Modular Forms on Noncongruence Subgroups and Atkin–Swinnerton-Dyer Relations

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We give new examples of noncongruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ whose space of weight-3 cusp forms $S_3(\Gamma)$ admits a basis satisfying the Atkin–Swinnerton-Dyer congruence relations with respect to a weight-3 newform for a certain congruence subgroup.

1. INTRODUCTION

A finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is a *noncongruence* subgroup if it does not contain $\Gamma(N)$ for any $N \geq 1$. The study of modular forms on such subgroups was initiated by Atkin and Swinnerton-Dyer, who discovered experimentally the congruences now bearing their names [Atkin et al. 71]. Subsequently, Scholl proved congruences satisfied by the coefficients of modular forms on noncongruence subgroups [Scholl 85a, Scholl 85b, Scholl 87, Scholl 88, Scholl 93]. A refined conjecture has recently been put forward by Atkin, Li, Long, and Yang [Li et al. 05a, Atkin et al. 08, Long 08]. See [Li et al. 05b] for a general survey.

In this paper we give new examples of noncongruence subgroups having a basis of cuspidal modular forms satisfying the Atkin–Swinnerton-Dyer (ASwD) congruences. We give only experimental evidence of our results, obtained using Magma, Mathematica, and Pari. In a later publication, we will give a detailed treatment of one of our examples.

1.1 Notation

We assume familiarity with the action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half complex plane $\mathbb{H}$, with congruence subgroups such as $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma^0(N)$, $\Gamma^1(N)$, and with $M_k(\Gamma)$ and $S_k(\Gamma)$ the finite-dimensional vector spaces of modular forms and cusp forms for Γ, and $S_k(\Gamma_0(N), \chi)$ the space of cusp forms with character $\chi : (\mathbb{Z}/N)^* \to \mathbb{C}^*$.

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It is well known (see [Shimura 71] for details) that $S_k(\Gamma_0(N),\chi)$ has a basis of Hecke eigenforms, which have q-expansions

$$f(z)=\sum_{n\geq 1} a_n(f)q^n, \quad \text{where } q=\exp(2\pi i z),$$

with a_n satisfying the relations

$$a_{np}-a_p a_n+\chi(p)p^{k-1}a_{n/p}=0, \quad a_n=a_n(f), \quad (1\text{–}1)$$

for all positive integers n and primes $p\nmid N$, taking $a_{n/p}=0$ if $p\nmid n$.

1.2 Atkin–Swinnerton-Dyer Congruences

If Γ is a *noncongruence* subgroup, then in general $S_k(\Gamma)$ has no basis of forms satisfying (1–1). Instead, it is conjectured that certain congruences hold, as in the following definition.

Definition 1.1. [Li et al. 05a] Suppose that the noncongruence subgroup Γ has cusp width μ at infinity, and that $h\in S_k(\Gamma)$ has an M-integral $q^{1/\mu}$-expansion $h=\sum a_n(h)q^{n/\mu}$ for some $M\in\mathbb{Z}$. (cf. [Scholl 85a, Proposition 5.2]). Let $f=\sum c_n(f)q^n$ be a normalized newform of weight k, level N, character χ. The forms h and f are said to satisfy the Atkin–Swinnerton-Dyer congruence relation if for all primes p not dividing MN and for all $n\geq 1$,

$$(a_{np}(h)-c_p(f)a_n(h)+\chi(p)p^{k-1}a_{n/p}(h))/(np)^{k-1} \quad (1\text{–}2)$$

is integral at some place dividing p.

Definition 1.2. We say that $S_k(\Gamma)$ has an ASwD basis if there exist a basis $h_1,\ldots,h_n$ of $S_k(\Gamma)$ and normalized newforms $f_1,\ldots,f_n$ such that each pair (h_i,f_i) satisfies the ASwD congruence relation in Definition 1.1.

Note that in the above definition, the choices of $h_1,\ldots,h_n$ and $f_1,\ldots,f_n$ may depend on the prime number p. There are examples known for which the same h_i and f_j work for every prime p (actually all but a finite number of exceptional primes). On the other hand, there are examples known for which the choice of the ASwD basis depends on the value of p modulo some modulus N (see the examples in Tables 2 and 3).

2. STATEMENT OF RESULTS

2.1 Tables

For the noncongruence subgroups Γ considered, there are two main issues addressed:

1. Modularity of the l-adic Scholl's representation $S_3(\Gamma)$ attached to the cusp forms of weight 3.
2. Giving a basis of $S_3(\Gamma)$ that satisfies ASwD congruences.

In our cases the dimension of $S_3(\Gamma)$ is 2, so the l-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is 4-dimensional. We find that this 4-dimensional representation breaks up into two 2-dimensional λ-adic representations, each of which is isomorphic to the 2-dimensional representations that Deligne constructed for Hecke eigenforms f on congruence subgroups. Thus, each $S_3(\Gamma)$ should be associated to a pair f_1,f_2 of Hecke eigenforms on *congruence* subgroups. In the examples, these are one and the same form, or conjugate forms or base extensions of one form to a quadratic extension of $\mathbb{Q}$.

There are four main cases that we study. The data on these is summarized in Tables 1 2, 4, and 5. These cases are further divided into subcases labeled 1a, 1b, 2a, 2b, 2c, 2d, 3a, 3b, 4a, and 4b. In those tables we define modular forms h_1, h_2, and f, where h_1 and h_2 span $S_3(\Gamma)$ for the noncongruence subgroup Γ given in Definition 3.1, and f is a weight-3 Hecke eigenform for some congruence subgroup. For each group we give a basis (h_1,h_2) of $S_3(\Gamma)$, in some cases depending on the prime p, and a newform f with (h_i,f) satisfying the ASwD congruence relation. Most forms are given in terms of the Dedekind eta function,

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \text{where } q=e^{2\pi i z}. \quad (2\text{–}1)$$

Our experiments support the following theorem:

Theorem 2.1. *Let ρ be the l-adic representation constructed by Scholl for $S_3(\Gamma)$ for an appropriate choice of $\mathbb{Q}$-model of the curve X_Γ. For the L-function of the corresponding representations we have*

$$\begin{aligned} L(s,\rho)&=L(s,f)L(s,f) \quad \text{for 1a, 1b,}\\ L(s,\rho)&=L(s,f)L(s,\overline{f}) \quad \text{for 3a, 3b, 4a, 4b.}\end{aligned}$$

A complete proof for cases 1a and 1b exists. We do not reproduce it here, since it is very similar to other published examples. The L-function for examples 2a, 2b exhibits new and interesting features and will be discussed in a future work.

2.2 The Examples

All the noncongruence subgroups Γ discussed in this paper are of index three inside a congruence subgroup G

that itself is one of the index-12 genus-0 subgroups considered by Beauville. Each of these gives rise to a family of elliptic curves $E_G \to X_G = (G \setminus \mathbb{H})^* \cong \mathbf{P}^1(\mathbb{C})$ with ramification over the four cusps of G. For each of these, we select two of the cusps of G to construct a subgroup Γ such that the corresponding covering

$$X_\Gamma \cong \mathbf{P}^1(\mathbb{C}) \longrightarrow X_G \cong \mathbf{P}^1(\mathbb{C})$$

branches only over the two chosen cusps. We describe these coverings in the form $r^3 = m(t)$, where r (respectively t) is a generator of the function field of X_Γ (respectively X_G), i.e., a Hauptmodul, which exists since these curves have genus 0. See Table 12. We have also considered arithmetic twists of a given covering obtained by varying some of the constants in the expression of $m(t)$. This leads to different models of Scholl's l-adic representation attached to $S_3(\Gamma)$, i.e., representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that become isomorphic as representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ for a finite extension $K/\mathbb{Q}$. It is an important point that in contrast to the case of classical modular curves for congruence subgroups, there are no *canonical* models defined over a number field. Scholl's construction of his l-adic representations depends on a choice of a model. Moreover, this choice is subject to a number of hypotheses: generally that there should be a model defined over $\mathbb{Q}$, and a cusp that is $\mathbb{Q}$-rational. This cusp is used for the expansions of modular forms whose coefficients satisfy ASwD congruences.

The l-adic representations that Scholl constructs that are associated to $S_k(\Gamma)$ for noncongruence subgroups Γ have very different properties from the corresponding representations constructed by Deligne for congruence Γ. The main point is that in the congruence case, the Hecke algebra acts and commutes with the Galois action so that the $2d$-dimensional representation ($d = \dim S_k(\Gamma)$) splits into 2-dimensional λ-adic representations. This is no longer the case in general for noncongruence subgroups. It is the case in our examples that the 4-dimensional representations attached to $S_3(\Gamma)$ factor into 2-dimensional pieces. Geometrically this is due to the presence of extra symmetries given by involutions and/or isogenies of our elliptic surfaces.

2.3 Outline

In Section 3 we define the congruence and noncongruence subgroups we will be working with. Section 4 gives the method we use to construct the noncongruence forms h_1, h_2. Section 5 explains how we computed the traces of Frobenius elements in the l-adic Scholl's representation attached to our group Γ. The main point is to count the number of rational points over $\mathbf{F}_p$ and $\mathbf{F}_{p^2}$ of the elliptic modular surface E_Γ. In Section 6 we discuss involutions and isogenies of these elliptic surfaces. Finally, in Section 7 we provide the experimental evidence for the ASwD congruences.

3. DESCRIPTION OF THE NONCONGRUENCE SUBGROUPS

3.1 Beauville's Families

We start with certain index-12 genus-0 torsion-free congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$, listed in Table 6 [Sebbar 01]. Figure 1 shows corresponding fundamental domains and generating matrices. Note that these matrices generate index-24 subgroups of $\mathrm{SL}_2(\mathbb{Z})$, though the projectivizations have index 12 in $\mathrm{PSL}_2(\mathbb{Z})$.

Table 6 gives equations for the associated families of elliptic curves [Beauville 82]. Table 7 gives the $a_1, \ldots, a_6$ of the Weierstrass form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. The Hauptmodul $t(z)$ listed in Table 6 is such that $j(E_{t(z)}) = j(z)$.

3.2 The Noncongruence Subgroups

We will work with certain index-3 normal subgroups of $\Gamma_1(6)$ and $\Gamma_0(8) \cap \Gamma_1(4)$. The case $\Gamma_1(5)$ has been studied in [Li et al. 05a]. The fundamental domain of Γ is a union of three copies of a fundamental domain for G, corresponding to the three cosets of Γ in G. The coverings are completely determined by the choice of two of the four cusps, taken to be points of ramification of the cubic cover. This choice determines the cusp widths for the index-3 subgroups. These data are given in Table 8.

Since $\binom{4}{2} = 6$, there are six index-3 subgroups in each case; however, for $\Gamma_1(6)$, the subgroup ramified at $0, \frac{1}{2}$ is $\Gamma_1(6) \cap \Gamma_0(18)$, and the subgroup ramified at $\infty, \frac{1}{3}$ is $\Gamma_1(6) \cap \Gamma^0(3)$, which are congruence subgroups, so only four subgroups are listed in this case.

From the fundamental domains, shown in Figures 1, 2, 3, and 4, we obtain generators and cusp widths [Kulkarni 91], allowing us to make the following definition.

Definition 3.1. We let $\Gamma_{24.6.1^6}$, $\Gamma_{8^3.6.3.1^3}$, $\Gamma_{24.3.2^3.1^3}$, $\Gamma_{8^3.2^3.3^2}$, $\Gamma_{24.3.2^3.1^3B}$, $\Gamma_{8^3.2^3.3^2B}$ be index-3 genus-0 subgroups of $\Gamma_0(8) \cap \Gamma_1(4)$, and $\Gamma_{18.6.3^3.1^3}, \Gamma_{9.6^4.1^3}, \Gamma_{9.6^3.3.2^3}$, $\Gamma_{18.3^4.2^3}$ index-3 genus-0 subgroups of $\Gamma_1(6)$, defined by their generators as given in Table 9.

By comparing cusp widths in Tables 14 and 16 with possible cusp widths of congruence subgroups in Table 10, we obtain the following result.

1a. Basis of $S_3(\Gamma_{24.6.1^6})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(4z)^{20}}{\eta(2z)^6}} = q - \frac{4}{3}q^2 + \frac{8}{9}q^3 - \frac{176}{81}q^4 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(4z)^{16}\eta(2z)^6}{\eta(z)^4}} = q + \frac{4}{3}q^2 + \frac{8}{9}q^3 + \frac{176}{81}q^4 + \cdots$$

Associated newform in $S_3(\Gamma_0(48), \chi)$, where $\chi(\mathrm{Frob}_p) = \left(\frac{-3}{p}\right)\left(\frac{-4}{p}\right)$:

$$f(z) = \frac{\eta(4z)^9\eta(12z)^9}{\eta(2z)^3\eta(6z)^3\eta(8z)^3\eta(24z)^3} = q + 3q^3 - 2q^7 + 9q^9 - 22q^{13} + \cdots$$

The ASwD basis is h_1, h_2.

1b. Basis of $S_3(\Gamma_{8^3.2^3.3^2})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2z)^{20}\eta(8z)^4}{\eta(4z)^6}} = q^{1/3} - \frac{20}{3}q^{4/3} + \frac{128}{9}q^{7/3} - \frac{400}{81}q^{10/3} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2z)^{16}\eta(4z)^6}{\eta(8z)^4}} = q^{2/3} - \frac{16}{3}q^{14/3} + \frac{38}{9}q^{26/3} + \frac{1696}{81}q^{38/3} + \cdots$$

The associated newform is a twist $f \otimes \chi$ of the f in case 1a.
The ASwD basis is h_1, h_2.

TABLE 1. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

2a. Basis of $S_3(\Gamma_{8^3.6.3.1^3})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(2z)^{10}\eta(8z)^8}{\eta(4z)^4}} = q - \frac{4}{3}q^2 - \frac{40}{9}q^3 + \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^8\eta(4z)^{10}\eta(8z)^4}{\eta(2z)^4}} = q - \frac{8}{3}q^2 + \frac{8}{9}q^3 + \frac{32}{81}q^4 - \frac{82}{243}q^5 + \cdots$$

Associated newform in $S_3(\Gamma_0(432), \chi)$, where $\chi(\mathrm{Frob}_p) = \left(\frac{-4}{p}\right)$:

$f(z) = f_1(12z) + 6\sqrt{2}f_5(12z) + \sqrt{-3}f_7(12z) + 6\sqrt{-6}f_{11}(12z)$,
where

$f_1(z) = \frac{\eta(2z)^3\eta(3z)}{\eta(6z)\eta(z)}E_6(z) \quad f_5(z) = \frac{\eta(z)\eta(2z)^3\eta(3z)^3}{\eta(6z)}$

$f_7(z) = \frac{\eta(6z)^3\eta(z)}{\eta(2z)\eta(3z)}E_6(z) \quad f_{11}(z) = \frac{\eta(3z)\eta(z)^3\eta(6z)^3}{\eta(2z)}$

and $E_6(z) = 1 + 12\sum_{n\geq 1}(\sigma(3n) - 3\sigma(n))q^n$, where $\sigma(n) = \sum_{d|n} d$.

Atkin–Swinnerton-Dyer basis:

if $p \equiv 1 \bmod 3$	basis is	h_1, h_2
if $p \equiv 2 \bmod 3$	basis is	$h_1 \pm \alpha h_2$, $\alpha^{12} = 4$.

2b. Basis of $S_3(\Gamma_{24.3.2^3.1^3})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2z)^{22}\eta(8z)^8}{\eta(z)^4\eta(4z)^8}} = q + \frac{4}{3}q^2 - \frac{40}{9}q^3 - \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2z)^{20}\eta(4z)^2\eta(8z)^4}{\eta(z)^8}} = q + \frac{8}{3}q^2 + \frac{8}{9}q^3 - \frac{32}{81}q^4 - \frac{82}{243}q^5 + \cdots$$

The associated newform and the ASwD basis
are given in exactly the same way as in case 2a.

TABLE 2. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

Theorem 3.2. *The groups in Definition 3.1 are noncongruence subgroups.*

3.3 Hauptmoduls and Covering Maps

Throughout this paper we fix our choice of identification of $X(\Gamma_0(8)\cap\Gamma_1(4))$ and $X(\Gamma_1(6))$ with the projective line $\mathbb{P}^1$, with parameters t_8 and t_6 respectively. As functions of z in the upper half complex plane, $t_8(z)$ and $t_6(z)$ are given in terms of the Dedekind eta function, as listed in the last column of Table 7:

$$t_8(z) = \frac{\eta(z)^8\eta(4z)^4}{\eta(2z)^{12}} \quad \text{and} \quad t_6(z) = \frac{1}{9}\frac{\eta(6z)^4\eta(z)^8}{\eta(3z)^8\eta(2z)^4}.$$

2c. Basis of $S_3(\Gamma_{8^3.6.3.1^3 B})$; $r = q^{1/3}$:

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^8\eta(4z)^{22}}{\eta(8z)^4\eta(8z)^8}} = r^2 - \frac{8}{3}r^5 + \frac{20}{9}r^8 - \frac{256}{81}r^{11} - \frac{64}{243}r^{14} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^4\eta(2z)^2\eta(4z)^{20}}{\eta(8z)^8}} = r - \frac{4}{3}r^4 - \frac{16}{9}r^7 + \frac{112}{81}r^{10} + \cdots$$

The associated newform is as in case 2a. The ASwD basis is:

if $p \equiv 1 \bmod 3$	basis is	h_1, h_2
if $p \equiv 5 \bmod 12$	basis is	$h_1 \pm \frac{\sqrt{2}}{\sqrt[3]{2}}h_2$
if $p \equiv 11 \bmod 12$	basis is	$h_1 \pm \frac{\sqrt{-2}}{\sqrt[3]{2}}h_2$

2d. Basis of $S_3(\Gamma_{24.3.2^3.1^3 B})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2z)^{14}\eta(4z)^{16}}{\eta(z)^4\eta(8z)^8}} = q^{1/3} + \frac{4}{3}q^{4/3} - \frac{16}{9}q^{7/3} - \frac{112}{81}q^{10/3} - \frac{1534}{243}q^{13/3} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2z)^{16}\eta(4z)^{14}}{\eta(z)^8\eta(8z)^4}} = q^{2/3} + \frac{8}{3}q^{5/3} + \frac{20}{9}q^{8/3} + \frac{256}{81}q^{11/3} - \frac{64}{243}q^{14/3} + \cdots$$

The associated newform is as in case 2a, and the ASwD basis is as in case 2c.

TABLE 3. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

3a. Basis of $S_3(\Gamma_{18.6.3^3.1^3})$

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(2z)^7\eta(6z)^{11}}{\eta(3z)^4}} = q - \frac{4}{3}q^2 - \frac{31}{9}q^3 + \frac{400}{81}q^4 + \frac{104}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(3z)^4\eta(6z)^7\eta(2z)^{11}}{\eta(z)^4}} = q + \frac{4}{3}q^2 - \frac{7}{9}q^3 - \frac{112}{81}q^4 - \frac{616}{243}q^5 + \cdots$$

Newform in $S_3(\Gamma_0(243), \chi)$, where $\chi(\mathrm{Frob}_p) = \left(\frac{-3}{p}\right)$.

$f(z) = q + 3iq^2 - 5q^4 + 6iq^5 + 11q^7 - 3iq^8 - 18q^{10} + \cdots$

Atkin–Swinnerton-Dyer basis:

if $p \equiv 1 \bmod 3$	basis is	h_1, h_2
if $p \equiv 2 \bmod 3$	basis is	$h_1 \pm i\sqrt[3]{3}h_2$

3b. Basis of $S_3(\Gamma_{9.6^3.3.2^3})$; $r = q^{1/3}$.

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^7\eta(2z)^4\eta(3z)^{11}}{\eta(6z)^4}} = r - \frac{7}{3}r^4 - \frac{19}{9}r^7 + \frac{193}{81}r^{10} + \frac{2306}{243}r^{13} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^{11}\eta(3z)^7\eta(6z)^4}{\eta(2z)^4}} = r^2 - \frac{11}{3}r^5 + \frac{23}{9}r^8 - \frac{13}{81}r^{11} + \cdots$$

The associated newform and the ASwD basis are given in exactly the same way as in case 3a.

TABLE 4. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

The values of these functions at the cusps are as in Table 11.

Since the ramification points of the covering maps $\Gamma \setminus \mathbb{H} \to G \setminus \mathbb{H}$ are at cusps as in Table 8, the covering maps are given in each case by a map

$$r \mapsto r^3 = m(t),$$

where the maps m corresponding to each of our subgroups are as in Table 12.

4. CONSTRUCTING ELEMENTS OF $S_3(\Gamma)$

4.1 Dimension

For odd k, [Shimura 71, Theorem 2.25] gives the following formula for $\dim S_k(\Gamma)$ for a genus-g subgroup $-I \notin \Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$:

$$\dim S_k(\Gamma) = (k-1)(g-1) + \frac{1}{2}(k-2)u + \frac{1}{2}(k-1)u' + \sum_{i=1}^{r} k\frac{e_i - 1}{2e_i}.$$

4a. Basis of $S_3(\Gamma_{9.6^4.1^3})$

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^{13}\eta(6z)^{14}}{\eta(2z)^2\eta(3z)^7}} = q - \frac{13}{3}q^2 + \frac{32}{9}q^3 + \frac{670}{81}q^4 - \frac{3577}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^{14}\eta(6z)^{13}}{\eta(2z)^7\eta(3z)^2}} = q - \frac{14}{3}q^2 + \frac{56}{9}q^3 - \frac{58}{81}q^4 + \frac{266}{243}q^5 + \cdots$$

Associated newform in $S_3(\Gamma_0(486), \chi)$, where $\chi(\mathrm{Frob}_p) = \left(\frac{-3}{p}\right)$.

$$f(z) = q - \sqrt{-2}q^2 - 2q^4 + 3\sqrt{-2}q^5 - 7q^7 + 2\sqrt{-2}q^8 + 6q^{10} - 3\sqrt{-2}q^{11} + 5q^{13}$$

Atkin–Swinnerton-Dyer basis:

if $p \equiv 1 \bmod 3$	basis is	h_1, h_2
if $p \equiv 2 \bmod 3$	basis is ?	

4b. Basis of $S_3(\Gamma_{18.3^4.2^3})$; $r = q^{1/3}$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2z)^{13}\eta(3z)^{14}}{\eta(6z)^7\eta(z)^2}} = r + \frac{2}{3}r^4 - \frac{28}{9}r^7 - \frac{482}{81}r^{10} - \frac{736}{243}r^{13} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2z)^{14}\eta(3z)^{13}}{\eta(6z)^2\eta(z)^7}} = r^2 + \frac{7}{3}r^5 + \frac{14}{9}r^8 - \frac{148}{81}r^{11} - \frac{1708}{243}r^{14} + \cdots$$

The associated newform is the same as in case 4a. The ASwD basis is:

if $p \equiv 1 \bmod 3$	basis is	h_1, h_2
if $p \equiv 2 \bmod 3$	basis is	$h_1 \pm \sqrt{-2}\sqrt[3]{3}h_2$

TABLE 5. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

Group	Elliptic Family	j-Invariant
$\Gamma(3)$	$(x^3 + y^3 + z^3) = txyz$	$\frac{t^3(t^3+216)^3}{(t^3-27)^3}$
$\Gamma(2) \cap \Gamma_1(4)$	$x(x^2 + z^2 + 2zy) = tz(x^2 - y^2)$	$\frac{(t^4-t^2+1)^3}{t^4(t-1)^2(t+1)^2}$
$\Gamma^1(5)$	$x(x - z)(y - z)t = y(y - x)z$	$-\frac{(t^4+12t^3+14t^2-12t+1)^3}{t^5(t^2+11t-1)}$
$\Gamma_1(6)$	$(xy + yx + zx)(x + y + z) = txyz$	$\frac{(3t-1)^3(3t^3-3t^2+9t-1)^3}{(t-1)^3t^6(9t-1)}$
$\Gamma_0(8) \cap \Gamma_1(4)$	$(x + y)(xy + z^2)t = 4xyz$	$-16\frac{(t^4-16t^2+16)^3}{t^8(t+1)(t-1)}$
$\Gamma_0(9) \cap \Gamma_1(3)$	$(x^2y + y^2z + z^2x) = txyz$	$\frac{t^3(t^3-24)^3}{t^3-27}$

TABLE 6. Data for Beauville's elliptic surfaces.

Level	Coefficients of Weierstrass Form					t as a Hauptmodul
	a_1	a_2	a_3	a_4	a_6	
3	0	t^2	0	$-72t$	$-8(4t^2 + 27)$	$\frac{\eta\left(\frac{1}{3}z\right)^3}{\eta(3z)^3} + 3$
4	0	$4 + 4t^2$	0	$16t^2$	0	$\frac{1}{2}\frac{\eta(z)^{12}}{\eta(2z)^8\eta\left(\frac{1}{2}z\right)^4}$
5	$t + 1$	t	t	0	0	$q^{\frac{1}{5}}\prod_{\substack{n=0\\ e=1,-1}}^{\infty}\left(\frac{\left(1-q^{n+e\frac{1}{5}}\right)}{\left(1-q^{n+e\frac{2}{5}}\right)}\right)^5$
6	$t + 1$	$t - t^2$	$t - t^2$	0	0	$\frac{1}{9}\frac{\eta(6z)^4\eta(z)^8}{\eta(3z)^8\eta(2z)^4}$
8	4	t^2	$4t^2$	0	0	$\frac{\eta(z)^8\eta(4z)^4}{\eta(2z)^{12}}$
9	0	t^2	0	$8t$	16	$27\frac{\eta(9z)^3}{\eta(z)^3} + 3$

TABLE 7. Weierstrass equations for Beauville's elliptic families.

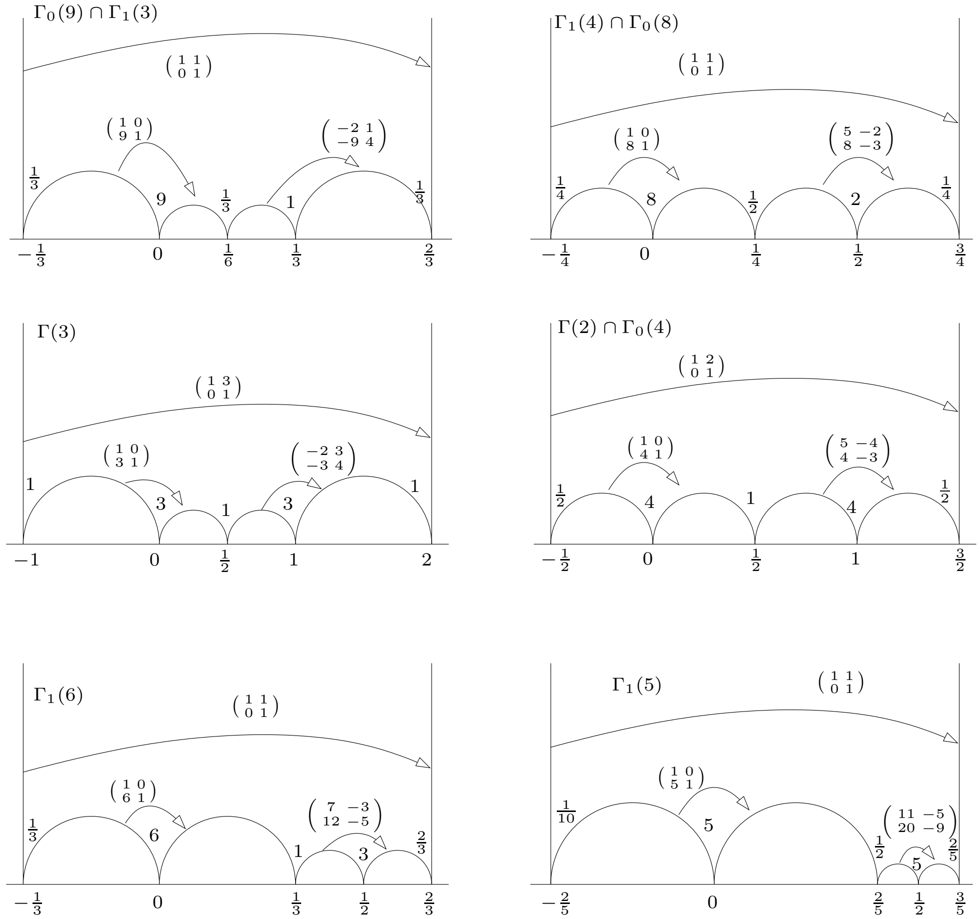

FIGURE 1. Fundamental domains for torsion-free index-24 congruence subgroups in $\mathrm{SL}_2(\mathbb{Z})$.

cusps and subgroups of $\Gamma_0(8)\cap\Gamma_1(4)$				
cusp z	∞	0	$\frac{1}{2}$	$\frac{1}{4}$
width	1	8	2	1
subgroup	ramified cusps indicated by ✓			
$\Gamma_{24.6.1^6}$		✓	✓	
$\Gamma_{8^3.2^3.3^2}$	✓			✓
$\Gamma_{8^3.6.3.1^3}$			✓	✓
$\Gamma_{24.3.2^3.1^3}$		✓		✓
$\Gamma_{8^3.6.3.1^3B}$	✓		✓	
$\Gamma_{24.3.2^3.1^3B}$	✓	✓		

cusps and subgroups of $\Gamma_1(6)$				
cusp z	∞	0	$\frac{1}{2}$	$\frac{1}{3}$
width	1	6	3	2
subgroup	ramified cusps indicated by ✓			
$\Gamma_{18.6.3^3.1^3}$		✓		✓
$\Gamma_{9.6^3.3.2^3}$	✓		✓	
$\Gamma_{9.6^4.1^3}$			✓	✓
$\Gamma_{18.3^4.2^3}$	✓	✓		

TABLE 8. Ramification points for triple covers of$X(\Gamma_0(8)\cap\Gamma_1(4))$ and $X(\Gamma_1(6))$, with corresponding subgroups.

The e_i are orders of elliptic points, u is the number of regular cusps, and u' the number of irregular cusps. Using this formula, we find that

$$\dim S_3(\Gamma) = 2,$$

for Γ equal to any of the groups in Definition 3.1.

4.2 Method of Constructing Elements of $S_3(\Gamma)$

Suppose that Γ has index 3 in G, one of the groups in Table 6, and that the corresponding covering is ramified at cusps c_1 and c_2.

Let t be a Hauptmodul for G, e.g., as in [Conway and Norton 79]. By a transformation, take t with $t(c_1) = 0$

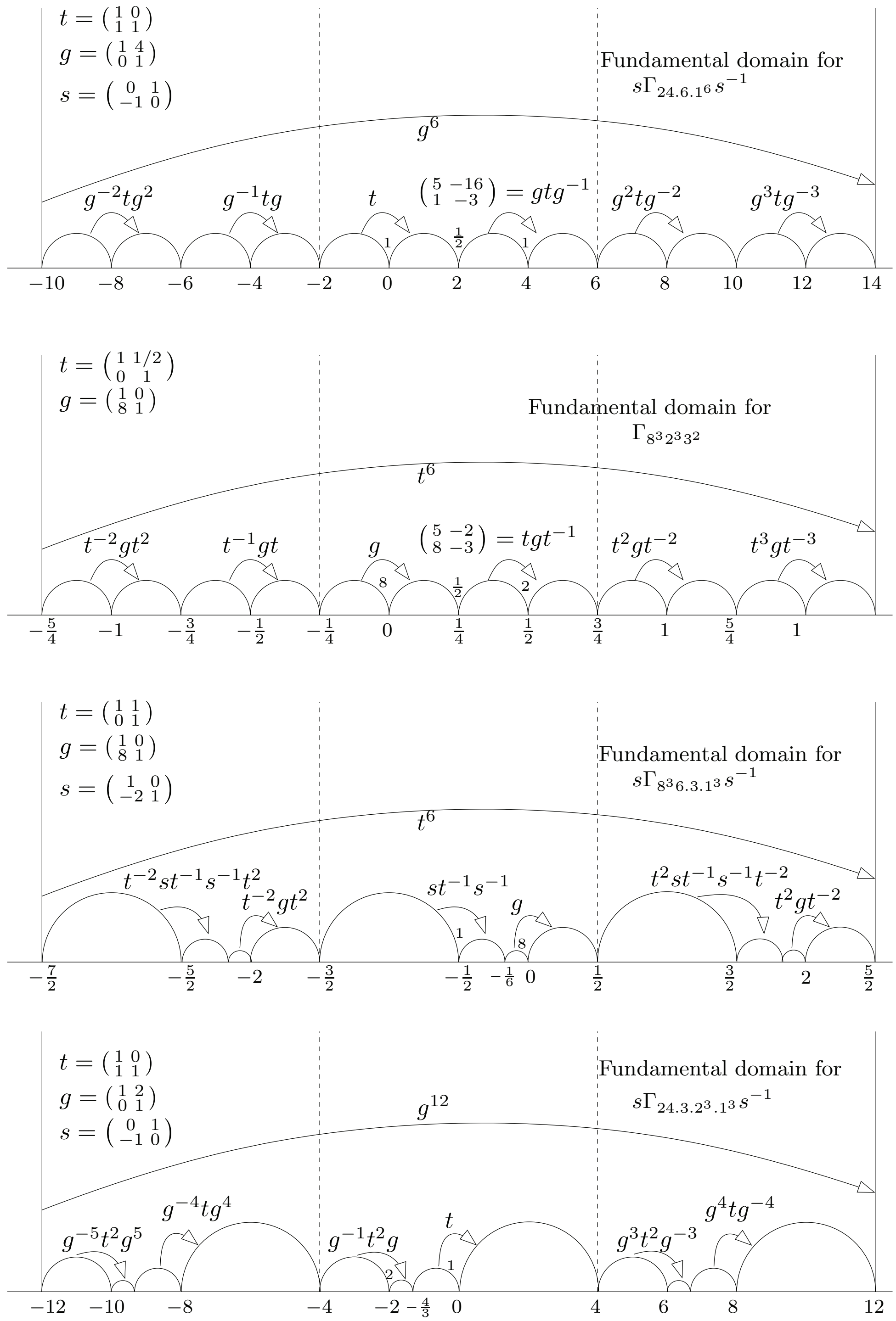

FIGURE 2. Fundamental domains for conjugates of some index-3 subgroups of $\Gamma_0(8) \cap \Gamma_1(4)$.

and $t(c_2) = \infty$. Then $\sqrt[3]{t}$ is a Hauptmodul for Γ. Let $f \in M_3(G)$. Then $\sqrt[3]{t}f \in A_3(\Gamma)$, the space of automorphic modular forms of weight 3 for Γ. If f has zeros of high enough order where t has poles, then $\sqrt[3]{t}f$ and $\sqrt[3]{t^2}f$ are in $S_3(\Gamma)$.

We give modular forms in terms of the Dedekind eta function, using the data given in [Martin 96]. Explicit details of the forms and their poles and zeros at all cusps are given in Tables 13, 14, 15, and 16. Note that since the f and t in all cases are eta products, the only possible

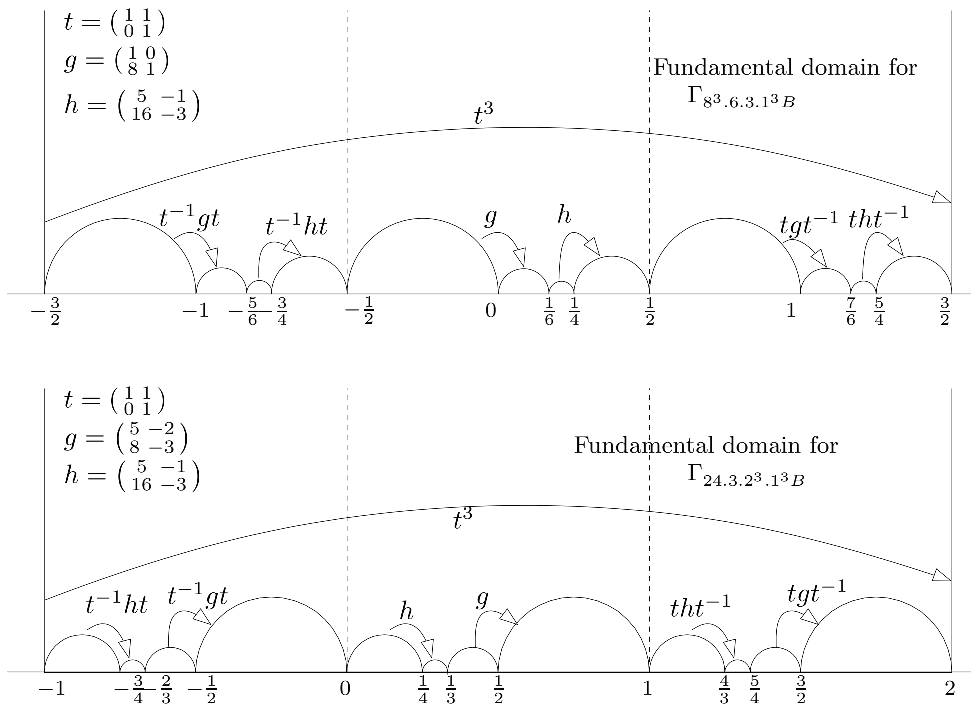

FIGURE 3. Fundamental domains for conjugates of some index-3 subgroups of $\Gamma_0(8) \cap \Gamma_1(4)$.

Γ	Generators
$\Gamma_{24.6.1^6}$	$\left(\begin{smallmatrix}1&0\\24&1\end{smallmatrix}\right), \left(\begin{smallmatrix}9&-1\\64&-7\end{smallmatrix}\right), \left(\begin{smallmatrix}5&-1\\16&-3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-3&-1\\16&5\end{smallmatrix}\right), \left(\begin{smallmatrix}-7&-1\\64&9\end{smallmatrix}\right), \left(\begin{smallmatrix}-11&-1\\144&13\end{smallmatrix}\right).$
$\Gamma_{8^3.2^3.3^2}$	$\left(\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-7&-8\\8&9\end{smallmatrix}\right), \left(\begin{smallmatrix}-3&-2\\8&5\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\8&1\end{smallmatrix}\right), \left(\begin{smallmatrix}5&-2\\8&-3\end{smallmatrix}\right), \left(\begin{smallmatrix}9&-8\\8&-7\end{smallmatrix}\right), \left(\begin{smallmatrix}13&-18\\8&-11\end{smallmatrix}\right).$
$\Gamma_{8^3.6.3.1^3}$	$\left(\begin{smallmatrix}-11&6\\-24&13\end{smallmatrix}\right), \left(\begin{smallmatrix}41&-25\\64&-39\end{smallmatrix}\right), \left(\begin{smallmatrix}49&-32\\72&-47\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\8&1\end{smallmatrix}\right), \left(\begin{smallmatrix}25&-9\\64&-23\end{smallmatrix}\right), \left(\begin{smallmatrix}81&-32\\200&-79\end{smallmatrix}\right).$
$\Gamma_{24.3.2^3.1^3}$	$\left(\begin{smallmatrix}1&0\\24&1\end{smallmatrix}\right), \left(\begin{smallmatrix}21&-2\\200&-19\end{smallmatrix}\right), \left(\begin{smallmatrix}9&-1\\64&-7\end{smallmatrix}\right), \left(\begin{smallmatrix}5&-2\\8&-3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-11&-2\\72&13\end{smallmatrix}\right), \left(\begin{smallmatrix}-7&-1\\64&9\end{smallmatrix}\right).$
$\Gamma_{8^3.6.3.1^3B}$	$\left(\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\8&1\end{smallmatrix}\right), \left(\begin{smallmatrix}5&-1\\16&-3\end{smallmatrix}\right), \left(\begin{smallmatrix}-7&-8\\8&9\end{smallmatrix}\right), \left(\begin{smallmatrix}-11&-9\\16&13\end{smallmatrix}\right), \left(\begin{smallmatrix}9&-8\\8&-7\end{smallmatrix}\right), \left(\begin{smallmatrix}53&-4\\40&-3\end{smallmatrix}\right).$
$\Gamma_{24.3.2^3.1^3B}$	$\left(\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\8&1\end{smallmatrix}\right), \left(\begin{smallmatrix}5&-2\\8&-3\end{smallmatrix}\right), \left(\begin{smallmatrix}-7&-8\\8&9\end{smallmatrix}\right), \left(\begin{smallmatrix}-3&-2\\8&5\end{smallmatrix}\right), \left(\begin{smallmatrix}9&-8\\8&-7\end{smallmatrix}\right), \left(\begin{smallmatrix}53&-5\\32&-3\end{smallmatrix}\right).$
$\Gamma_{18.6.3^3.1^3}$	$\left(\begin{smallmatrix}1&0\\18&1\end{smallmatrix}\right), \left(\begin{smallmatrix}25&-3\\192&-23\end{smallmatrix}\right), \left(\begin{smallmatrix}7&-1\\36&-5\end{smallmatrix}\right), \left(\begin{smallmatrix}7&-3\\12&-5\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-11&-3\\48&13\end{smallmatrix}\right), \left(\begin{smallmatrix}-5&-1\\36&7\end{smallmatrix}\right).$
$\Gamma_{9.6^3.3.2^3}$	$\left(\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-5&-6\\6&7\end{smallmatrix}\right), \left(\begin{smallmatrix}-11&-8\\18&13\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right), \left(\begin{smallmatrix}7&-2\\18&-5\end{smallmatrix}\right), \left(\begin{smallmatrix}7&-6\\6&-5\end{smallmatrix}\right), \left(\begin{smallmatrix}25&-32\\18&-23\end{smallmatrix}\right).$
$\Gamma_{9.6^4.1^3}$	$\left(\begin{smallmatrix}-17&6\\-54&19\end{smallmatrix}\right), \left(\begin{smallmatrix}127&-49\\324&-125\end{smallmatrix}\right), \left(\begin{smallmatrix}61&-24\\150&-59\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right), \left(\begin{smallmatrix}91&-25\\324&-89\end{smallmatrix}\right), \left(\begin{smallmatrix}85&-24\\294&-83\end{smallmatrix}\right).$
$\Gamma_{18.3^4.2^3}$	$\left(\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-11&-8\\18&13\end{smallmatrix}\right), \left(\begin{smallmatrix}-5&-3\\12&7\end{smallmatrix}\right), \left(\begin{smallmatrix}7&-2\\18&-5\end{smallmatrix}\right), \left(\begin{smallmatrix}7&-3\\12&-5\end{smallmatrix}\right), \left(\begin{smallmatrix}25&-32\\18&-23\end{smallmatrix}\right), \left(\begin{smallmatrix}19&-27\\12&-17\end{smallmatrix}\right).$

TABLE 9. Generators for Table 3.1.

$$
\begin{gathered}
6-6-6-6-3-3-3-3\\
9-9-9-3-3-1-1-1\\
9-9-3-3-3-3-3-3\\
10-10-5-5-2-2-1-1\\
18-9-2-2-2-1-1-1\\
27-3-1-1-1-1-1-1
\end{gathered}
$$

TABLE 10. Possible cusp widths of index-36 genus-zero torsion-free subgroups of $\mathrm{PSL}_2(\mathbb{Z})$, taken from [Sebbar 01, Section 7, Table 2].

Values of t_8				
cusp c	∞	0	$\frac{1}{2}$	$\frac{1}{4}$
$t_8(c)$	1	0	∞	-1

Values of t_6				
cusp c	∞	0	$\frac{1}{2}$	$\frac{1}{3}$
$t_8(c)$	$\frac{1}{9}$	0	1	∞

TABLE 11. Values of Hauptmoduls at cusps.

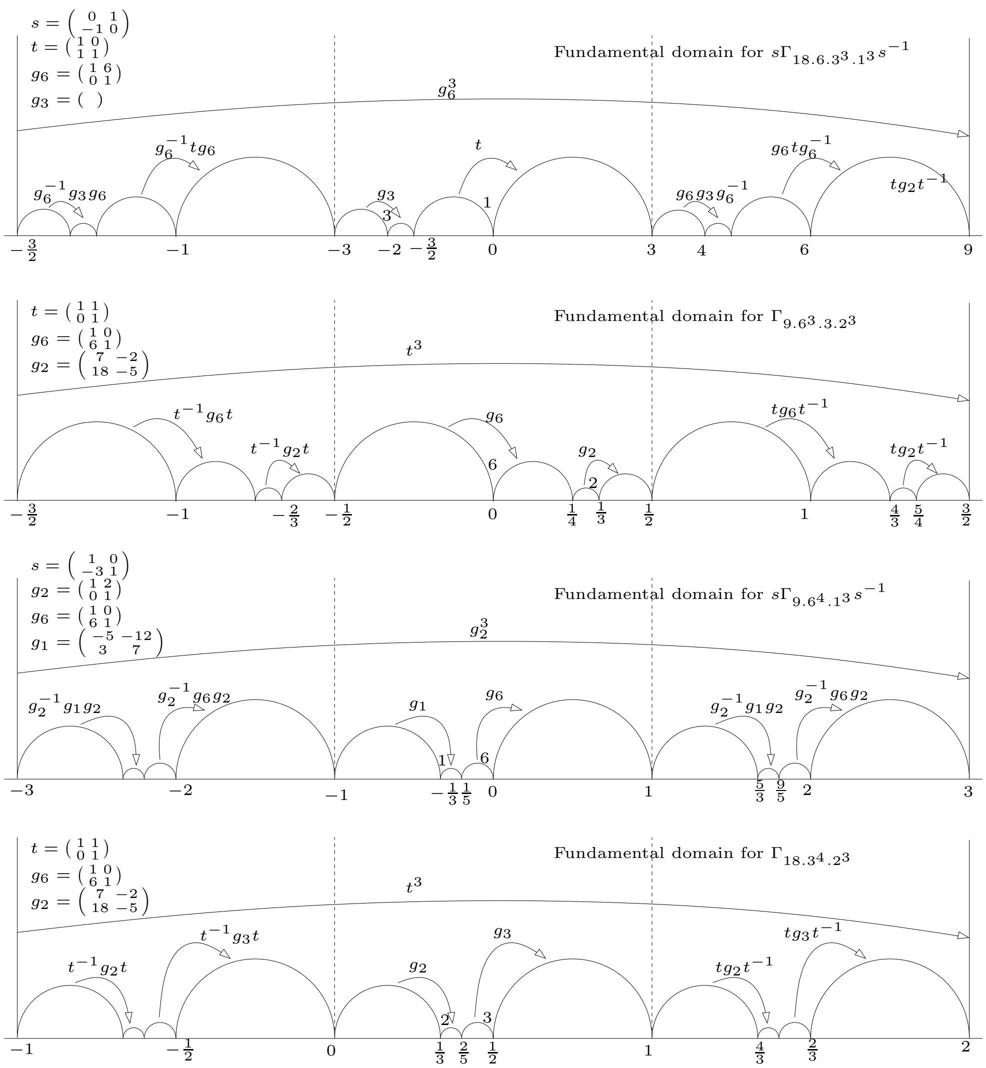

FIGURE 4. Fundamental domains for conjugates of some index-3 subgroups of $\Gamma_1(6)$.

zeros and poles are at the cusps, so no other points need be considered.

In each case of the subgroups given in Tables 14 and 16, we list the forms $\sqrt[3]{t}$, f, $h_1 = \sqrt[3]{t}f$, $h_2 = \sqrt[3]{t^2}f$. In each case, the basis of the space of weight-3 forms is given by the forms denoted by h_1 and h_2. The q-expansions of the forms h_1 and h_2 are given in Tables 17 and 18.

5. TRACES AND POINT COUNTING

As described by Scholl, corresponding to each of these families, we have a representation on parabolic cohomology:

$$\rho = \rho_l : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to H^1(X(\Gamma), j_* R^1 f_* \mathbb{Q}_l). \qquad (5\text{–}1)$$

Here $E^\circ(\Gamma)$ is a family of elliptic curves over $Y(\Gamma)$, with

$$Y(\Gamma) = \Gamma \setminus \mathbf{H}, \quad X(\Gamma) = (\Gamma \setminus \mathbf{H})^*$$

and with natural maps

$$E^\circ(\Gamma) \xrightarrow{f} Y(\Gamma) \overset{j}{\hookrightarrow} X(\Gamma).$$

We let $\mathcal{F} = j_* R^1 f_* \mathbb{Q}_l$, an l-adic sheaf for the étale topology on $X(\Gamma)$. We computed the traces of the Frobenius elements of this representation via point counting, as in [Li et al. 05a] and [Atkin et al. 08].

5.1 Equations for Elliptic Surfaces Associated with the Noncongruence Subgroups

As in Section 3.1, associated to $\Gamma_0(8) \cap \Gamma_1(4)$ and $\Gamma_1(6)$, we have families of elliptic curves $E_8(t)$ and $E_6(t)$ as given

subgroup	$m(t)$	$m^{-1}(r^3)$
$\Gamma_{24.6.1^6}$	t	r^3
$\Gamma_{8^3.2^3.3^2}$	$\frac{1+t}{1-t}$	$\frac{r^3-1}{r^3+1}$
$\Gamma_{8^3.6.3.1^3}$	$\frac{t+1}{4}$	$4r^3-1$
$\Gamma_{24.3.2^3.1^3}$	$\frac{2(1+t)}{t}$	$\frac{2}{r^3-2}$
$\Gamma_{8^3.6.3.1^3B}$	$\frac{t-1}{4}$	$4r^3+1$
$\Gamma_{24.3.2^3.1^3B}$	$\frac{2(1-t)}{t}$	$\frac{2}{r^3+2}$

subgroup	$m(t)$	$m^{-1}(r^3)$
$\Gamma_{18.6.3^3.1^3}$	$t/9$	$9r^3$
$\Gamma_{9.6^3.3.2^3}$	$\frac{1-9t}{3-3t}$	$\frac{1-3r^3}{9-3r^3}$
$\Gamma_{9.6^4.1^3}$	$\frac{8}{3-3t}$	$1-\frac{8}{3r^3}$
$\Gamma_{18.3^4.2^3}$	$\frac{1-9t}{24t}$	$\frac{1}{24r^3+9}$

TABLE 12. Covering maps corresponding to subgroups of $\Gamma_0(8)\cap\Gamma_1(4)$ and $\Gamma_1(6)$.

cusps (and widths)		$\frac{1}{2}(2)$	$0(8)$	$\infty(1)$	$\frac{1}{4}(1)$
forms for $\Gamma_0(8)\cap\Gamma_1(4)$	weight	order of vanishing			
$t=\frac{\eta(z)^8\eta(4z)^4}{\eta(2z)^{12}}=1-8q+32q^2+\cdots$	0	-1	1	0	0
$\frac{t+1}{2}=\frac{\eta(z)^4\eta(4z)^{14}}{\eta(8z)^4\eta(2z)^{14}}=1-4q+16q^2+\cdots$	0	-1	0	0	1
$\frac{t+1}{2t}=\frac{\eta(4z)^{10}}{\eta(8z)^4\eta(2z)^2\eta(z)^4}=1+4q+16q^2+\cdots$	0	0	-1	0	1
$\frac{4(t+1)}{(1-t)}=\frac{\eta(4z)^{12}}{\eta(8z)^8\eta(2z)^4}=q^{-1}+4q+2q^3+\cdots$	0	0	0	-1	1
$\frac{1-t}{8}=\frac{\eta(z)^4\eta(4z)^2\eta(8z)^4}{\eta(2z)^{10}}=q-4q^2+12q^3+\cdots$	0	-1	0	1	0
$E_a=\frac{\eta(4z)^4\eta(2z)^6}{\eta(z)^4}$	3	1	0	1	1
$E_b=\left(\frac{2t}{t+1}\right)E_a=\frac{\eta(2z)^8\eta(8z)^4}{\eta(4z)^6}$	3	1	1	1	0
$E_c=\left(\frac{4(t+1)}{1-t}\right)E_b=\frac{\eta(4z)^6\eta(2z)^4}{\eta(8z)^4}$	3	1	1	0	1

TABLE 13. Orders of vanishing at cusps for forms for $\Gamma_0(8)\cap\Gamma_1(4)$.

in Table 7:

$$E_8(t): y^2+4xy+4t^2y=x^3+t^2x^2, \tag{5–2}$$
$$E_6(t): y^2+(t+1)xy+(t-t^2)y=x^3+(t-t^2)x^2. \tag{5–3}$$

Thus we have elliptic surfaces E_8 and E_6, with fibrations

$$f_8: E_8\to X(\Gamma_0(8)\cap\Gamma_1(4))$$

and

$$f_6: E_6\to X(\Gamma_1(6)),$$

with fibers given by $f_8^{-1}(t)=E_8(t)$ and $f_6^{-1}(t)=E_6(t)$.

By composing each of the covering maps given in Table 12 with the fibration f_8 or f_6, associated with our noncongruence subgroups we have the families of elliptic curves given in Table 19. Our notation is explained by example: The elliptic surface $E(\Gamma_{8^3.2^3.3^2})$ corresponding to $\Gamma_{8^3.2^3.3^2}$ has a fibration

$$f: E(\Gamma_{8^3.2^3.3^2})\to X(\Gamma_{8^3.2^3.3^2}),$$

with fiber $f^{-1}(r)$ having equation

$$y^2+4xy+4\left(\frac{r^3-1}{r^3+1}\right)^2y=x^3+4\left(\frac{r^3-1}{r^3+1}\right)^2x^2,$$

i.e., the t in (5–2) is replaced by $m^{-1}(r^3)=\frac{r^3-1}{r^3+1}$, where $m(t)=\frac{1+t}{1-t}$. This family of elliptic curves is denoted by $E_8\left(\frac{r^3-1}{r^3+1}\right)$. The other families are constructed and denoted in a similar way.

We computed the traces of the Frobenius by summing local terms using the following theorem:

Theorem 5.1.

$$\mathrm{Tr}(\mathrm{Frob}_q\,|H^1(X(\Gamma),\mathcal{F}))=-\sum_{x\in X(\mathbb{F}_q)}\mathrm{Tr}(\mathrm{Frob}_q\,|\mathcal{F}_x).$$

Proof: This follows from the Grothendieck–Lefschetz trace formula because the other terms $H^i(X(\Gamma),\mathcal{F})$, $i\neq 1$, are zero. □

The following theorem is also well known:

	cusps		$1/2$			0		$-\frac{1}{8}$	∞	$\frac{1}{8}$	$\frac{-1}{4}$	$\frac{1}{4}$	$\frac{1}{12}$
forms for	width		6			24		1	1	1	1	1	1
$\Gamma_{24.6.1^6}$	weight	order of vanishing of form at cusps											
$\sqrt[3]{t}$	0		-1			1		0	0	0	0	0	0
E_a	3		3			0		1	1	1	1	1	1
$h_1 = t^{1/3}E_a$	3		2			1		1	1	1	1	1	1
$h_2 = t^{2/3}E_a$	3		1			2		1	1	1	1	1	1
	cusps		$1/2$		$\frac{2}{5}$	0	$\frac{2}{3}$	$\frac{3}{8}$	∞	$\frac{5}{8}$		$1/4$	
forms for	width		6		8	8	8	1	1	1		3	
$\Gamma_{8^3.6.3.1^3}$	weight	order of vanishing of form at cusps											
$r_1 = \sqrt[3]{\frac{t+1}{2}}$	0		-1		0	0	0	0	0	0		1	
E_b	3		3		1	1	1	1	1	1		0	
$h_1 = r_1E_b$	3		2		1	1	1	1	1	1		1	
$h_2 = r_1^2E_b$	3		1		1	1	1	1	1	1		2	
	cusps	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{10}$		0		$-\frac{1}{8}$	∞	$\frac{1}{8}$		$1/4$	
forms for	width	2	2	2		24		1	1	1		3	
$\Gamma_{24.3.2^3.1^3}$	weight	order of vanishing of form at cusps											
$r_2 = \sqrt[3]{\frac{(t+1)}{2t}}$	0	0	0	0		-1		0	0	0		1	
E_b	3	1	1	1		3		1	1	1		0	
$h_1 = r_2E_b$	3	1	1	1		2		1	1	1		1	
$h_2 = r_2^2E_b$	3	1	1	1		1		1	1	1		2	
	cusps	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	-1	0	1		∞			$\frac{1}{4}$	
forms for	width	2	2	2	8	8	8		3			3	
$\Gamma_{8^3.2^3.3^2}$	weight	order of vanishing of form at cusps											
$r_3 = \sqrt[3]{\frac{4(t+1)}{(t-1)}}$	0	0	0	0	0	0	0		-1			1	
E_b	3	1	1	1	1	1	1		3			0	
$h_1 = r_3E_b$	3	1	1	1	1	1	1		2			1	
$h_2 = r_3^2E_b$	3	1	1	1	1	1	1		1			2	
	cusps		$1/2$		-1	0	1		∞		$\frac{-3}{4}$	$\frac{1}{4}$	$\frac{5}{4}$
forms for	width		6		8	8	8		3		1	1	1
$\Gamma_{8^3.6.3.1^3B}$	weight	order of vanishing of form at cusps											
$r_4 = \sqrt[3]{\frac{1-t}{8}}$	0		-1		0	0	0		1		0	0	0
E_c	3		3		1	1	1		0		1	1	1
$h_1 = r_4E_c$	3		2		1	1	1		1		1	1	1
$h_2 = r_4^2E_c$	3		1		1	1	1	1	1	1		2	
	cusps	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$		0			∞		$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{5}{4}$
forms for	width	2	2	2		24		1	1	1		3	
$\Gamma_{24.3.2^3.1^3B}$	weight	order of vanishing of form at cusps											
$r_5 = \sqrt[3]{\frac{(1-t)}{8t}}$	0	0	0	0		-1			1		0	0	0
E_c	3	1	1	1		3			0		1	1	1
$h_1 = r_5E_c$	3	1	1	1		2			1		1	1	1
$h_2 = r_5^2E_c$	3	1	1	1		1			2		1	1	1

TABLE 14. Orders of vanishing at cusps for forms for subgroups of $\Gamma_0(8) \cap \Gamma_1(4)$. The form t is as in Table 13. In each case, h_1, h_2 form a basis for the space of weight-3 cusp forms.

Theorem 5.2. *$\mathrm{Tr}(\mathrm{Frob}_q\,|\mathcal{F}_x)$ may be computed according to the following:*

1. *If the fiber E_x is smooth, then*

$$\begin{aligned}\mathrm{Tr}(\mathrm{Frob}_q|\mathcal{F}_x) &= \mathrm{Tr}(\mathrm{Frob}_q|H^1(E_x, \mathbb{Q}_l)) \\ &= q + 1 - \#E_x(\mathrm{F}_q).\end{aligned}$$

2. *If the fiber E_x is singular, then Tate's algorithm tells us that*

$$\mathrm{Tr}(\mathrm{Frob}_q|\mathcal{F}_x) = \begin{cases} 1 & \textit{if the fiber is split multiplicative,} \\ -1 & \textit{if the fiber is nonsplit multiplicative,} \\ 0 & \textit{if the fiber is additive.} \end{cases}$$

forms for $\Gamma_1(6)$	weight	$\infty(1)$	$0(6)$	$\frac{1}{2}(3)$	$\frac{1}{3}(2)$
cusps (and widths)		order of vanishing			
$a = \frac{\eta(z)\eta(6z)^6}{\eta(2z)^2\eta(3z)^3} = q - q^2 + q^3 + q^4 + \cdots$	1	1	0	0	0
$b = \frac{\eta(2z)\eta(3z)^6}{\eta(z)^2\eta(6z)^3} = 1 + 2q + 4q^2 + 2q^3 + \cdots$	1	0	0	0	1
$c = \frac{\eta(3z)\eta(2z)^6}{\eta(6z)^2\eta(z)^3} = 1 + 3q + 3q^2 + 3q^3 + \cdots$	1	0	0	1	0
$d = \frac{\eta(6z)\eta(z)^6}{\eta(3z)^2\eta(2z)^3} = 1 - 6q + 12q^2 - 6q^3 \cdots$	1	0	1	0	0
$r_0 = b/d = 1 + 8q + 40q^2 + 152q^3 + \cdots$	0	0	-1	0	1
$r_1 = b/c = 8\frac{r_0}{(9r_0-1)} = 1 - q + 4q^2 + \cdots$	0	0	0	-1	1
$r_2 = a/c = \frac{(r_0-1)}{(9r_0-1)} = q - 4q^2 + 10q^3 \cdots$	0	1	0	-1	0
$r_3 = a/d = \frac{1}{8}(r_0 - 1) = q + 5q^2 + 19q^3 \cdots$	0	1	-1	0	0
$acd = q - 4q^2 + q^3 + 16q^4 + \cdots$	3	1	1	1	0
$bcd = 1 - q - 5q^2 - q^3 + 11q^4 + \cdots$	3	0	1	1	1

TABLE 15. Orders of vanishing at cusps for forms for $\Gamma_1(6)$.

forms for $\Gamma_{18.6.3^3.1^3}$	cusps	$\frac{1}{6}$	∞	$-\frac{1}{6}$	0	$\frac{1}{8}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{3}$
	width	1	1	1	18	3	3	3	6
	weight	order of vanishing of form at cusps							
$\sqrt[3]{b/d}$	0	0	0	0	-1	0	0	0	1
acd	3	1	1	1	3	1	1	1	0
$h_1 = (\sqrt[3]{b/d})acd$	3	1	1	1	2	1	1	1	1
$h_2 = (\sqrt[3]{b/d})^2 acd$	3	1	1	1	1	1	1	1	2

forms for $\Gamma_{9.6^4.1^3}$	cusps	$\frac{5}{18}$	∞	$\frac{7}{18}$	$\frac{2}{5}$	0	$\frac{2}{7}$	$\frac{1}{2}$	$\frac{1}{3}$
	width	1	1	1	6	6	6	9	6
	weight	order of vanishing of form at cusps							
$\sqrt[3]{b/c}$	0	0	0	0	0	0	0	-1	1
acd	3	1	1	1	1	1	1	3	0
$h_1 = (\sqrt[3]{b/c})acd$	3	1	1	1	1	1	1	2	1
$h_2 = (\sqrt[3]{b/c})^2 acd$	3	1	1	1	1	1	1	1	2

forms for $\Gamma_{9.6^3.3.2^3}$	cusps	∞	-1	0	1	$\frac{1}{2}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{4}{3}$
	width	3	6	6	6	9	2	2	2
	weight	order of vanishing of form at cusps							
$\sqrt[3]{a/c}$	0	1	0	0	0	-1	0	0	0
bcd	3	0	1	1	1	3	1	1	1
$h_1 = (\sqrt[3]{a/c})bcd$	3	1	1	1	1	2	1	1	1
$h_2 = (\sqrt[3]{a/c})^2 bcd$	3	2	1	1	1	1	1	1	1

forms for $\Gamma_{18.3^4.2^3}$	cusps	∞	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{4}{3}$
	width	3	18	3	3	3	2	2	2
	weight	order of vanishing of form at cusps							
$\sqrt[3]{a/d}$	0	1	-1	0	0	0	0	0	0
bcd	3	0	3	1	1	1	1	1	1
$h_1 = (\sqrt[3]{a/d})bcd$	3	1	2	1	1	1	1	1	1
$h_2 = (\sqrt[3]{a/d})^2 bcd$	3	2	1	1	1	1	1	1	1

TABLE 16. Orders of vanishing at cusps for forms for subgroups of $\Gamma_1(6)$. The forms a, b, c, d are as in Table 15. In each case, h_1, h_2 form a basis for the space of weight-3 cusp forms.

3. *If E is a singular curve over a field with characteristic not 2 or 3, given by an equation $E : y^2 = x^3 + ax + b$, then the reduction type of E is determined as follows:*

   additive if $-2ab$ is 0 in k,
   split multiplicative if $-2ab$ is a nonzero square in k,
   nonsplit multiplicative if $-2ab$ is not a square in k.

In order to apply part (3) of the above result, we need to transform $E_8(t)$ and $E_6(t)$ into the simplified Weierstrass form $y^2 = x^3 + ax + b$. We obtain the following

$\Gamma_{24.6.1^6}$	
$\sqrt[3]{\eta(z)^{-4}\eta(2z)^{6}\eta(4z)^{16}}$	$= q + \frac{4}{3}q^2 + \frac{8}{9}q^3 + \frac{176}{81}q^4 - \frac{850}{243}q^5 - \frac{3488}{729}q^6 - \frac{5968}{6561}q^7 + \cdots$
$\sqrt[3]{\eta(z)^{4}\eta(2z)^{-6}\eta(4z)^{20}}$	$= q - \frac{4}{3}q^2 + \frac{8}{9}q^3 - \frac{176}{81}q^4 - \frac{850}{243}q^5 + \frac{3488}{729}q^6 - \frac{5968}{6561}q^7 + \cdots$
$\Gamma_{8^3.6.3.1^3}$	
$\sqrt[3]{\eta(z)^{4}\eta(2z)^{10}\eta(4z)^{-4}\eta(8z)^{8}}$	$= q - \frac{4}{3}q^2 - \frac{40}{9}q^3 + \frac{400}{81}q^4 + \frac{1454}{243}q^5 - \frac{1888}{729}q^6 - \frac{13168}{6561}q^7 + \cdots$
$\sqrt[3]{\eta(z)^{8}\eta(2z)^{-4}\eta(4z)^{10}\eta(8z)^{4}}$	$= q - \frac{8}{3}q^2 + \frac{8}{9}q^3 + \frac{32}{81}q^4 - \frac{82}{243}q^5 + \frac{5440}{729}q^6 - \frac{24400}{6561}q^7 + \cdots$
$\Gamma_{24.3.2^3.1^3}$	
$\sqrt[3]{\eta(z)^{-4}\eta(2z)^{22}\eta(4z)^{-8}\eta(8z)^{8}}$	$= q + \frac{4}{3}q^2 - \frac{40}{9}q^3 - \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \frac{1888}{729}q^6 - \frac{13168}{6561}q^7 + \cdots$
$\sqrt[3]{\eta(z)^{-8}\eta(2z)^{20}\eta(4z)^{2}\eta(8z)^{4}}$	$= q + \frac{8}{3}q^2 + \frac{8}{9}q^3 - \frac{32}{81}q^4 - \frac{82}{243}q^5 - \frac{5440}{729}q^6 - \frac{24400}{6561}q^7 + \cdots$
$\Gamma_{8^3.2^3.3^2}$	
$\sqrt[3]{\eta(2z)^{20}\eta(4z)^{-6}\eta(8z)^{4}}$	$= q^{2/3} - \frac{20}{3}q^{8/3} + \frac{128}{9}q^{14/3} - \frac{400}{81}q^{20/3} + \cdots$
$\sqrt[3]{\eta(2z)^{16}\eta(4z)^{6}\eta(8z)^{-4}}$	$= q^{1/3} - \frac{16}{3}q^{7/3} + \frac{38}{9}q^{13/3} + \frac{1696}{81}q^{19/3} + \cdots$
$\Gamma_{8^3.6.3.1^3B}$	
$\sqrt[3]{\eta(z)^{8}\eta(4z)^{22}\eta(8z)^{-4}\eta(2z)^{-8}}$	$= q^{2/3} - \frac{8}{3}q^{5/3} + \frac{20}{9}q^{8/3} - \frac{256}{81}q^{11/3} - \frac{64}{243}q^{14/3} + \cdots$
$\sqrt[3]{\eta(z)^{4}\eta(2z)^{2}\eta(4z)^{20}\eta(8z)^{-8}}$	$= q^{1/3} - \frac{4}{3}q^{4/3} - \frac{16}{9}q^{7/3} + \frac{112}{81}q^{10/3} + \cdots$
$\Gamma_{24.3.2^3.1^3B}$	
$\sqrt[3]{\eta(z)^{-4}\eta(2z)^{14}\eta(4z)^{16}\eta(8z)^{-8}}$	$= q^{1/3} + \frac{4}{3}q^{4/3} - \frac{16}{9}q^{7/3} - \frac{112}{81}q^{10/3} - \frac{1534}{243}q^{13/3} + \cdots$
$\sqrt[3]{\eta(z)^{-8}\eta(2z)^{16}\eta(4z)^{14}\eta(8z)^{-4}}$	$= q^{2/3} + \frac{8}{3}q^{5/3} + \frac{20}{9}q^{8/3} + \frac{256}{81}q^{11/3} - \frac{64}{243}q^{14/3} + \cdots$

TABLE 17. q-expansions of basis of forms for $S_3(\Gamma)$ for six subgroups of $\Gamma_0(8) \cap \Gamma_1(4)$.

$\Gamma_{18.6.3^3.1^3}$	
$ab^{1/3}cd^{2/3}$	$= \sqrt[3]{\eta(z)^{4}\eta(2z)^{7}\eta(3z)^{-4}\eta(6z)^{11}} = q - \frac{4}{3}q^2 - \frac{31}{9}q^3 + \frac{400}{81}q^4 + \frac{104}{243}q^5 + \cdots$
$ab^{2/3}cd^{1/3}$	$= \sqrt[3]{\eta(z)^{-4}\eta(2z)^{11}\eta(3z)^{4}\eta(6z)^{7}} = q + \frac{4}{3}q^2 - \frac{7}{9}q^3 - \frac{112}{81}q^4 - \frac{616}{243}q^5 + \cdots$
$\Gamma_{9.6^4.1^3}$	
$ab^{1/3}c^{2/3}d$	$= \sqrt[3]{\eta(z)^{13}\eta(2z)^{-2}\eta(3z)^{-7}\eta(6z)^{14}} = q - \frac{13}{3}q^2 + \frac{32}{9}q^3 + \frac{670}{81}q^4 - \frac{3577}{243}q^5 + \cdots$
$ab^{2/3}c^{1/3}d$	$= \sqrt[3]{\eta(z)^{14}\eta(2z)^{-7}\eta(3z)^{-2}\eta(6z)^{13}} = q - \frac{14}{3}q^2 + \frac{56}{9}q^3 - \frac{58}{81}q^4 + \frac{266}{243}q^5 + \cdots$
$\Gamma_{9.6^3.3.2^3}$	
$a^{1/3}bc^{2/3}d$	$= \sqrt[3]{\eta(z)^{7}\eta(2z)^{4}\eta(3z)^{11}\eta(6z)^{-4}} = q^{\frac{1}{3}} - \frac{7}{3}q^{\frac{4}{3}} - \frac{19}{9}q^{\frac{7}{3}} + \frac{193}{81}q^{\frac{10}{3}} + \frac{2306}{243}q^{\frac{13}{3}} + \cdots$
$a^{2/3}bc^{1/3}d$	$= \sqrt[3]{\eta(z)^{11}\eta(2z)^{-4}\eta(3z)^{7}\eta(6z)^{4}} = q^{\frac{2}{3}} - \frac{11}{3}q^{\frac{5}{3}} + \frac{23}{9}q^{\frac{8}{3}} - \frac{13}{81}q^{\frac{11}{3}} + \frac{2495}{243}q^{\frac{14}{3}} + \cdots$
$\Gamma_{18.3^4.2^3}$	
$a^{1/3}bcd^{2/3}$	$= \sqrt[3]{\eta(z)^{-2}\eta(2z)^{13}\eta(3z)^{14}\eta(6z)^{-7}} = q^{\frac{1}{3}} + \frac{2}{3}q^{\frac{4}{3}} - \frac{28}{9}q^{\frac{7}{3}} - \frac{482}{81}q^{\frac{10}{3}} - \frac{736}{243}q^{\frac{13}{3}} + \cdots$
$a^{2/3}bcd^{1/3}$	$= \sqrt[3]{\eta(z)^{-7}\eta(2z)^{14}\eta(3z)^{13}\eta(6z)^{-2}} = q^{\frac{2}{3}} + \frac{7}{3}q^{\frac{5}{3}} + \frac{14}{9}q^{\frac{8}{3}} - \frac{148}{81}q^{\frac{11}{3}} - \frac{1708}{243}q^{\frac{14}{3}} + \cdots$

TABLE 18. Basis of weight-3 cusp forms for some index-3 subgroups of $\Gamma_1(6)$. Here a, b, c, d are eta products as in Table 15.

Group	Family of Curves
$\Gamma_{24.6.1^6}$	$E_8(r^3)$
$\Gamma_{8^3.2^3.3^2}$	$E_8\left(\frac{r^3-1}{r^3+1}\right)$
$\Gamma_{8^3.6.3.1^3}$	$E_8(4r^3-1)$
$\Gamma_{24.3.2^3.1^3}$	$E_8\left(\frac{2}{r^3-2}\right)$

Group	Family of Curves
$\Gamma_{18.6.3^3.1^3}$	$E_6(9r^3)$
$\Gamma_{9.6^3.3.2^3}$	$E_6\left(\frac{1-3r^3}{9-3r^3}\right)$
$\Gamma_{9.6^4.1^3}$	$E_6\left(1-\frac{8}{9r^3}\right)$
$\Gamma_{18.3^4.2^3}$	$E_6\left(\frac{1}{9(8r^3+1)}\right)$

TABLE 19. Families of elliptic curves $E_n(m^{-1}(r^3))$ corresponding to certain noncongruence subgroups.

curves, isomorphic to the originals, over any field of characteristic not 2 or 3:

$$\begin{aligned}\widetilde{E}_8 : y^2 &= x^3 - 27(t^4 - 16t^2 + 16)x \\ &\quad + 54(t^2 - 2)(t^4 + 32t^2 - 32) \\ \widetilde{E}_6 : y^3 &= x^3 - 2^4 3^3(3t - 1)(3t^3 - 3t^2 + 9t - 1)x \\ &\quad - 2^7 3^3(3t^2 + 6t - 1) \\ &\qquad \times (9t^4 - 36t^3 + 30t^2 - 12t + 1).\end{aligned}$$

Thus one may compute values of the trace using the above result, for example with Magma. The results for a range of values of p and various covers of E_8 and E_6 are given in Table 20.

6. INVOLUTIONS AND ISOGENIES

6.1 Involutions

The four-dimensional representations on $H^1(X(\Gamma), \mathcal{F}_\Gamma)$ in fact split into two 2-dimensional Galois representations. We can achieve this splitting using an involution on $\Gamma \setminus \mathbb{H}$ that extends to either an automorphism or an isogeny on the elliptic surface.

For each family given in Table 19 by an equation $E_n(r)$ corresponding to a covering $r^3 = m(t)$, we have involutions i and ι of t and r, given in Table 21, such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "r\mapsto\iota(r)"] \arrow[d, "r\mapsto r^3=m(t)"'] & \mathbb{P}^1 \arrow[d, "r\mapsto r^3=m(t)"] \\ \mathbb{P}^1 \arrow[r, "t\mapsto i(t)"'] & \mathbb{P}^1 \end{tikzcd}$$

Furthermore, if c_1, c_2 are the ramified cusps of the map $r \mapsto r^3 = m(t)$, and c_3, c_4 are the unramified cusps, then i fixes the sets $\{c_1, c_2\}$ and $\{c_3, c_4\}$. This means that the involution i lifts to an involution ι of r, as indicated in Table 21. To check that these are the correct maps, one just needs to verify that $(\iota(\sqrt[3]{m(t)}))^3 = m(i(t))$, which is simple algebra.

6.2 Isogenies

The involutions i of modular curves given in Table 21 lift to maps

$$\begin{aligned}&\tilde{i} : E_n \to E_n, \\ &\tilde{i} : (t, x, y) \in E_n(t) \mapsto (i(t),\, i_x(t, x, y),\, i_y(t, x, y)),\end{aligned}$$

where $n = 8$ or 6, which restrict to isogenies between the fibers of the corresponding family of elliptic curves (given by (5–2) and (5–3)). From the isogenies of the families $E_6(t)$ and $E_8(t)$, one can obtain the isogenies on the families $E_6(m^{-1}(r^3))$ and $E_8(m^{-1}(r^3))$ lifting ι to $\tilde{\iota}$. These isogenies will give rise to involutions on the level of cohomology.

To show that two curves $E(t)$ and $E(i(t))$ are isogenous by an isogeny of degree d, it suffices to show that

$$\Phi_d(j(E(t)), j(E(i(t)))) = 0,$$

where Φ_d is the dth modular polynomial. The isogeny can be explicitly determined by Velu's method from a subgroup of order d in $E(t)$. Although the algorithms involved are well known and not difficult theoretically, in practice they should be carried out with the help of a computer program, such as Magma, because of the large number of terms in the polynomials involved. For example, Φ_8 is a polynomial in two variables of degree 20 with 141 terms; Φ_n can be found in a Magma database using the command `ClassicalModularPolynomial(n)` for $1 \le \mathtt{n} \le 17$.

Although it is not important to know the isogeny exactly, we do need to know the field over which the map is defined. This information was computed with the assistance of Magma, and is given in Table 22. The polynomials given in this table are such that their roots are the x-coordinates of points in the kernel of the isogeny.

6.3 Isogenous Relationships between Families

In the previous section we showed how involutions give rise to isogenies on the fibers, which will result in involutions on the cohomology of each family. There are also isogenous maps between families, which explain our grouping into pairs of cases, which was originally based on the relationships between traces seen in Table 20.

Combining the relations between curves we already have, we find that

$$\begin{aligned}\Phi_8\left(j\left(E_6\left(\frac{t-1}{t+1}\right)\right), j\left(E_8\left(\phi_1(t)\right)\right)\right) &= 0, \\ \Phi_8\left(j(E_8(4t-1)), j\left(E_8\left(\frac{2}{\phi_2(t)-2}\right)\right)\right) &= 0, \\ \Phi_6\left(j\left(E_6\left(\frac{1-3t}{9-3t}\right)\right), j\left(E_6\left(9\phi_3(t)\right)\right)\right) &= 0, \\ \Phi_3\left(j\left(E_6\left(1-\frac{8}{3t}\right)\right), j\left(E_6\left(\frac{1}{9-24\phi_4(t)}\right)\right)\right) &= 0,\end{aligned}$$

where $\phi_1(t) = \phi(t) = 1/t$, $\phi_3(t) = t/3, \phi_4(t) = -1/t$. This may also be checked directly with Magma. Thus the maps ϕ_i between the bases lift to isogenies on the fibers between families. Replacing t by r^3 in these equations does not change the relationships, so this also holds

Group	Equation	p	5	7	11	13	17	19	23	73
$\Gamma_{24.6.1^6}$	$E_8(r^3)$	Tr_p	0	4	0	-44	0	52	0	-92
		Tr_{p^2}	100	-188	484	292	1156	-92	2116	-17084
$\Gamma_{8^3.2^3.3^2}$	$E_8\left(\frac{r^3-1}{r^3+1}\right)$	Tr_p	0	-4	0	-44	0	-52	0	-92
		Tr_{p^2}	100	-188	484	292	1156	-92	2116	-17084
$\Gamma_{8^3.6.3.1^3}$	$E_8(r^3-1)$	Tr_p	0	-3	0	13	0	33	0	-71
		Tr_{p^2}	-44	-95	52	169	1012	-359	-1772	5617
	$E_8(2r^3-1)$	Tr_p	0	3	0	13	0	-33	0	-71
		Tr_{p^2}	-44	-95	52	169	1012	-359	-1772	5617
	$E_8(4r^3-1)$	Tr_p	0	0	0	-26	0	0	0	142
		Tr_{p^2}	-44	190	52	-338	1012	718	-1772	-11234
$\Gamma_{24.3.2^3.1^3}$	$E_8\left(\frac{2}{r^3-2}\right)$	Tr_p	0	0	0	-26	0	0	0	142
		Tr_{p^2}	-44	190	52	-338	1012	718	-1772	-11234
$\Gamma_{18.6.3^3.1^3}$	$E_6(3r^3)$	Tr_p	0	-11	0	-5	0	19	0	76
		Tr_{p^2}	28	-23	196	313	508	361	316	-18428
$\Gamma_{18.6.3^3.1^3}$	$E_6(9r^3)$	Tr_p	0	22	0	10	0	-38	0	76
		Tr_{p^2}	28	46	196	-626	508	-722	316	-18428
$\Gamma_{9.6^3.3.2^3}$	$E_6\left(\frac{1-3r^3}{9-3r^3}\right)$	Tr_p	0	22	0	10	0	-38	0	76
		Tr_{p^2}	28	46	196	-626	508	-722	316	-18428
$\Gamma_{9.6^4.1^3}$	$E_6\left(1-\frac{24}{r^3}\right)$	Tr_p	0	7	0	-5	0	-17	0	-248
		Tr_{p^2}	64	49	448	313	-140	433	1972	9436
	$E_6\left(1-\frac{8}{3r^3}\right)$	Tr_p	0	-14	0	10	0	34	0	-248
		Tr_{p^2}	64	-98	448	-626	-140	-866	1972	9436
$\Gamma_{18.3^4.2^3}$	$E_6\left(\frac{1}{24r^3+9}\right)$	Tr_p	0	-14	0	10	0	34	0	-248
		Tr_{p^2}	64	-98	448	-626	-140	-866	1972	9436

TABLE 20. Table of $\mathrm{Tr}\,\rho^*(\mathrm{Frob}_p)$.

for the covers, and these maps induce isomorphisms on the level of cohomology. Refer to Table 19 for which cover corresponds to which group.

7. EXPERIMENTAL DATA FOR THE ATKIN–SWINNERTON-DYER CONGRUENCES

The strategy for finding an ASwD basis is the following: For our noncongruence subgroup Γ, we have found a basis h_1, h_2 for $S_3(\Gamma)$. We have also found a Hecke eigenform $f \in S_3(\Gamma_0, \chi)$ for some congruence subgroup Γ_0. Let a_n and b_n respectively be the expansion coefficients of h_1 and h_2. Let A_n be the expansion coefficients of f. We consider two possible situations.

7.1 Case 1

In the simplest case, h_1, h_2 is already an ASwD basis. This case occurs in Section 7.3. So for good primes p and integers n with $p \nmid n$,

$$a_{pn} \equiv A_p a_n \bmod p^2 \quad \text{and} \quad b_{pn} \equiv A_p b_n \bmod p^2,$$

which implies, for p fixed and n varying with $a_n \neq 0$ and $b_n \neq 0$,

$$\frac{a_{pn}}{a_n} \equiv \text{constant mod } p^2 \text{ and } \frac{b_{pn}}{b_n} \equiv \text{constant mod } p^2. \tag{7–1}$$

So, our test for whether h_1, h_2 is an ASwD basis is to check whether a_{pn}/a_n and b_{pn}/b_n take constant values

Involutions i of $X(\Gamma_0(8)\cap\Gamma_1(4))$, and ι of $X(\Gamma)$, for $\Gamma\subset\Gamma_0(8)\cap\Gamma_1(4)$					
subgroup Γ	values of z and t where cover ramifies		$r^3 = m(t)$	involutions of t and r	
				$i : t \mapsto$	$\iota : r \mapsto$
	z	$t(z)$			
$\Gamma_{24.6.1^6}$	$1/2, 0$	$\infty, 0$	t	$-t$	$-r$
$\Gamma_{8^2.2^3.3^2}$	$\infty, 1/4$	$1, -1$	$\frac{t+1}{1-t}$	$1/t$	$-r$
$\Gamma_{8^3.6.3.1^3}$	$1/2, 1/4$	$\infty, -1$	$\frac{t+1}{4}$	$\frac{1-t}{1+t}$	$\frac{1}{2r}$
$\Gamma_{24.3.2^3.1^3}$	$0, 1/4$	$0, -1$	$\frac{2(1+t)}{t}$	$\frac{t+1}{t-1}$	$\frac{2}{r}$
Involutions i of $X(\Gamma_1(6))$, and ι of $X(\Gamma)$ for $\Gamma\subset\Gamma_1(6)$					
subgroup Γ	values of z and t where cover ramifies		$r^3 = m(t)$	involutions of t and r	
				$i : t \mapsto$	$\iota : r \mapsto$
	z	$t(z)$			
$\Gamma_{18.6.3^3.1^3}$	$1/3, 0$	$\infty, 0$	$t/9$	$\frac{1}{9t}$	$\frac{1}{9r}$
$\Gamma_{9.6^3.3.2^3}$	$\infty, 1/2$	$\frac{1}{9}, 1$	$\frac{1-9t}{3(1-t)}$	$\frac{1}{9t}$	$\frac{1}{r}$
$\Gamma_{9.6^4.1^3}$	$1/2, 1/3$	$1, \infty$	$\frac{8}{3(1-t)}$	$\frac{1-9t}{9-9t}$	$\frac{2}{r}$
$\Gamma_{18.3^4.2^3}$	$\infty, 0$	$\frac{1}{9}, 0$	$\frac{1-9t}{24t}$	$\frac{1-9t}{9-9t}$	$\frac{1}{2r}$

TABLE 21. Involutions of modular curves $\Gamma\setminus\mathbb{H}$. For $\Gamma_0(8)\cap\Gamma_1(4)$, $t(z) = \frac{\eta(z)^8\eta(4z)^4}{\eta(2z)^{12}}$, and for $\Gamma_1(6)$, $t(z) = \frac{1}{9}\frac{\eta(6z)^4\eta(z)^8}{\eta(3z)^8\eta(2z)^4}$, as in Tables 7, 13, and 15.

subgroup	$i(t)$	d	polynomial defining kernel of isogeny	$\tilde{\iota}$'s field of definition
Level-8 cases				
$\Gamma_{24.6.1^6}$	$-t$	1	–	$\mathbb{Q}$
$\Gamma_{8^2.2^3.3^2}$	$1/t$	4	$(x+t^2)x$	$\mathbb{Q}$
$\Gamma_{8^3.6.3.1^3}$	$\frac{1-t}{1+t}$	8	$(x^2-4tx-4t^3)(x+t^2)x$	$\mathbb{Q}[\sqrt{-1}]$
$\Gamma_{24.3.2^3.1^3}$	$\frac{t+1}{t-1}$	8	$(x^2+4tx+4t^3)(x+t^2)x$	$\mathbb{Q}[\sqrt{-1}]$
Level-6 cases				
$\Gamma_{18.6.3^3.1^3}, \Gamma_{9.6^4.1^3}$	$\frac{1}{9t}$	3	$x-t^2+t$	$\mathbb{Q}[\sqrt{-3}]$
$\Gamma_{9.6^3.1^3}, \Gamma_{18.3^4.2^3}$	$\frac{1-9t}{9-9t}$	6	$(x-t^2+t)x(x+t)$	$\mathbb{Q}[\sqrt{-3}]$

TABLE 22. Data concerning involutions i and ι of Table 21, lifted to maps $\tilde{\iota}$ of families of curves, defining isogenies of degree d on fibers. In particular, $\Phi_d(j(E_n(i(t))), j(E_n(t))) = 0$, where n is the level and Φ_d is the dth modular polynomial.

for fixed p and varying n, with np less than some fixed bound. If this holds, then we also consider this to be evidence that h_1, h_2 is an ASwD basis. We can draw this conclusion regardless of whether f is known.

In the case $n = 1$, since $a_1 = b_1 = 1$, (7–1) implies that

$$a_p \equiv A_p \bmod p^2 \quad \text{and} \quad b_p \equiv A_p \bmod p^2. \tag{7–2}$$

In order to determine the associated congruence modular form, we test whether (7–2) holds for small primes for the candidate form f. This is what is done in Section 7.3.1.

In some cases, to obtain congruences, f needs to be replaced by $f\otimes\chi$ for some character χ. Then A_p will be replaced by $A_p\chi(p)$ in (7–2), so this phenomenon can be recognized by checking whether A_p/a_p and A_p/b_p are roots of unity. This is done in Section 7.3.2. However, we have not worked out what the character χ is.

7.2 Case 2

In most of our examples, it turns out that the ASwD basis depends on the congruence class of the prime p modulo some small integer. For some primes, (7–1) holds for the values tested, in which case h_1, h_2 is assumed to be the ASwD basis, but for other primes, this does not hold.

If (7–1) does not hold for some prime p, then we will assume that for this prime, an ASwD basis consists of linear combinations of the form $h_1 + \alpha h_2$, where α is an algebraic number of small degree, such that for integers n with $p \nmid n$, the expansion coefficients satisfy

$$a_{pn} + \alpha b_{pn} \equiv A_p(a_n + \alpha b_n) \bmod p^2. \qquad (7\text{–}3)$$

A priori, α depends on p, though we will see that in the examples we are considering, evidence suggests that it depends only on the congruence class of p modulo a small integer.

For (7–3) to hold, it is sufficient, but not necessary, that

$$a_{pn} \equiv A_p \alpha b_n \bmod p^2 \quad \text{and} \quad \alpha b_{pn} \equiv A_p a_n \bmod p^2, \qquad (7\text{–}4)$$

which, assuming that all the terms are nonzero, implies that $a_{pn}/b_n = A_p\alpha_p$ and $b_{pn}/a_n = A_p/\alpha_p$.

So if (7–1) does not hold as n varies, we test whether

$$\frac{a_{np}}{b_n} \equiv \text{constant} \bmod p^2 \quad \text{and} \quad \frac{b_{np}}{a_n} \equiv \text{constant} \bmod p^2. \qquad (7\text{–}5)$$

If this holds, the values of α and A_p modulo p^2, up to sign, are determined by

$$\alpha^2 \equiv \frac{a_{np}}{b_n} \Big/ \frac{b_{np}}{a_n} \bmod p^2 \quad \text{and} \quad A_p^2 \equiv \frac{a_{np}}{b_n}\frac{b_{np}}{a_n} \bmod p^2. \qquad (7\text{–}6)$$

For p for which (7–5) holds, there are two solutions to (7–6) for α, and the ASwD basis has the form $h_1 + \alpha h_2, h_1 - \alpha h_2$.

We expect that α depends only on p modulo some small integer. Since α is expected to be an algebraic integer, but not an integer, it may be difficult to guess the value of α from $\alpha \bmod p^2$. So we also look at powers of $\alpha \bmod p^2$, and if for some small power these are constant as p varies, then we deduce a value of α.

Once α is determined, $A_p \bmod p^2$ is determined, and if this agrees with the coefficients of our congruence modular form, then we take this as evidence that $h_1+\alpha h_2, h_1-\alpha h_2$ is an ASwD basis with f the associated newform. As for Case 1, we will also test whether the A_p must be multiplied by some root of unity, presumably the value $\chi(p)$ for some character χ, though again, we have not determined the character in question.

7.3 Examples Associated with a Newform in $S_3(\Gamma_0(48), \chi)$

For $\Gamma_{24.6.1^6}$ and $\Gamma_{8^3.2^3.3^2}$, evidence suggests that the associated congruence form is as follows, with the first few A_p as in Table 23:

$$\begin{aligned} f(z) &= \frac{\eta(4z)^9\eta(12z)^9}{\eta(2z)^3\eta(6z)^3\eta(8z)^3\eta(24z)^3} \qquad (7\text{–}7)\\ &= q + 3q^3 - 2q^7 + 9q^9 - 22q^{13} - 26q^{19} - 6q^{21} \\ &\quad + 25q^{25} + \cdots. \end{aligned}$$

7.3.1 ASwD Congruences for $\Gamma_{24.6.1^6}$. We have shown previously that $S_3(\Gamma_{24.6.1^6})$ has a basis

$$\begin{aligned} h_1(z) &= \sqrt[3]{\frac{\eta(z)^4\eta(4z)^{20}}{\eta(2z)^6}} = q - \frac{4}{3}q^2 + \frac{8}{9}q^3 - \frac{176}{81}q^4 \\ &\quad - \frac{850}{243}q^5 + \ldots, \qquad (7\text{–}8)\\ h_2(z) &= \sqrt[3]{\frac{\eta(4z)^{16}\eta(2z)^6}{\eta(z)^4}} = q + \frac{4}{3}q^2 + \frac{8}{9}q^3 + \frac{176}{81}q^4 \\ &\quad - \frac{850}{243}q^5 + \cdots. \qquad (7\text{–}9) \end{aligned}$$

The first few prime coefficients of these forms are as given in Table 24.

Since the ratios a_{np}/a_n and b_{np}/b_n, given in Table 25, appear to be constant, and the numbers in Tables 23 and 25 agree modulo p^2, we conclude that the ASwD basis of $S_3(\Gamma_{24.6.1^6})$ is h_1, h_2, as given by (7–8) and (7–9) for all primes, with f in (7–7) being the associated congruence form.

7.3.2 ASwD Congruences for $\Gamma_{8^3.2^3.3^2}$. A basis of $S_3(\Gamma_{8^3.2^3.3^2})$, written in terms of $r = q^{1/3}$ and $s = q^{2/3}$, is

$$\begin{aligned} h_1(z) &= \sqrt[3]{\frac{\eta(2z)^{20}\eta(8z)^4}{\eta(4z)^6}} = \sum_{n\ge1} a_n s^n \\ &= s - \frac{20}{3}s^4 + \frac{128}{9}s^7 - \frac{400}{81}s^{10} + \cdots, \\ h_2(z) &= \sqrt[3]{\frac{\eta(2z)^{16}\eta(4z)^6}{\eta(8z)^4}} = \sum_{n\ge1} b_n r^n \\ &= r - \frac{16}{3}r^7 + \frac{38}{9}r^{13} + \frac{1696}{81}r^{19} + \cdots. \end{aligned}$$

The first few prime coefficients are

p	2	3	5	7	11	13	17	19
a_p	0	0	0	$\frac{128}{9}$	0	$-\frac{3454}{243}$	0	$-\frac{38656}{6561}$
b_p	0	0	0	$-\frac{16}{3}$	0	$\frac{38}{9}$	0	$\frac{1696}{81}$

p	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67
a_p	0	-2	0	-22	0	-26	0	0	46	26	0	22	0	0	0	74	-122

TABLE 23. The first few coefficients A_p for the newform for $S_3(\Gamma_0(48), \chi)$.

p	2	3	5	7	11	13	17	19
a_p	$-\frac{4}{3}$	$\frac{8}{9}$	$-\frac{850}{243}$	$-\frac{5968}{6561}$	$-\frac{35104520}{4782969}$	$\frac{952141694}{129140163}$	$-\frac{206256733102}{31381059609}$	$\frac{60201506159720}{2541865828329}$
b_p	$\frac{4}{3}$	$\frac{8}{9}$	$-\frac{850}{243}$	$-\frac{5968}{6561}$	$-\frac{35104520}{4782969}$	$\frac{952141694}{129140163}$	$-\frac{206256733102}{31381059609}$	$\frac{60201506159720}{2541865828329}$

TABLE 24. First few prime coefficients of basis for $S_3(\Gamma_{24.6.1^6})$.

p	5	7	11	13	17	19	23	29	31	37	41	43	47
$a_{np}/a_n \bmod p^2$	0	47	0	147	0	335	0	0	46	26	0	22	0
$b_{np}/b_n \bmod p^2$	0	47	0	147	0	335	0	0	46	26	0	22	0

TABLE 25. Values of $\frac{a_{np}}{a_n}$ and $\frac{b_{np}}{b_n}$ for primes $p \geq 5$ and integers n, with $pn \leq 500$. These agree modulo p^2 with values in Table 23.

Our computations show that the ratios $\frac{a_{np}}{a_n}$ and $\frac{b_{np}}{b_n}$ remain constant for fixed p, for values of pn up to 500. We can write these ratios in terms of ω, a sixth root of 1 modulo p^2, as in Table 26. In this table we also tabulate ω and the order of ω as an element of $(\mathbb{Z}/p^2\mathbb{Z})^\times$.

Since the values of a_{np}/a_n and b_{np}/b_n are constant over the ranges computed, we conjecture that h_1, h_2 is an ASwD basis for all primes. Comparing these values with the coefficients of f, we conjecture that the associated congruence form is $f \otimes \chi$, where χ is a certain Hecke character.

7.4 Examples Associated with a Newform in $S_3(\Gamma_0(432), \chi)$

For $\Gamma_{8^3.6.3.1^3}$ and $\Gamma_{24.3.2^3.1^3}$, evidence suggests that the associated congruence form is

$$\begin{aligned} f(z) &= q + 6\sqrt{2}q^5 + \sqrt{-3}q^7 + 6\sqrt{-6}q^{11} + 13q^{13} \\ &\quad - 6\sqrt{2}q^{17} + 11\sqrt{-3}q^{19} - 18\sqrt{-6}q^{23} \\ &\quad + 47q^{25} - 24\sqrt{2}q^{29} + \cdots . \end{aligned} \tag{7–10}$$

The first few A_p are given in Table 27, where they are divided by one of 1, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{-6}$ for easy readability.

The form f can be given in terms of eta products and an Eisenstein series as follows:

$$\begin{aligned} f(z) &= f_1(12z) + 6\sqrt{2}f_5(12z) + \sqrt{-3}f_7(12z) \\ &\quad + 6\sqrt{-6}f_{11}(12z), \end{aligned} \tag{7–11}$$

where

$$\begin{aligned} f_1(z) &= \frac{\eta(2z)^3\eta(3z)}{\eta(6z)\eta(z)}E_6(z), \\ f_5(z) &= \frac{\eta(z)\eta(2z)^3\eta(3z)^3}{\eta(6z)}, \\ f_7(z) &= \frac{\eta(6z)^3\eta(z)}{\eta(2z)\eta(3z)}E_6(z), \\ f_{11}(z) &= \frac{\eta(3z)\eta(z)^3\eta(6z)^3}{\eta(2z)}, \end{aligned}$$

where

$$E_6(z) = 1 + 12\sum_{n\geq 1}(\sigma(3n) - 3\sigma(n))q^n$$

and $\sigma(n) = \sum_{d|n} d$.

7.4.1 ASwD Congruences for $\Gamma_{8^3.6.3.1^3}$.

We have seen that a basis of $S_3(\Gamma_{8^3.6.3.1^3})$ can be given by

$$\begin{aligned} h_1(z) &= \sqrt[3]{\frac{\eta(z)^4\eta(2z)^{10}\eta(8z)^8}{\eta(4z)^4}} = \sum_{n\geq 1} a_n q^n \\ &= q - \frac{4}{3}q^2 - \frac{40}{9}q^3 + \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \cdots , \\ h_2(z) &= \sqrt[3]{\frac{\eta(z)^8\eta(4z)^{10}\eta(8z)^4}{\eta(2z)^4}} = \sum_{n\geq 1} b_n q^n \\ &= q - \frac{8}{3}q^2 + \frac{8}{9}q^3 + \frac{32}{81}q^4 - \frac{82}{243}q^5 + \cdots . \end{aligned}$$

The first few prime coefficients of h_1 and h_2 are given in Table 28.

For $p \equiv 1 \bmod 3$, our data suggest that a_{pn}/a_p and b_{pn}/b_n remain constant as n varies, with values as in

p	$\frac{a_{np}}{a_n} \bmod p^2$		$\frac{b_{np}}{b_n} \bmod p^2$		ω	$o(\omega)$
7	36	$= -2\omega$	11	$= -2\omega^{-1}$	31	6
11	0		0			
13	168	$= -22\omega$	23	$= -22\omega^{-1}$	146	3
17	0		0			
19	11	$= -26\omega$	324	$= -26\omega^{-1}$	69	6
23	0		0			
29	0		0			
31	915	$= 46\omega$	915	$= 46\omega^{-1}$	-1	2
37	47	$= 26\omega$	1296	$= 26\omega^{-1}$	581	3
41	0		0			
43	1827	$= 22\omega$	1827	$= 22\omega^{-1}$	-1	2
47	0		0			

TABLE 26. Values of $\frac{a_{np}}{a_n}$ and $\frac{b_{np}}{b_n}$ for $\Gamma_{8^3.2^3.3^2}$, for primes $p \geq 5$ and integers n, with $pn \leq 500$, in terms of a sixth root of unity ω of order $o(\omega)$. Compare with values in Table 23.

p	5	7	11	13	17	19	23	29	31	37	41	43	47
A_p				13						35			
$A_p/\sqrt{2}$	6				-6			-24			0		
$A_p/\sqrt{-3}$		1				11			24			-24	
$A_p/\sqrt{-6}$			6				-18						6

TABLE 27. Coefficients of f in (7–10) and (7–11).

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p	2	3	5	7	11	13	17	19
a_p	$-\frac{4}{3}$	$-\frac{40}{9}$	$\frac{1454}{243}$	$-\frac{13168}{6561}$	$\frac{38671144}{4782969}$	$-\frac{2230795138}{129140163}$	$-\frac{418720079278}{31381059609}$	$\frac{30660416258552}{2541865828329}$
b_p	$-\frac{8}{3}$	$\frac{8}{9}$	$-\frac{82}{243}$	$-\frac{24400}{6561}$	$\frac{16345336}{4782969}$	$\frac{1236747902}{129140163}$	$\frac{842483994194}{31381059609}$	$-\frac{34758650729368}{2541865828329}$

TABLE 28. First few prime coefficients of basis for $S_3(\Gamma_{8^3.6.3.1^3})$.

Table 29. This means we are in Case 1, described in Section 7.1. Experimentally, we noted that for these p we always have $\left(\frac{a_{pn}}{a_p}/\frac{b_{pn}}{b_n}\right)^6 \equiv 1 \bmod p^2$ (excluding the case $p = 13$, when $a_{pn} \equiv b_{pn} \equiv 0 \bmod 13$). We also checked that $\frac{a_{pn}}{a_p} \times \frac{b_{pn}}{b_n} \equiv A_p^2 \bmod p^2$, where the A_p are as in Table 27.

The first observation indicates that these two forms correspond to congruence forms that are twists of each other by a character of order 6, and the second observation indicates that the congruence form is the f given by (7–10). Using these two observations, we write the ratios a_{np}/a_n and b_{np}/b_n in the factored forms in Table 29. The values of ω, a sixth root of 1, and the values used for $\sqrt{3} \bmod p^2$ are also tabulated.

Based on these experiments, we conjecture that the Atkin–Swinnerton-Dyer basis of $S_3(\Gamma_{8^3.6.3.1^3})$ when $p \equiv 1 \bmod 3$ is h_1, h_2, and the associated congruence forms are $f \otimes \chi$ and $f \otimes \chi^{-1}$ for a certain Hecke character.

From the data in Table 30, following the explanation of Section 7.2, the Atkin–Swinnerton-Dyer basis of $S_3(\Gamma_{8^3.6.3.1^3})$ when $p \equiv 1 \bmod 3$ should be h_1, h_2, and when $p \equiv 2 \bmod 3$, it should consist of forms of the form $h_1 + \alpha h_2$ with $\alpha^{12} = 4$.

7.4.2 ASwD Congruences for $\Gamma_{24.3.2^3.1^3}$.

A basis of $S_3(\Gamma_{24.3.2^3.1^3})$ is

$$h_1(z) = \sqrt[3]{\frac{\eta(2z)^{22}\eta(8z)^8}{\eta(z)^4\eta(4z)^8}} = q + \frac{4}{3}q^2 - \frac{40}{9}q^3 - \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \frac{1888}{729}q^6 - \frac{13168}{6561}q^7 + \cdots,$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2z)^{20}\eta(4z)^2\eta(8z)^4}{\eta(z)^8}} = q + \frac{8}{3}q^2 + \frac{8}{9}q^3 - \frac{32}{81}q^4 - \frac{82}{243}q^5 - \frac{5440}{729}q^6 - \frac{24400}{6561}q^7 + \cdots.$$

The first few prime coefficients are given in Table 31. Note that up to sign these are identical to the coefficients of the forms given for the $\Gamma_{8^3.6.3.1^3}$ case, and so the ASwD basis is expected to be the same as in the $\Gamma_{8^3.6.3.1^3}$ case, namely h_1, h_2 when $p \equiv 1 \bmod 3$ and $h_1 + \alpha h_2$ with $\alpha^{12} = 4$ when $p \equiv 2 \bmod 3$. See Tables 38 and 32.

p	$\frac{a_{np}}{a_n}$ mod p^2		$\frac{b_{np}}{b_n}$ mod p^2		$\sqrt{-3}$	ω
7	17	$=\omega^{-4}\sqrt{-3}$	29	$=\omega^{-2}\sqrt{-3}$	37	$\sqrt[4]{-18}$
13	52	$=\omega^{-2}13$	130	$=\omega^{2}13$		$\sqrt{23}$
19	48	$=\omega^{-2}11\sqrt{-3}$	346	$=\omega^{-4}11\sqrt{-3}$	137	$\sqrt{69}$
31	915	$=\omega^{6}24\sqrt{-3}$	46	$=24\sqrt{-3}$	82	$\sqrt[6]{-1}$
37	165	$=\omega^{-4}35$	1169	$=\omega^{4}35$		$\sqrt[4]{581}$
43	11	$=-\omega^{6}24\sqrt{-3}$	1838	$=-24\sqrt{-3}$	1002	$\sqrt[6]{-1}$

TABLE 29. Values of a_{np}/a_n and b_{np}/b_n for $p \equiv 1 \bmod 3$, for h_1 and h_2, for $\Gamma_{8^3.6.3.1^3}$, in terms of A_p in Table 27.

p	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$ mod p^2	$(\frac{a_{np}}{b_n}/\frac{b_{np}}{a_n})^6 \equiv \alpha^{12}$	$\frac{a_{np}}{b_n}\frac{b_{np}}{a_n} \equiv A_p^2$
5	3	1	4	$-2\cdot 6^2$
11	84	32	4	$-6\cdot 6^2$
17	278	243	4	$-2\cdot 6^2$
23	335	130	4	$-6\cdot 18^2$
29	272	441	4	$-2\cdot 24^2$
41	0	0		
47	302	760	4	$-6\cdot 6^2$

TABLE 30. Values of a_{np}/b_n and b_{np}/a_n for $p \equiv 2 \bmod 3$, for h_1 and h_2, for $\Gamma_{8^3.6.3.1^3}$, with α as in (7–6), and A_p as in Table 27.

p	2	3	5	7	11	13	17	19
a_p	$\frac{4}{3}$	$-\frac{40}{9}$	$\frac{1454}{243}$	$-\frac{13168}{6561}$	$\frac{38671144}{4782969}$	$-\frac{2230795138}{129140163}$	$-\frac{418720079278}{31381059609}$	$\frac{30660416258552}{2541865828329}$
b_p	$\frac{8}{3}$	$\frac{8}{9}$	$-\frac{82}{243}$	$-\frac{24400}{6561}$	$\frac{16345336}{4782969}$	$\frac{1236747902}{129140163}$	$\frac{842483994194}{31381059609}$	$-\frac{34758650729368}{2541865828329}$

TABLE 31. First few prime coefficients and basis for $S.3(\Gamma_{24.3.2^3.1^3})$.

p	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$ mod p^2
5	3	1
11	84	32
17	278	243
23	335	130
29	272	441
41	0	0
47	302	760

TABLE 32. Values of a_{np}/b_n and b_{np}/a_n for $p \equiv 2 \bmod 3$, for h_1 and h_2 for $S_3(\Gamma_{24.3.2^3.1^3})$. These values are the same as those in Table 30.

p	2	3	5	7	11	13	17	19	23	29	31
a_p	1	0	$\frac{8}{3}$	0	$\frac{256}{81}$	0	$-\frac{7984}{729}$	0	$-\frac{172544}{19683}$	$\frac{18907736}{1594323}$	0
b_p	0	0	0	$-\frac{16}{9}$	0	$-\frac{1534}{243}$	0	$\frac{78560}{6561}$	0	0	$-\frac{126424784}{4782969}$

TABLE 33. First few prime coefficients and basis for $S.3(\Gamma_{24.3.2^3.1^3})$.

p	a_{np}/a_n		b_{np}/b_n		ω	$\sqrt{-3}$
7	32	$=-\sqrt{-3}\cdot\omega^2$	20	$=\sqrt{-3}\cdot\omega$	18	12
13	52	$=-13\cdot\omega$	130	$=-13\cdot\omega^2$	22	45
19	313	$=11\sqrt{-3}\cdot\omega$	15	$=-11\sqrt{-3}\cdot\omega^2$	68	137
31	46	$=24\sqrt{-3}$	915	$=-24\sqrt{-3}$	439	82
37	165	$=35\cdot\omega^2$	1169	$=35\cdot\omega$	581	
43	1838	$=24\sqrt{-3}$	11	$=-24\sqrt{-3}$	423	847

TABLE 34. Values of a_{np}/a_n and b_{np}/b_n for $p \equiv 1 \bmod 3$, for h_1 and h_2 for $S_3(\Gamma_{24.3.2^3.1^3B})$, written in terms of the coefficients in Table 27.

p	a_{np}/b_n		b_{np}/a_n		i or $\sqrt{3}$	$\sqrt[3]{2}$
5	14	$= 6\sqrt{-2} \cdot \frac{\sqrt{2}}{2\sqrt[3]{2}}$	2	$= 6\sqrt{-2} \cdot \frac{2\sqrt[3]{2}}{\sqrt{2}}$	$i = 7$	3
11	79	$= 6\sqrt{-6} \cdot \frac{\sqrt{-2}}{2\sqrt[3]{2}}$	57	$= 6\sqrt{-6} \cdot \frac{2\sqrt[3]{2}}{\sqrt{-2}}$	$\sqrt{3} = 27$	73
17	139	$= 6\sqrt{-2} \cdot \frac{\sqrt{2}}{2\sqrt[3]{2}}$	197	$= 6\sqrt{-2} \cdot \frac{2\sqrt[3]{2}}{\sqrt{2}}$	$i = 38$	195
23	97	$= -18\sqrt{-6} \cdot \frac{\sqrt{-2}}{2\sqrt[3]{2}}$	269	$= -18\sqrt{-6} \cdot \frac{2\sqrt[3]{2}}{\sqrt{-2}}$	$\sqrt{3} = 223$	384
29	136	$= -24\sqrt{-2} \cdot \frac{\sqrt{2}}{2\sqrt[3]{2}}$	41	$= -24\sqrt{-2} \cdot \frac{2\sqrt[3]{2}}{\sqrt{2}}$	$i = 800$	403
41	0		0			
47	2058	$= 6\sqrt{-6} \cdot \frac{\sqrt{-2}}{2\sqrt[3]{2}}$	689	$6\sqrt{-6} \cdot \frac{2\sqrt[3]{2}}{\sqrt{-2}}$	$\sqrt{3} = 270$	1854

TABLE 35. Values of a_{np}/b_n and b_{np}/a_n for $p \equiv 2 \bmod 3$, for h_1 and h_2 for $S_3(\Gamma_{24.3.2^3.1^3B})$, written in terms of the coefficients in Table 27. Note that the terms $\sqrt{-6} \cdot \frac{\sqrt{-2}}{2\sqrt[3]{2}}$ and $\sqrt{-2} \cdot \frac{\sqrt{2}}{2\sqrt[3]{2}}$ can be rewritten as $2i/\sqrt[3]{2}$ and $2\sqrt{3}/\sqrt[3]{2}$, hence the tabulation of i, $\sqrt{3}$, and $\sqrt[3]{2}$.

p	5	7	11	13	17	19	23	29	31	37
$\tilde{A}_p$	$6i$	11	$12i$	5	$-18i$	-19	$-30i$	$48i$	-13	17

TABLE 36. Coefficients of $f \in S_3(\Gamma_0(243), \chi)$.

p	2	3	5	7	11	13	17	19
a_p	$-\frac{4}{3}$	$-\frac{31}{9}$	$\frac{104}{243}$	$\frac{44018}{6561}$	$-\frac{38654696}{4782969}$	$-\frac{1857609346}{129140163}$	$\frac{362933655200}{31381059609}$	$-\frac{33243449873158}{2541865828329}$
b_p	$\frac{4}{3}$	$-\frac{7}{9}$	$-\frac{616}{243}$	$-\frac{15886}{6561}$	$\frac{43656424}{4782969}$	$-\frac{343807618}{129140163}$	$-\frac{100695940768}{31381059609}$	$\frac{19258418018042}{2541865828329}$

TABLE 37. First few prime coefficients and basis for $S.3(\Gamma_{18.6.3^3.1^3})$.

7.4.3 ASwD Congruences for $\Gamma_{24.3.2^3.1^3B}$. This is a conjugate of the $S_3(\Gamma_{24.3.2^3.1^3})$ example by the involution

$$W_8 = \begin{pmatrix} 0 & -1 \\ 8 & 0 \end{pmatrix}.$$

A basis of $S_3(\Gamma_{24.3.2^3.1^3B})$ in terms of $r = q^{1/3}$ is

$$h_1(z) = \sqrt[3]{\frac{\eta(2z)^{14}\eta(4z)^{16}}{\eta(z)^4\eta(8z)^8}} = \sum_{n\geq 1} a_n r^n = r^1 + \frac{4}{3}r^4 - \frac{16}{9}r^7 - \frac{112}{81}r^{10} - \frac{1534}{243}r^{13} + \cdots,$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2z)^{16}\eta(4z)^{14}}{\eta(z)^8\eta(8z)^4}} = \sum_{n\geq 1} b_n r^n = r^2 + \frac{8}{3}r^5 + \frac{20}{9}r^8 + \frac{256}{81}r^{11} - \frac{64}{243}r^{14} + \cdots.$$

The first few prime coefficients are given in Table 33.

The ratios when terms are nonzero are given in Tables 34 and 35.

An Atkin–Swinnerton-Dyer basis is

$$h_1, h_2 \text{ if } p \equiv 1 \bmod 3,$$

$$h_1 \pm \frac{\sqrt{2}}{2\sqrt[3]{2}}h_2 \text{ if } p \equiv 5 \bmod 12,$$

$$h_1 \pm \frac{\sqrt{-2}}{2\sqrt[3]{2}}h_2 \text{ if } p \equiv 11 \bmod 12.$$

p	$\frac{a_{np}}{a_n}$	$\frac{b_{np}}{b_n}$
7	17	29
13	52	130
19	48	346
31	915	46
37	165	1169
43	11	1838

TABLE 38. Values of a_{np}/a_n and b_{np}/b_n for $p \equiv 1 \bmod 3$, for h_1 and h_2 for $S_3(\Gamma_{24.3.2^3.1^3})$. These values are the same as those in Table 29.

7.5 Examples Associated with a Newform in $S_3(\Gamma_0(243), \chi)$

We consider

$$f(z) = q + 3iq^2 - 5q^4 + 6iq^5 + 11q^7 - 3iq^8 - 18q^{10} + 12iq^{11} + \cdots,$$

where i is a root of $x^2+1=0$. Note that the corresponding Galois representation is a twist of the representation corresponding to $E_6(3r^3)$.

The first few prime coefficients $\tilde{A}_p$ of this form are as in Table 36.

7.5.1 ASwD Congruences for $\Gamma_{18.6.3^3.1^3}$.

A basis of $S_3(\Gamma_{18.6.3^3.1^3})$ is

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(2z)^7\eta(6z)^{11}}{\eta(3z)^4}} = \sum_{n\geq 1} a_n q^n = q - \frac{4}{3}q^2 - \frac{31}{9}q^3 + \frac{400}{81}q^4 + \frac{104}{243}q^5 + \cdots,$$

$$h_2(z) = \sqrt[3]{\frac{\eta(3z)^4\eta(6z)^7\eta(2z)^{11}}{\eta(z)^4}} = \sum_{n\geq 1} b_n q^n = q + \frac{4}{3}q^2 - \frac{7}{9}q^3 - \frac{112}{81}q^4 - \frac{616}{243}q^5 + \cdots,$$

with a_n and b_n the coefficients of the noncongruence forms given above.

The first few prime coefficients are given in Table 37.

The ratios in Table 39, all computed modulo p^2, appear to be constant as n varies, for the given p's. The table shows the constants; if no entry is shown, this means that the ratio is not constant in this case.

Case I: $p \equiv 1 \bmod 3$. These ratios are a special case of a relation of Atkin–Swinnerton-Dyer type, e.g., $a_{7n}/a_n \equiv 36 \bmod 7^2$ can be written as

$$a_{7n} - 36a_n + 7^2 a_{n/p} \equiv 0 \bmod 7^2.$$

So, for $p \equiv 1 \bmod 3$, it looks as though h_1 and h_2 form an Atkin–Swinnerton-Dyer basis.

Note that for p in Table 39 with $p \equiv 1 \bmod p$, except for the case $p = 19$, we have

$$(\frac{a_{np}}{a_n}/\frac{b_{np}}{b_n})^3 \equiv 1 \bmod p^2.$$

It is not surprising that this relation holds, since the ratios ought to be the values of A_p in Table 36, which, as we can see, should always be ω or ω^2 in these cases, including for $p = 19$.

The reason the congruence does not hold for $p = 19$ is that in this case we have $\omega, \omega^2 \equiv 68$ and $292 \bmod 19^2$, and $\alpha_1 = -19\omega, \alpha_2 = -19\omega^2 \equiv 152$ and $228 \bmod 19^2$, so we have only that $\alpha_1/19 \equiv \omega \bmod 19$, $\alpha_2/19 \equiv \omega^2 \bmod 19$, i.e., the ratio satisfies $(\frac{a_{19n}}{a_n}/\frac{b_{19n}}{b_n})^3 \equiv 1 \bmod 19$, which we can check is true.

Case II: $p \equiv 2 \bmod 3$. We observe that when $p \equiv 2 \bmod 3$, we always have $(\frac{a_{np}}{b_n}/\frac{b_{np}}{a_n})^3 \equiv -9 \bmod p^2$.

Suppose that the Atkin–Swinnerton-Dyer basis is $h_1 + \alpha h_2$. Then (writing $\alpha_p = \alpha \bmod p^2$) we would have

$$a_{pn} + \alpha_p b_{pn} \equiv A_p(a_n + \alpha_p b_n) \bmod p^2.$$

And suppose that we in fact have

$$a_{pn} \equiv A_p\alpha_p b_n \bmod p^2 \quad \text{and} \quad \alpha_p b_{pn} \equiv A_p a_n \bmod p^2.$$

Then this implies that $a_{pn}/b_n = A_p\alpha_p$ and $b_{pn}/a_n = A_p/\alpha_p$, so $\alpha_p^2 \equiv \frac{a_{np}}{b_n}/\frac{b_{np}}{a_n}$, so from the above observation we expect $\alpha^6 \equiv -9 \bmod p^2$, i.e., $\alpha \equiv \sqrt[3]{3}i \bmod p^2$, so it seems that for $p \equiv 2 \bmod 3$ we should have an Atkin–Swinnerton-Dyer basis consisting of forms of the form $h_1 + \alpha h_2$, where $\alpha^6 = -9$.

The value of A_p is given by $A_p \equiv \pm\sqrt{\frac{a_{np}}{b_n}\frac{b_{np}}{a_n}} \bmod p^2$, whereas the values for $p \equiv 1 \bmod 3$ are those already in Table 36. From the values in Table 37, we obtain the values of A_p in Table 40, with no particular order given to the two possible values. In this table, we write, e.g., $A_p \equiv 6i \bmod 25$ to mean that $A_p^2 \equiv -36 \bmod 25$, etc., and ω means $\omega^2 + \omega + 1 \equiv 0 \bmod p^2$.

p	$\frac{a_{np}}{a_n}$	$\frac{b_{np}}{b_n}$	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$ mod p^2
5			3	13
7	36	2		
11			13	82
13	54	110		
17			279	148
19	228	152		
23			130	400
29			296	515
31	915	59		
37	1058	294		

TABLE 39. Values of a_{np}/b_n and b_{np}/a_n mod p^2 for h_1 and h_2 for $S_3(\Gamma_{18.6.3^3.1^3})$.

p	5	7	11	13	17	19	23	29	31	37
A_p	$6i$	11ω	$12i$	5ω	$18i$	-19ω	$30i$	$48i$	-13ω	17ω
modp^2	$-6i$	$11\omega^2$	$-12i$	$5\omega^2$	$-18i$	$-19\omega^2$	$-30i$	$-48i$	$-13\omega^2$	$17\omega^2$

TABLE 40. Coefficients of congruence forms associated to $S_3(\Gamma_{18.6.3^3.1^3})$.

7.5.2 ASwD Congruences for $\Gamma_{9.6^3.3.2^3}$. A basis of $S_3(\Gamma_{9.6^3.3.2^3})$ in terms of $r = q^{1/3}$ is

$$\begin{aligned} h_1(z) &= \sqrt[3]{\frac{\eta(z)^7\eta(2z)^4\eta(3z)^{11}}{\eta(6z)^4}} = \sum_{n\geq 1} a_n r^n \\ &= r - \frac{7}{3}r^4 - \frac{19}{9}r^7 + \frac{193}{81}r^{10} + \frac{2306}{243}r^{13} + \cdots, \\ h_2(z) &= \sqrt[3]{\frac{\eta(z)^{11}\eta(3z)^7\eta(6z)^4}{\eta(2z)^4}} = \sum_{n\geq 1} b_n r^n \\ &= r^2 - \frac{11}{3}r^5 + \frac{23}{9}r^8 - \frac{13}{81}r^{11} + \cdots. \end{aligned}$$

The first few prime coefficients are

p	2	3	5	7	11	13	17	19
a_p	0	0	0	$-\frac{19}{9}$	0	$\frac{2306}{243}$	0	$-\frac{151696}{6561}$
b_p	1	0	$-\frac{11}{3}$	0	$-\frac{13}{81}$	0	$-\frac{7130}{729}$	0

Notice that these are either zero or the same as in the $\Gamma_{18.6.3^3.1^3}$ case. Table 41 shows the ratios of coefficients (when all terms are nonzero), all numbers given modulo p^2. When $p \equiv 2 \bmod 3$, there is a unique cube root modulo p^2 of any integer, so the given value of $\sqrt[3]{3}$ is unique. Here i means the square root of -1.

Table 41 indicates that when $p \equiv 1 \bmod 3$, we have $a_{np} - A_p\omega^2 a_n \equiv 0 \bmod p^2$ and $b_{np} - A_p\omega b_n \equiv 0 \bmod p^2$ for certain A_p, indicating that h_1, h_2 is an ASwD basis in this case.

Note that this relation holds only when terms are nonzero, e.g., $b_1 = 0$, so we cannot have $b_p + A_p b_1 \equiv 0 \bmod p$ for any p with $b_p \neq 0$.

For $p \equiv 2 \bmod 3$, Table **??** indicates that we have

$$\begin{aligned} \left(a_{np} + i\sqrt[3]{3}b_{np}\right) + iA_p\left(a_n + i\sqrt[3]{3}b_n\right) &\equiv 0 \bmod p^2, \\ \left(a_{np} - i\sqrt[3]{3}b_{np}\right) - iA_p\left(a_n - i\sqrt[3]{3}b_n\right) &\equiv 0 \bmod p^2, \end{aligned}$$

so $h_1 + i\sqrt[3]{3}h_2$, $h_1 - i\sqrt[3]{3}h_2$ should be the ASwD basis in this case.

7.6 Examples Associated with a Newform in $S_3(\Gamma_0(486), \chi)$

We consider

$$\begin{aligned} f(z) = q &- \sqrt{-2}q^2 - 2q^4 + 3\sqrt{-2}q^5 - 7q^7 + 2\sqrt{-2}q^8 \\ &+ 6q^{10} - 3\sqrt{-2}q^{11} + 5q^{13} + 7\sqrt{-2}q^{14} + 4q^{16} \\ &- 18\sqrt{-2}q^{17} + 17q^{19} - 6\sqrt{-2}q^{20} - 6q^{22} \\ &- 6\sqrt{-2}q^{23} + 7q^{25} - 5\sqrt{-2}q^{26} + 14q^{28} \\ &- 39\sqrt{-2}q^{29} + 59q^{31} - 4\sqrt{-2}q^{32} - 36q^{34} + \cdots. \end{aligned}$$

The first few coefficients A_p appear in Table 42, divided by either 1 or $3\sqrt{-2}$ for easy readability.

p	a_{np}/a_n		b_{np}/b_n		a_{np}/b_n		b_{np}/a_n		ω	ω^2	$\sqrt[3]{3}$
5					3	$= 6i \cdot i\sqrt[3]{3}$	13	$= 6i/i\sqrt[3]{3}$			12
7	36	$= 11 \cdot \omega^2$	2	$= 11 \cdot \omega$					18	30	
11					13	$= 12i \cdot i\sqrt[3]{3}$	82	$= 12i/i\sqrt[3]{3}$			9
13	54	$= 5 \cdot \omega^2$	110	$= 5 \cdot \omega$					22	146	
17					279	$= -18i \cdot i\sqrt[3]{3}$	148	$= -18i/i\sqrt[3]{3}$			160
19	228	$= -19 \cdot \omega^2$	152	$= -19 \cdot \omega$					68	292	
23					130	$= -30i \cdot i\sqrt[3]{3}$	400	$= -30i/i\sqrt[3]{3}$			357
29					296	$= 48i \cdot \sqrt[3]{3}$	515	$= 48i/i\sqrt[3]{3}$			134
31	915	$= -13 \cdot \omega^2$	59	$= -13 \cdot \omega$					439	521	
37	1058	$= 17 \cdot \omega^2$	294	$= 17 \cdot \omega$					581	787	
41					1384	$= -30i \cdot i\sqrt[3]{3}$	869	$= -30i/i\sqrt[3]{3}$			1503
43	1173	$= 29 \cdot \omega^2$	647	$= 29 \cdot \omega$					1425	423	
47					155	$= -24i \cdot i\sqrt[3]{3}$	1906	$= -24i/i\sqrt[3]{3}$			1203

TABLE 41. Values of a_{np}/b_n and b_{np}/a_n mod $p^2$for h_1 and h_2 for $S_3(\Gamma_{9.6^3.3.2^3})$.

p	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67
A_p		-7		5		17			59	-19		47				-4	-46
$\frac{A_p}{3\sqrt{-2}}$	1		-1		-6		-2	-13			13		-19	9	-5		

TABLE 42. Coefficients of $f \in S_3(\Gamma_0(486), \chi)$.

p	2	3	5	7	11	13	17	19
a_p	$-\frac{13}{3}$	$\frac{32}{9}$	$-\frac{3577}{243}$	$\frac{38780}{6561}$	$\frac{97488844}{4782969}$	$-\frac{198000616}{129140163}$	$\frac{1030071452831}{31381059609}$	$-\frac{91038813695632}{2541865828329}$
b_p	$-\frac{14}{3}$	$\frac{56}{9}$	$\frac{266}{243}$	$-\frac{1036}{6561}$	$\frac{24235144}{4782969}$	$-\frac{2216727472}{129140163}$	$-\frac{894269035558}{31381059609}$	$\frac{97467805305080}{2541865828329}$

TABLE 43. First few prime coefficients of basis for $S_3(\Gamma_{9.6^4.1^3})$.

p	$\frac{a_{np}}{a_n}$	$\frac{b_{np}}{b_n}$	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$ mod p^2	$(\frac{a_{np}}{a_n}/\frac{b_{np}}{b_n})^3$	$\frac{a_{np}}{a_n}\frac{b_{np}}{b_n}$
7	35	21			1	0
13	54	110			1	5^2
19	271	73			1	17^2
31	948	915			1	59^2
37	106	1282			1	19^2
43	1391	411			1	47^2

TABLE 44. Ratios of coefficients modulo p^2 for h_1 and h_2 for $S_3(\Gamma_{9.6^4.1^3})$.

p	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$ mod p^2	$\frac{a_{np}}{b_n}/\frac{b_{np}}{a_n}$	$\frac{a_{np}}{b_n}\frac{b_{np}}{a_n}$
5	11	12	3	-18
11	94	41	20	-18
17	10	282	205	$-18 \cdot 6^2$
23	503	369	20	$-18 \cdot 2^2$
29	661	101	273	$-18 \cdot 13^2$
41	1463	1587	968	$-18 \cdot 13^2$
47	2117	887	2052	$-18 \cdot 19^2$

TABLE 45. Ratios of coefficients modulo p^2 for h_1 and h_2 for $S_3(\Gamma_{9.6^4.1^3})$.

p	a_{np}/a_n		b_{np}/b_n		a_{np}/b_n		b_{np}/a_n		ω	ω^2	$\sqrt[3]{3}$
5					3	$= -1 \cdot 6\sqrt[3]{3}$	19	$= 1 \cdot 3/\sqrt[3]{3}$			12
7	35	$= -7 \cdot \omega^2$	21	$= -7 \cdot \omega$					18	30	
11					54	$= 1 \cdot 6\sqrt[3]{3}$	40	$= -1 \cdot 3/\sqrt[3]{3}$			9
13	54	$= 5 \cdot \omega^2$	110	$= 5 \cdot \omega$					22	146	
17					269	$= 6 \cdot 6\sqrt[3]{3}$	148	$= -6 \cdot 3/\sqrt[3]{3}$			160
19	271	$= 17 \cdot \omega^2$	73	$= 17 \cdot \omega$					68	292	
23					52	$= 2 \cdot 6\sqrt[3]{3}$	80	$= -2 \cdot 3/\sqrt[3]{3}$			357
29					360	$= 13 \cdot 6\sqrt[3]{3}$	370	$= -13 \cdot 3/\sqrt[3]{3}$			134
31	948	$= 59 \cdot \omega^2$	915	$= 59 \cdot \omega$					439	521	
37	106	$= -19 \cdot \omega^2$	1282	$= -19 \cdot \omega$					581	787	
41					436	$= -13 \cdot 6\sqrt[3]{3}$	47	$= 13 \cdot 3/\sqrt[3]{3}$			1503
43	1391	$= 47 \cdot \omega^2$	411	$= 47 \cdot \omega$					1425	423	
47					184	$= 19 \cdot 6\sqrt[3]{3}$	661	$= -19 \cdot 3/\sqrt[3]{3}$			1203

TABLE 46. Ratios of coefficients mod p^2 for h_1 and h_2 for $S_3(\Gamma_{18.3^4.2^3})$.

7.6.1 ASwD Congruences for $\Gamma_{9.6^4.1^3}$.

A basis of $S_3(\Gamma_{9.6^4.1^3})$ is

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^{13}\eta(6z)^{14}}{\eta(2z)^2\eta(3z)^7}} = \sum_{n\geq 1} a_n q^n$$
$$= q - \frac{13}{3}q^2 + \frac{32}{9}q^3 + \frac{670}{81}q^4 - \frac{3577}{243}q^5 + \cdots,$$
$$h_2(z) = \sqrt[3]{\frac{\eta(z)^{14}\eta(6z)^{13}}{\eta(2z)^7\eta(3z)^2}} = \sum_{n\geq 1} b_n q^n$$
$$= q - \frac{14}{3}q^2 + \frac{56}{9}q^3 - \frac{58}{81}q^4 + \frac{266}{243}q^5 + \cdots.$$

The first few prime coefficients are given in Table 43.

Tables 44 and 45 show the ratios of coefficients of h_1 and h_2 modulo p^2. From this table we see that the ASwD basis has the following form (for $p \geq 5$):

if $p \equiv 1 \bmod 3$ then ASwD basis is h_1, h_2,
if $p \equiv 2 \bmod 3$ then ASwD basis is $h_1 \pm \alpha_p h_2$.

However, although the value of $\alpha_p \equiv a_{np}a_n/b_nb_{np}$ remains constant as p is fixed and n varies, we could not find a pattern allowing us to be more specific about the value of α_p for $p \equiv 2 \bmod 3$.

7.6.2 ASwD Congruences for $\Gamma_{18.3^4.2^3}$.

A basis of $S_3(\Gamma_{18.3^4.2^3})$ in terms of $r = q^{1/3}$ is

$$h_1(z) = \sqrt[3]{\frac{\eta(2z)^{13}\eta(3z)^{14}}{\eta(6z)^7\eta(z)^2}} = \sum_{n\geq 1} a_n r^n$$
$$= r + \frac{2}{3}r^4 - \frac{28}{9}r^7 - \frac{482}{81}r^{10} - \frac{736}{243}r^{13} + \cdots,$$
$$h_2(z) = \sqrt[3]{\frac{\eta(2z)^{14}\eta(3z)^{13}}{\eta(6z)^2\eta(z)^7}} = \sum_{n\geq 1} b_n q^n$$
$$= r^2 + \frac{7}{3}r^5 + \frac{14}{9}r^8 - \frac{148}{81}r^{11} - \frac{1708}{243}r^{14} + \cdots.$$

The first few prime coefficients are

p	2	3	5	7	11	13	17	19
a_p	0	0	0	$-\frac{28}{9}$	0	$-\frac{736}{243}$	0	$\frac{120680}{6561}$
b_p	1	0	$\frac{7}{3}$	0	$-\frac{148}{81}$	0	$-\frac{4529}{729}$	0

From Table 46 we can see that when $p \equiv 1 \bmod 3$, the ASwD basis should be h_1, h_2.

For $p \equiv 2 \bmod 3$, the congruences (which hold only when all terms are nonzero)

$$a_{np}/b_p \equiv -\alpha_p \cdot 6\sqrt[3]{3} \quad \text{and} \quad b_{np}/a_p \equiv \alpha_p \cdot 3/\sqrt[3]{3}$$

should be rewritten in terms of u, where $u^2 = -2$, writing $-6 = 3u \cdot u$, so we have

$$a_{np}/b_p \equiv \alpha_p 3u \cdot u\sqrt[3]{3} \quad \text{and} \quad b_{np}/a_p \equiv \alpha_p 3u/u\sqrt[3]{3}.$$

These imply that

$$a_{np} \equiv \alpha_p 3u \cdot u\sqrt[3]{3} b_p$$

and

$$u\sqrt[3]{3}b_{np} \equiv \alpha_p 3ua_p,$$

so

$$a_{np} + u\sqrt[3]{3}b_{np} \equiv \alpha_p 3u \cdot \left(u\sqrt[3]{3}b_p + a_p\right),$$

which holds for u replaced with $-u$, so the ASwD basis should be $h_1 \pm \sqrt{-2}\sqrt[3]{3}h_2$.

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