Rationality of Moduli Spaces of Plane Curves of Small Degree

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CONTENTS

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We prove that the moduli space $C(d)$ of plane curves of degree d (with respect to projective equivalence) is rational except possibly if $d = 6, 7, 8, 11, 12, 14, 15, 16, 18, 20, 23, 24, 26, 32, 48$.

1. INTRODUCTION

Let $C(d) := \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ be the moduli space of plane curves of degree d. As a particular instance of the general question of rationality for invariant function fields under actions of connected linear algebraic groups (see [Dolgachev 87] for a survey), one can ask whether $C(d)$ is always a rational space. The main results obtained in this direction in the past can be summarized as follows:

- $C(d)$ is rational for $d \equiv 0 \pmod 3$ and $d \geq 210$ [Katsylo 89].
- $C(d)$ is rational for $d \equiv 1 \pmod 3$, $d \geq 37$, and for $d \equiv 2 \pmod 3$, $d \geq 65$ [Böhning and Bothmer 08a].
- $C(d)$ is rational for $d \equiv 1 \pmod 4$ [Shepherd-Barron 88].

Apart from these general results, rationality of $C(d)$ was known for some sporadic smaller values of d, for which the problem, however, can be very hard (see, for example, [Katsylo 92b, Katsylo 96]).

In this paper, using methods of computer algebra, we improve these results substantially so that only 15 values of d remain for which rationality of $C(d)$ is open. This is the content of our main result, Theorem 4.1.

In Section 2 we discuss the algorithms used to improve the result that $C(d)$ is rational for $d \equiv 0 \pmod 3$ and $d \geq 210$ (see above) to the degree that $C(d)$ is rational for $d \equiv 0 \pmod 3$ and $d \geq 30$ with the possible exception of $d = 48$. This is the hardest part computationally. We use the *double bundle method* of [Bogomolov and Katsylo 85]

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and an algorithm to find matrix representatives for certain $\mathrm{SL}_3(\mathbb{C})$-equivariant bilinear maps

$$\psi : V \times U \to W$$

(V, U, W being $\mathrm{SL}_3(\mathbb{C})$-representations) in a fast and algorithmically efficient way. It is described in Section 2, and ultimately based on writing a homogeneous polynomial as a sum of powers of linear forms. An immense speedup of our software was achieved with the help of the FFPACK library [Dumas et al. 08] for linear algebra over finite fields.

In Section 3 we describe the methods and algorithms to improve the degree bounds for $d \equiv 1 \pmod 3$ and $d \equiv 2 \pmod 3$ mentioned above: we obtain rationality of $C(d)$ for $d \equiv 1 \pmod 3$ and $d \geq 19$ (for $d \equiv 1$ (mod 9), $d \geq 19$, rationality had been proven in [Shepherd-Barron 88]), and for $d \equiv 2 \pmod 3$, $d \geq 35$. This uses techniques introduced in [Böhning and Bothmer 08a] and is ultimately based on the *method of covariants*, which appeared for the first time in [Shepherd-Barron 88], as well as on writing a homogeneous polynomial as a sum of powers of linear forms and interpolation.

In Section 4 we summarize these results, and combine them with the known results for $C(d)$ for smaller d and with the proofs of rationality for $C(10)$ and $C(27)$ (the method to prove rationality for $C(10)$ was suggested in [Bogomolov and Katsylo 85]).

2. THE DOUBLE BUNDLE METHOD: ALGORITHMS

In this section we give a brief account of the double bundle method, and then describe the algorithms pertaining to it that we use in our applications. The main technical point is the "no-name lemma":

Lemma 2.1. *Let G be a linear algebraic group with an almost free action on a variety X. Let $\pi : \mathcal{E} \to X$ be a G-vector bundle of rank r on X. Then one has the following commutative diagram of G-varieties*

$$\begin{tikzcd} \mathcal{E} \arrow[r, dashed, "f"] \arrow[dr, "\pi"'] & X \times \mathbb{A}^r \arrow[d, "\mathrm{pr}_1"] \\ & X \end{tikzcd}$$

where G acts trivially on $\mathbb{A}^r$, pr_1 is the projection onto X, and the rational map f is birational.

If X embeds G-equivariantly in $\mathbb{P}(V)$, V a G-module, G is reductive, and X contains stable points of $\mathbb{P}(V)$, then this is an immediate application of descent theory and the fact that a vector bundle in the étale topology is a vector bundle in the Zariski topology. The result appears in [Bogomolov and Katsylo 85]. A proof without the previous technical restrictions is given in [Chernousov et al. 06, Section 4.3]. The following result [Bogomolov and Katsylo 85, Katsylo 89] is the form in which Lemma 2.1 is most often applied, since it allows one to extend its scope to irreducible representations.

Theorem 2.2. *Let G be a linear algebraic group, and let U, V and W, K be (finite-dimensional) G-representations. Assume that the stabilizer in general position of G in U, V, and K is equal to one and the same subgroup H in G, which is also assumed to equal the ineffectiveness kernel in these representations (so that the action of G/H on U, V, K is almost free). The relations* $\dim U - \dim W = 1$ *and* $\dim V - \dim U > \dim K$ *are required to hold.*

Suppose, moreover, that there are a G-equivariant bilinear map

$$\psi : V \times U \to W$$

and a point $(x_0, y_0) \in V \times U$ with $\psi(x_0, y_0) = 0$ and $\psi(x_0, U) = W$, $\psi(V, y_0) = W$. Then if K/G is rational, the same holds for $\mathbb{P}(V)/G$.

Proof: We set $\Gamma := G/H$ and let pr_U and pr_V be the projections of $V \times U$ to U and V. By the genericity assumption on ψ, there is a unique irreducible component X of $\psi^{-1}(0)$ passing through (x_0, y_0), and there are nonempty open Γ-invariant sets $V_0 \subset V$ and $U_0 \subset U$ on which Γ acts with trivial stabilizer, and the respective fibers $X \cap \mathrm{pr}_V^{-1}(v)$ and $X \cap \mathrm{pr}_U^{-1}(u)$ have the expected dimensions $\dim U - \dim W = 1$ and $\dim V - \dim W$. Thus

$$\mathrm{pr}_V^{-1}(V_0) \cap X \to V_0, \quad \mathrm{pr}_U^{-1}(U_0) \cap X \to U_0,$$

are Γ-equivariant bundles, and by Lemma 2.1 one obtains vector bundles

$$(\mathrm{pr}_V^{-1}(V_0) \cap X)/\Gamma \to V_0/\Gamma, \quad (\mathrm{pr}_U^{-1}(U_0) \cap X)/\Gamma \to U_0/\Gamma$$

of rank 1 and $\dim V - \dim W$, and there is still a homothetic $T := \mathbb{C}^* \times \mathbb{C}^*$-action on these bundles. By a well-known theorem of Rosenlicht [Rosenlicht 56], the action of the torus T on the respective base spaces of these bundles has a section over which the bundles are trivial; thus we get

$$\begin{aligned} \mathbb{P}(V)/\Gamma &\sim (\mathbb{P}(U)/\Gamma) \times \mathbb{P}^{\dim V - \dim W - 1} \\ &= (\mathbb{P}(U)/\Gamma) \times \mathbb{P}^{\dim V - \dim U}. \end{aligned}$$

On the other hand, one may view $U \oplus K$ as a Γ-vector bundle over both U and K; hence, again by Lemma 2.1,

$$U/\Gamma \times \mathbb{P}^{\dim K} \sim K/\Gamma \times \mathbb{P}^{\dim U}.$$

Since U/Γ is certainly stably rationally equivalent to $\mathbb{P}(U)/\Gamma$ of level at most one, the inequality $\dim V - \dim U > \dim K$ ensures that $\mathbb{P}(V)/\Gamma$ is rational as K/Γ is rational. □

In [Katsylo 89] this is used to prove the rationality of the moduli spaces $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ of plane curves of degree $d \equiv 0 \pmod 3$ and $d \geq 210$. A clever inductive procedure is used there to reduce the genericity requirement for the occurring bilinear maps ψ to a purely numerical condition on the labels of highest weights of irreducible summands in V, U, W. This method is applicable only if d is large. We will obtain rather comprehensive results for $d \equiv 0 \pmod 3$ and d smaller than 210 by explicit computer calculations.

In the following we put $G := \mathrm{SL}_3(\mathbb{C})$ and denote as usual by $V(a, b)$ the irreducible G-module whose highest weight has numerical labels a, b with respect to the fundamental weights ω_1, ω_2 determined by the choice of the torus T of diagonal matrices and the Borel subgroup B of upper triangular matrices. In addition, we set

$$S^a := \mathrm{Sym}^a(\mathbb{C}^3), \quad D^b := \mathrm{Sym}^b(\mathbb{C}^3)^\vee$$

and introduce dual bases e_1, e_2, e_3 in $\mathbb{C}^3$ and x_1, x_2, x_3 in $(\mathbb{C}^3)^\vee$. Recall that $V(a, b)$ is the kernel of the G-equivariant operator

$$\Delta : S^a \otimes D^b \to S^{a-1} \otimes D^{b-1}, \quad \Delta = \sum_{i=1}^{3} \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i}$$

(we will always view $V(a, b)$ realized in this way in the following), and there is also the G-equivariant operator

$$\delta : S^{a-1} \otimes D^{b-1} \to S^a \otimes D^b, \quad \delta = \sum_{i=1}^{3} e_i \otimes x_i.$$

In particular,

$$S^a \otimes D^b = \bigoplus_{i=0}^{\min(a,\, b)} V(a-i,\, b-i)$$

as G-modules.

In the vast majority of cases in which we apply Theorem 2.2 we will have

$$U := V(e,\, 0), \quad V := V(0,\, f),$$
$$W := V(e-i_1,\, f-i_1) \oplus \cdots \oplus V(e-i_m,\, f-i_m)$$

for some nonnegative integers e and f and integers $0 \leq i_1 < i_2 < \cdots < i_m \leq M := \min(e,\, f)$. We need a fast method to compute the G-equivariant map

$$\psi : U \otimes V \to W. \tag{2–1}$$

Remark 2.3. If we know how to compute the map ψ in formula (2–1), in the sense, say, that upon choosing bases $u_1, \ldots, u_r$ in U, $v_1, \ldots, v_s$ in V, $w_1, \ldots, w_t$ in W, we know the t matrices

$$M^1, \ldots, M^t$$

of size $r \times s$, given by

$$(M^k)_{ij} := (w_k)^\vee(\psi(u_i,\, v_j)),$$

then the map

$$\tilde{\psi} : W^\vee \otimes V \to U^\vee,$$
$$\tilde{\psi}(l_W,\, v)(u) = l_W(\psi(u,\, v)) \;\; l_W \in W^\vee, \quad v \in V,\, u \in U,$$

induced by ψ has a similar representation by r matrices

$$N^1, \ldots, N^r$$

of size $t \times s$ in terms of the bases $w_1^\vee, \ldots, w_t^\vee$ of $W^\vee$, $v_1, \ldots, v_s$ of V, and $u_1^\vee, \ldots, u_r^\vee$ of $U^\vee$. In fact,

$$(N^i)_{kj} = (\tilde{\psi}(w_k^\vee,\, v_j))(u_i) = w_k^\vee(\psi(u_i,\, v_j)) = (M^k)_{ij}.$$

The map $\tilde{\psi}$ is occasionally convenient to use instead of ψ.

We now describe how we compute ψ by writing elements of $U \otimes V$ as sums of pure tensor products of powers of linear forms. We start by proving some helpful formulas:

Lemma 2.4. *Let $u \in \mathbb{C}^3$ and $v \in (\mathbb{C}^3)^\vee$. Then*

1. $\Delta(u^e \otimes v^f) = efv(u)u^{e-1} \otimes v^{f-1}$;
2. $\Delta^i(u^e \otimes v^f) = \frac{e!}{(e-i)!}\frac{f!}{(f-i)!}v(u)^i u^{e-i} \otimes v^{f-i}$.

Proof: We can assume that $v(u) \neq 0$, for otherwise, $\Delta(u^e \otimes v^f) = 0$. We put

$$u_1 := \frac{u}{v(u)},$$

so that $v(u_1) = 1$, and complete $v_1 := v$ and u_1 to dual bases $u_1,\ u_2,\ u_3$ in $\mathbb{C}^3$ and $v_1,\ v_2,\ v_3$ in $(\mathbb{C}^3)^\vee$. Then

$$\begin{aligned}
&\Delta(u^e \otimes v^f)\\
&= \left(\frac{\partial}{\partial u_1} \otimes \frac{\partial}{\partial v_1} + \frac{\partial}{\partial u_2} \otimes \frac{\partial}{\partial v_2} + \frac{\partial}{\partial u_3} \otimes \frac{\partial}{\partial v_3}\right)(u^e \otimes v^f)\\
&= f\frac{\partial}{\partial u_1}\left((v(u)u_1)^e\right) \otimes v^{f-1}\\
&= fe(v(u))^e u_1^{e-1} \otimes v^{f-1}\\
&= efv(u)u^{e-1} \otimes v^{f-1}.
\end{aligned}$$

This gives the first formula. Iterating it gives the second one. $\square$

Lemma 2.5. *Let $\pi_{e,\,f,\,i}$ be the equivariant projection*

$$\pi_{e,\,f,\,i} : S^e \otimes D^f \to V(e-i,\ f-i) \subset S^e \otimes D^f.$$

Then one has

$$\pi_{e,\,f,i} = \sum_{j=0}^{\min(e,f)} \mu_{i,j}\delta^j\Delta^j$$

for certain $\mu_{i,j} \in \mathbb{Q}$.

Proof: Set $\pi_{e,f} := \pi_{e,f,0}$ and look at the following diagram:

$$\begin{tikzcd}
S^e \otimes D^f \arrow[r, "\Delta^i"] & S^{e-i} \otimes D^{f-i} \arrow[d, "\pi_{e-i,\,f-i}"] \\
& V(e-i,\ f-i) \subset S^{e-i} \otimes D^{f-i} \arrow[ul, "\delta^i"]
\end{tikzcd}$$

By Schur's lemma,

$$\pi_{e,\,f,\,i} = \lambda_i \delta^i \pi_{e-i,\,f-i}\Delta^i \tag{2–2}$$

for some nonzero constants λ_i. On the other hand,

$$\pi_{e,\,f} = \mathrm{id} - \sum_{i=1}^{\min(e,\,f)} \pi_{e,\,f,\,i}\,.$$

Therefore, since the assertion of the lemma holds trivially if one of e and f is zero, the general case follows by induction on i. $\square$

Note that to compute the $\mu_{i,j}$ in the expression of $\pi_{e,\,f,i}$ in Lemma 2.5, it suffices to calculate the λ_i in formula (2–2), which can be done by the rule

$$\frac{1}{\lambda_i}\left(e_1^{e-i} \otimes x_3^{f-i}\right) = \left(\pi_{e-i,\,f-i} \circ \Delta^i \circ \delta^i\right)\left(e_1^{e-i} \otimes x_3^{f-i}\right).$$

Notice that applying $\delta^i \circ \Delta^i$ to a decomposable element can still yield a bihomogeneous polynomial with very many terms. A final improvement in the complexity of calculating ψ is obtained by representing these bihomogeneous polynomials not by a sum of monomials but rather by their value on many points of $\mathbb{C}^3 \times (\mathbb{C}^3)^\vee$. Indeed, such values can be calculated easily:

Lemma 2.6. *Let $a, b \geq 0$ be integers, $u \in \mathbb{C}^3$, $v \in (\mathbb{C}^3)^\vee$, $p \in (\mathbb{C}^3)^\vee$, and $q \in \mathbb{C}^3$. Then*

$$\begin{aligned}
&\left(\delta^i \circ \Delta^i(u^a \otimes v^b)\right)(p,q)\\
&\quad = \frac{a!}{(a-i)!}\frac{b!}{(b-i)!}(\delta(p,q))^i v(u)^i u(p)^{a-i} v(q)^{b-i}.
\end{aligned}$$

Proof: By Lemma 2.4 we have

$$\begin{aligned}
&\delta^i \circ \Delta^i(u^a \otimes v^b)(p,q)\\
&\quad = \left(\delta^i(v(u)^i \frac{a!}{(a-i)!}\frac{b!}{(b-i)!} u^{a-i} \otimes v^{b-i})\right)(p,q).
\end{aligned}$$

Evaluation gives the above formula. $\square$

Corollary 2.7. *Let $\psi\colon V \otimes U \to W$ be as above and assume $e \leq f$. Then there exists a homogeneous polynomial $\chi \in \mathbb{Q}[x,y]$ of degree e such that*

$$\psi(u^e \otimes v^f)(p,\ q) = v(q)^{f-e}\chi\big(\delta(p,q)v(u), u(p)v(q)\big)$$

holds for all $u \in \mathbb{C}^3$, $v \in (\mathbb{C}^3)^\vee$, $p \in (\mathbb{C}^3)^\vee$, and $q \in \mathbb{C}^3$.

Proof: We have

$$\psi = (\pi_{e,f,i_1} + \cdots + \pi_{e,f,i_m}).$$

Using that

$$\pi_{e,f,i} = \sum_{j=0}^{e} \lambda_{i,j}\delta^j\Delta^j$$

for certain $\lambda_{i,j}$, we obtain

$$\begin{aligned}
&\psi(u^e \otimes v^f)(p,q)\\
&\quad = \left(\sum_{\alpha=1}^{m}\sum_{j=0}^{e} \lambda_{i_\alpha,j}\delta^j\Delta^j(u^e \otimes v^f)\right)(p,q)\\
&\quad = \sum_{\alpha=1}^{m}\sum_{j=0}^{e} \lambda_{i_\alpha,j}(\delta(p,q))^j v(u)^j\\
&\qquad\qquad \times \frac{e!}{(e-j)!}\frac{f!}{(f-j)!}u(p)^{e-j}v(q)^{f-j}
\end{aligned}$$

$$= v(q)^{f-e} \sum_{\alpha=1}^{m} \sum_{j=0}^{e} \lambda_{i_\alpha,j} \frac{e!}{(e-j)!} \frac{f!}{(f-j)!} \big(\delta(p,q)v(u)\big)^j \big(u(p)v(q)\big)^{e-j}$$
$$= v(q)^{f-e} \chi\big(\delta(p,q)v(u), u(p)v(q)\big).$$

□

Now we are in a position to check the important genericity conditions of Theorem 2.2 efficiently:

Proposition 2.8. *Let n be a positive integer, $u_i \in \mathbb{C}^3$, $v_i \in (\mathbb{C}^3)^\vee$, $p_i \in (\mathbb{C}^3)^\vee$, and $q_i \in \mathbb{C}^3$ for $0 \le i \le n$. Set $x_0 = \sum_{i=0}^n \xi_i u_i^e$ and consider the $n \times n$ matrix M with entries*

$$M_{j,k} = \sum_{i=0}^{n} \xi_i \psi(u_i^e \otimes v_j^f)(p_k, q_k).$$

If $\operatorname{rank} M = \dim W$*, then $\psi(x_0, V) = W$. Similarly, if $y_0 = \sum_{j=0}^n \eta_j v_j^f$ and N is the $n \times n$ matrix with entries*

$$N_{i,k} = \sum_{j=0}^{n} \eta_j \psi(u_i^e \otimes v_j^f)(p_k, q_k),$$

then $\operatorname{rank} N = \dim W$ *implies $\psi(U, y_0) = W$.*

Proof: Since ψ is bilinear, it follows that $\psi(x_0, v_j^f) = \sum \xi_i \psi(u_i^e, v_j^f)$. Therefore the jth row of M contains the values of $\psi(x_0, v_j^f)$ at the points (p_k, q_k) for all k. Therefore $\operatorname{rank} M \le \dim \psi(x_0, V) \le \dim W$. If $\operatorname{rank} M = \dim W$, the claim follows. The second claim follows similarly. □

Remark 2.9. Notice the following:

1. The rank condition of Proposition 2.8 can also be checked over a finite field.
2. Over a finite field all possible values of the polynomial χ can be precomputed and stored in a table.
3. Since $\psi(u^e \otimes v^f)(p,q)$ can be evaluated quickly using Corollary 2.7, we do not have to store the n^3 values of this expression used in Proposition 2.8. It is enough to store the $2n^2$ entries of M and N. This is fortunate, since n must be at least 20,000 for $d = 217$, and in this case $n^3 = 8 \times 10^{12}$ values would consume about 8 GB of memory.
4. Given the polynomial χ, the formula of Corollary 2.7 becomes so simple that it can easily be implemented in C++. See, for example, our program `nxnxn` at [Böhning et al. 08].
5. Calculating the rank of a 20,000 × 20,000 matrix is still difficult and takes several weeks on current computers if implemented naively. Using FFPACK we could take advantage of a multicore system and of optimized linear-algebra algorithms. For example, the case $d = 210$ (the largest we computed) required 23.8 hours run time on a cluster node with two Quad-Core Xeon-E5472 CPUs.
6. The algorithm presented here is related to the one presented in [Böhning and Bothmer 08c] with the substantial improvements that the elements of U and V are represented as sums of powers of linear forms and that the elements of W are represented by their values. This eliminates the need to calculate with big bihomogeneous polynomials.

3. THE METHOD OF COVARIANTS: ALGORITHMS

Virtually all the methods for addressing the rationality problem are based on introducing some fibration structure over a stably rational base in the space for which one wants to prove rationality; with the double bundle method, the fibers are linear, but it turns out that fibrations with nonlinear fibers can also be useful if rationality of the generic fiber of the fibration over the function field of the base can be proven. The *method of covariants* (see [Shepherd-Barron 88]) accomplishes this by inner linear projection of the generic fiber from a very singular center.

Definition 3.1. If V and W are G-modules for a linear algebraic group G, then a *covariant* φ of degree d from V with values in W is a G-equivariant polynomial map of degree d:

$$\varphi : V \to W.$$

In other words, φ is an element of $\operatorname{Sym}^d(V^\vee) \otimes W$.

The method of covariants phrased in a way that we find useful is contained in the following theorem.

Theorem 3.2. *Let G be a connected linear algebraic group whose semisimple part is a direct product of groups of type* SL *or* Sp*. Let V and W be G-modules, and suppose that the action of G on W is generically free. Let Z be the ineffectivity kernel of the action of G on W, and assume that the action of $\bar{G} := G/Z$ is generically free on $\mathbb{P}(W)$ and that Z acts trivially on $\mathbb{P}(V)$.*

Let

$$\varphi : V \to W$$

be a (nonzero) covariant of degree d. Suppose the following assumptions hold:

(a) *$\mathbb{P}(W)/G$ is stably rational of level $\leq \dim \mathbb{P}(V) - \dim \mathbb{P}(W)$.*

(b) *If we view φ as a map $\varphi : \mathbb{P}(V) \dashrightarrow \mathbb{P}(W)$ and denote by B the base* scheme *of φ, then there is a linear subspace $L \subset V$ such that $\mathbb{P}(L)$ is contained in B together with its full infinitesimal neighborhood of order $(d-2)$, i.e.,*

$$\mathcal{I}_B \subset \mathcal{I}^{d-1}_{\mathbb{P}(L)}.$$

Denote by π_L the projection $\pi_L : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V/L)$ away from $\mathbb{P}(L)$ to $\mathbb{P}(V/L)$.

(c) *Consider the diagram*

$$\begin{tikzcd}
\mathbb{P}(V) \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\pi_L"] & \mathbb{P}(W) \\
\mathbb{P}(V/L) &
\end{tikzcd}$$

and assume that one can find a point $[\bar{p}] \in \mathbb{P}(V/L)$ such that

$$\varphi|_{\mathbb{P}(L+\mathbb{C}p)} : \mathbb{P}(L + \mathbb{C}p) \dashrightarrow \mathbb{P}(W)$$

is dominant.

Then $\mathbb{P}(V)/G$ is rational.

Proof: By assumption, the group G is *special* (cf. [Serre 58]), and thus $W \dashrightarrow W/G$, which is generically a principal G-bundle in the étale topology, is a principal bundle in the Zariski topology. Combining this with Rosenlicht's theorem on torus sections [Rosenlicht 56], we get that the projection $\mathbb{P}(W) \dashrightarrow \mathbb{P}(W)/G$ has a rational section σ. Observe that property (c) implies that the generic fiber of π_L maps dominantly to $\mathbb{P}(W)$ under φ, which means that the generic fiber of φ maps dominantly to $\mathbb{P}(V/L)$ under π_L, too. Note also that the map φ becomes linear on a fiber $\mathbb{P}(L + \mathbb{C}g)$ because of property (b) and that thus the generic fiber of φ is birationally a vector bundle via π_L over the base $\mathbb{P}(V/L)$. Thus, if we introduce the graph

$$\Gamma = \overline{\{([q], [\bar{q}], [f]) \mid \pi_L([q]) = [\bar{q}], \varphi([q]) = [f]\}} \subset \mathbb{P}(V) \times \mathbb{P}\big(V/L\big) \times \mathbb{P}(W)$$

and look at the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, dashed, "1:1", "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbb{P}(V) \arrow[r, dashed] & \mathbb{P}(V)/\bar{G} \arrow[dd, dashed, "\bar{\varphi}"] \\
\mathbb{P}\big(V/L\big) \times \mathbb{P}(W) \arrow[d] & & \\
\mathbb{P}(W) \arrow[rr, dashed] & & \mathbb{P}(W)/\bar{G} \arrow[ll, dashed, bend left, "\bar{\sigma}"]
\end{tikzcd}$$

we find that the projection pr_{23} is dominant and makes Γ birationally into a vector bundle over $\mathbb{P}(V/L) \times \mathbb{P}(W)$. Hence Γ is birational to a succession of vector bundles over $\mathbb{P}(W)$ or has a *ruled structure* over $\mathbb{P}(W)$. Since $\bar{G}$ acts generically freely on $\mathbb{P}(W)$, the generic fibers of φ and $\bar{\varphi}$ can be identified, and we can pull back this ruled structure via σ (possibly replacing σ by a suitable translate). Hence $\mathbb{P}(V)/\bar{G}$ is birational to $\mathbb{P}(W)/\bar{G} \times \mathbb{P}^N$ with $N = \dim \mathbb{P}(V) - \dim \mathbb{P}(W)$. Thus by property (a), $\mathbb{P}(V)/G$ is rational. □

In [Shepherd-Barron 88], essentially this method is used to prove the rationality of the moduli spaces of plane curves of degrees $d \equiv 1 \pmod 9$, $d \geq 19$. In [Böhning and Bothmer 08a], it is the basis of the proof that for $d \equiv 1 \pmod 3$, $d \geq 37$, and $d \equiv 2 \pmod 3$, $d \geq 65$, these moduli spaces are rational. We improve these bounds here substantially and now recall the results from [Böhning and Bothmer 08a], which we use in our algorithms.

In that paper we used Theorem 3.2 with the following data; G is $\mathrm{SL}_3(\mathbb{C})$ throughout:

- For $d = 3n + 1$, $n \in \mathbb{N}$, and $V = V(0,\, d) = \mathrm{Sym}^d(\mathbb{C}^3)^\vee$, we take $W = V(0,\, 4)$ and produce covariants
$$S_d : V(0,\, d) \to V(0,\, 4)$$
of degree 4. We show that property (b) of Theorem 3.2 holds for the space
$$L_S = x_1^{2n+3} \cdot \mathbb{C}[x_1, x_2, x_3]_{n-2} \subset V(0,\, d).$$
Moreover, $\mathbb{P}(V(0,\, 4))/G$ is stably rational of level 8. So for particular values of d, it suffices to check property (c) by explicit computation. We give the details as to how this is done below.

- For $d = 3n + 2$, $n \in \mathbb{N}$, and $V = V(0,\, d) = \mathrm{Sym}^d(\mathbb{C}^3)^\vee$, we take $W = V(0,\, 8)$ and produce covariants
$$T_d : V(0,\, d) \to V(0,\, 8),$$

again of degree 4. In this case, property (b) of Theorem 3.2 can be shown to be true for the subspace

$$L_T = x_1^{2n+5} \cdot \mathbb{C}[x_1, x_2, x_3]_{n-3} \subset V(0,\, d).$$

Moreover, $\mathbb{P}(V(0, 8))/G$ is stably rational of level 8, too, and hence again everything comes down to checking property (c) of Theorem 3.2.

Remark 3.3. As was pointed out to us by the referee, the quotients $V(0,\, 4)/G$ and $V(0,\, 8)/G$ are even stably rational of level 3, which can be seen by considering $V = \mathbb{C}^3 \oplus W$, where $W = V(0, 4)$, $V(0, 8)$, or indeed any generically free representation of $\mathrm{SL}_3(\mathbb{C})$. The action of $\mathrm{SL}_3(\mathbb{C})$ on V has a P-section, where P is a parabolic subgroup with semisimple part $\mathrm{SL}_2(\mathbb{C})$. This can be used to reduce to the rationality result for $\mathrm{SL}_2(\mathbb{C})$-quotients [Bogomolov and Katsylo 85, Katsylo 83, Katsylo 84]. Unfortunately, this sharper bound does not allow us to improve our results.

We recall from [Böhning and Bothmer 08a] how some elements of L_S (respectively L_T) can be written as sums of powers of linear forms, which is very useful for evaluating S_d (respectively T_d) easily. Let K be a positive integer.

Definition 3.4. Let $\mathbf{b} = (b_1, \ldots, b_K) \in \mathbb{C}^K$ be given. Then we denote by

$$p_i^{\mathbf{b}}(c) := \prod_{\substack{j \neq i \\ 1 \leq j \leq K}} \frac{c - b_j}{b_i - b_j} \tag{3–1}$$

for $i = 1, \ldots, K$ the interpolation polynomials of degree $K-1$ with respect to $\mathbf{b}$ in the one variable c.

Then we have the following easy lemma (see [Böhning and Bothmer 08a, Lemma 5.2] for a proof).

Lemma 3.5. *Let $\mathbf{b} = (b_1, \ldots, b_K) \in \mathbb{C}^K$, $b_i \neq b_j$ for $i \neq j$, and set $x = x_1$, $y = \lambda x_2 + \mu x_3$, $(\lambda,\, \mu) \neq (0,\, 0)$. Suppose $d > K$ and put $l_i := b_i x + y$. Then for each $c \in \mathbb{C}$ with $c \neq b_i$, $\forall i$,*

$$f(c) = p_1^{\mathbf{b}}(c) l_1^d + \cdots + p_K^{\mathbf{b}}(c) l_K^d - (cx + y)^d \tag{3–2}$$

is nonzero and divisible by x^K.

So for $K = 2n+3$ we obtain elements in $f(c) \in L_S$, and for $K = 2n+5$ elements, $f(c) \in L_T$. We now check property (c) of Theorem 3.2 computationally in the following way. We choose a fixed $g \in V(0,\, d)$, which we write as a sum of powers of linear forms

$$g = m_1^d + \cdots + m_{\mathrm{const}}^d,$$

where const is a positive integer. We choose a random vector $\mathbf{b}$, random λ and μ, and a random c, and use [Böhning and Bothmer 08a, formula (30)], which reads

$$S_d(f(c) + g) = \sum_{i,j,k,p} p_i^{\mathbf{b}}(c) I(l_i,\, m_j,\, m_k,\, m_p)^n l_i m_j m_k m_p + S_d\left(-(cx+y)^d + g\right),$$

to evaluate S_d. Here I is a function on quadruples of linear forms to $\mathbb{C}$: if in coordinates,

$$L_\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

and L_β, L_γ, L_δ are linear forms defined analogously, and if we moreover abbreviate

$$(\alpha\, \beta\, \gamma) := \det \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}, \quad \text{etc.},$$

as in the symbolic method of [Grace and Young 03], then

$$I(L_\alpha,\, L_\beta,\, L_\gamma,\, L_\delta) := (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\delta)(\beta\gamma\delta)\,.$$

For T_d we have by an entirely analogous computation

$$T_d(f(c) + g) = \sum_{i,j,k,p} p_i^{\mathbf{b}}(c) I(l_i,\, m_j,\, m_k,\, m_p)^n l_i^2 m_j^2 m_k^2 m_p^2 + T_d\left(-(cx+y)^d + g\right). \tag{3–3}$$

So we can evaluate T_d similarly. Thus for each particular value of d we can produce points in $\mathbb{P}(V(0,\, 4))$, for $d = 3n+1$, or $\mathbb{P}(V(0, 8))$, for $d = 3n+2$, that are in the image of the restriction of S_d to a fiber of π_{L_S} (respectively in the image of the restriction of T_d to a fiber of π_{L_T}). We then check that these span $\mathbb{P}(V(0,\, 4))$ (respectively $\mathbb{P}(V(0,\, 8))$) to check condition (c) of Theorem 3.2.

4. APPLICATIONS TO MODULI OF PLANE CURVES

The results on the moduli spaces of plane curves $C(d)$ of degree d that we obtain are described below. We organize them according to the method employed.

4.1 Double Bundle Method

As we mentioned above, Katsylo obtained in [Katsylo 89] the rationality of $C(d)$, $d \equiv 0 \pmod 3$ and $d \geq 210$. Using the computational scheme of Section 2 and our program `nxnxn`, we obtain the rationality of all $C(d)$ with $d \equiv 0 \pmod 3$ and $d \geq 30$ except for $d = 48, 54, 69$. Moreover,

Degree d of Curves	Result and Method of Proof/Reference
1	rational (trivial)
2	rational (trivial)
3	rational (moduli space affine j-line)
4	rational, [Katsylo 92b], [Katsylo 96]
5	rational, two-form trick [Shepherd-Barron 88]
6	rationality unknown
7	rationality unknown
8	rationality unknown
9	rational, two-form trick [Shepherd-Barron 88]
10	rational, double bundle method, this article
11	rationality unknown
12	rationality unknown
13	rational, two-form trick [Shepherd-Barron 88]
14	rationality unknown
15	rationality unknown
16	rationality unknown
17	rational, two-form trick [Shepherd-Barron 88]
18	rationality unknown
19	covariants, [Shepherd-Barron 88] and this article
20	rationality unknown
21	rational, two-form trick [Shepherd-Barron 88]
22	covariants, this article
23	rationality unknown
24	rationality unknown
25	rational, two-form trick [Shepherd-Barron 88]
26	rationality unknown
27	rational, this article (method cf. above)
28	covariants, [Shepherd-Barron 88] and this article
29	rational, two-form trick [Shepherd-Barron 88]
30	double bundle method, this article
31	covariants, this article
32	rationality unknown
≥ 33 (excl. 48)	rational, this article, [Böhning and Bothmer 08a], [Katsylo 89]

TABLE 1. Table of known rationality results for $C(d)$.

we obtain rationality for $d = 10$ and $d = 21$ (the latter was known before, since by the results of [Shepherd-Barron 88], $C(d)$ is rational for $d \equiv 1 \pmod 4$). A table of U, V, and W used in each case can be found at [Böhning et al. 08], `UVW.html`. We found these combinatorially using our program `alldimensions2.m2`, also at [Böhning et al. 08].

For $d = 69$ the result is known by [Shepherd-Barron 88], since $69 \equiv 1 \pmod 4$. For the cases $d = 27$ and $d = 54$ we need more special U, V, W and use the methods from our article [Böhning and Bothmer 08c].

4.2 The case $d = 27$

We establish the rationality of $C(27)$ as follows: there is a bilinear, $\mathrm{SL}_3(\mathbb{C})$-equivariant map

$$\psi : V(0,\,27) \times (V(11,\,2) \oplus V(15,\,0)) \to V(2,\,14),$$

and

$$\dim V(0,\,27) = 406, \quad \dim V(11,\,2) = 270,$$
$$\dim V(15,\,0) = 136, \quad \dim V(2,\,14) = 405\,.$$

We compute ψ by the method of [Böhning and Bothmer 08c] and find that $\psi = \omega^2\beta^{11} \oplus \beta^{13}$ in the notation of that article. For a random $x_0 \in V(0,\,27)$, the kernel of $\psi(x_0, \cdot)$ turns out to be one-dimensional, generated by y_0, say, and $\psi(\cdot,\,y_0)$ has likewise one-dimensional kernel generated by x_0 (See[Böhning et al. 08], `degree27.m2`, for a Macaulay script for doing this calculation.) It follows that the map induced by ψ,

$$\mathbb{P}(V(0,\,27)) \dashrightarrow \mathbb{P}(V(11,\,2) \oplus V(15,\,0)),$$

is birational, and it is sufficient to prove rationality of $\mathbb{P}(V(11,\,2) \oplus V(15,\,0))/\mathrm{SL}_3(\mathbb{C})$. But $\mathbb{P}(V(11,\,2) \oplus$

$V(15, 0))$ is birationally a vector bundle over $\mathbb{P}(V(15, 0))$, and $\mathbb{P}(V(15, 0))/\mathrm{SL}_3(\mathbb{C})$ is stably rational of level 19, so $\mathbb{P}(V(11, 2) \oplus V(15, 0))/\mathrm{SL}_3(\mathbb{C})$ is rational by the no-name lemma (Lemma 2.1).

4.3 The case $d = 54$

We establish the rationality of $C(54)$ as follows: there is a bilinear, $\mathrm{SL}_3(\mathbb{C})$-equivariant map

$$\psi\colon V(0, 54) \times \big(V(11, 8) \oplus V(6, 3) \oplus V(5, 2) \oplus V(3, 0)\big) \to V(0, 51)$$

with

$$\begin{aligned} \dim V(0,\ 54) &= 1540, & \dim V(11,\ 8) &= 1134, \\ \dim V(6,\ 3) &= 154, & \dim V(5,\ 2) &= 81, \\ \dim V(3,\ 0) &= 10, & \dim V(0, 51) &= 1378. \end{aligned}$$

Since $1134 + 154 + 81 + 10 = 1379 = 1378 + 1$ and $1540 - 1379 > 19$, we need to check only the genericity condition of Theorem 2.2 to prove rationality. For this we compute ψ by the method of [Böhning and Bothmer 08c] and find that $\psi = \beta^{11} \oplus \beta^6 \oplus \beta^5 \oplus \beta^3$ in the notation of that article.

For a random $x_0 \in V(0,\ 54)$, the kernel of $\psi(x_0, \cdot)$ turns out to be one-dimensional, generated by y_0, say, and $\psi(\cdot,\ y_0)$ has full rank 1378, and therefore $\psi(V(0, 54),\ y_0) = V(0, 51)$, as required. See [Böhning et al. 08], `degree54.m2`, for a Macaulay script for doing this calculation.

4.4 Method of Covariants

According to [Böhning and Bothmer 08a], $C(d)$ is rational for $d \equiv 1 \pmod 3$, $d \geq 37$, and $d \equiv 2 \pmod 3$, $d \geq 65$ (for $d \equiv 1 \pmod 9$, $d \geq 19$, rationality was proven before in [Shepherd-Barron 88]). By the method of Section 3, we improve this and obtain that $C(d)$ is rational for $d \equiv 1 \pmod 3$, $d \geq 19$, which uses the covariants S_d of Section 3, and rational for $d \equiv 2 \pmod 3$, $d \geq 35$, which uses the family of covariants T_d of Section 3. See [Böhning et al. 08], `interpolation.m2`, for a Macaulay script for doing this calculation.

Combining what was stated above with the known rationality results for $C(d)$ for small values of d, we can summarize the current state of knowledge in Table 1. Thus we obtain our main theorem:

Theorem 4.1. *The moduli space $C(d)$ of plane curves of degree d is rational except possibly for one of the values in the following list:*

$d = 6, 7, 8, 11, 12, 14, 15, 16, 18, 20, 23, 24, 26, 32, 48.$

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