Minimal Regulators for Rank-2 Subgroups of Rational and $K3$ Elliptic Surfaces

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CONTENTS

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We determine the smallest possible regulator $R(P, Q)$ for a rank-2 subgroup $\mathbb{Z}P \oplus \mathbb{Z}Q$ of an elliptic curve E over $\mathbb{C}(t)$ of discriminant degree $12n$ for $n = 1$ (a rational elliptic surface) and $n = 2$ (a $K3$ elliptic surface), exhibiting equations for all (E, P, Q) attaining the minimum. The minimum $R(P, Q) = 1/36$ for a rational elliptic surface was known [Oguiso and Shioda 91], but a formula for (E, P, Q) was not, nor was the fact that this is the minimum for an elliptic curve of discriminant degree 12 over a function field of any genus. For a $K3$ surface, both the minimal regulator $R(P, Q) = 1/100$ and the explicit equations are new. We also prove that 1/100 is the minimum for an elliptic curve of discriminant degree 24 over a function field of any genus. The optimal (E, P, Q) are uniquely characterized by having mP and $m'Q$ integral for $m \leq M$ and $m' \leq M'$, where $(M, M') = (3, 3)$ for $n = 1$ and $(M, M') = (6, 3)$ for $n = 2$. In each case MM' is maximal. We use the connection with integral points to find explicit equations for the curves. As an application we use the $K3$ surface to produce, in a new way, the elliptic curves $E/\mathbb{Q}$ with nontorsion points of smallest known canonical height. These examples appeared previously in [Elkies 02].

1. INTRODUCTION

Let K be a function field of genus g over a field k of characteristic 0. Let E be a nonconstant elliptic curve over K of discriminant degree $d = 12n$ (or equivalently an elliptic surface of arithmetic genus n). When $g = 0$ and $n = 1$, E is a rational elliptic surface; when $g = 0$ and $n = 2$, E is a $K3$ elliptic surface. Let

$$\hat{h} : E(K) \longrightarrow [0, \infty) \cap \mathbb{Q}$$

be the canonical height function. The canonical height is a quadratic form on the group $E(K)$ and gives $E(K)/E(K)_{\text{tors}}$ the structure of a lattice in a Euclidean space. If one fixes the discriminant degree d, the set of numbers that can occur as the canonical height of a point is discrete, because the bound on d imposes an upper bound on the denominator. One may ask, what is

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the smallest positive height of a nontorsion point on an elliptic curve over K of discriminant degree d? More generally, one can ask, what is the smallest possible regulator for a rank-r sublattice of the lattice $E(K)/E(K)_{\text{tors}}$?

Let $R^r_{\min}(g, 12n)$ denote the minimal regulator for a rank-r subgroup of an elliptic curve of discriminant degree $12n$ over a function field of genus g. Let $R^r_{\min}(12n)$ denote the minimal regulator for a rank-r subgroup of an elliptic curve of discriminant degree $12n$. In the case of $r = 1$, these values denote the smallest positive height of a nontorsion point. The value $R^1_{\min}(0, 12) = 1/30$ was determined in [Oguiso and Shioda 91], and $R^1_{\min}(0, 24) = 11/420$ was determined in [Nishiyama 96]. Nishiyama used arguments specific to $K3$ surfaces, and was unable to write down any examples of curves attaining the minima. Explicit methods were used in [Elkies 06a] to determine $R^1_{\min}(0, 12n)$ for $n = 1, 2, 3$ and $R^1_{\min}(12n)$ for $n = 1, 2$. In each case Elkies exhibited equations for all (E, P) attaining the minima. He also proved that the (E, P) attaining the minima have the property that mP is an integral point for each $m = 1, \ldots, M$, where $M = 6, 8, 9$ for $n = 1, 2, 3$, and that this M is maximal in each case. In this paper we generalize Elkies' results to sublattices of rank 2.

1.1 Results

We determine $R^2_{\min}(0, 12n)$ and $R^2_{\min}(12n)$ for $n = 1, 2$. Let $\langle\cdot,\cdot\rangle$ denote the bilinear form on $E(K)$ induced by $\hat{h}$. By a reduced basis for a rank-2 sublattice $\mathbb{Z}P \oplus \mathbb{Z}Q \subset E(K)$, we mean a basis (P, Q) such that $0 \le 2|\langle P, Q\rangle| \le \hat{h}(P)$ and $0 < \hat{h}(P) \le \hat{h}(Q)$. We write (E, P, Q) for a triple consisting of an elliptic curve E/K and points $P, Q \in E(K)$ that form a reduced basis for a rank-2 sublattice. Let $R(P, Q)$ denote the volume of the sublattice. We obtain the following results:

Theorem 1.1.

(i) [Oguiso and Shioda 91] $R^2_{\min}(0, 12) = 1/36$.

(ii) $R^2_{\min}(12) = 1/36$.

Furthermore, let E be an elliptic curve of discriminant degree $d = 12$ over a complex function field K, and let (P, Q) be a reduced basis for a rank-two subgroup of the lattice $E(K)$. The the following are equivalent:

(a) $R(P, Q) = 1/36$;

(b) *Each of P, $2P$, $3P$, Q, $2Q$, $3Q$ is an integral point on E;*

(c) *$K \cong \mathbb{C}(t)$, and (E, P, Q) is equivalent to the curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2$, where*

$$a_1 = (2q - q' + 2)s + (q' + 1)s',$$
$$a_2 = q(q' - q - 1)s^2 - qq'ss',$$
$$a_3 = -s'\big((q' - q - 1)s^2 - q's's\big),$$

over the $(s : s')$ line with the independent rational points $P : (x, y) = (0, 0)$ and $Q : (x, y)$, where

$$x = (q' - q - 1)s^2 - q'ss',$$
$$y = s((q' - q - 1)s - q's')^2,$$

for some $q, q' \in \mathbb{C}$ other than 0 or 1.

The Mordell–Weil lattice of a rational elliptic surface in the family above is rectangular, i.e., $\langle P, Q\rangle = 0$, with $\hat{h}(P) = \hat{h}(Q) = 1/6$. The surface has the symmetry $s \leftrightarrow s'$, which corresponds to interchanging Q and $-P - Q$ on E. This symmetry is inherited by the symmetry of the moduli space in which we recover the family (see Section 5). The surface has multiplicative fibers of type I_3 at $(1 : 0)$ and $(q : q' - q - 1)$, and type I_2 at $(1 : 1)$ and $(0 : 1)$.

The existence of this surface was known [Shioda 92a], and the lattice structure of the surface is number 40 on the list of [Oguiso and Shioda 91]. However, the formulas, the result over higher-genus curves, and the connection with integral multiples are new. Any surface in the above family has 28 integral points in the rank-2 subgroup $\mathbb{Z}P \oplus \mathbb{Z}Q$. No rational elliptic surface outside of this family has a rank-2 subgroup with as many integral points (see Section 4.2).

The minimum for $d = 24$ is attained by a single $K3$ surface (E, P, Q) with regulator $R(P, Q) = 1/100$:

Theorem 1.2.

(i) $R^2_{\min}(0, 24) = 1/100$.

(ii) $R^2_{\min}(24) = 1/100$.

Furthermore, let E be an elliptic curve of discriminant degree $d = 24$ over a complex function field K, and let (P, Q) be a reduced basis for a rank-two subgroup of the lattice $E(K)$. Then the following are equivalent:

(a) $R(P, Q) = 1/100$.

(b) *Each of $P, \ldots, 6P, Q, 2Q, 3Q$ is integral.*

(c) *$K \cong \mathbb{C}(t)$, and (E, P, Q) is equivalent to the curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2$, where*

$$a_1 = 3\left(4ss' - s^2 - s'^2\right),$$
$$a_2 = 3ss'\left(5ss' - 2s'^2 - 2s^2\right),$$
$$a_3 = 3ss'\left(10ss' - 3s^2 - 3s'^2\right)\left(5ss' - 2s'^2 - 2s^2\right),$$

over the $(s : s')$ line with the independent rational points $P : (x, y) = (0, 0)$ and $Q : (x, y)$, where

$$x = 3ss'(2s - s')(s - 3s'),$$
$$y = 9s^2s'^2(s' - 2s)(s - 3s').$$

The optimal $K3$ elliptic surface is defined over $\mathbb{Q}$ and has semirectangular Mordell–Weil lattice, with $\hat{h}(P) = 1/15$ and $\hat{h}(Q) = 1/6$ and $\langle P, Q\rangle = -1/30$. The surface has the symmetry $s \leftrightarrow s'$, which corresponds to interchanging Q and $-P - Q$ on E. Again this symmetry is inherited by the symmetry of the moduli space in which we recover the curve (Section 5). The surface has multiplicative fibers of type I_5 at $(0 : 1)$ and $(1 : 0)$, I_3 at $(2 : 1)$ and $(1 : 2)$, and I_2 at $(3 : 1)$ and $(1 : 3)$.

At $(1 : -1)$ the surface has a fiber of type IV. The discriminant group of the Néron–Severi lattice of E is cyclic of order 27, and its Picard number is $\rho = 20$.[1] Among elliptic surfaces of discriminant degree 24, the above surface has a rank-2 subgroup with the greatest possible number of integral points. It has a total of 52 integral points in the subgroup $\mathbb{Z}P \oplus \mathbb{Z}Q$ (see Section 4.2).

Finally, the elliptic surfaces in Theorems 1.1 and 1.2 have the following maximality property with regard to consecutive integral multiples: The points mP and $m'Q$ are integral for $m \leq M$ and $m' \leq M'$, where $(M, M') = (3, 3)$ for $n = 1$ and $(M, M') = (6, 3)$ for $n = 2$. In each case the product MM' is maximal.

1.2 Methods

If the discriminant degree $d = 12n$ of E is fixed, the condition

$$d = \sum_v d_v$$

on the Kodaira types of singular fibers (see Section 2.5) implies that there will be only finitely many possible collections of fibers. A small section can meet each collection in only finitely many ways, and thus determining $R^r_{\min}(g, 12n)$ or $R^r_{\min}(12n)$ of a rank-r subgroup is a finite problem. The difficulty lies in eliminating those collections of fibers and components that will not correspond to an elliptic surface E and sections on it.

We place several conditions on the collections of fibers and sections (Section 4.1), and compute lower bounds for $R^2_{\min}(0, 12n)$ in the cases $n = 1$ and $n = 2$. The lower bound for $R^2_{\min}(0, 12)$ is attained by the two-parameter family of rational elliptic surfaces in Theorem 1.1. The lower bound for $R^2_{\min}(0, 24)$ is attained only by the single $K3$ elliptic surface in Theorem 1.2.

In Section 5, we find equations for the surfaces in Theorems 1.1 and 1.2. We use an adaptation of Tate's trick [Tate 74] for computing the generic elliptic curve with an N-torsion point. This adaptation is due to Elkies, who parameterizes the moduli space of elliptic curves with a point (E, P) such that $P, \ldots, 4P$ are integral by an open subset of $\mathbb{P}^2$ and recovers the equations for the elliptic surfaces attaining $R^1_{\min}(12n)$ as curves of degree n in this $\mathbb{P}^2$ [Elkies 06b].

We use the technique to parameterize the moduli space of elliptic curves with two independent points (E, P, Q) such that P, $2P$, Q, $P + Q$, $P - Q$, and $2P + Q$ are all integral by an open subset of $\mathbb{P}^3$. We then recover the optimal elliptic surfaces in Theorems 1.1 and 1.2 as curves in this $\mathbb{P}^3$. The symmetries exhibited by these surfaces are a consequence of the fact that they can be recovered in this way. There is a linear involution of this $\mathbb{P}^3$ that corresponds to interchanging the point Q with the point $-Q - P$, and all elliptic surfaces recovered as curves in this moduli space exhibit this symmetry.

Note that the equations could have been found in other ways. The $K3$ surface in Theorem 1.2 could have been obtained by Elkies' equations for the one-parameter family of rational elliptic surfaces attaining $R^1_{\min}(12) = 1/30$: Changing base gives a two-parameter family of $K3$ elliptic surfaces with a section of height $1/15$. We can use one free parameter to force ramification over $(1 : -1)$, and another parameter to impose another section of height $1/6$. However, the methods in Section 5 are general, and can be used to find explicit equations for elliptic surfaces with various patterns of integral points.

In Section 6 we prove that $R^2_{\min}(12n) = R^2_{\min}(0, 12n)$. We analyze configurations over higher-genus curves, where the Euler characteristic $\chi = 2 - 2g$ can be negative. In this case we must use different techniques to eliminate several configurations of small conductor degree.

The approach in this paper generalizes to any rank-r subgroup, though most likely the techniques in Section 5

[1] Such $K3$ surfaces are called singular. Elkies points out that the surface in Theorem 1.2 is another elliptic model for the CM(-27) surface, i.e., the $K3$ surface obtained by starting with an elliptic $E/\mathbb{Q}$ with complex multiplication by the imaginary quadratic order of discriminant -27, and applying the Shioda–Inose construction [Shioda and Inose 77] to the Kummer surface $E \times E/\{-1, 1\}$.

would have to be modified. We do not expect the curves attaining $R^r_{\min}(0, 12n)$ for $r \geq 3$ to have rank-2 subgroups with the pattern of integral points of the curves in our $\mathbb{P}^3$ model, and hence most likely it will be necessary to construct other models to find explicit equations in this manner.

The approach in this paper also generalizes, in theory, to any discriminant degree $12n$. Heuristics show, however, that the lower bounds we obtain by this approach will most likely not be attained for $n > 2$ (Section 4.4). For $n = 1$, one can deduce from the theory of rational elliptic surfaces [Shioda 92a] that any configuration of fibers and sections satisfying the conditions in Section 4.1 will be realized by a rational elliptic surface. For $n = 2$, our heuristics tell us that any configuration of fibers and sections that satisfy the conditions listed in Section 4.1 should be realized by a $K3$ elliptic surface. In Section 7 we give a counterexample to these heuristics, exhibiting a local obstruction to the realization of a particular configuration.

In Section 8 we illustrate, as an application, how to use the equations for the $K3$ elliptic surface in Theorem 1.2 to produce elliptic curves over $\mathbb{Q}$ with a nontorsion point of small canonical height. In fact, specializations of this $K3$ elliptic fibration yield the elliptic curves with points whose canonical heights are the five smallest known over $\mathbb{Q}$. These curves were constructed earlier in different ways in [Elkies 02] and by others.

1.3 Further Directions

One can use similar techniques to compute the minimum regulator for a rank-r subgroup for $r > 2$, or in some restricted class of elliptic surfaces. Elsewhere we determine the smallest possible canonical height $R^1_{\min}(g, 12n, G)$ for a nontorsion point on an elliptic surface with torsion subgroup $G = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ for $n = 1, 2, 3$. The optimal surfaces are again characterized by their patterns of integral points. For each torsion subgroup G we construct a suitable moduli space of elliptic curves (E, P, G) with a nontorsion point P and a torsion point of order $|G|$ such that some $\mathbb{Z}$-linear combinations of P and the torsion point are integral. We again recover the optimal elliptic surfaces as curves in the moduli space, proving results analogous to Theorems 1.1 and 1.2.

One may also explore the asymptotic behavior of $R^r_{\min}(g, d)$ for fixed g as $d \to \infty$. In [Hindry and Silverman 88] it is proved that there exists a constant $C > 0$ such that

$$R^1_{\min}(g, 12n) \geq Cn - O_g(1),$$

proving effectively a conjecture from [Lang 78] in the function-field case. The error terms $O_g(1)$ vanish for $g = 0, 1$. Hindry and Silverman computed an explicit $C \approx 7 \cdot 10^{-10}$. Elkies improved the value to $C \approx 5 \cdot 10^{-4}$, and conjectured what the best possible value for C should be [Elkies 01]. In a future paper, we examine the asymptotic behavior of $R^2_{\min}(g, d)$ as $d \to \infty$. More generally, we describe the region in the three-dimensional space of reduced binary forms that is asymptotically obtainable by Mordell–Weil lattices in rank 2, proving that the boundary of the region is cut out by algebraic equations.

2. BACKGROUND

In this section we review the necessary background from the theory of elliptic surfaces. In the first section we state basic facts about lattices. In Section 2.2, we define an elliptic surface and recall the connection between the Néron–Severi group of an elliptic surface and the Mordell–Weil group of the corresponding elliptic curve. We recall the definitions of the naive height and integral points on elliptic surfaces in Section 2.3, and we state some useful inequalities involving the naive height. In Section 2.4 we state the definition of the canonical height pairing. In Section 2.5 we discuss the special fibers of a minimal elliptic fibration. In Section 2.6 we discuss local height corrections, stating explicit formulas depending only on the fiber type and the component of the fiber meeting the section. We follow [Elkies 06a, Shioda 90] and [Silverman 94, Chapter 3] closely.

2.1 Lattices

In this section we list some of the basic facts we will need about lattices. By a *lattice* we mean a finitely generated free abelian group L, equipped with a bilinear pairing

$$B : L \longrightarrow \mathbb{Q}.$$

The lattice L is said to be *integral* if B takes its values in $\mathbb{Z}$, and *even* if B takes its values in $2\mathbb{Z}$. The *signature* of the lattice is the real signature $(r_+,\ r\ , r_0)$ of the pairing B, where r_+, r_-, and r_0 denote the numbers of positive, negative, and zero eigenvalues of B. The lattice L is said to be *nondegenerate* if $r_0 = 0$. The *discriminant* of L is $|\det B|$. The lattice is called *unimodular* if $|\det B| = 1$.

Suppose that L is an integral lattice. The dual lattice of L is the group $L^* = \mathrm{Hom}(L, \mathbb{Z})$, equipped with the pairing induced by B. There is a natural embedding of $L \hookrightarrow L^*$ given by $v \mapsto \langle \cdot, v \rangle$. We call the quotient L^*/L the *discriminant group* of the lattice. The discriminant $|\det B|$ of L is equal to $|L^*/L|$. We write $L(\alpha)$ for the lattice with the same underlying group as L and bilinear form αB.

2.2 Elliptic Surfaces

Let C be a curve of genus g over a field k of characteristic 0. By an elliptic surface $\mathcal{E}$ over C we mean the following: $\mathcal{E}$ is a smooth projective surface over k with a relatively minimal elliptic fibration $\pi : \mathcal{E} \to C$ such that:

(i) The generic fiber is an elliptic curve.

(ii) No fibers contain exceptional curves of the first kind.

(iii) The surface $\mathcal{E}$ is not constant, i.e., $\mathcal{E} \neq E \times C$ for an elliptic curve E over k.

Let E be a nonconstant elliptic curve over the function field $K = k(C)$. We write $\pi : \mathcal{E} \to C$ for the Kodaira–Néron model of E/K. This is the associated minimal elliptic surface whose generic fiber is E. The Kodaira–Néron model exists and is unique up to isomorphism [Kodaira 63a, Kodaira 63b, Néron 64]. The global sections of $\pi : \mathcal{E} \to C$ are in one-to-one correspondence with the K-rational points of E, and we use the notation $E(K)$ to denote both the Mordell–Weil group of E/K and the group of sections of π defined over k. The Mordell–Weil theorem holds in this setting, and thus $E(K)$ is a finitely generated abelian group. We let r denote the Mordell–Weil rank of $E(K)$.

The Néron–Severi group $\mathrm{NS}(\mathcal{E})$ of an algebraic surface $\mathcal{E}$ is defined as the group of divisors modulo algebraic equivalence. This group has a symmetric bilinear pairing induced by the intersection pairing on the second cohomology:

$$H^2(\mathcal{E}, \mathbb{Z}) \times H^2(\mathcal{E}, \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

The Néron–Severi group embeds into $H^2(\mathcal{E}, \mathbb{Z})$, and this embedding gives $\mathrm{NS}(\mathcal{E})$ the structure of an integral lattice, which we call the *Néron–Severi lattice* of $\mathcal{E}$. We define the *Picard number* ρ of $\mathcal{E}$ to be the rank of this lattice. For an elliptic surface, $\mathrm{NS}(\mathcal{E})$ embeds into the cohomology group $H^{1,1}(\mathcal{E}, \mathbb{Z})$, and hence[2]

$$\rho \leq 10n + 2g = \mathrm{rk}\, H^{1,1}(\mathcal{E}, \mathbb{Z}).$$

Given E/K and the associated elliptic surface $\pi : \mathcal{E} \to C$, we write $E_v = \pi^{-1}(v)$ for the fiber over $v \in C$. Each reducible fiber can be decomposed as a sum

$$E_v = c_{v,0} + \sum_{i=1}^{m_v - 1} a_{v,i} c_{v,i},$$

where the $c_{v,i}$ are the irreducible components of the fiber E_v, and $c_{v,0}$ is the component of E_v that meets the identity section of E. The Picard number ρ is given by

$$\rho = r + 2 + \sum_v (m_v - 1).$$

In addition, we have the following theorem:

Theorem (Hodge index theorem.) *The Néron–Severi lattice of an algebraic surface is an indefinite lattice of signature* $(1, \rho - 1)$.

For an elliptic surface, the Néron–Severi group is finitely generated and torsion-free, and is very closely related to the group $E(K)$. Let $T \subset \mathrm{NS}(\mathcal{E})$ be the subgroup generated by the zero section s_0, a generic fiber, and the irreducible components $\{c_{v,i}\}$ of the reducible fibers E_v. We call T the *trivial sublattice* of $\mathrm{NS}(\mathcal{E})$. There is a natural isomorphism of groups [Shioda 90]

$$E(K) \cong \mathrm{NS}(\mathcal{E})/T,$$
$$P \longmapsto s_P \bmod T.$$

The intersection pairing allows us to define a naive height, which we discuss in the following section.

2.3 The Naive Height and Integral Points

Let $C = \mathbb{P}^1$, and let $E/k(C)$ be an elliptic curve given by a minimal Weierstrass equation

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

and let $P \in E(K)$. Each a_i is a homogeneous polynomial in two variables of degree $i \cdot n$. If $P = (X, Y)$ is a rational point, then X and Y are homogeneous rational functions of degree $2n$ and $3n$. We write X and Y in lowest terms so that the numerators and denominators have no common factors. The denominators will then, up to some scalar, be the square and cube of some polynomial δ. We define the *naive height* of the point P by $h(P) := \deg(\delta)$. We say that P is an *integral point* on E if $h(P) = 0$, i.e., if the coordinates of P are homogeneous polynomials.

If one changes coordinates

$$(x, y) \longmapsto \left(u^2(y + \alpha_2), u^3(y + \alpha_1 x + \alpha_3)\right) \qquad (u \neq 0),$$

then the Weierstrass equation above changes but the new curve is isomorphic to the original one. If $u \in k^*$ and α_i

[2]Without the hypothesis that k is of characteristic zero, $\mathrm{NS}(\mathcal{E})$ will in general not embed into $H^{1,1}(\mathcal{E}, \mathbb{Z})$. If k has positive characteristic, we have the weaker bound on the rank of the Néron–Severi group, $\rho \leq 10n + 4g + 2(n - 1)$.

is a homogeneous polynomial of degree $i{\cdot}n$, then both the discriminant degree of E and the set of integral points of E are preserved. We write s_0 and s_P for the zero section and the section corresponding to P of the Néron model $\mathcal{E} \to \mathbb{P}^1$ of E.

Since the roots of δ are the images on $\mathbb{P}^1$ of the intersection points of s_0 and s_P (counted with multiplicity), we see that $h(P) = 2s_0 \cdot s_P$.

Now let C be a curve of genus possibly bigger than zero, and let $K = \mathbb{C}(C)$. Then a_i is a global section of $\mathcal{O}(iL)$, where L is a divisor on C of degree n. The coordinates X and Y of P are meromorphic sections of $\mathcal{O}(2L)$ and $\mathcal{O}(3L)$. The pole divisors of X and Y are $2Z$ and $3Z$ for some effective divisor Z on C. We define the naive height of a point P as the degree of the pole divisor $2Z$. So P is integral if and only if X and Y are holomorphic sections of $2L$ and $3L$. A linear change of coordinates with $\alpha_i \in \Gamma(iL)$ for $i = 1, 2, 3$ and $\delta \in K^*$ will yield a curve with the same integral points.

Over a more general function field $K = k(C)$, the degree of Z is $s_0{\cdot}s_P$, and one can define the naive height via intersection theory. If s_0 is the zero section of $\mathcal{E}$, and s_P is the section corresponding to P, then these s_0 and s_P are distinct curves on $\mathcal{E}$. Their intersection number $s_P \cdot s_0$ is a nonnegative integer, and we define the naive height of the point P by $h(P) := 2s_0 \cdot s_P$. If the sections s_P and s_0 are disjoint, then $h(P) = 0$, and we say that P is an integral point.

The important fact that we will need about the naive height is the following:

Proposition 2.1. *Let P be a point on an elliptic curve $E/k(C)$, and suppose $m \in \mathbb{Z}$ such that $mP \neq 0$. Then we have the naive height inequality $h(P) \leq h(mP)$.*

Proof: If s_P intersects the zero section s_0 at a point, then s_{mP} will also meet the zero section at that point. This shows that $s_0 \cdot s_P \leq s_0 \cdot s_{mP}$. □

2.4 The Canonical Height

We know from Section 2.2 that the Mordell–Weil group $E(K)$ is canonically isomorphic to the quotient group $\mathrm{NS}(\mathcal{E})/T$. Ideally, one would like the intersection pairing $(\cdot,\cdot)$ on $\mathrm{NS}(\mathcal{E})$ to induce a canonical pairing on $E(K)$, giving $E(K)$ the structure of a lattice. Modulo a small "correction," the quotient map on $\mathrm{NS}(\mathcal{E})$ can be split. If $P \in E(K)$, one defines a divisor D_P by

$$D_P = s_P - s_0 + \Phi_P,$$

where Φ_P is a certain vertical divisor in $\mathrm{NS}(\mathcal{E}) \otimes \mathbb{Q}$ [Silverman 94]. The pairing on $E(K)$ defined by

$$E(K) \times E(K) \longrightarrow \mathbb{Q}, \quad \langle P, Q\rangle = -D_P \cdot D_Q,$$

gives $E(K)/E(K)_{\mathrm{tors}}$ the structure of a lattice in Euclidean space. We define the canonical height $\hat{h}$ by

$$\hat{h}(P) = \frac{1}{2}\langle P, P\rangle.$$

The canonical height $\hat{h}$ is a quadratic form that is positive definite on $E(K)/E(K)_{\mathrm{tors}}$. It can be described as a sum

$$\hat{h}(P) = h(P) + \sum_v \lambda_v(P)$$

of the naive height $h(P)$ and local correction terms $\lambda_v(P)$, where we sum over the singular fibers v. Each local correction term depends only on the type of singular fiber E_v and the component at which P meets the fiber. Explicit formulas for λ_v are known, and given in Section 2.6. In the next section we list all possible singular fibers.

2.5 Kodaira Fiber Types

Set d equal to the degree of the discriminant of E/K, considered as a divisor on C. It is known that $d = 12n$, where n is the arithmetic genus of the corresponding surface $\mathcal{E}/k$ fibered over C. We let N denote the degree of the conductor of E/K, also considered as a divisor on C. The conductor degree can be defined, equivalently, as the number of multiplicative fibers plus twice the number of additive fibers. We may write the discriminant degree d and the conductor N as sums of local terms:

$$12n = d = \sum_v d_v, \quad N = \sum_v N_v,$$

where the sum is taken over singular fibers. The possible singular fibers were first classified by Kodaira in [Kodaira 63a, Kodaira 63b]. Table 1, copied from [Elkies 06a], gives the local data for each possible Kodaira type E_v.

The following inequality from [Shioda 72] will be essential to the proofs of Theorems 1.1 and 1.2, where it will be used to eliminate configurations with too few fibers (Section 4.1).

Proposition 2.2. (Shioda's inequality.) *Let $\mathcal{E}/k$ be an elliptic surface fibered over a curve C/k of genus g, and let E/K be the corresponding elliptic curve over $K = k(C)$. Let $d = 12n$ be the discriminant degree and N the conductor degree. Then*

$$N \geq 2n + \chi(C) + r, \tag{2–1}$$

where r is the Mordell–Weil rank of the group $E(K)$.

Kodaira Type	I_ν $(\nu > 0)$	**II**	**III**	**IV**	I_ν^*	**IV***	**III***	**II***
d_ν	ν	2	3	4	$6+\nu$	8	9	10
N_ν	1	2	2	2	2	2	2	2
$E_\nu/(E_\nu)_0$	$\mathbb{Z}/\nu\mathbb{Z}$	$\{0\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$D_{4+\nu}^*/D_{4+\nu}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\{0\}$
root lattice	$A_{\nu-1}$	A_0	A_1	A_2	$D_{4+\nu}$	E_6	E_7	E_8

TABLE 1. Kodaira fiber types.

Proof: Let $T \subset \mathrm{NS}(\mathcal{E})$ be the trivial sublattice (Section 2.2). By taking the sum on the generic fiber E of $\mathcal{E} \to C$, we have an exact sequence of abelian groups

$$0 \longrightarrow T \longrightarrow \mathrm{NS}(\mathcal{E}) \longrightarrow E(K) \longrightarrow 0.$$

Taking ranks, we find that $\rho = \mathrm{rk}\, T + r$, and since $\mathrm{rk}\,\mathrm{NS}(\mathcal{E}) \leq 10n + 2g$ (Section 2.2), we obtain Shioda's inequality:

$$N \geq (d + 2 + r) - (10n + 2g) = 2n + (2 - 2g) + r.$$

□

2.6 Local Height Corrections

Let P be a point on an elliptic curve E over $k(C)$. The canonical height $\hat{h}(P)$ can be written as a sum of the naive height $h(P)$ and some local correction terms:

$$\hat{h}(P) = h(P) + \sum_{\upsilon} \lambda_\upsilon(P),$$

where the sum is taken over singular fibers υ. The local correction term $\lambda_\upsilon(P)$ depends only on the type of the singular fiber E_υ at υ and the component c_υ of E_υ that meets the section s_P corresponding to P. We list explicit formulas for the local correction terms for each possible singular fiber and component listed in Table 1. These formulas have been worked out in [Cox and Zucker 79].

- If the section s_P intersects the identity component of E_υ, then
$$\lambda_\upsilon(P) = d_\upsilon/6.$$
- If E_υ is an additive fiber of type III, IV, I_0^*, III^*, or IV^*, and s_P intersects a nonidentity component of E_υ, then $\lambda_\upsilon(P) = 0$.
- Suppose E_υ is an additive fiber of type I_ν^* $(\nu > 0)$ and s_P passes through a nonidentity component. If ν is odd and s_P meets the distinguished 2-torsion component, then $\lambda_\upsilon(P) = \nu/6$. Otherwise, we have $\lambda_\upsilon = -\nu/12$.
- Finally, if E_υ is a multiplicative fiber of type I_ν and s_P passes through component a, then
$$\lambda_\upsilon(P) = \nu B(a/\nu),$$
where $B(x) = \langle x\rangle^2 - \langle x\rangle + 1/6$ is the second Bernoulli function of x. The quantity $\langle x\rangle$ denotes the fractional part of x.

From the formulas above one sees that the local correction terms are bounded above and below:

$$-d_\upsilon/12 \leq \lambda_\upsilon(P) \leq d_\upsilon/6.$$

If E has discriminant degree $12n$, then summing over the reducible fibers, one immediately obtains the following bound on the difference between the naive height and the canonical height:

$$-n \leq \hat{h}(P) - h(P) \leq 2n. \tag{2–2}$$

If for some integer m the point mP is an integral point on E, i.e., $h(mP) = 0$, then we have

$$m^2\hat{h}(P) = \hat{h}(mP) = h(mP) + \sum_{\upsilon} \lambda_\upsilon(mP)$$
$$= \sum_{\upsilon} \lambda_\upsilon(mP) \leq \frac{d}{6} = 2n,$$

and we find that $m^2\hat{h}(P) \leq 2n$. This implies that

$$\hat{h}(P) \leq 2n/m^2$$

if mP is an integral point.

3. REDUCTION TO THE SEMISTABLE CASE

To the elliptic curve E/K we may associate a relatively minimal elliptic surface $\mathcal{E}/k$ fibered over C whose generic fiber is E, i.e., the Kodaira–Néron model of E. The global sections of the Kodaira–Néron model are in one-to-one correspondence with the K-rational points of E. Thus there is no ambiguity when we refer to E/K as an elliptic surface, and $P, Q \in E(K)$ as sections of E. We reduce the problem of finding lower bounds for the regulator $R(P, Q)$ to the case in which the elliptic curve has everywhere semistable reduction.

3.1 Notation

Define the following equivalence relation on $\mathbb{Z}^3$: set $(x, y \mid z) \sim (u, v \mid w)$ when $xw = uz$ and $yw = vz$. Each equivalence class has a unique representative $[x, y \mid z]$ with $\gcd(x, y, z) = 1$ and $z \geq 0$. Let $\mathcal{A}$ denote the quotient of $\mathbb{Z}^3$ by this equivalence relation. The infinite dihedral group D_∞ acts on $\mathcal{A}$ by $[x, y \mid z] \mapsto [z-x, z-y \mid z]$ and $[x, y \mid z] \leftrightarrow [x+z, y+z \mid z]$. Let $\mathcal{G}$ be the group of formal $\mathbb{Z}$-linear combinations of orbits of $\mathcal{A}$ under the action of D_∞.

We will associate to each triple (E, P, Q) of a curve E/K and points $P, Q \in E(K)$ an element $\gamma \in \mathcal{G}$. From this γ one can calculate the discriminant degree d of E, an upper bound for the conductor degree N of E, and the local height corrections for $mP + m'Q$. Conversely, for any $\gamma \in \mathcal{G}$, we can compute quantities that in the case that γ is associated to some (E, P, Q), coincide with the local height correction terms, discriminant degree, and conductor bound for (E, P, Q).

3.2 Curves with Semistable Reduction

Suppose that E/K is semistable, i.e., all of its singular fibers are of type I_ν (Section 2.5), and that P and Q are in $E(K)$. Suppose that E_v is a fiber of E of type I_ν lying over v. Then the group of multiplicity-one components $E_v/(E_v)_0$ is isomorphic to $\mathbb{Z}/\nu(v)\mathbb{Z}$.

Suppose that P and Q intersect the fiber at the components $a(v), b(v) \in \mathbb{Z}/\nu(v)\mathbb{Z}$ respectively. Set $\gamma \in \mathcal{G}$ equal to

$$\gamma = \sum_v \gamma_v = \sum_v \gcd(a(v), b(v), \nu(v)) \cdot [a(v), b(v) \mid \nu(v)].$$

Suppose that $[a, b \mid \nu] \in \mathcal{A}$ with $\nu > 0$ and $\gcd(a, b, \nu) = 1$. We define homomorphisms $\lambda_{(m,m')} : \mathcal{G} \longrightarrow \mathbb{Q}$ and $\mathbf{d}, \mathbf{N} : \mathcal{G} \longrightarrow \mathbb{Z}$ on the generators of $\mathcal{G}$ and extend linearly:

$$\lambda_{(m,m')}([a, b \mid \nu]) := \nu B_2((ma + m'b)/\nu),$$
$$\mathbf{d}([a, b \mid \nu]) := \nu, \quad \mathbf{N}([a, b \mid \nu]) := 1.$$

The height corrections $\hat{h}(mP + m'Q) - h(mP + m'Q)$ and the discriminant degree are the images of γ under the homomorphisms $\lambda_{(m,m')}$ and $\mathbf{d}$. Each v contributes 1 to the conductor N. Since $\gcd(a(v), b(v), \nu(v)) = 1$, the conductor degree N is less than or equal to $\mathbf{N}(\gamma)$.

We obtain formula (3–1) easily from the formulas for the height correction terms $\lambda_v(P)$ in Section 2.6. For the conductor inequality, we assume that the rank r is greater than or equal to 2. We then have

$$\hat{h}(mP + m'Q) = h(mP + m'Q) + \lambda_{(m,m')}(\gamma) \quad (3\text{–}1)$$

for $(m, m') \in \mathbb{Z}^2 \setminus (0, 0)$ and

$$12n = d = \mathbf{d}(\gamma).$$
$$\mathbf{N}(\gamma) \geq N \geq d/6 + (2 - 2g) + r \geq \frac{1}{6}\mathbf{d}(\gamma) + 4 - 2g.$$

If $g = \gcd(a(v), b(v), \nu(v)) > 1$, then we are replacing the I_ν fiber at v with g fibers of type $\mathrm{I}_{\nu/g}$, and the values of $\lambda_{(m,m')}$, $\mathbf{d}$, and $\mathbf{N}$ do not change.

3.3 A Replacement Table

Suppose that E is not necessarily semistable, and that E_v is an additive fiber. Suppose that $a(v), b(v) \in E_v/(E_v)_0$ are the components of the fiber that contain P and Q, respectively. To each triple $(E_v, a(v), b(v))$ we may associate an element γ_v of $\mathcal{G}$ whose images under $\lambda_{(m,m')}$ and $\mathbf{d}$ are the same as $\lambda_v(mP + m'Q)$ and d_v.

The following proposition describes this replacement. In the case that E_v is a I_ν^* fiber, we let t denote the distinguished component of $E_v/(E_v)_0$ that is adjacent to the identity component. We write s and s' for the two "far" components on $E_v/(E_v)_0$. In the case that both P and Q meet the identity component of E_v, we set $\gamma_v = d_v[0, 0 \mid 1] = d_v[0]$.

Proposition 3.1. *Let E be an elliptic curve over a function field K of genus g and $P, Q \in E(K)$. For each singular fiber E_v define an element $\gamma_v \in \mathcal{G}$ according to Table 2. Then*

(i) $\lambda_v(mP + m'Q) = \lambda_{(m,m')}(\gamma_v)$ *for each* $(m, m') \in \mathbb{Z}^2 \setminus (0, 0)$,

(ii) $d_v = \mathbf{d}(\gamma_v)$,

(iii) $N_v \leq \mathbf{N}(\gamma_v)$.

Proof: The first statement can be verified by comparing the values arrived at in Table 2 to the values arrived at using the formulas in Section 2.6. The other two statements are immediate. □

3.4 Remark

Note that this "replacement" table does not ensure that $R^r_{\min}(12n)$ will be attained by an elliptic surface with only semistable reduction. In fact, this is not the case. For example, the elliptic surface attaining $R^2_{\min}(24) = 1/100$ has a fiber of type IV. In this case, the point P meets the identity component of the type-IV fiber and Q meets a nonidentity component of this fiber. However, all of the local data for P and Q that are important to us

E_v	$E_v/(E_v)_0$	$(a(v),\ b(v))$	d_v	γ_v
III	$\mathbb{Z}/2\mathbb{Z}$	$(0,1)$	3	$[0,1 \mid 2] + [0,0 \mid 1]$
	$\mathbb{Z}/2\mathbb{Z}$	$(1,1)$	3	$[1,1 \mid 2] + [0,0 \mid 1]$
IV	$\mathbb{Z}/3\mathbb{Z}$	$(0,1)$	4	$[0,1 \mid 3] + [0,0 \mid 1]$
	$\mathbb{Z}/3\mathbb{Z}$	$(1,1)$	4	$[1,1 \mid 3] + [0,0 \mid 1]$
	$\mathbb{Z}/3\mathbb{Z}$	$(1,2)$	4	$[1,2 \mid 3] + [0,0 \mid 1]$
IV*	$\mathbb{Z}/3\mathbb{Z}$	$(0,1)$	8	$2[0,1 \mid 3] + 2[0,0 \mid 1]$
	$\mathbb{Z}/3\mathbb{Z}$	$(1,1)$	8	$2[1,1 \mid 3] + 2[0,0 \mid 1]$
	$\mathbb{Z}/3\mathbb{Z}$	$(1,2)$	8	$2[1,2 \mid 3] + 2[0,0 \mid 1]$
III*	$\mathbb{Z}/2\mathbb{Z}$	$(0,1)$	9	$3[0,1 \mid 2] + 3[0,0 \mid 1]$
	$\mathbb{Z}/2\mathbb{Z}$	$(1,1)$	9	$3[1,1 \mid 2] + 3[0,0 \mid 1]$
I_ν^*	$D_{4+\nu}^*/D_{4+\nu}$	$(0,t)$	$6+\nu$	$2[0,1 \mid 2] + (\nu+2)[0,0 \mid 1]$
	$D_{4+\nu}^*/D_{4+\nu}$	(t,t)	$6+\nu$	$2[1,1 \mid 2] + (\nu+2)[0,0 \mid 1]$
$\mathrm{I}_{2\mu}^*$	$D_{4+\nu}^*/D_{4+\nu}$	$(0,s)$	$6+\nu$	$(\mu+2)[0,1 \mid 2] + 2[0,0 \mid 1]$
	$D_{4+\nu}^*/D_{4+\nu}$	(t,s)	$6+\nu$	$(\mu+1)[0,1 \mid 2] + [1,1 \mid 2] + [1,0 \mid 2]$
	$D_{4+\nu}^*/D_{4+\nu}$	(s,s)	$6+\nu$	$(\mu+2)[1,1 \mid 2] + 2[0,0 \mid 1]$
	$D_{4+\nu}^*/D_{4+\nu}$	(s,s')	$6+\nu$	$[1,0 \mid 2] + (\mu+1)[1,1 \mid 2] + [0,1 \mid 2]$
$\mathrm{I}_{2\mu+1}^*$	$D_{4+\nu}^*/D_{4+\nu}$	$(0,s)$	$6+\nu$	$[0,1 \mid 4] + (\mu+1)[0,1 \mid 2] + [0,0 \mid 1]$
	$D_{4+\nu}^*/D_{4+\nu}$	(t,s)	$6+\nu$	$[2,1 \mid 4] + (\mu+1)[1,1 \mid 2] + [0,0 \mid 1]$
	$D_{4+\nu}^*/D_{4+\nu}$	(s,s)	$6+\nu$	$[1,1 \mid 4] + (\mu+1)[1,1 \mid 2] + [0,0 \mid 1]$
	$D_{4+\nu}^*/D_{4+\nu}$	(s,s')	$6+\nu$	$[1,3 \mid 4] + (\mu+1)[1,1 \mid 2] + [0,0 \mid 1]$

TABLE 2. Replacement table.

are the same as the local data we would have in the case that E had a type-I_3 and type-I_1 fiber instead of the fiber of type IV, with P meeting the identity component of the type-I_3 fiber and Q meeting the nonidentity component of the I_3 fiber. The element $\gamma \in \mathcal{G}$ corresponding to E does not tell us whether E will have a type-IV fiber or fibers of types I_3 and I_1. However, when we recover E in our moduli space (Section 5.3), we see that symmetry forces a fiber of type IV.

4. THE SEARCH

In the previous section, we assigned to each elliptic surface and pair of independent sections (E, P, Q) an element γ of the free abelian group $\mathcal{G}$ from which we could compute the local height correction terms for $mP+m'Q$, as well as the discriminant degree and an upper bound for the conductor. We first defined this assignment for E with semistable reduction, and then defined the assignment for E with additive fibers in Proposition 3.1.

In this section we compute lower bounds for the regulator over all $\gamma \in \mathcal{G}$ of fixed discriminant degree $\mathbf{d}(\gamma) = 12n$, which in turn gives us lower bounds for the regulator $R(P,Q)$ over all E of discriminant degree $12n$. In Section 4.1 we place several conditions on the elements $\gamma \in \mathcal{G}$ to eliminate those configurations that cannot be attained by elliptic surfaces.

In the case $n = 1$, the lower bound is attained by the two-parameter family of rational elliptic surfaces listed in Theorem 1.1. In the case $n = 2$, the lower bound is attained by the single $K3$ surface in Theorem 1.2.

4.1 The Combinatorial Conditions

To eliminate several configurations that cannot be attained by elliptic surfaces, we generalize the approach of [Elkies 06a]. Let (E, P, Q) be an elliptic surface with two independent sections P and Q forming a reduced basis. Let $S \in \mathbb{Z}P \oplus \mathbb{Z}Q$. We observe that the following conditions hold:

(1) $\hat{h}(mS) = m^2\hat{h}(S)$ for all $m \in \mathbb{Z}$.

(2) The naive height $h(mS)$ takes values in $\{0, 2, 4, 6, \dots\}$.

(3) If $mS \neq 0$ then

$$\hat{h}(mS) = h(mS) + \sum_v \lambda_v(mS),$$

where the sum is taken over places $v \in \mathbb{C}(C)$ at which the fiber E_v is reducible.

(4) The local correction $\lambda_v(mS)$ depends only on the Kodaira type of fiber E_v, and the component c_v of E_v meeting S.

(5) $\hat{h}(mP + m'Q) > 0$ for $(m, m') \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

(6) The pair (P, Q) forms a reduced basis: $2|\langle P, Q\rangle| \leq \hat{h}(P) \leq \hat{h}(Q)$.

(7) If $mS \neq 0$, then $h(m'S) \leq h(mS)$ for $m' \mid m$.

(8) (Shioda's inequality.) The conductor degree N of E is at least

$$N \geq (d/6) + \chi(C) + r,$$

where r is the Mordell–Weil rank of $E(K)$.

Each of the conditions above is explained in Section 2. For curves with small discriminant degree d over $C = \mathbb{P}^1$, conditions (1)–(8) will suffice in eliminating almost all configurations that cannot be realized. Over curves of higher genus, the Euler characteristic $\chi(C)$ in (8) could be negative, and Shioda's inequality does not give a lower bound for the conductor degree of an attainable configuration.

To eliminate configurations with small conductor degree, and hence prove that $R^2_{\min}(12n) = R^2_{\min}(0, 12n)$ for $n = 1, 2$, we must use a different approach (see Section 6). In Algorithm 4.1 we write down precisely how the conditions above translate into eliminating elements of $\mathcal{G}$ that will not correspond to elliptic surfaces. We implement the algorithm in GP [PARI 08].

Algorithm 4.1. (Eliminating elements of $\mathcal{G}$.)

1. Choose a partition $\{\nu_i\}_{i=1}^N$ of $12n$ of length $N \geq 2n+4$.
2. Choose $0 \leq a_i \leq \nu_i/2$ and $b_i \leq \nu$ such that $\sum_{i=1}^N [a_i, b_i \mid \nu_i] \in \mathcal{G}$.
3. Compute correction terms $\lambda_{(m,n)}(\gamma)$.
4. Select naive heights $h_{(1,0)}, h_{(0,1)}, h_{(1,1)}$ in $\{0, 2, 4, 6, \ldots\}$ for P, Q, $P + Q$ less than some bound.
5. Set $\hat{h}_{(1,0)}(\gamma) = h_{(1,0)} + \lambda_{(1,0)}(\gamma)$.
6. Set $\hat{h}_{(0,1)}(\gamma) = h_{(0,1)} + \lambda_{(0,1)}(\gamma)$.
7. Set $\hat{h}_{(1,1)}(\gamma) = h_{(1,1)} + \lambda_{(1,1)}(\gamma)$.
8. Calculate inner product $\langle\gamma\rangle = \frac{1}{2}(\hat{h}_{(1,1)} - \hat{h}_{(1,0)} - \hat{h}_{(1,0)})$.
9. Check condition for reduced basis, i.e., $2|\langle\gamma\rangle| \leq \hat{h}_{(1,0)}(\gamma) \leq \hat{h}_{(0,1)}(\gamma)$ and $\hat{h}_{(1,0)}(\gamma) > 0$.
10. Compute height matrix: $\hat{h}_{(m,m')}(\gamma) = m^2\hat{h}_{(1,0)}(\gamma) + m'^2\hat{h}_{(0,1)}(\gamma) + 2mm'\langle\gamma\rangle$.
11. Check that height matrix has no nonzero entries.
12. Calculate naive height matrix h: $h_{(m,n)}(\gamma) = \hat{h}_{(m,n)}(\gamma) - \lambda_{(m,n)}(\gamma)$.
13. Check that entries of h satisfy the naive height inequality $h_{(m,m')}(\gamma) \leq h_{(km,km')}(\gamma)$.
14. Compute the regulator $R(\gamma) = \hat{h}_{(1,0)}(\gamma)\hat{h}_{(0,1)}(\gamma) - \langle\gamma\rangle^2$.

4.2 Rational Elliptic Surfaces

Let E be an elliptic curve of discriminant degree 12 over $K = \mathbb{C}(t)$, i.e., E is a rational elliptic surface. Suppose (P, Q) is a reduced basis for a rank-2 subgroup of E. If $h(P) \geq 2$, then by (2–2), $\hat{h}(P) \geq 1$. In this case the regulator satisfies

$$R(P, Q) = \hat{h}(P)\hat{h}(Q) - \langle P, Q\rangle^2 \geq \frac{3}{4}\hat{h}(P)^2 \geq \frac{3}{4}.$$

Hence we assume that the naive height of P is zero.

If $h(Q) \geq 4$, then again from (2–2) we have that $\hat{h}(Q) \geq 3$. Because we know from [Oguiso and Shioda 91] that the minimal height of a nontorsion point on a rational elliptic surface is 1/30, we have a lower bound $\hat{h}(P) \geq 1/30$. Then

$$R(P, Q) = \hat{h}(P)\hat{h}(Q) - \langle P, Q\rangle^2 \geq \hat{h}(P)\hat{h}(Q) - \frac{1}{4}\hat{h}(P)^2$$
$$\geq \frac{1}{30}\left(\hat{h}(Q) - \frac{1}{120}\right).$$

If $\hat{h}(Q) \geq 3$ then $R(P, Q) \geq 359/3600$. Hence if we bound $\hat{h}(Q) < 3$, then we search through at least all configurations that could correspond to a triple (E, P, Q) with $R(P, Q) < 359/3600$. Hence we assume that the naive height of Q is either 0 or 2, which includes all cases in which $h(Q) < 3$.

The bounds $h(P) = 0$ and $h(Q) \leq 2$ give us a bound on $h(P + Q)$. Since

$$-1 \leq \hat{h}(P + Q) - h(P + Q) \leq 2$$

by (2–2), and

$$\hat{h}(P + Q) = \hat{h}(P) + \hat{h}(Q) + 2\langle P, Q\rangle$$
$$\leq 2\hat{h}(P) + \hat{h}(Q) \leq 6,$$

it follows that $h(P + Q) \leq 7$ also. Since the naive height is even, it follows that $h(P + Q) \leq 6$.

We program all of the conditions listed in Section 4.1, allowing the naive heights in our algorithm to be $h_{(1,0)} = 0$, $h_{(0,1)} = 0, 2$, and $h_{(1,1)} = 0, 2, 4, 6$. From this we generate a list of configurations γ.

We find that a lower bound for the regulator $R(P, Q)$ occurs for the unique configuration

$$\gamma = [1, 1 \mid 3] + [1, 2 \mid 3] + [0, 1 \mid 2] + [1, 0 \mid 2] + 2[0], \quad (4\text{–}1)$$

with $R(\gamma) = 1/36$, $\hat{h}_{(1,0)}(\gamma) = \hat{h}_{(0,1)}(\gamma) = 1/6$, and $\langle\gamma\rangle = 0$. There are no other configurations with $R(\gamma) = 1/36$.

The naive height matrix has $h_{(m,m')}(\gamma) = 0$ for $m, m' = 0, 1, 2$, and for $(3, 0)$, $(0, 3)$. Since $\mathbf{N}(\gamma) = 6$, we expect this configuration to be attained by a two-parameter family (E, P, Q) of rational elliptic surfaces (see Section 4.4, where we discuss parameter-counting heuristics). We search through our list of configurations for other γ that could correspond to (E, P, Q) with mP and $m'Q$ integral for $m \le M = 3$ and $m' \le M' = 3$. We find that (4–1) is the unique configuration over $\mathbb{P}^1$ that could give rise to a curve with these integral multiples. We also find that the value $MM' = 9$ is maximal, i.e., any other configuration that could correspond to an (E, P, Q) with mP and $m'Q$ integral for $m \le N$ and $m' \le N'$ has $NN' < 9$.

Examining the two-parameter family of curves in part (c) of Theorem 1.1, we find that it has the fiber configuration γ and this exact pattern of integral points. This proves all parts of Theorem 1.1 except implications (a), (b) $\Longrightarrow$ (c) of part (ii). In Section 5.2, we prove that the configuration γ uniquely determines the equations for this curve. Since γ is the only configuration with that pattern of integral points, this will prove the implications (a), (b) $\Longrightarrow$ (c).

The rank-2 subgroup $\mathbb{Z}P \oplus \mathbb{Z}Q$ has a total of 28 integral points. The list of configurations that we generated must include any γ corresponding to an (E, P, Q) with P and Q integral. Because of the naive height inequality, we know then that this list will contain all configurations corresponding to rational elliptic surfaces (E, P, Q) that contain rank-2 subgroups $\mathbb{Z}P \oplus \mathbb{Z}Q$ with P and Q integral. Examining our list, we also find that this configuration yields the greatest number of integral points within a rank-2 subgroup of a rational elliptic surface.

4.3 $K3$ Elliptic Surfaces

Let E be an elliptic surface of discriminant degree 24 over $K = \mathbb{C}(t)$, i.e., a $K3$ elliptic surface. Let (P, Q) form a reduced basis for $E(K)$. If the naive height $h(P)$ is greater than or equal to 4, then it follows that $\hat{h}(P) \ge 2$. In this case we can bound the regulator:

$$R(P, Q) = \hat{h}(P)\hat{h}(Q) - \langle P, Q\rangle^2 \ge \frac{3}{4}\hat{h}(P)^2 \ge \frac{3}{2}.$$

Hence we assume that the naive height of P is 0 or 2.

Suppose that the naive height $h(Q)$ is greater than or equal to 4. Then again by (2–2) we know that $\hat{h}(Q) \ge 2$. We know from [Nishiyama 96] that the minimal height of a nontorsion point on a $K3$ elliptic surface is 11/420, so we bound $\hat{h}(P) \ge 11/420$. It follows that

$$\begin{aligned} R(P, Q) &= \hat{h}(P)\hat{h}(Q) - \langle P, Q\rangle^2 \\ &\ge \hat{h}(P)\hat{h}(Q) - \frac{1}{4}\hat{h}(P)^2 \\ &\ge \frac{11}{420}\left(\hat{h}(Q) - \frac{11}{1680}\right). \end{aligned}$$

Thus if $h(Q) \ge 4$, it follows that $R(P, Q) \ge 36839/705600$. Thus if we bound $h(Q) \le 2$, we search through at least all configurations γ that could correspond to a triple (E, P, Q) with $R(P, Q) < 36839/705600$.

As in the previous section, the bounds $h(P) \le 2$ and $h(Q) \le 2$ automatically give us the bound $h(P + Q) \le 20$. We program all of the conditions listed in Section 4.1, allowing the naive heights in our algorithm to be $h_{(1,0)} = 0, 2$, $h_{(0,1)} = 0, 2$, and $h_{(1,1)} = 0, 2, \ldots, 20$. From this we generate a list of configurations γ.

We find that a lower bound for $R(P, Q)$ occurs for the configuration

$$\begin{aligned} \gamma = {} & [1, 1 \mid 5] + [1, 3 \mid 5] + [0, 1 \mid 3] + [1, 0 \mid 3] + [1, 2 \mid 3] \\ & + [1, 0 \mid 2] + [1, 1 \mid 2] + [0], \end{aligned}$$

with $R(\gamma) = 1/100$, $\hat{h}_{(1,0)}(\gamma) = 1/15$, $\hat{h}_{(0,1)}(\gamma) = 1/6$, and $\langle\gamma\rangle = -1/30$. There are no other configurations with $R(\gamma) = 1/100$. The naive height matrix has $h_{(m,0)}(\gamma) = h_{(0,m')}(\gamma) = 0$ for $m \le M = 6$ and $m' \le M' = 3$, as well as $h_{(m,\pm m)}(\gamma) = 0$ for $m \le 3$, and $h_{(m,m')}(\gamma) = 0$ for $(m, m') = (2, \pm 1)$, $(3, \pm 1)$, $(1, 3)$, $(2, 3)$, $(3, 2)$, $(4, 1)$, $(4, 2)$, $(6, 3)$, $(1, -2)$.

We search through our list for other γ that could correspond to (E, P, Q) with $P, \ldots, 6P$ and $Q, \ldots, 3Q$ integral, and find none. We again find that the value $MM' = 18$ is maximal among rank-two subgroups of $K3$ elliptic surfaces: any other configuration that could correspond to an (E, P, Q) with mP and $m'Q$ integral for $m \le N$ and $m' \le N'$ has $NN' < 18$.

Examining the $K3$ elliptic surface in part (c) of Theorem 1.2, we find that the surface has the fiber configuration γ above, and the same pattern of integral points. This proves all parts of Theorem 1.2, except that $R_{\min}(24, 2) = 1/100$ and (a), (b) $\Longrightarrow$ (c) of part (ii).

The rank-2 subgroup $\mathbb{Z}P \oplus \mathbb{Z}Q$ has a total of 52 integral points. Again, because of the naive height inequality, we know that our list will contain all configurations corresponding to $K3$ elliptic surfaces (E, P, Q) that contain a rank-2 subgroup $\mathbb{Z}P + \mathbb{Z}Q$ with P and Q integral. This $K3$ elliptic surface has the greatest number of integral points within a rank-2 subgroup.

4.4 Parameter-Counting Heuristics

Let E be an elliptic curve over $\mathbb{C}(t)$ of discriminant degree $12n$, with minimal Weierstrass equation

$$y^2 = x^3 + a_4(t)x + a_6(t).$$

Here $a_4(t)$ and $a_6(t)$ are coprime polynomials of respective degrees $4n$ and $6n$. The discriminant $\Delta(t)$ is given by

$$\Delta(t) = a_4(t)^3 - 27a_6(t)^2,$$

and is a polynomial of degree $12n$. Varying $a_4(t)$ and $a_6(t)$, we have $10n + 2 - 4 = 10n - 2$ free parameters, where we subtract 4 to account for the four dimensions of symmetry.

Suppose now that E has a nontorsion section $P = (X(t), Y(t))$, so $X(t)$ and $Y(t)$ are polynomials of respective degrees $2n$ and $3n$. Varying the coefficients of $X(t)$ and $Y(t)$, we have $(2n + 1) + (3n + 1) = 5n + 2$ free parameters. The minimal Weierstrass equation, however, is of degree $6n$ and hence imposes $6n + 1$ conditions on the section P. So in order that E have an integral nontorsion section P, we should have to impose $n - 1$ conditions on $a_4(t)$ and $a_6(t)$. So in this case, varying $a_4(t)$ and $a_6(t)$, we have $(10n - 2) - (n - 1) = 9n - 1$ free parameters. Imposing the condition that E have another section Q brings us down to $8n$ free parameters. Hence we expect to be able to force $\Delta(t)$ to have $12n - 8n = 4n$ distinct roots, but no fewer.

This gives a heuristic improvement of Shioda's inequality (2–1) in the case that E has two independent sections and $n > 2$:

$$N \geq 4n.$$

For $n = 1$ this inequality is weaker than Shioda's inequality, and for $n = 2$ it has the same strength. More importantly, however, these heuristics tell us that for configurations γ with $\mathbf{d}(\gamma) = 12n$ and $\mathbf{N}(\gamma) \geq 4n$, we expect to be able to find an $(\mathbf{N}(\gamma) - 4n)$-parameter family of elliptic surfaces corresponding to γ. For the optimal rational and $K3$ surfaces, our heuristic is correct. In Section 7, we give an example of a γ with $\mathbf{D}(\gamma) = 24$ and $\mathbf{N}(\gamma) = 8$ that satisfies all of the conditions in Section 4.1, but cannot be realized by a $K3$ elliptic surface.

For the case that $n > 2$, these heuristics explain the breakdown of the methods of this paper. As n grows, there will be several configurations of conductor degree smaller than $4n$ that probably will not be realized. Shioda's inequality, however, will not be enough to eliminate these configurations.

5. INTEGRAL POINTS AND MODULAR PARAMETERIZATIONS

In this section we discuss how we found the equations in Theorems 1.1 and 1.2. We begin by parameterizing the set of elliptic curves E with independent rational points P and Q such that P, $2P$, Q, $P+Q$, $P-Q$, and $2P+Q$ are all integral by an open subset in $\mathbb{P}^3$. A curve of degree n in this $\mathbb{P}^3$ corresponds to an elliptic surface of discriminant degree $12n$ with sections P and Q satisfying these integrality conditions. We recover the equations in Theorems 1.1 and 1.2 by finding the curves of degree 1 and 2 in this moduli space that correspond to the optimal configurations.

5.1 Parameterization of Moduli Space

We begin with an elliptic curve E with a rational point $P = (0, 0)$ placed at the origin:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x.$$

The point $2P$ is integral precisely when the slope of the tangent to E at P is integral. The slope of this line is a_4/a_3, and if this slope is integral we may translate y by $(a_4/a_3)x$ to make the slope of this line zero. This gives an equivalent condition for $2P$ being integral, namely $a_4 = 0$. Our curve is now of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2.$$

Let $Q = (X, Y)$ be another integral point of E that is independent of P. The point $P + Q$ is integral when the slope of the secant line through P and Q has integral slope. This slope is Y/X, and we write $Y = cX$ for a constant c.

Next we impose the condition that $2P+Q$ be integral, forcing the slope of the secant line through P and $P+Q$ to be integral. This line has slope

$$(-a_1 - c) + \frac{a_3}{X - c^2 + a_2 - ca_1},$$

and we write $a_3 = a(X - ca_1 + a_2 - c^2)$. Writing the Weierstrass equation that the coordinates of Q must satisfy, we obtain

$$X(X - ac)(X - c^2 - a_1c + a_2) = 0.$$

If $X = 0$, then $Q = P$. If $X = c^2 + a_1 c - a_2$, then $a_3 = 0$, and E is a singular curve. This forces $X = ac$.

Finally, we impose the condition that $P - Q$ be integral, or that the line through P and $-Q$ have integral slope. This line has slope $-a_2/c + a$, and we set $a_2 = cd$. Changing variables $a_1 = a + b + d - c$ gives us

$$y^2 + (a+b+d-c)xy - abcy = x^3 + cdx^2,$$
$$P = (0,0), \quad Q = (ac, ac^2),$$

with discriminant $\Delta = a^2b^2c^3 f(a,b,c)$, where f is a homogeneous quintic polynomial. This parameterizes the set of elliptic curves E with rational points P and Q such that P, $2P$,Q, $P+Q$, $P-Q$, $2P+Q$ are all integral.

Changing (a,b,c,d) to $(\lambda a, \lambda b, \lambda c, \lambda d)$ yields an isomorphic curve, and we have a parameterization of the (E,P,Q) moduli space by an open subset in $\mathbb{P}^3$. There is an automorphism sending Q to $-P-Q$ that arises from the linear involution of $\mathbb{P}^3$ that interchanges a and b.

A generic curve C of degree n in this moduli space corresponds to an elliptic surface $\mathcal{E}/k$ fibered over C of discriminant degree $12n$. The surface will have a configuration γ of fibers and sections P, Q given by

$$\gamma = \underbrace{n[1,1 \mid 2]}_{a=0} + \underbrace{n[1,0 \mid 2]}_{b=0} + \underbrace{n[1,1 \mid 3]}_{c=0} + \underbrace{5n[0]}_{f=0},$$

corresponding to the n points at which C meets each plane $a = 0$, $b = 0$, and $c = 0$, as well as the $5n$ points at which C meets the surface $f(a,b,c) = 0$.

We recover equations for the optimal rank-2 surfaces from this $\mathbb{P}^3$ model. All surfaces recovered as curves in this $\mathbb{P}^3$ exhibit the symmetry corresponding to switching a and b. This is where the symmetries described in Section 1.1 come from.

5.2 The Optimal Rational Elliptic Surface

We saw in Section 4.2 that the optimal $n = 1$ configuration is

$$\gamma = [1,1 \mid 3] + [1,2 \mid 3] + [0,1 \mid 2] + [1,0 \mid 2] + 2[0],$$

with $R(P,Q) = 1/36$, $\hat{h}(P) = 1/6$, $\hat{h}(Q) = 1/6$, and $\langle P,Q \rangle = 0$. This configuration has $h(mP) = 0$ and $h(mQ) = 0$ for $m = 1,\ldots,3$, $h(mP \pm m'Q) = 0$ for $m = 1,2$ and $m' = \pm 1, \pm 2$, and the negatives of these points. The configuration has conductor $\mathbf{N}(\gamma) = 6$, and hence should be attained by a two-parameter family of lines in our $\mathbb{P}^3$ model.

Suppose that

$$\lambda : \mathbb{P}^1 \longrightarrow \mathbb{P}^3,$$
$$(s : s') \longmapsto (A : B : C : D),$$

denotes a line in $\mathbb{P}^3$ that gives rise to such a rational elliptic surface E. Each A, B, C, D is a homogeneous linear form in s, s'. Without loss of generality, we may assume that $A = s$ and $B = s'$. The point $3P$ is integral on E if and only if the slope of the line through $-P$ and $-2P$ is integral. This slope is $-AB/D$, and hence either $D = qA$ or $D = qB$. Switching A and B corresponds to the symmetry interchanging Q and $-P-Q$; hence we may assume that $D = qA$.

The condition that $2Q$ is integral says that the slope of the tangent to E at Q is integral. This slope is

$$\frac{C((q+2)A + C - B)}{(q+1)A + C},$$

which implies that $(q+1)s + C \mid s - s'$. Thus we may set $C = q'(s-s') - (q+1)s$. We obtain a two-parameter family of lines

$$\lambda(q,q') : \mathbb{P}^1 \longrightarrow \mathbb{P}^3,$$
$$(s : s') \longmapsto (s : s' : (q' - q - 1)s - q's' : qs).$$

Each line in this family gives rise to a rational elliptic surface E with sections P and Q such that $\hat{h}(P) = 1/6$, $\hat{h}(Q) = 1/6$, and $\langle P,Q \rangle = 0$, except when $q = 0$ or $q' = 0$. This gives us the equations in Theorem 1.1.

5.3 The Optimal $K3$ Elliptic Surface

We find equations for the single $K3$ surface yielding the minimum regulator $R(P,Q) = 1/100$. The optimal configuration is

$$\gamma = [1,1 \mid 5] + [1,3 \mid 5] + [0,1 \mid 3] + [1,0 \mid 3] + [1,2 \mid 3]$$
$$+ [1,0 \mid 2] + [1,1 \mid 2] + [0],$$

with $R(P,Q) = 1/100$, $\hat{h}(P) = 1/15$, $\hat{h}(Q) = 1/6$, and $\langle P,Q \rangle = -1/30$. The naive height is zero for

$$P,\ \ldots,\ 6P,\ Q,\ 2Q,\ 3Q,\ P+Q,\ P+3Q,$$
$$2P+Q,\ 2P+2Q,\ 2P+3Q,\ 3P+Q,\ 3P+2Q,$$
$$3P+3Q,\ 4P+Q,\ 4P+2Q,\ 6P+3Q,\ P-Q,$$
$$P-2Q,\ 2P-Q,\ 2P-2Q,\ 3P-Q,\ 3P-3Q,$$

and the negatives of these points. We recover the configuration above as a conic in the $\mathbb{P}^3$ model.

Let

$$\lambda : \mathbb{P}^1 \longrightarrow \mathbb{P}^3,$$
$$(s : s') \longmapsto (A : B : C : D),$$

denote a curve of degree 2 in $\mathbb{P}^3$ that gives rise to such a $K3$ elliptic surface. Each A, B, C, D is a homogeneous form of degree 2 in s, s'.

A generic curve C of degree 2 in this moduli space will give rise to a surface having configuration γ of fibers and sections P, Q given by

$$\gamma = 2[1,1 \mid 2] + 2[1,0 \mid 2] + 2[1,1 \mid 3] + 10[0].$$

We will arrive at this family by forcing the curve λ to go through some specific points in the $\mathbb{P}^3$ model.

We first use the condition that $3P$ is integral, which says that $D \mid AB$. We set $A = (s' - u)v$, $B = u(s - v)$, and $D = uv$. Next we use the condition that $3P + Q$ is integral, which says that the slope of the line between $2P + Q$ and $P + Q$ is integral. This slope is

$$-A - B - D + CD/(B + D),$$

which implies that $(B + D) \mid CD$. Because $2P - Q$ is also integral, and the symmetry of the moduli space interchanging a and b corresponds to $P \leftrightarrow -P - Q$, this implies that $(A + D) \mid CD$ as well. We set $C = qs's$. This brings our configuration down to

$$\gamma = \underbrace{[1,1 \mid 4]}_{s=0} + \underbrace{[1,2 \mid 4]}_{s'=0} + \underbrace{[1,1 \mid 3]}_{v=0} + \underbrace{[1,0 \mid 3]}_{u=0} + \underbrace{[0,1 \mid 2]}_{s'=u} + \underbrace{[1,0 \mid 2]}_{s=v} + 6[0],$$

and the discriminant Δ_E of the elliptic curve to

$$\Delta_E = s^4 s'^4 u^3 v^3 (s' - u)^2 (s - v)^2 F(u, v, s, s'),$$

where F is a homogeneous polynomial of degree 6 in four variables. Since we want a fiber of type I_5 at $s = 0$ and $s' = 0$, we force the polynomial F to meet $s = 0$ and $s' = 0$ to order one.

We write $F(u, v, s, s')$ as a polynomial in s' and compute its degree-zero term:

$$\begin{aligned} F(u,v,s,s')|_{s'=0} &= q^3u^3v^3 + \left(sq^4 - 3sq^3\right)u^3v^2 \\ &\quad + \left(-2s^2q^4 + 3s^2q^3\right)u^3v \\ &\quad + \left(s^3q^4 - s^3q^3\right)u^3 \\ &= q^3u^3(s - v)^2((q - 1)s + v). \end{aligned}$$

This must divisible by s'. Clearly $s' \nmid u$, for otherwise, at $u = s' = 0$ we would have a fiber of type I_6. Similarly, $s' \mid (s - v)$; otherwise, we would have a fiber of type I_6. Hence it must be the case that $s' \mid (q - 1)s + v$. A symmetric argument shows that $s \mid (q - 1)s' + u$. Thus we may set $v = (1 - q)s + q's'$ for some q', and $u = (1 - q)s' + q''s$ for some q''.

Next we write the condition that $2P + 2Q$ is integral, which says that $B + C + D \mid C(A - B)$, or that

$$q\frac{q''s'^3 - q's's^2}{q''s + s'}$$

is integral. This implies that $q's + s' \mid s + q''s'$, and hence $q'' = 1/q'$. Up to this point we have

$$\begin{aligned} A &= -ss'q^2 + \left(q's^2 + ss' + \frac{1}{q'}s'^2\right)q + \left(-q's^2 - s's\right), \\ B &= -s'sq^2 + \left(q's^2 + s's + \frac{1}{q'}s'^2\right)q + \left(-s's - \frac{-1}{q'}s'^2\right), \\ C &= qs's, \\ D &= s'sq^2 + \left(-q's^2 - 2s's - \frac{-1}{q'}s'^2\right)q \\ &\quad + \left(q's^2 + 2s's + \frac{1}{q'}s'^2\right). \end{aligned}$$

Examining the above equations, we see that we can replace s' by $q's'$ and eliminate q'. We are left with a one-parameter family of $K3$ elliptic surfaces, with fiber configuration

$$\begin{aligned} \gamma = {} & [1,1 \mid 5] + [1,3 \mid 5] + [0,1 \mid 2] + [1,0 \mid 3] + [1,2 \mid 3] \\ & + [1,0 \mid 2] + [1,1 \mid 2] + 2[0]. \end{aligned}$$

Generically, a $K3$ surface in this family has regulator $R(P, Q) = 19/900$, with $\hat{h}(P) = 1/15$, $\hat{h}(Q) = 1/3$, and $\langle P, Q \rangle = -1/30$.

Finally, we attempt to collide the I_2 fiber at $s' = -s$ with a fiber of type I_1, forcing $u + v$ to divide the remaining quadratic factor of the discriminant. This quadratic factor is

$$\begin{aligned} & s'sq^3 + \left(-s^2 - 11s's - s'^2\right)q^2 + \left(2s^2 + 31s's + 2s'^2\right)q \\ & - 27s's. \end{aligned}$$

At $s' = -s$ this becomes $(3 - q)^3 s^3$, and forces $q = 3$. This brings us down to the desired configuration, and yields the equations in Theorem 1.2. Note that at $s' = -s$ we end up with a fiber of type IV instead of a multiplicative fiber of type I_3. This is forced by the symmetry $s \leftrightarrow s'$ of the $K3$ surface, of which $s = -s'$ is a fixed point. This symmetry comes from interchanging a and b in the moduli space. This gives us the equations for the $K3$ elliptic surface in Theorem 1.2.

6. ELLIPTIC SURFACES OVER HIGHER-GENUS CURVES

If C is a curve of genus $g > 0$, then the Euler characteristic $\chi(C)$ can be negative. In this case, the lower bound on the conductor obtained from Shioda's inequality is trivial, and we must search through all configurations γ with $\mathbf{N}(\gamma) \geq 1$. In the case $n = 1$, no new configurations arise. Hence we automatically get that $R^2_{\min}(12) = 1/36$.

Configuration	$R(P,Q)$
$[2,4 \mid 7]+[3,2 \mid 7]+[0,1 \mid 5]+[1,0 \mid 5]$	4/1225
$[3,1 \mid 7]+[1,1 \mid 6]+[1,3 \mid 4]+[0,1 \mid 4]+[1,2, \mid 3]$	5/1008
$[1,2 \mid 5]+[2,1 \mid 5]+[0,1 \mid 4]+[1,0 \mid 4]+[1,2, \mid 3]+[1,1 \mid 2]+[0]$	7/1200
$[2,3 \mid 7]+[3,5 \mid 7]+[1,0 \mid 5]+[0,1 \mid 4]+[0]$	3/490
$[1,2 \mid 8]+[1,6 \mid 7]+[2,1 \mid 5]+[2,3 \mid 4]$	1/160
$[3,3 \mid 8]+[1,5 \mid 6]+[0,2 \mid 5]+[2,0 \mid 5]$	1/150
$[2,3 \mid 7]+[0,1 \mid 5]+[1,0 \mid 5]+[1,2 \mid 4]+[1,2, \mid 3]$	1/150
$[1,0 \mid 5]+[0,3 \mid 5]+[2,4 \mid 5]+[1,1 \mid 4]+[1,2, \mid 3]+[1,1 \mid 2]$	1/150
$[2,2 \mid 7]+[1,5 \mid 6]+[2,4 \mid 5]+[0,1 \mid 4]+[1,0, \mid 2]$	1/140
$[3,1 \mid 7]+[1,1 \mid 6]+[1,3 \mid 4]+[0,1 \mid 3]+[1,2, \mid 3]+[0]$	1/126
$[1,0 \mid 8]+[1,2 \mid 5]+[1,3 \mid 5]+[2,1 \mid 4]+[1,1 \mid 2]$	7/800
$[1,5 \mid 6]+[0,2 \mid 5]+[2,0 \mid 5]+[2,2 \mid 5]+[1,1, \mid 3]$	2/225
$[1,4 \mid 7]+[2,5 \mid 7]+[2,1 \mid 4]+[0,2 \mid 3]+[1,0 \mid 3]$	4/441
$[4,2 \mid 9]+[1,2 \mid 7]+[1,6 \mid 7]+[0]$	4/441
$[1,0 \mid 5]+[2,1 \mid 5]+[1,3 \mid 4]+[0,2 \mid 3]+[1,1, \mid 3]+[0,1 \mid 2]+[1,1 \mid 2]$	17/1800
$[2,1 \mid 7]+[1,5 \mid 6]+[2,1 \mid 5]+[0,1 \mid 4]+[1,1, \mid 2]$	1/105
$[1,1 \mid 6]+[2,1 \mid 5]+[0,1 \mid 4]+[1,3 \mid 4]+[1,2, \mid 3]+[1,0 \mid 2]$	7/720
$[3,3 \mid 8]+[1,5 \mid 6]+[2,0 \mid 5]+[0,2 \mid 3]+[0,1 \mid 2]$	7/720
$[2,3 \mid 8]+[0,1 \mid 5]+[2,3 \mid 5]+[2,0 \mid 5]+[0]$	1/100

TABLE 3. Configurations with small $\mathbf{N}$.

In the case $n = 2$, however, several new configurations satisfy all combinatorial conditions. We first use a strong naive height inequality to eliminate several configurations. The inequality (2.1) can be strengthened:

Proposition 6.1. [Elkies 06a] *Let P be a point on $E/k(C)$, and suppose $m \in \mathbb{Z}$ such that $mP \neq 0$. Then*

$$\sum_{m'|m} \mu(m/m')h(m'P) \geq 0.$$

Proof: The sum can be interpreted as twice the number of points of C, counted with multiplicity, for which $mP = 0$ but $m'P \neq 0$ for each proper factor m' of m. □

We add this inequality to the list of conditions in Section 4.1 and run our search again. We find twenty configurations γ with $\mathbf{N}(\gamma) < 8$ that if realizable, would yield elliptic surfaces (E,P,Q) of discriminant degree 24, with regulator $R(P,Q) \leq 1/100$. These configurations are listed in Table 3. All configurations have $h_{(1,0)}(\gamma) = h_{(0,1)}(\gamma) = h_{(1,1)}(\gamma) = 0$.

Consider the configuration

$$\gamma = [2,4 \mid 7] + [3,2 \mid 7] + [0,1 \mid 5] + [1,0 \mid 5].$$

This configuration has $\mathbf{N}(\gamma) = 4$, satisfies all of the combinatorial conditions, and gives a regulator $R(P,Q) = 4/1225$. If realized, γ would yield an (E,P,Q) with many integral points, including P, $2P$, Q, $P \pm Q$, and $2P+Q$. However, if such an (E,P,Q) existed, it could be recovered by a curve of degree 2 in the $\mathbb{P}^3$ model we constructed. Since a curve of degree 2 in $\mathbb{P}^3$ is rational, one could fiber this surface over $\mathbb{P}^1$ to obtain a $K3$ elliptic surface (E',P',Q') with $R(P',Q') = 4/1225$. This contradicts part (i) of Theorem 1.2, which says that $R^2_{\min}(0,24) = 1/100$, and which was proven in Section 4.3.

We conduct a similar analysis of each of the configurations in Table 3. We find that each configuration, if realized, would give rise to an elliptic surface with P, $2P$, Q, $P \pm Q$, and $2P+Q$ all integral. This would imply that they are $K3$ surfaces, violating part (i) of Theorem 1.2. This proves that $R^2_{\min}(24) = 1/100$.

7. AN EXAMPLE OF A $K3$ OBSTRUCTION

In both the case of a rational elliptic surface and the case of a $K3$ elliptic surface, the lower bound for $R(P,Q)$ computed in Section 4 is attained. From a configuration γ satisfying the conditions in Section 4.1, we can write down the Néron–Severi lattice of the elliptic surface that should correspond to γ. For a rational elliptic surface, if the configuration satisfies the combinatorial conditions, then it must exist. For a $K3$ elliptic surface, however, there are examples of configurations that satisfy all of the combinatorial conditions but cannot be realized.

In this section we show an obstruction to the existence of a $K3$ elliptic surface with the fiber configuration listed below:

$$\gamma = [3, 4 \mid 8] + [1, 0 \mid 6] + [1, 2 \mid 4] + [1, 0 \mid 2] + 4[0]. \quad (7\text{–}1)$$

This configuration satisfies all of the combinatorial conditions listed in Section 4.1, and by our heuristics in Section 4.4 we expect γ to be attained by a $K3$ elliptic surface (E, P, Q), with $R(P, Q) = 1/24$, and $\hat{h}(P) = 1/24$, $\hat{h}(Q) = 1$.

7.1 $K3$ Lattices

Let U denote the hyperbolic plane, i.e., the rank-2 lattice with bilinear form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let E_8 denote the unique positive definite unimodular lattice of rank 8:

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

For a $K3$ surface $\mathcal{E}$, the second cohomology group $H^2(\mathcal{E}, \mathbb{Z})$ equipped with the cup product pairing is the unique unimodular lattice of signature $(3, 19)$, and is isometric to $H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ (see [Barth et al. 04, Chapter 7]). We call this lattice the $K3$ lattice. The Néron–Severi lattice $\mathrm{NS}(\mathcal{E})$ is a primitive sublattice of the $K3$ lattice, i.e., $H^2(\mathcal{E}, \mathbb{Z}) / \mathrm{NS}(\mathcal{E})$ is a free abelian group.[3]

The precise conditions for a lattice to have a primitive embedding into the $K3$ lattice are worked out in [Nikulin 79]. Below we write one of those conditions, which we use to show a local obstruction to the existence of a $K3$ surface corresponding to the configuration (7–1).

Theorem (Nikulin.) *The discriminant group of the Néron–Severi lattice of a $K3$ surface contains at most p factors of $\mathbb{Z}/p\mathbb{Z}$.*

[3] For elliptic surfaces that are not $K3$ or rational, we know the structure of the lattice $H^2(\mathcal{E}, \mathbb{Z})$, since it is unimodular of known signature and parity. Very little, however, is known about $\mathrm{NS}(\mathcal{E})$ in this case.

7.2 Lattice Computation

Suppose that the configuration γ in (7–1) is realized by a $K3$ elliptic surface $\mathcal{E}/k$ fibered over $\mathbb{P}^1$, and let E/K be the corresponding elliptic curve. Then the surface $\mathcal{E}$ has one I_ν fiber for each of $\nu = 8, 6, 4, 2$ and four I_1 fibers. We write a matrix M whose rows generate the essential sublattice T of the Néron–Severi lattice $\mathrm{NS}(\mathcal{E})$ (Section 2.2). For each reducible I_ν fiber, we write the $\nu - 1$ rows of M corresponding to the irreducible nonidentity components of the fiber. This gives us the first $7+5+3+1 = 16$ rows of M. The last two rows correspond to the Mordell–Weil generators of $E(K)$ of height $1/24$ and 1. We write the coordinates of the last two rows R_P and R_Q so that the intersection of P and Q is as in (7–1). We need three extra columns, since we cannot write $1/24$ as the sum of two squares. The rows of this matrix, along with the standard dot product, give us a rank-18 sublattice of the Néron–Severi lattice of $\mathcal{E}/k$. This sublattice M is the orthogonal complement in $\mathrm{NS}(\mathcal{E})$ of the rank-2 sublattice generated by a generic fiber and the zero section. This matrix M appears as Figure 1.

We let $G = M \cdot M^t$ be the Gram matrix of M. We compute the Smith normal form of G, which gives us the discriminant group of the Néron-Severi lattice:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Hence the discriminant group is

$$\mathrm{NS}(\mathcal{E})^* / \mathrm{NS}(\mathcal{E}) \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

However, for a $K3$ surface the discriminant group $\mathrm{NS}(\mathcal{E})^* / \mathrm{NS}(\mathcal{E})$ can contain at most p factors of $\mathbb{Z}/p\mathbb{Z}$. Otherwise, the Néron–Severi lattice will not admit a primitive embedding into the $K3$ lattice. It follows that the configuration (7–1) cannot be realized.

Remark 7.1. It would be interesting to examine whether adding the condition that there be no "local obstruction" (i.e., that L^*/L not contain too many factors of $\mathbb{Z}/p\mathbb{Z}$) to the set of conditions in Section 4.1 would eliminate all configurations that do not correspond to $K3$ elliptic surfaces. We believe that the addition of this condition will guarantee the existence of a $K3$ with the given configurations. If this is the case, then one might be able

$$M = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{5}{8} & \frac{5}{8} & \frac{5}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{6}
\end{pmatrix}$$

FIGURE 1. The matrix M defined in Section 7.2.

to use the techniques in this paper to write down (without equations) all $K3$ elliptic surfaces with sections of naive height less than some fixed bound. This is essentially what Nishiyama did in [Nishiyama 96] for the case in which the Mordell–Weil rank is 1.

8. POINTS OF LOW CANONICAL HEIGHT OVER $\mathbb{Q}$

In this section we search through specializations of the $K3$ surface in Theorem 1.2 to find elliptic curves over $\mathbb{Q}$ with a point of low canonical height. We write the equations over the t line, where $t = s/s'$. The points P and Q have integral multiples, and hence we expect the corresponding specializations P_t and Q_t to have low canonical height on the elliptic curve E_t over $\mathbb{Q}$.

There are 28 pairs of points on E with naive height 2 that have the form $mP + m'Q$, with m and m' both nonzero. In this case the x-coordinate of $mP + m'Q$ will have a double pole for some values of t, i.e., the denominator of $x(mP+m'Q)$ is the square of a linear polynomial in t. We find the values of t that force this denominator to be zero, which in turn forces $mP_t + m'Q_t = 0$ on the curve E_t. This will yield a point P_g generating $\mathbb{Z}P_t \oplus \mathbb{Z}Q_t$ of potentially small canonical height on E_t:

$$\hat{h}(P_g) = \frac{\gcd(m', m)^2}{m^2}\hat{h}(P_t).$$

We expect this height to be small, both because several multiples of the point P_g will be integral on E_t, and also because of various specialization theorems of Silverman and Tate (cf. [Silverman 94, Chapter III, Section 11]).

For example, the point $7P+Q$ has naive height 2, and we find that

$$x(7P+Q) = \frac{72t^6 + 426t^5 - 501t^4 - 1233t^3 - 198t^2 + 216t}{49t^2 + 84t + 36},$$

which has denominator $(7t+6)^2$. Specializing the elliptic curve E to $t = -6/7$ then gives us an elliptic curve E_t over $\mathbb{Q}$ and a nontorsion point P_t, which we write in minimal Weierstrass form:

$$y^2 + xy + y = x^3 - 125615x + 61201397,$$
$$P_t = (7107, -602054).$$

This is, in fact, the elliptic curve over $\mathbb{Q}$ of conductor 3990 with the point of smallest known canonical height, $\hat{h}(P_t) \approx 0.00445716$.

In fact, specializing this $K3$ elliptic fibration at such values of t yields the five smallest known canonical heights. The data from our search is compiled in Table 4. We let N_E denote the conductor of E, and P_g a generator for the subgroup $\mathbb{Z}P \oplus \mathbb{Z}Q$. A much more comprehensive list including these curves was written down

t	$E : (a_1, a_2, a_3)$	N_E	$\hat{h}(P_g)$
$-6/7$	$(-759/49, 6840/343, -4617000/16807)$	3990	0.00445716
$2/9$	$(-13/27, -160/243, 4000/6561)$	3630	0.00451934
$-3/4$	$(-219/16, 495/32, -96525/512)$	1430	0.00486993
$3/4$	$(69/16, 45/32, 2025/512)$	1470	0.00498205
$-1/3$	$(-22/3, 35/9, -700/27)$	280	0.00563876
$-1/2$	$(-39/4, 15/2, -525/8)$	1890	0.00603439
$2/3$	$(11/3, 8/9, 56/27)$	350	0.00642340
1	$(6, 3, 12)$	216	0.01562106
$-8/9$	$(-433/27, 5200/243, -2002000/6561)$	30030	0.01722760
$-2/3$	$(-37/3, 112/9, -1232/9)$	462	0.01970553

TABLE 4. Specializations yielding small heights over $\mathbb{Q}$.

by Elkies in 2002, and can be found in [Elkies 02]. We use the implementation of Silverman's algorithm [Silverman 88] in GP to compute approximate values for the canonical heights listed in Table 4.

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