is the Mordell–Weil group of E/K, and for a finitely generated abelian group A, $\mathrm{rank}_{\mathbb{Z}} A$ is the $\mathbb{Z}$-rank of A, i.e., the number of copies of $\mathbb{Z}$ in the decomposition of A into cyclic factors.

The Birch & Swinnerton-Dyer conjecture also asserts that the first non-vanishing derivative satisfies

$$\frac{L^{(r_K)}(E/K,1)}{r_K!} = \frac{\Omega_K |\text{Ш}(K)| R_K}{\sqrt{|d(K)|}|E(K)_{\mathrm{Tors}}|^2} \prod_{\mathfrak{p}} c_{\mathfrak{p}}$$

where $r_K = \mathrm{rank}_{\mathbb{Z}}\, E(K)$, where Ω_K is a product of periods of E, $d(K)$ is the discriminant of K and R_K denotes the elliptic regulator, *i.e.* R_K is the absolute value of the determinant of the height pairing matrix of $E(K)$. Also $\text{Ш}(K)$ is the Tate-Shafarevich group of E/K, and the $c_{\mathfrak{p}}$ are the Tamagawa numbers of E/K modified by a rational factor as in [Do 10].

If $K/\mathbb{Q}$ is an abelian extension with $\mathrm{Gal}(K/\mathbb{Q}) = G$, then there is a factorization

$$L(E/K,s) = \prod_{\chi \in \widehat{G}} L(E,s,\chi),$$

where $\widehat{G}$ is the group of primitive Dirichlet characters associated with the extension $K/\mathbb{Q}$. Then $L^{(r_K)}(E/K,1)$ can be computed in terms of the derivatives $L^{(i)}(E,1,\chi)$, for $i \le r_K$ and for all $\chi \in \widehat{G}$.

Let Ω_χ be the real period Ω^+ of $E/\mathbb{Q}$ if $\chi(-1) = 1$ and the imaginary part Ω^- of the complex period of E if $\chi(-1) = -1$. Then [Mazur 79] defines the *algebraic part* $L^{\mathrm{alg}}(E,1,\chi)$ of $L(E,1,\chi)$ by

$$L(E,1,\chi) = \frac{\tau(\chi)}{2f_\chi} \Omega_\chi L^{\mathrm{alg}}(E,1,\chi).$$

Mazur shows that

$$L^{\mathrm{alg}}(E,1,\chi) = \sum_{a \bmod f_\chi} \overline{\chi}(a) \Lambda(a,m),$$

where $\Lambda(a,m) \in \frac{1}{n_E}\mathbb{Z}$ if E is the "strong curve" in its isogeny class, and where n_E is an integer depending on E but not on the character χ as long as $\mathfrak{f}_\chi$ and N_E are coprime.

It is then clear that $L^{\mathrm{alg}}(E,1,\chi)^\gamma = L^{\mathrm{alg}}(E,1,\chi^\gamma)$ for any $\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, where $\chi^\gamma(\sigma) = \gamma(\chi(\sigma)))$ for all $\sigma \in G$. Therefore $L(E,1,\chi) = 0$ for some $\chi \in \widehat{G}$ if and only if $L(E,1,\chi^\gamma) = 0$ for all $\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$.

Remark 1.1. The conjecture of Deligne–Gross asserts in a special case that [Deligne 79, p. 323]

$$\mathrm{ord}_{s=1} L(E,s,\chi^\gamma) = \mathrm{ord}_{s=1} L(E,s,\chi)$$

for all $\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. It is shown in [Rohrlich 90] that if we assume the Birch–Swinnerton-Dyer conjecture, in addition to the above-mentioned conjecture of Deligne–Gross, then the order of vanishing of $L(E/\mathbb{Q},s,\chi)$ at $s = 1$ is the multiplicity of χ in the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acting on $\overline{\mathbb{Q}} \otimes_{\mathbb{Z}} E(\overline{\mathbb{Q}})$.

In this article we consider the case that $K/\mathbb{Q}$ is a cyclic extension of odd prime degree, $[K:\mathbb{Q}] = \ell \ge 3$. Fix a generator $\sigma_0 \in G = \mathrm{Gal}(K/\mathbb{Q})$.

We examine the case that the functions $L(E,s,\chi)$ have simple zeros at $s = 1$ for $\chi \ne \chi_0$, and we call this the χ-rank-1 case.

Then the Birch–Swinnerton-Dyer conjecture in the χ-rank-1 case predicts that

$$\mathrm{rank}_{\mathbb{Z}}\, E(K) = \mathrm{rank}_{\mathbb{Z}}\, E(\mathbb{Q}) + (\ell - 1).$$

Hence one expects that

$$\frac{L^{(r_K)}(E/K,1)}{r_K!} = \frac{L^{(r_{\mathbb{Q}})}(E/\mathbb{Q},1)}{r_{\mathbb{Q}}!} \cdot \prod_{\chi_0 \ne \chi \in \widehat{G}} L'(E,1,\chi),$$

and therefore that

$$\frac{\Omega_K |\text{Ш}(K)| R_K}{\sqrt{|d(K)|}|E(K)_{\mathrm{Tors}}|^2} \prod_{\mathfrak{p}} c_{\mathfrak{p}} = \frac{\Omega^+ |\text{Ш}(\mathbb{Q})| R_{\mathbb{Q}}}{|E(\mathbb{Q})_{\mathrm{Tors}}|^2} \prod_p c_p \times \prod_{\chi_0 \ne \chi \in \widehat{G}} L'(E,1,\chi), \tag{1–2}$$

where $\Omega_K = (\Omega^+)^\ell$, since K is a totally real field.

Numerical computations, which we describe below in Section 9, suggest that for nonprincipal (nontrivial) characters $\chi \ne \chi_0$,

$$L'(E,1,\chi) = \frac{\tau(\chi)}{\mathfrak{f}_\chi} \Omega^+ \lambda_\chi(P) \alpha_\chi^+(P) z_\chi,$$

where $P \in E_{\mathrm{Tr}}(K)$ is a point of infinite order in $E(K)$ with trace 0 to $\mathbb{Q}$ (i.e., $\mathrm{Tr}_{K/\mathbb{Q}}(P) = 0$); $\lambda_\chi(P) = \sum_{\sigma \in G} \chi(\sigma)\langle P, P^\sigma\rangle$, where $\langle\cdot,\cdot\rangle$ is the Néron–Tate canonical height pairing on $E(\overline{\mathbb{Q}})$; and where $\alpha_\chi^+(P)$ and z_χ are algebraic numbers in $\mathbb{Q}(\chi)$.

These results can be considered as evidence in support of the equivariant Tamagawa number conjectures as formulated in [Burns and Flach 01] and [Burns 10, Theorem 5.1.1 (i)].

2. CYCLOTOMIC ANALOGUE

Let $L(s,\chi)$ denote a Dirichlet L-function with primitive Dirichlet character χ of conductor $\mathfrak{f}_\chi$. Then

$$L(1,\chi)=\begin{cases}-\frac{\tau(\chi)}{\mathfrak{f}_\chi}\sum_a \overline{\chi}(a)\log|1-\zeta^a| & \text{if } \chi(-1)=1,\\ -\frac{\tau(\chi)}{\mathfrak{f}_\chi}\frac{\pi}{i\mathfrak{f}_\chi}\sum_a \overline{\chi}(a)a & \text{if } \chi(-1)=-1,\end{cases}$$

where $\zeta=\exp(2\pi i/\mathfrak{f}_\chi)$ and the sums are taken over integers $1\le a\le \mathfrak{f}_\chi$ with $(a,\mathfrak{f}_\chi)=1$. Using the functional equation for Dirichlet L-functions, this translates at $s=0$ to the following:

$$L'(0,\chi)=-\frac{1}{2}\sum_a \chi(a)\log|1-\zeta^a|,\qquad \text{if } \chi(-1)=1,$$

$$L(0,\chi)=-\frac{1}{\mathfrak{f}_\chi}\sum_a \chi(a)a,\qquad \text{if } \chi(-1)=-1.$$

In this article, we think of $L(0,\chi)$ for odd characters as the analogue of $L(E,1,\chi)$ in the case of χ-rank 0, and $L'(0,\chi)$ for even characters as analogous to $L'(E,1,\chi)$ in the case of χ-rank 1.

3. THE ARGUMENT OF $L^{(r)}(E,1,\chi)$

Suppose that E is an elliptic curve defined over $\mathbb{Q}$ with conductor N_E and let $L(E,s)$ denote its L-function.

Proposition 3.1. *Let χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ coprime to N_E and of order at least 3. Let r_χ be the order of vanishing of $L(E,s,\chi)$ at $s=1$. Then there is an algebraic integer $z_\chi\in\mathbb{Q}(\chi)$ such that*

$$\frac{L^{(r_\chi)}(E,1,\chi)}{\tau(\chi)z_\chi}$$

is real. Furthermore, for all $\gamma\in\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, z_χ satisfies

$$z_{\chi^\gamma}=\gamma(z_\chi).$$

Proof. Differentiating the functional equation (1–1) r_χ times and evaluating at $s=1$, we obtain

$$L^{(r_\chi)}(E,1,\chi)=(-1)^{r_\chi}w_E\chi(N_E)\tau(\chi)^2\mathfrak{f}_\chi^{-1}L^{(r_\chi)}(E,1,\overline{\chi}). \tag{3–1}$$

Noting that $\mathfrak{f}_\chi=\tau(\chi)\overline{\tau(\chi)}$, we can rewrite (3–1) as

$$\frac{L^{(r_\chi)}(E,1,\chi)}{\tau(\chi)}=(-1)^{r_\chi}w_E\chi(N_E)\frac{\overline{L^{(r_\chi)}(E,1,\chi)}}{\overline{\tau(\chi)}}. \tag{3–2}$$

Therefore (3–2) has the form

$$z=\zeta\overline{z},$$

where $z\in\mathbb{C}^*$ and $\zeta=(-1)^{r_\chi}w_E\chi(N_E)$ is a root of unity whose order divides twice the order of χ.

Choose

$$z_\chi=\begin{cases}1+\zeta & \text{if } \zeta=(-1)^{r_\chi}w_E\chi(N_E)\neq -1,\\ \chi(\sigma_0)-\overline{\chi}(\sigma_0) & \text{if } \zeta=(-1)^{r_\chi}w_E\chi(N_E)=-1.\end{cases}$$

Then

$$z_\chi=\zeta\overline{z_\chi},$$

and therefore

$$\frac{L^{(r_\chi)}(E,1,\chi)}{\tau(\chi)z_\chi}=\frac{\overline{L^{(r_\chi)}(E,1,\chi)}}{\overline{\tau(\chi)z_\chi}}$$

is real. Clearly, for this choice of z_χ we have $z_{\chi^\gamma}=\gamma(z_\chi)$ for all $\gamma\in\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. $\square$

Remark 3.2. We note that z_χ is not unique. If ρ_χ is chosen in the real subfield $\mathbb{Q}(\chi)^+\subset\mathbb{Q}(\chi)$ such that $\rho_{\chi^\gamma}=\gamma(\rho_\chi)$ for all $\gamma\in G$, then $z'_\chi=z_\chi\rho_\chi$ will work as well.

4. REGULATORS AND HEIGHT PAIRINGS

Suppose that E is an elliptic curve defined over $\mathbb{Q}$ and that $K/\mathbb{Q}$ is an arbitrary finite extension. Let $h: E(\overline{\mathbb{Q}})\to\mathbb{R}$ denote the Néron–Tate canonical height function on E and

$$\langle\cdot,\cdot\rangle: E(\overline{\mathbb{Q}})\times E(\overline{\mathbb{Q}})\to\mathbb{R}$$

the canonical height pairing on E (see [Silverman 86, Section 8.8]). Then the height pairing $\langle\cdot,\cdot\rangle$ is a positive definite bilinear form on $E(\overline{\mathbb{Q}})$ satisfying the following properties for $P,Q\in E(\overline{\mathbb{Q}})$:

$$\begin{aligned} h(P)\ge 0,\ \text{and } h(P)=0 \text{ iff } P\in E(\overline{\mathbb{Q}})_{\mathrm{Tors}},\\ \langle P,P\rangle=h(P)\ge 0,\\ \langle P,Q\rangle=\langle Q,P\rangle,\\ \langle P^\sigma,Q^\sigma\rangle=\langle P,Q\rangle,\end{aligned}$$

for all $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

It follows that $\langle Q,P^\sigma\rangle=\langle Q,P\rangle$ for $Q\in E(\mathbb{Q})$ and for all $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Suppose that $P_1,P_2,\dots,P_r$ generate a subgroup $E_{\{P\}}\subseteq E(K)$ of rank r. The regulator matrix and the regulator (determinant) are defined as

$$\begin{aligned} RM(P_1,P_2,\dots,P_r)&:=\left(\langle P_i,P_j\rangle\right)_{1\le i,j\le r},\\ R(P_1,P_2,\dots,P_r)&:=|\det\left(\langle P_i,P_j\rangle\right)|.\end{aligned}$$

If $E_{\{P\}}$ and $E(K)_{\mathrm{Tors}}$ generate all of $E(K)$, then $R(P_1,P_2,\dots,P_r)=R_K$.

If $Q_1, Q_2, \ldots, Q_r$ generate a subgroup $E_{\{Q\}}$ of finite index m in $E_{\{P\}}$, then

$$R(Q_1, Q_2, \ldots, Q_r) = m^2 R(P_1, P_2, \ldots, P_r).$$

Therefore passing to subgroups of finite index introduces only integral square factors in the regulators.

Let $E_{\mathrm{Tr}}(K) \subseteq E(K)$ be the subgroup of points of trace 0 to $\mathbb{Q}$, i.e.,

$$E_{\mathrm{Tr}}(K) = \{P \in E(K) \mid \mathrm{Tr}_{K/\mathbb{Q}}(P) = 0\}.$$

Since $E(K)$ is a finitely generated abelian group, it follows from the exact sequence

$$0 \longrightarrow E_{\mathrm{Tr}}(K) \longrightarrow E(K) \longrightarrow \mathrm{Tr}_{K/\mathbb{Q}}(E(K)) \longrightarrow 0$$

that

$$\mathrm{rank}_{\mathbb{Z}}\, E(K) = \mathrm{rank}_{\mathbb{Z}}\, \mathrm{Tr}_{K/\mathbb{Q}}(E(K)) + \mathrm{rank}_{\mathbb{Z}}\, E_{\mathrm{Tr}}(K).$$

On the other hand, we have

$$[K : Q]E(\mathbb{Q}) \subset \mathrm{Tr}_{K/\mathbb{Q}}(E(K)) \subset E(\mathbb{Q}).$$

Hence

$$\mathrm{rank}_{\mathbb{Z}}\, E(\mathbb{Q}) = \mathrm{rank}_{\mathbb{Z}}\, \mathrm{Tr}_{K/\mathbb{Q}}(E(K)),$$

and since

$$E(\mathbb{Q}) \cap E_{\mathrm{Tr}}(K) \subset E(\mathbb{Q})_{\mathrm{Tors}},$$

we see that the subgroup

$$E' = E_{\mathrm{Tr}}(K) \cdot E(\mathbb{Q})$$

has finite index in $E(K)$, say $[E(K) : E'] = m < \infty$. It follows that the regulator $R(E')$ is equal to $m^2 R_K$.

We also see that $E_{\mathrm{Tr}}(K)$ is orthogonal to $E(\mathbb{Q})$ with respect to the canonical height pairing, since for $Q \in E(\mathbb{Q})$ and $P \in E_{\mathrm{Tr}}(K)$ we have

$$\begin{aligned}\langle Q, P\rangle &= \frac{1}{[K:\mathbb{Q}]}\Big\langle Q, \sum_{\sigma: K \to \overline{\mathbb{Q}}} P^{\sigma}\Big\rangle \\ &= \frac{1}{[K:\mathbb{Q}]}\left\langle Q, \mathrm{Tr}_{K/\mathbb{Q}}(P)\right\rangle = 0.\end{aligned}$$

Therefore, the regulator matrix of $E'(K)$ (up to finite index) can be written as

$$\begin{pmatrix} RM(E(\mathbb{Q})) & 0 \\ 0 & RM(E_{\mathrm{Tr}}(K)) \end{pmatrix}.$$

Hence for some integer m, we have

$$R_K = R_{\mathbb{Q}} \times R(E_{\mathrm{Tr}}(K)) \times m^2.$$

5. $K/\mathbb{Q}$ CYCLIC OF PRIME DEGREE

We assume now that $K/\mathbb{Q}$ is a cyclic extension of odd prime degree $[K : \mathbb{Q}] = \ell \geq 3$ with Galois group $G = \mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma_0\rangle$.

Then $E_{\mathrm{Tr}}(K)$ is a module over the ring $\mathbb{Z}[G]/(\mathrm{Tr}) \simeq \mathbb{Z}[\zeta_\ell] = \mathcal{O}_\ell$, where $\mathrm{Tr} = 1 + \sigma_0 + \sigma_0^2 + \cdots + \sigma_0^{\ell-1} \in \mathbb{Z}[G]$. Since $\mathcal{O}_\ell$ is a Dedekind domain, there is an ideal $\mathfrak{A} \subseteq \mathcal{O}_\ell$ such that there is an $\mathcal{O}_\ell$-isomorphism

$$E_{\mathrm{Tr}}(K) \simeq E_{\mathrm{Tr}}(K)_{\mathrm{Tors}} \times \mathcal{O}_\ell^{d-1} \times \mathfrak{A}.$$

Hence there is a free $\mathcal{O}_\ell$-submodule $E^*(K) \subseteq E_{\mathrm{Tr}}(K)$ of finite index with $E^*(K) \simeq \mathcal{O}_\ell^d$. The index $[E_{\mathrm{Tr}}(K) : E^*(K)]$ depends on the order of $E_{\mathrm{Tr}}(K)_{\mathrm{Tors}}$ and on the index of the largest principal ideal of $\mathcal{O}_\ell$ contained in $\mathfrak{A}$. By the finiteness of class number and by Merel's theorem [Merel 96] on the boundedness of torsion, the minimal such index depends only on ℓ (and not on the curve E or on the character χ of order ℓ). It is clear that the $\mathbb{Z}$-rank r_0 of $E_{\mathrm{Tr}}(K)$ satisfies

$$r_0 = d(\ell - 1).$$

Under the χ-rank-1 assumption that $\mathrm{ord}_{s=1} L(E, s, \chi^\gamma) = 1$ for $\chi \neq \chi_0 \in \widehat{G}$ and for all $\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, it is a consequence of the Birch–Swinnerton-Dyer conjecture that $d = 1$ and $E^*(K) \simeq \mathcal{O}_\ell$.

Therefore there is a subgroup $E'(K) \subseteq E(K)$ of bounded index, and there is a point of trace zero $P \in E_{\mathrm{Tr}}(K)$ such that

$$E'(K) \simeq E(\mathbb{Q}) \oplus \left\langle P, \sigma_0(P), \sigma_0^2(P), \ldots, \sigma_0^{\ell-2}(P)\right\rangle.$$

Since $E(\mathbb{Q})$ and $E_{\mathrm{Tr}}(K)$ are orthogonal with respect to the canonical height pairing, we see that the regulator matrix of $E'(K)$ can be written as

$$\begin{pmatrix} RM(E(\mathbb{Q})) & 0 & \cdots & 0 \\ 0 & \langle P, P\rangle & \cdots & \langle P, P^{\sigma_0^{\ell-2}}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \langle P^{\sigma_0^{\ell-2}}, P\rangle & \cdots & \langle P^{\sigma_0^{\ell-2}}, P^{\sigma_0^{\ell-2}}\rangle \end{pmatrix},$$

so that there is an integer m such that

$$R_K = R_{\mathbb{Q}} \times R(P, \sigma_0(P), \sigma_0^2(P), \ldots, \sigma_0^{\ell-2}(P)) \times m^2. \tag{5–1}$$

But $R(P, \sigma_0(P), \sigma_0^2(P), \ldots, \sigma_0^{\ell-2}(P))$ is a group determinant (see [Washington 82, Lemma 5.26]) and can therefore be computed as

$$R(P, \sigma_0(P), \sigma_0^2(P), \ldots, \sigma_0^{\ell-2}(P)) = \frac{1}{|G|}\prod_{\chi \neq \chi_0} \lambda_\chi(P), \tag{5–2}$$

where $\lambda_\chi(P) = \sum_{\sigma \in G} \chi(\sigma)\langle P, P^\sigma\rangle$.

6. THE FUNCTION λ_χ

Let E be an elliptic curve defined over the rational field $\mathbb{Q}$, and in this section we let $K/\mathbb{Q}$ be an arbitrary finite Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) = G$. For $P \in E(K)$ and $\chi \in \widehat{G}$ a 1-dimensiona1 character of G, we use the canonical height pairing to define

$$\lambda_\chi(P) := \sum_{\sigma\in G} \chi(\sigma)\langle P, P^\sigma\rangle.$$

In order to derive the properties of $\lambda_\chi(\cdot)$, we introduce a two-variable generalization and define the function $\lambda_\chi(\cdot,\cdot) : E(K) \times E(K) \longrightarrow \mathbb{C}$ by

$$\lambda_\chi(P,Q) := \sum_{\sigma\in G} \chi(\sigma)\langle P, Q^\sigma\rangle.$$

Proposition 6.1. *The function $\lambda_\chi(P,Q)$ is bilinear for $P,Q \in E(K)$ and satisfies the following elementary equivariance properties:*

1. *If P or Q belongs to $E(K)_{\mathrm{Tors}}$, then $\lambda_\chi(P,Q) = 0$.*
2. *$\lambda_\chi(P,P) = \lambda_\chi(P)$.*
3. *$\lambda_\chi(P^\tau,Q) = \chi(\tau)\lambda_\chi(P,Q)$ for $\tau \in G$.*
4. *$\lambda_\chi(P,Q^\tau) = \overline{\chi}(\tau)\lambda_\chi(P,Q)$ for $\tau \in G$.*
5. *$\overline{\lambda_\chi(P,Q)} = \lambda_{\overline{\chi}}(P,Q) = \lambda_\chi(Q,P)$.*

Proof. The bilinearity is immediate from the corresponding property of the canonical height pairing. To prove assertion 3, we use equivariance properties of the pairing

$$\begin{aligned}\lambda_\chi(P^\tau,Q) &= \sum_{\sigma\in G}\chi(\sigma)\langle P^\tau, Q^\sigma\rangle = \sum_{\sigma\in G}\chi(\sigma)\langle P, Q^{\sigma\tau^{-1}}\rangle\\ &= \sum_{\rho\in G}\chi(\rho\tau)\langle P, Q^\rho\rangle = \chi(\tau)\lambda_\chi(P,Q).\end{aligned}$$

The other statements are proved similarly. $\square$

Corollary 6.2. *If $P \in E(K)$, then $\overline{\lambda_\chi(P)} = \lambda_\chi(P) = \lambda_{\overline{\chi}}(P)$ is real. Furthermore, if $\xi = \sum_{\sigma\in G} a_\sigma\sigma$ and $\eta = \sum_{\sigma\in G} b_\sigma\sigma$ are elements of the integral group ring $\mathbb{Z}[G]$, then*

$$\begin{aligned}\lambda_\chi(\xi(P)) &= \chi(\xi)\overline{\chi}(\xi)\lambda_\chi(P) = |\chi(\xi)|^2\lambda_\chi(P)\\ \lambda_\chi(\xi(P),\eta(Q)) &= \chi(\xi)\overline{\chi}(\eta)\lambda_\chi(P,Q),\end{aligned}$$

where $\chi(\xi) = \sum_{\sigma\in G} a_\sigma\chi(\sigma)$ and $\chi(\eta) = \sum_{\sigma\in G} b_\sigma\chi(\sigma)$.

Proof. This follows immediately from Proposition 6.1. $\square$

Assume now that $K/\mathbb{Q}$ is a cyclic extension of order ℓ with $\mathrm{Gal}(K/\mathbb{Q}) = G = \langle\sigma_0\rangle$.

Proposition 6.3. *Suppose that $P \in E(K)$ has infinite order. If $P \notin E(\mathbb{Q})$, then $\lambda_\chi(P)$ is positive for all nonprincipal (nontrivial) characters $\chi_0 \neq \chi \in \widehat{G}$.*

Proof. Consider the $\ell\times\ell$ group matrix

$$A = \begin{pmatrix} \langle P,P\rangle & \cdots & \left\langle P, P^{\sigma_0^{\ell-1}}\right\rangle\\ \vdots & \vdots & \vdots\\ \left\langle P^{\sigma_0^{\ell-1}}, P\right\rangle & \cdots & \left\langle P^{\sigma_0^{\ell-1}}, P^{\sigma_0^{\ell-1}}\right\rangle\end{pmatrix}.$$

Then a result of Dedekind–Frobenius (see [Washington 82, Lemma 5.26]) shows that the matrix A has eigenvalues $\lambda_\chi(P) = \sum_\sigma \chi(\sigma)\langle P, P^\sigma\rangle$ and corresponding eigenvectors

$$v_\chi = \begin{pmatrix} 1\\ \chi(\sigma_0)\\ \vdots\\ \chi(\sigma_0^{\ell-1})\end{pmatrix}.$$

If $\mathrm{Tr}_{K/\mathbb{Q}}(P)$ has infinite order in $E(\mathbb{Q})$, then A is the matrix of the quadratic form induced by the Néron–Tate height-pairing on $L(P)\otimes\mathbb{R}$, where

$$L(P) = \left\langle P, \sigma_0(P), \sigma_0^2(P), \ldots, \sigma_0^{\ell-1}(P)\right\rangle \subseteq E(K)$$

is the subgroup of $E(K)$ generated by all the conjugates of P. By a result of Cassels (see [Silverman 86, Section 8.9]), A is a positive definite matrix whose eigenvalues $\lambda_\chi(P)$ are therefore positive.

Suppose now that $\mathrm{Tr}_{K/\mathbb{Q}}(P) = 0$. Since $\lambda_{\overline{\chi}}(P) = \overline{\lambda_\chi(P)} = \lambda_\chi(P)$ is real, the vector $\frac{1}{i}(v_\chi - v_{\overline{\chi}})$ is also a real eigenvector for $\lambda_\chi(P)$. But then a simple calculation shows that the vector

$$w_\chi = \frac{1}{i}\cdot\begin{pmatrix}\chi(\sigma_0) - \overline{\chi}(\sigma_0)\\ \vdots\\ \chi(\sigma_0^{\ell-1}) - \overline{\chi}(\sigma_0^{\ell-1})\end{pmatrix}$$

is a real eigenvector of the $(\ell-1)\times(\ell-1)$ matrix

$$\begin{aligned}B &= \begin{pmatrix} \langle P,P\rangle & \cdots & \left\langle P, P^{\sigma_0^{\ell-2}}\right\rangle\\ \vdots & \vdots & \vdots\\ \left\langle P^{\sigma_0^{\ell-2}}, P\right\rangle & \cdots & \left\langle P^{\sigma_0^{\ell-2}}, P^{\sigma_0^{\ell-2}}\right\rangle\end{pmatrix}\\ &= \begin{pmatrix} \langle P^{\sigma_0},P^{\sigma_0}\rangle & \cdots & \left\langle P^{\sigma_0}, P^{\sigma_0^{\ell-1}}\right\rangle\\ \vdots & \vdots & \vdots\\ \left\langle P^{\sigma_0^{\ell-1}}, P^{\sigma_0}\right\rangle & \cdots & \left\langle P^{\sigma_0^{\ell-1}}, P^{\sigma_0^{\ell-1}}\right\rangle\end{pmatrix}\end{aligned}$$

with eigenvalue $\lambda_\chi(P)$. But as above, B is the matrix of the quadratic form induced by the Néron–Tate height-pairing on $L'(P) \otimes \mathbb{R}$, where

$$L'(P) = \langle P, \sigma_0(P), \sigma_0^2(P), \ldots, \sigma_0^{\ell-2}(P) \rangle \subseteq E(K)$$

is the subgroup of $E(K)$ generated by all the conjugates of P. Hence by Cassels' result, B is a positive definite matrix. Since $\lambda_\chi(P)$, $\chi \neq \chi_0$, is an eigenvalue of B, it is therefore positive. $\square$

Remark 6.4. It was pointed out to us by R. Loewy that the positivity could also be proved using the Cauchy interlacing theorem relating the eigenvalues of a real symmetric matrix A and those of a principal minor B.

7. χ-Rank-1 Critical Values

Let $K/\mathbb{Q}$ be a cyclic extension of odd prime degree $[K : \mathbb{Q}] = \ell \geq 3$ with Galois group $G = \mathrm{Gal}(K/\mathbb{Q}) = \langle \sigma_0 \rangle$. Suppose that $E/\mathbb{Q}$ is an elliptic curve defined over $\mathbb{Q}$ for which the twisted L-functions $L(E, s, \chi)$ have χ-rank 1 for all nontrivial $\chi \in \widehat{G}$, i.e., $\mathrm{ord}_{s=1} L(E, s, \chi) = 1$ for $\chi_0 \neq \chi \in \widehat{G}$.

If we let z_χ be as in Proposition 3.1, we know that

$$\frac{L'(E, 1, \chi)}{\tau(\chi) z_\chi} \in \mathbb{R}.$$

Then it follows from Corollary 6.2 that if $P \in E_{\mathrm{Tr}}(K)$ has infinite order, then

$$\alpha_\chi^+(P) := \frac{L'(E, 1, \chi)\mathfrak{f}_\chi}{\tau(\chi) z_\chi \Omega^+ \lambda_\chi(P)}$$

is also real. Then

$$L'(E, 1, \chi) = \frac{\tau(\chi)}{\mathfrak{f}_\chi} \Omega^+ \lambda_\chi(P) z_\chi \alpha_\chi^+(P) \tag{7–1}$$

with $\alpha_\chi^+(P) \in \mathbb{R}$.

We note that if $Q \in E_{\mathrm{Tr}}(K)$ is any other point of infinite order, then the rank-1 assumption implies that $nQ = \xi(P)$ for some $n \in \mathbb{Z}$, and $\xi \in \mathbb{Z}[G]$. Therefore using Corollary 6.2, we can write

$$L'(E, 1, \chi) = \frac{\tau(\chi)}{\mathfrak{f}_\chi} \Omega^+ \lambda_\chi(Q) z_\chi \alpha_\chi^+(Q),$$

where

$$\alpha_\chi^+(Q) = \frac{n^2}{|\chi(\xi)|^2} \alpha_\chi^+(P). \tag{7–2}$$

Remark 7.1. From the tabulated computations in Section 9 (particularly the septic tables), it is clear that there are several instance for which the signs of $\alpha_\chi^+(P)$ are not constant as χ varies in $\widehat{G}$, and therefore we conclude that there is no point $Q \in E_{\mathrm{Tr}}(K)$ for which $\alpha_\chi^+(Q) = 1$ for all $\chi \in \widehat{G}$. That is, given a choice of $z_\chi \in \mathbb{Q}(\chi)$, there need not exist a point $Q \in E_{\mathrm{Tr}}(K)$ for which

$$L'(E, 1, \chi) = \frac{\tau(\chi)}{\mathfrak{f}_\chi} \Omega^+ \lambda_\chi(Q) z_\chi$$

for all $\chi \in \widehat{G}$. This is in contrast to the analogous case of cyclotomic fields.

A particularly striking example occurs for the curve 61A in Cremona's tables [Cremona 92]. For this curve we have $z_\chi = 1$ for the quintic characters χ_{691} and χ_{761} of conductors 691 and 761 respectively. The sign patterns of $\alpha_\chi^+(P)$ for $\{\chi, \chi^2\}$ are $\{-,-\}$ and $\{-,+\}$. Since for any $Q \in E_{\mathrm{Tr}}(K)$ the sign of $\alpha_\chi^+(Q)$ is the same as that for $\alpha_\chi^+(P)$ by (7–2), there can be no point $Q \in E_{\mathrm{Tr}}(K)$ for which $\alpha_\chi^+(Q) = \alpha_{\chi^2}^+(Q) = 1$.

Proposition 7.2. *Assume the Birch–Swinnerton-Dyer conjecture. Then for any $P \in E_{\mathrm{Tr}}(K)$, the product*

$$\prod_{\chi_0 \neq \chi \in \widehat{G}} \alpha_\chi^+(P) \in \mathbb{Q}$$

is rational.

Proof. The Birch–Swinnerton-Dyer conjecture implies (1–2) and therefore

$$\prod_{\chi_0 \neq \chi \in \widehat{G}} L'(E, 1, \chi) = \frac{R_K}{R_\mathbb{Q}} \times (\Omega^+)^{\ell-1} \times q,$$

where $q \in \mathbb{Q}$ rational. By (5–1) and (5–2),

$$\frac{R_K}{R_\mathbb{Q}} = \prod_{\chi_0 \neq \chi \in \widehat{G}} \lambda_\chi(P) \times q'$$

with $q' \in \mathbb{Q}$ rational. Dividing and appealing to (7–1), we obtain

$$\prod_{\chi_0 \neq \chi \in \widehat{G}} \frac{\tau(\chi)}{\mathfrak{f}_\chi} z_\chi \alpha_\chi^+(P) = qq' \in \mathbb{Q}.$$

Since K is totally real, [Washington 82, Corollary 4.6] proves that

$$\prod_{\chi \in \widehat{G}} \tau(\chi) = \sqrt{|d(K)|},$$

where $d(K)$ is the discriminant of K. But since $d(K)$ is a square and $\tau(\chi_0) = 1$, we see that $\prod_{\chi_0 \neq \chi \in \widehat{G}} \tau(\chi)$ is rational. Also

$$\prod_{\chi_0 \neq \chi \in \widehat{G}} z_\chi = \prod_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} z_{\chi^\gamma} = \prod_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \gamma(z_\chi)$$

is rational, and therefore

$$\prod_{\chi_0 \neq \chi \in \widehat{G}} \alpha_\chi^+(P) \in \mathbb{Q}.$$

$\square$

8. RANK-1 ALGEBRAICITY AND GALOIS EQUIVARIANCE

The results of the computations described in Section 9 suggest that the numbers $\alpha_\chi^+(P)$ are algebraic and are in the real subfield $\mathbb{Q}(\chi)^+$ of the cyclotomic field $\mathbb{Q}(\chi)$.

More precisely, we suppose that ℓ is an odd prime ($\ell = 3, 5, 7, 11$ in our computed examples) and that χ is a Dirichlet character of order ℓ and conductor $\mathfrak{f}_\chi$ coprime to ℓ. As above, $K/\mathbb{Q}$ is the cyclic extension of degree ℓ corresponding to χ and $G = \mathrm{Gal}(K/\mathbb{Q}) = \langle \sigma_0 \rangle$. Then for any nonprincipal character $\chi_1 \in \widehat{G}$,

$$\{\chi \in \widehat{G} | \chi \neq \chi_0\} = \{\chi_1^\gamma | \gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})\}.$$

Then it is clear that $z_{\chi^\gamma} = \gamma(z_\chi)$, and by taking the extension of $\gamma \in G$ to the automorphism $\gamma_0 \in \mathrm{Gal}(\mathbb{Q}(\chi, \exp(2\pi i/\mathfrak{f}_\chi))/\mathbb{Q})$ such that γ_0 is the identity on $\mathbb{Q}((\exp(2\pi i/\mathfrak{f}_\chi)))$, we have $\tau(\chi^\gamma) = \gamma_0(\tau(\chi))$.

Then the computations of Section 9 suggest the following:

$$\alpha_\chi^+(P) \in \mathbb{Q}(\chi)^+$$

and

$$\alpha_{\chi^\gamma}^+(P) = \gamma(\alpha_\chi^+(P)) \text{ for all } \gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}).$$

A more precise statement is predicted by the equivariant Tamagawa conjecture in [Burns 10].

9. NUMERICAL DATA

It is our intention in this section to provide numerical data in support of the conjectures of Section 8. All curves are identified using the notation in [Cremona 92].

The tables below are expressed to six decimal places for ease of presentation. The original calculations were performed to between 15 and 30 decimal places. Most of the work was performed using PARI, with MAGMA being used for the height calculations.[1]

We first searched all elliptic curves of conductor $N < 10000$ for which there is a Dirichlet character χ of prime order ≤ 11 and prime conductor $\mathfrak{f}_\chi < 10000/\sqrt{N}$ such that $L(E, 1, \chi) = 0$. These candidate curves were then scanned for points over the corresponding number fields K using a coarse search (looking for points in a small box). This usually failed but nevertheless succeeded sufficiently often to provide illustrative data.

[1]More complete data may be found at http://www.dms.umontreal.ca/~jack/local/.

9.1 Computational Methodology

For a successful search, we have the following data: an elliptic curve $E/\mathbb{Q}$, a Dirichlet character χ of order ℓ and conductor $\mathfrak{f}_\chi$ for which $L(E, 1, \chi) = 0$, and a point $P \in E(K)$ for the corresponding cyclic extension $K/\mathbb{Q}$ of degree ℓ. In the tables below it is important to fix precisely the characters and the Galois action.

Let g be the smallest positive primitive root modulo $\mathfrak{f}_\chi$. (This is possible because we are considering only characters χ with prime conductor.) Then the Galois group $\mathrm{Gal}(\mathbb{Q}(\exp(2\pi i/\mathfrak{f}_\chi)/\mathbb{Q})$ is generated by σ_0:

$$\sigma_0 : \exp(2\pi i/\mathfrak{f}_\chi) \mapsto \exp(2\pi i g/\mathfrak{f}_\chi).$$

The field $K = K_{\mathfrak{f}_\chi}$ is the fixed field of σ_0^ℓ and is generated by

$$t = \mathrm{Tr}_{\mathbb{Q}(\exp(2\pi i/\mathfrak{f}_\chi))/K}(\exp(2\pi i/\mathfrak{f}_\chi)).$$

Fix the character $\chi_{\mathfrak{f}_\chi}$ by setting $\chi_{\mathfrak{f}_\chi}(\sigma_0) = \exp(2\pi i/\ell)$.

By abuse of notation we denote $\sigma_0|K$ by σ_0. For example, for the quintic subfield of conductor 11 we have $g = 2$, $t = 2\cos(2\pi/11)$, $\sigma_0(t) = t^2 - 2$, and $\chi(\sigma_0) = \exp(2\pi i/5)$.

Similarly, to fix the action of $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, let b denote the smallest positive primitive root modulo ℓ and define $\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ by $\gamma(\zeta) = \zeta^b$ for all ℓth roots of unity ζ. Then $\chi^\gamma(\sigma) = \gamma(\chi(\sigma))$ for all $\chi \in \widehat{G}$ and all $\sigma \in G$.

9.2 Cyclic Cubic Extensions

The number fields correspond to Dirichlet characters of conductors 7, 13, and 19 with the relevant properties shown in Table 1.

Table 2 shows the basic properties of the elliptic curves, the values of the derivatives of the critical L-functions, and the values of the Gauss sums.

The data on the trace-zero points appear in Table 3, where t is a root of the minimal polynomial indicated above. It should be noted that the point may not necessarily be the generator of the Mordell–Weil group. An asterisk appearing in certain rows of the table indicates the case in which, in the notation of Section 3, $(-1)^{r_\chi} w_E \chi(N_E) = -1$.

Conductor	Minimal Polynomial	Galois Action
7	x^3+x^2-2x-1	$t \mapsto -t^2-t+1$
13	x^3+x^2-4x+1	$t \mapsto -t^2-2t+2$
19	x^3+x^2-6x-7	$t \mapsto -t^2+4$

TABLE 1. Cyclic cubic fields.

Curve	Coefficients	$\mathfrak{f}_\chi$	$L'(E,1,\chi_{\mathfrak{f}_\chi})$	$\tau(\chi)$
38B1	$[1,1,1,0,1]$	13	$2.33706-0.610180i$	$0.910836+3.48861i$
40A1	$[0,0,0,-7,-6]$	7	$2.92521+0.185634i$	$2.37047-1.17511i$
42A1	$[1,1,1,-4,5]$	19	$2.52981-2.29354i$	$-1.33281+4.15013i$
44A1	$[0,1,0,3,-1]$	13	$2.62549-0.685485i$	$0.910836+3.48861i$
50B1	$[1,1,1,-3,1]$	19	$3.69275+1.18593i$	$-1.33281+4.15013i$
54A1	$[1,-1,0,12,8]$	7	$3.40885+0.216326i$	$2.37047-1.17511i$
54B1	$[1,-1,1,1,-1]$	7	$2.84964+0.180839i$	$2.37047-1.17511i$
54B1	$[1,-1,1,1,-1]$	13	$1.80693+1.83025i$	$0.910836+3.48861i$

TABLE 2. Elliptic L-values, cubic case.

Curve	$\mathfrak{f}_\chi$	Point	$\langle P, P^\sigma \rangle$	$\lambda_\chi(P)$	$\alpha_\chi^+(P)$
38B1*	13	$[-t, 2t-2]$	-0.40916	1.22748	-1
40A1	7	$[t^2-t-2, 1]$	-0.50270	1.50811	2
42A1	19	$[t^2+2t-2, 2t^2+4t+2]$	-0.82421	2.47263	-1
44A1*	13	$[2t^2-3t+1, 10t^2-15t+3]$	-0.78000	2.33999	-1
50B1*	19	$[t^2+3t+3, 4t^2+12t+7]$	-0.68008	2.04025	-1
54A1	7	$[2t+3, 4t+7]$	-0.82633	2.47898	1
54B1	7	$[-t+2, 3t-4]$	-0.47027	1.41082	1
54B1	13	$[t^2-t+1, 3t^2-3t-1]$	-0.57726	1.73178	1

TABLE 3. Algebraic L-values, cubic case.

9.3 Cyclic Quintic Extensions

The number fields correspond to Dirichlet characters of conductors 11 and 31 with the relevant properties shown in Table 4.

Table 5 shows the basic properties of the elliptic curves, the values of the derivatives of the critical L-functions, and the values of the Gauss sums.

The data on the trace-zero points appear in Table 6, where t is a root of the minimal polynomial indicated above.

It should be noted that the point may not necessarily be the generator of the Mordell–Weil group. The asterisk in the fourth row of the table indicates the case in which in the notation of Section 3, $(-1)^{r_\chi} w_E \chi(N_E) = -1$.

The values of $\alpha_\chi^+(P)$ for the above five curves are (up to our numerical accuracy) roots of

$$\begin{aligned} X^2-5X+5&=0 \quad \text{for } 35\text{A1},\\ X^2-5X-5&=0 \quad \text{for } 106\text{A1},\\ X^2-5X+5&=0 \quad \text{for } 162\text{B1},\\ X^2+5X+5&=0 \quad \text{for } 208\text{C1},\\ X^2-10X+20&=0 \quad \text{for } 246\text{E1}. \end{aligned}$$

All these roots lie in the maximal real subfield of fifth roots of unity.

9.4 Cyclic Septic Extensions

The number fields correspond to Dirichlet characters of conductor 29 with minimal polynomial

$$x^7+x^6-12x^5-7x^4+28x^3+14x^2-9x+1$$

Conductor	**Minimal Polynomial**	**Galois Action**
11	$x^5+x^4-4x^3-3x^2+3x+1$	$t \mapsto t^2-2$
31	$x^5+x^4-12x^3-21x^2+x+5$	$t \mapsto (3t^4-t^3-33t^2-24t+15)/5$

TABLE 4. Cyclic quintic fields.

Curve	**Coefficients**	$\mathfrak{f}_\chi$	$L'(E,1,\chi_{\mathfrak{f}_\chi})$ $L'(E,1,\chi_{\mathfrak{f}_\chi}^\gamma)$	$\tau(\chi)$ $\tau(\chi^\gamma)$
35A1	$[0,1,1,9,1]$	11	$3.44288-1.02884i$ $3.22913+2.12745i$	$2.63611+2.01270i$ $2.07016+2.59122i$
106A1	$[1,0,0,1,1]$	11	$6.26214+2.20060i$ $0.619026-2.25071i$	$2.63611+2.01270i$ $2.07016+2.59122i$
162B1	$[1,-1,1,-5,5]$	11	$2.64258+3.82523i$ $3.85740-0.176636i$	$2.63611+2.01270i$ $2.07016+2.59122i$
208C1	$[0,0,0,1,10]$	11	$2.52073-3.30149i$ $3.72533-2.97621i$	$2.63611+2.01270i$ $2.07016+2.59122i$
246E1	$[1,0,0,-9,9]$	31	$4.70188+1.60514i$ $2.55763+2.00991i$	$4.55242-3.20554i$ $5.22658+1.91908i$

TABLE 5. Elliptic L-values, quintic case.

Curve	$\mathfrak{f}_\chi$	**Point**	$\langle P,P^\sigma\rangle$ $\langle P,P^{\sigma^2}\rangle$	$\lambda_\chi(P)$ $\lambda_{\chi^\gamma}(P)$	$\alpha_\chi^+(P)$ $\alpha_{\chi^\gamma}^+(P)$
35A1	11	$[t^2-t,t^3-t^2+2t-2]$	-0.58449 -0.14401	1.32877 2.31371	3.61803 1.38197
106A1	11	$[-t^3+t^2+t,-2t^4+2t^3+3t^2-1]$	-0.59455 0.08276	0.522218 2.03674	5.85410 -0.854102
162B1	11	$[t+2,-t^4+4t^2+t-2]$	-0.45014 0.04935	0.443523 1.56043	3.61803 1.38197
208C1*	11	$[t,t^4-4t^2+2t+4]$	-0.198200 -0.441207	1.87021 1.32683	-1.38197 -3.61803
246E1	31	$[t^2+2t+1,t^3+3t^2+t-3]$	-0.341737 -0.300348	1.55894 1.65149	7.23607 2.76393

TABLE 6. Algebraic L-values, quintic case.

and Galois action

$$t \mapsto (30t^6+42t^5-350t^4-350t^3+785t^2+700t-160)/17$$

and conductor 43 with minimal polynomial

$$x^7+x^6-18x^5-35x^4+38x^3+104x^2+7x-49$$

and Galois action

$$t \mapsto (-6t^6+3t^5+105t^4+52t^3-330t^2-153t+231)/7.$$

Table 7 shows the basic properties of the elliptic curves, the values of the derivatives of the critical L-functions, and the values of the Gauss sums.

The data on the points appear in Table 8, where t is a root of the minimal polynomial indicated above. It should be noted that the point may not necessarily be the generator of the Mordell–Weil group. It should be noted as well that each point Q in Table 8 is not, in fact, a trace-zero point. The height and subsequent calculations are based on the point $P=Q-Q^{\sigma_0}$, which does have zero trace.

The values of $\alpha_\chi^+(P)$ for each of the four curves are (up to our numerical accuracy) roots of

$$\begin{aligned} X^3-196X-3927&=0 \quad \text{for 674B1},\\ X^3-21X^2+98X-49&=0 \quad \text{for 856A1},\\ X^3+21X^2+98X-49&=0 \quad \text{for 1329A1},\\ X^3-126X^2+3528X-10584&=0 \quad \text{for 1876B1}.\end{aligned}$$

Curve	Coefficients	$\mathfrak{f}_\chi$	$L'(E,1,\chi_{\mathfrak{f}_\chi})$ $L'(E,1,\chi_{\mathfrak{f}_\chi}^\gamma)$ $L'(E,1,\chi_{\mathfrak{f}_\chi}^{\gamma^2})$	$\tau(\chi)$ $\tau(\chi^\gamma)$ $\tau(\chi^{\gamma^2})$
674B1	$[1,-1,1,-6,5]$	29	$5.32925-1.96500i$ $1.03488-4.61662i$ $5.89043+5.60476i$	$4.60683+2.78875i$ $1.21868+5.24546i$ $-4.48716+2.97748i$
856A1	$[0,1,0,-3,2]$	43	$0.0301342-4.92124i$ $3.22624-1.68817i$ $0.515405+7.05781i$	$5.15175-4.05703i$ $3.91562-5.26003i$ $6.48233+0.989674i$
1329A1	$[1,0,1,2,5]$	43	$-0.0558510+9.12108i$ $3.50665-1.83489i$ $-0.270492-3.70404i$	$5.15175-4.05703i$ $3.91562-5.26003i$ $6.48233+0.989674i$
1876B1	$[0,1,0,-29,44]$	43	$5.67782-7.03101i$ $-0.0943741-2.83435i$ $4.00073+13.0582i$	$5.15175-4.05703i$ $3.91562-5.26003i$ $6.48233+0.989674i$

TABLE 7. Elliptic L-values, septic case.

Curve	$\mathfrak{f}_\chi$	Point(Q)	$\langle P,P^\sigma\rangle$ $\langle P,P^{\sigma^2}\rangle$ $\langle P,P^{\sigma^3}\rangle$	$\lambda_\chi(P)$ $\lambda_{\chi^\gamma}(P)$ $\lambda_{\chi^{\gamma^2}}(P)$	$\alpha^+_\chi(P)$ $\alpha^+_{\chi^\gamma}(P)$ $\alpha^+_{\chi^{\gamma^2}}(P)$
674B1	29	$[t+2,(4t^6+9t^5-41t^4-75t^3+82t^2+99t-44)/17]$	-0.984527 -0.419174 0.248767	0.820468 3.45051 3.81356	14.9095 -2.04354 -12.8659
856A1	43	$[2t+3,(-2t^6+4t^5+34t^4-26t^3-144t^2+8t+119)/7]$	-1.288629 0.458858 -0.297146	0.978168 5.28029 1.62996	5.97823 0.567040 14.4547
1329A1	43	$[t^2+3t+1,(-2t^6+33t^4+45t^3-38t^2-73t-21)/7]$	-2.721459 1.252466 -0.587921	1.22222 10.8412 2.33500	-13.4330 0.454731 -8.02177
1876B1	43	$[t^2-t-2,-t^6+18t^4+16t^3-50t^2-39t+28]$	-0.567253 0.180598 -0.157113	0.582918 2.40482 0.818646	35.8694 3.40224 86.7284

TABLE 8. Algebraic L-values, septic case.

All of these roots lie in the maximal real subfield of seventh roots of unity.

The zero-trace points mentioned above are listed here for reference purposes. For the curve 674B1, the point is

```
P=[2338/493*t^6 + 2770/493*t^5 - 27356/493*t^4
- 21236/493*t^3 + 59642/493*t^2 + 43475/493*t -
10248/493, 2946/493*t^6 + 3645/493*t^5 - 34523/493*t^4
- 29015/493*t^3 + 75540/493*t^2 + 60104/493*t -
12873/493]
```

For the curve 856A1, it is

```
P = [347/301*t^6 - 2/43*t^5 - 6016/301*t^4 -
5823/301*t^3 + 15455/301*t^2 + 15287/301*t - 655/43,
836/301*t^6 + 373/301*t^5 - 2049/43*t^4 - 21166/301*t^3
+ 27745/301*t^2 + 53014/301*t + 1758/43]
```

For the curve 1329A1, it is

```
P = [13941526/34184269*t^6 + 6823816/34184269*t^5
- 272120413/34184269*t^4 - 300304659/34184269*t^3
+ 892749859/34184269*t^2 + 15348359/794983*t -
87127871/4883467, -1275747511/11520098653*t^6 -
2333533755/1645728379*t^5 + 1452560596/267909271*t^4
+ 224356347972/11520098653*t^3 -
306167740201/11520098653*t^2 -
404336548491/11520098653*t + 36986710930/1645728379]
```

For the curve 1876B1, it is

```
P = [t^2 - t - 2, -t^6 + 18*t^4 + 16*t^3 - 50*t^2 -
39*t + 28]
```

Curve	**Coefficients**	$\mathfrak{f}_\chi$	$L'(E,1,\chi_{\mathfrak{f}_\chi})$ $L'(E,1,\chi_{\mathfrak{f}_\chi}^{\gamma})$ $L'(E,1,\chi_{\mathfrak{f}_\chi}^{\gamma^2})$ $L'(E,1,\chi_{\mathfrak{f}_\chi}^{\gamma^3})$ $L'(E,1,\chi_{\mathfrak{f}_\chi}^{\gamma^4})$	$\tau(\chi)$ $\tau(\chi^{\gamma})$ $\tau(\chi^{\gamma^2})$ $\tau(\chi^{\gamma^3})$ $\tau(\chi^{\gamma^4})$
5906B1	$[1,1,0,-32,58]$	23	$8.43740+1.56399i$ $14.7072-7.16721i$ $2.25530+4.78838$ $17.2140+8.63991i$ $8.76334-2.79430i$	$0.489832+4.77075i$ $4.09821+2.49092i$ $-0.0564765-4.79550i$ $1.83051+4.43274i$ $4.73001-0.791844i$

TABLE 9. Elliptic L-values, degree 11 case.

Curve	$\mathfrak{f}_\chi$	$\langle P,P^{\sigma}\rangle$ $\langle P,P^{\sigma^2}\rangle$ $\langle P,P^{\sigma^3}\rangle$ $\langle P,P^{\sigma^4}\rangle$ $\langle P,P^{\sigma^5}\rangle$	$\lambda_\chi(P)$ $\lambda_{\chi^\gamma}(P)$ $\lambda_{\chi^{\gamma^2}}(P)$ $\lambda_{\chi^{\gamma^3}}(P)$ $\lambda_{\chi^{\gamma^4}}(P)$	$\alpha_\chi^+(P)$ $\alpha_{\chi^\gamma}^+(P)$ $\alpha_{\chi^{\gamma^2}}^+(P)$ $\alpha_{\chi^{\gamma^3}}^+(P)$ $\alpha_{\chi^{\gamma^4}}^+(P)$
5906B1	23	-1.648409 -0.978593 1.129189 -0.858499 0.422566	0.273081 2.56805 10.2034 5.15487 3.07181	62.8439 6.63948 -0.321316 2.78561 1.70448

TABLE 10. Algebraic L-values, degree 11 case.

9.5 A Cyclic Extension of Degree 11

One curve was discovered with a point in an eleventh-degree extension. The curve is 5906B1 with coefficients $[1,1,0,-32,58]$. The point was found over a field of conductor 23 with minimal polynomial

$$x^{11}+x^{10}-10x^9-9x^8+36x^7+28x^6-56x^5-35x^4 +35x^3+15x^2-6x-1.$$

The Galois action in this field is

$$t\mapsto t^5-5t^3+5t.$$

The point Q discovered in the search was

$$[-t^3-2t^2+4t+1,-3t^{10}-2t^9+31t^8+15t^7-115t^6 -34t^5+179t^4+23t^3-96t^2-2t+8],$$

and the point $P=Q-Q^{\sigma_0}$ was used in the subsequent calculations.

Table 9 shows the basic properties of the elliptic curve, the values of the derivatives of the critical L-functions, and the value of the Gauss sums.

The data for the trace-zero point are shown in Table 10. It should be noted that the point may not necessarily be the generator of the Mordell–Weil group.

The points $\alpha_\chi^+(P)$ are (up to our numerical accuracy) the roots of

$$23X^5-1694X^4+16335X^3-45254X^2 +29282X+14641=0$$

and lie in the maximal real subfield of the eleventh roots of unity.

The point P used in this analysis is

```
P = [8546182/50807*t^10 - 4840142/50807*t^9
- 77988935/50807*t^8 + 45093287/50807*t^7 +
10354887/2209*t^6 - 132803588/50807*t^5 -
274532428/50807*t^4 + 128641774/50807*t^3 +
102744505/50807*t^2 - 29991223/50807*t - 5420004/50807,
15797792218/2387929*t^10 - 8729122400/2387929*t^9 -
6276343301/103823*t^8 + 81888886062/2387929*t^7 +
441046362414/2387929*t^6 - 242101414169/2387929*t^5
- 507362284341/2387929*t^4 + 234319824740/2387929*t^3
+ 187634372590/2387929*t^2 - 54371726550/2387929*t -
10166772313/2387929]
```

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