Secant Varieties of Segre–Veronese Varieties $\mathbb{P}^m \times \mathbb{P}^n$ Embedded by $\mathcal{O}(1,2)$

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CONTENTS

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Let $X_{m,n}$ be the Segre–Veronese variety $\mathbb{P}^m \times \mathbb{P}^n$ embedded by the morphism given by $\mathcal{O}(1,2)$. In this paper, we provide two functions $\underline{s}(m,n) \leq \overline{s}(m,n)$ such that the sth secant variety of $X_{m,n}$ has the expected dimension if $s \leq \underline{s}(m,n)$ or $\overline{s}(m,n) \leq s$. We also present a conjecturally complete list of defective secant varieties of such Segre–Veronese varieties.

1. INTRODUCTION

Let $X \subset \mathbb{P}^N$ be an irreducible nonsingular variety of dimension d. Then the sth *secant variety* of X, denoted by $\sigma_s(X)$, is defined to be the Zariski closure of the union of the linear spans of all s-tuples of points of X.

The study of secant varieties has a long history. Interest in this subject goes back to the Italian school at the turn of the twentieth century. This topic has received renewed interest over the past several decades, mainly due to its increasing importance in an ever widening collection of disciplines including algebraic complexity theory [Bürgisser et al. 97, Landsberg 06, Landsberg 08], algebraic statistics [Garcia et al. 05, Eriksson et al. 05, Aoki et al. 07], and combinatorics [Sturmfels and Sullivant 06, Sullivant 08].

The major questions surrounding secant varieties revolve around finding invariants of those objects such as dimension. A simple dimension count suggests that the expected dimension of $\sigma_s(X)$ is $\min\{N, s(d+1)-1\}$. We say that X has a *defective* s*th secant variety* if $\sigma_s(X)$ does not have the expected dimension. In particular, X is said to be *defective* if X has a defective sth secant variety for some s. For instance, the Veronese surface X in $\mathbb{P}^5$ is defective, because the dimension of $\sigma_2(X)$ is four, while its expected dimension is five. A well-known classification of the defective Veronese varieties was completed in a series of papers by Alexander and Hirschowitz [Alexander and Hirschowitz 95] (see also [Brambilla and Ottaviani 08]).

There are corresponding conjecturally complete lists of defective Segre varieties [Abo et al. 09b] and defective Grassmann varieties [Baur et al. 07]. However, secant varieties of Segre–Veronese varieties are less well understood. In recent years, considerable efforts have been made to develop techniques to study secant varieties of such varieties (see, for example, [Catalisano et al. 05, Carlini and Chipalkatti 03, Carlini and Catalisano 07, Ottaviani 06, Catalisano et al. 08, Ballico 06, Abrescia 08]). But even the classification of defective two-factor Segre–Veronese varieties is still far from complete.

In order to classify defective Segre–Veronese varieties, a crucial step is to prove the existence of a large family of nondefective such varieties. A powerful tool to establish nondefectiveness of large classes of Segre–Veronese varieties is the inductive approach based on specialization techniques, which consist in placing a certain number of points on a chosen divisor. For a given $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, we write $\mathbb{P}^{\mathbf{n}}$ for $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Let $X_{\mathbf{n}}^{\mathbf{a}}$ be the Segre–Veronese variety $\mathbb{P}^{\mathbf{n}}$ embedded by the morphism given by $\mathcal{O}(\mathbf{a})$ with $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$. As we shall see in Section 2, the problem of determining the dimension of $\sigma_s(X_{\mathbf{n}}^{\mathbf{a}})$ is equivalent to the problem of determining the value of the Hilbert function $h_{\mathbb{P}^{\mathbf{n}}}(Z, \cdot)$ of a collection Z of s general double points in $\mathbb{P}^{\mathbf{n}}$ at $\mathbf{a}$, i.e.,

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) = \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}(\mathbf{a})) - \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{I}_Z(\mathbf{a})).$$

Suppose that $a_k \geq 2$. Denote by $\mathbf{n}'$ and $\mathbf{a}'$ the k-tuples $(n_1, n_2, \dots, n_k - 1)$ and $(a_1, a_2, \dots, a_k - 1)$ respectively. Given $\mathbb{P}^{\mathbf{n}'} \subset \mathbb{P}^{\mathbf{n}}$, we have an exact sequence

$$0 \to \mathcal{I}_{\widetilde{Z}}(\mathbf{a}') \to \mathcal{I}_Z(\mathbf{a}) \to \mathcal{I}_{Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbb{P}^{\mathbf{n}'}}(\mathbf{a}) \to 0,$$

where $\widetilde{Z}$ is the residual scheme of Z with respect to $\mathbb{P}^{\mathbf{n}'}$. This exact sequence gives rise to the so-called Castelnuovo inequality

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) \geq h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}') + h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbf{a}).$$

Thus, we can conclude that

- if $h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}')$ and $h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbf{a})$ are the expected values and
- if the degrees of $\widetilde{Z}$ and $Z \cap \mathbb{P}^{\mathbf{n}'}$ are both less than or both greater than $\dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}(\mathbf{a}'))$ and $\dim H^0(\mathbb{P}^{\mathbf{n}'}, \mathcal{O}(\mathbf{a}))$ respectively,

then $h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a})$ is also the expected value.

By semicontinuity, the Hilbert function of a general collection of s double points in $\mathbb{P}^{\mathbf{n}}$ has the expected value at $\mathbf{a}$. This enables one to check whether $\sigma_s(X_{\mathbf{n}}^{\mathbf{a}})$ has the expected dimension by induction on $\mathbf{n}$ and $\mathbf{a}$.

To apply this inductive approach, we need some initial cases regarding either dimensions or degrees. The class of secant varieties of two-factor Segre–Veronese varieties embedded by the morphism given by $\mathcal{O}(1,2)$ can be viewed as one such initial case. In fact, in this case the above-mentioned specialization technique would involve secant varieties of two-factor Segre varieties, most of which are known to be defective, and thus we cannot apply this technique to find $\dim \sigma_s(X_{\mathbf{n}}^{\mathbf{a}})$ for $\mathbf{n} = (m, n)$ and $\mathbf{a} = (1, 2)$. To sidestep this problem, we therefore need an ad hoc approach.

This paper is devoted to studying secant varieties of Segre–Veronese varieties $\mathbb{P}^m \times \mathbb{P}^n$ embedded by the morphism given by $\mathcal{O}(1,2)$. Let

$$q(m,n) = \left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor.$$

Our main goal is to prove the following theorem:

Theorem 1.1. *Let $\mathbf{n} = (m, n)$ and $\mathbf{a} = (1, 2)$. If n is sufficiently large, then $\sigma_s(X_{\mathbf{n}}^{\mathbf{a}})$ has the expected dimension for $s = q(m, n)$.*

A straightforward consequence of this theorem is the following:

Corollary 1.2. *Let $\mathbf{n} = (m, n)$ and $\mathbf{a} = (1, 2)$. If n is sufficiently large, then $\sigma_s(X_{\mathbf{n}}^{\mathbf{a}})$ has the expected dimension for all $s \leq q(m, n)$.*

In order to prove Theorem 1.1, we show that if $m \leq n + 2$, then $\sigma_{\underline{s}(m,n)}(X_{\mathbf{n}}^{\mathbf{a}})$ has the expected dimension, where

$$\underline{s}(m,n) = \begin{cases} (m+1)\lfloor n/2 \rfloor - \frac{(m-2)(m+1)}{2} & n \text{ even;} \\ (m+1)\lfloor n/2 \rfloor - \frac{(m-3)(m+1)}{2} & m, n \text{ odd;} \\ (m+1)\lfloor n/2 \rfloor - \frac{(m-3)(m+1)+1}{2} & m \text{ even, } n \text{ odd.} \end{cases}$$

Theorem 1.1 then follows immediately, because $\underline{s}(m,n) = q(m,n)$ for a sufficiently large n (an explicit bound for n can be found just before Corollary 3.14).

To prove that $\sigma_{\underline{s}(m,n)}(X_{\mathbf{n}}^{\mathbf{a}})$ has the expected dimension, we will use double induction on m and n. More precisely, we will prove the following two claims:

(i) Let $\mathbf{n} = (n+1, n)$. Then the secant variety $\sigma_{\underline{s}(n+1,n)}(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension. Note that the case $\mathbf{n} = (n+2, n)$ is trivial, since $\underline{s}(n+2, n) = 0$.

(ii) Let $\mathbf{n}' = (m, n-2)$ and $\mathbf{n} = (m, n)$. If $\sigma_{\underline{s}(m,n-2)}(X^{\mathbf{a}}_{\mathbf{n}'})$ has the expected dimension, then $\sigma_{\underline{s}(m,n)}(X^{\mathbf{a}}_{\mathbf{n}})$ also has the expected dimension.

Claim (i) can be proved by an inductive approach that specializes a certain number of points on a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ of the form $\mathbb{P}^{m'} \times \mathbb{P}^n$ (see Section 2 for more details). Note that a similar approach has been successfully applied to the study of secant varieties of Segre varieties (see, for example, [Abo et al. 09b]).

The proof of (ii) relies on a different specialization technique that allows one to place a certain number of points on a two-codimensional subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ of the form $\mathbb{P}^m \times \mathbb{P}^{n-2}$ (see Section 3 for more details). This approach can be regarded as a modification of the approach introduced in [Brambilla and Ottaviani 08] that simplifies the proof of the Alexander–Hirschowitz theorem for cubic Veronese varieties. We also would like to mention that the same approach was extended to secant varieties of Grassmannians of planes in [Abo et al. 09a].

In Section 4, we will modify the above techniques to prove the following theorem:

Theorem 1.3. *Let $\mathbf{n} = (m, n)$, $\mathbf{a} = (1, 2)$, and*

$$\bar{s}(m,n) = \begin{cases} (m+1)\lfloor n/2 \rfloor + 1 & \text{if } n \text{ is even;} \\ (m+1)\lfloor n/2 \rfloor + 3 & \text{otherwise.} \end{cases}$$

Then $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension for any $s \geq \bar{s}(m,n)$.

Theorems 1.1 and 1.3 complete the classification of defective Segre–Veronese varieties $X^{1,2}_{m,n}$ for $m = 1, 2$. To be more precise, the following is an immediate consequence of these theorems:

Corollary 1.4. *Let $\mathbf{n} = (m, n)$ and $\mathbf{a} = (1, 2)$.*

(i) *If $m = 1$, then $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension for any s.*

(ii) *If $m = 2$, then $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension unless $(n, s) = (2k+1, 3k+2)$ with $k \geq 1$.*

Note that (i) is well known; see, for example, [Carlini and Chipalkatti 03]. We also mention that [Baur and Draisma 07, Theorem 1.3] gives a complete classification of the case $m = 1$, $n = 2$ for any degree $\mathbf{a} = (d_1, d_2)$, where $d_1, d_2 \geq 1$. On the other hand, to our best knowledge, (ii) was previously unknown. The defectivity of the $(3k+2)$th secant variety of $X^{1,2}_{2,2k+1}$ has already been established (see [Carlini and Chipalkatti 03, Ottaviani 06] for the proofs). Thus Corollary 1.4(ii) completes the classification of defective secant varieties of $X^{1,2}_{2,n}$.

In Section 5, we will give a conjecturally complete list of defective secant varieties of $X^{1,2}_{m,n}$. Evidence for the conjecture is provided by results in [Catalisano et al. 05, Carlini and Chipalkatti 03, Ottaviani 06]. Further evidence in support of the conjecture was obtained via the computational experiments we carried out. Thus the first part of this section will be devoted to explaining these experiments, which were done with the computer algebra system Macaulay2, developed by Dan Grayson and Mike Stillman.[1]

The proofs of Lemmas 3.10 and 4.5 are also based on computations in Macaulay2.

2. SPLITTING THEOREM

Let V be an $(m+1)$-dimensional vector space over $\mathbb{C}$ and let W be an $(n+1)$-dimensional vector space over $\mathbb{C}$. For simplicity, we write $\mathbb{P}^{m,n}$ for $\mathbb{P}^m \times \mathbb{P}^n = \mathbb{P}(V) \times \mathbb{P}(W)$. In this section, for simplicity we indicate by $X_{m,n}$ the Segre–Veronese variety $\mathbb{P}^{m,n}$ embedded by the morphism $\nu_{1,d}$ given by $\mathcal{O}(1,d)$. Let $T_p(X_{m,n})$ be the affine cone over the tangent space $\mathbb{T}_p(X_{m,n})$ to $X_{m,n}$ at a point $p \in X_{m,n}$.

For each $p \in X_{m,n}$, there are two vectors $u \in V \setminus \{0\}$ and $v \in W \setminus \{0\}$ such that $p = [u \otimes v^d] \in \mathbb{P}(V \otimes S_d(W))$. In this way, p can be identified with $([u],[v]) \in \mathbb{P}^{m,n}$ through $\nu_{1,d}$. Thus p is also denoted by $([u],[v])$. Let $p = [u \otimes v^d] \in X_{m,n}$. Then $T_p(X_{m,n}) = V \otimes v^d + u \otimes v^{d-1}W$. We denote by $Y_p(X_{m,n})$ (or just by Y_p) the $(m+1)$-dimensional subspace $V \otimes v^d$ of $V \otimes S_d(W)$.

Definition 2.1. Let $p_1, \ldots, p_s, q_1, \ldots, q_t$ be general points of $X_{m,n}$ and let $U_{m,n}(s,t)$ be the subspace of $V \otimes S_d(W)$ spanned by $\sum_{i=1}^{s} T_{p_i}(X_{m,n})$ and $\sum_{i=1}^{t} Y_{q_i}(X_{m,n})$. Then $U_{m,n}(s,t)$ is expected to have dimension

$$\min\left\{ s(m+n+1) + t(m+1), (m+1)\binom{n+d}{d} \right\}.$$

[1] Available online (http://www.math.uiuc.edu/Macaulay2/). All the Macaulay2 scripts needed to carry out these computations are available at http://www.webpages.uidaho.edu/~abo/programs.html.

We say that $S(m,n;1,d;s;t)$ *is true* if $U_{m,n}(s,t)$ has the expected dimension. For simplicity, we denote $S(m,n;1,d;s;0)$ by $T(m,n;1,d;s)$.

Note that $U_{m,n}(s,0)$ is the affine cone of $\sigma_s(X_{m,n})$.

Remark 2.2. Let $q_1,\dots,q_t$ be general points of $X_{m,n}$ and let $\sigma_s(X_{m,n})$ be the sth secant variety of $X_{m,n}$. By Terracini's lemma [Terracini 11], the span of the tangent spaces to $X_{m,n}$ at s generic points is equal to the tangent space to $\sigma_s(X_{m,n})$ at the generic z point in the linear subspace spanned by the s points. Thus the vector space $U_{m,n}(s,t)$ can be thought of as the affine cone over the tangent space to the join $J(\mathbb{P}(Y_{q_1}),\dots,\mathbb{P}(Y_{q_t}),\sigma_s(X_{m,n}))$ of $\mathbb{P}(Y_{q_1}),\dots,\mathbb{P}(Y_{q_t})$ and $\sigma_s(X_{m,n})$ at a general point in the linear subspace spanned by $q_1,\dots,q_t$ and z. Therefore, $S(m,n;1,d;s;t)$ is true if and only if $J(\mathbb{P}(Y_{q_1}),\dots,\mathbb{P}(Y_{q_t}),\sigma_s(X_{m,n}))$ has the expected dimension. In particular, $\sigma_s(X_{m,n})$ has the expected dimension if and only if $S(m,n;1,d;s;0)$ is true.

Remark 2.3. Let $N = (m+1)\binom{n+d}{d}$. Then $H^0(\mathbb{P}^{m,n},\mathcal{O}(1,d))$ can be identified with the set of hyperplanes in $\mathbb{P}^{N-1}$. Since the condition that a hyperplane $H\subset\mathbb{P}^N$ contains $\mathbb{T}_p(X_{m,n})$ is equivalent to the condition that H intersects $X_{m,n}$ in the first infinitesimal neighborhood of p, the elements of $H^0(\mathbb{P}^{m,n},\mathcal{I}_{p^2}(1,d))$ can be viewed as hyperplanes containing $\mathbb{T}_p(X_{m,n})$.

Let $q\in X_{m,n}$. A similar argument shows that the elements of $H^0(\mathbb{P}^{m,n},\mathcal{I}_{q^2|_{\mathbb{P}(Y_q)}}(1,d))$ can be identified with hyperplanes containing Y_q, where $q^2|_{\mathbb{P}(Y_q)}$ is a zero-dimensional subscheme of $X_{m,n}$ of length $m+1$.

Let $p_1,\dots,p_s,q_1,\dots,q_t\in X_{m,n}$ and let

$$Z=\{p_1^2,\dots,p_s^2,q_1^2|_{\mathbb{P}(Y_{q_1})},\dots,q_t^2|_{\mathbb{P}(Y_{q_t})}\}.$$

Recall that Terracini's lemma says that the linear subspace spanned by $\mathbb{T}_{p_1}(X_{m,n}),\dots,\mathbb{T}_{p_s}(X_{m,n})$ is the tangent space to $\sigma_s(X_{m,n})$ at a general point in the linear subspace spanned by $p_1,\dots,p_s$. This implies that $\dim J(\mathbb{P}(Y_{q_1}),\dots,\mathbb{P}(Y_{q_t}),\sigma_s(X_{m,n}))$ equals the value of the Hilbert function $h_{\mathbb{P}^{m,n}}(Z,\cdot)$ of Z at $(1,d)$, i.e.,

$$h_{\mathbb{P}^{m,n}}(Z,(1,d))=\dim H^0(\mathbb{P}^{m,n},\mathcal{O}(1,d)) - \dim H^0(\mathbb{P}^{m,n},\mathcal{I}_Z(1,d)).$$

In particular,

$$h_{\mathbb{P}^{m,n}}(Z,(1,d))=\min\{s(m+n+1)+t(m+1),N\}$$

if and only if $S(m,n;1,d;s;t)$ is true.

Definition 2.4. A sextuple $(m,n;1,d;s;t)$ is called *subabundant* if

$$s(m+n+1)+t(m+1)\le(m+1)\binom{n+d}{d},$$

and it is called *superabundant* if

$$s(m+n+1)+t(m+1)\ge(m+1)\binom{n+d}{d}.$$

We say that $(m,n;1,d;s;t)$ is *equiabundant* if it is both subabundant and superabundant. For brevity, we will write the quintuple $(m,n;1,d;s)$ instead of the sextuple $(m,n;1,d;s;0)$.

Assume that $S(m,n;1,d;s;t)$ is true. Note that when $(m,n;1,d;s;t)$ is superabundant, $U_{m,n}(s,t)$ coincides with the whole space $V\otimes S_d(W)$, whereas for subabundant $(m,n;1,d;s;t)$, $U_{m,n}(s,t)$ can be a proper subspace of the whole space.

Remark 2.5. Given two vectors (s,t) and (s',t'), we say that $(s,t)\ge(s',t')$ if $s\ge s'$ and $t\ge t$. Suppose that $S(m,n;1,2;s;t)$ is true and that $(m,n;1,2;s;t)$ is subabundant (respectively superabundant). Then $S(m,n;1,2;s';t')$ is true for any choice of s' and t' with $(s,t)\ge(s',t')$ (respectively with $(s,t)\le(s',t')$).

Remark 2.6. Suppose that $m=0$. We make the following simple remarks:

(i) Let $q\in X_{0,n}$. Then $\mathbb{P}(Y_q(X_{0,n}))$ is just q itself. If $q_1,\dots,q_t$ are general points of $X_{0,n}$ and if $(0,n;1,d;s;t)$ is subabundant, then $S(0,n;1,d;s;t)$ is true if and only if $T(0,n;1,d;s)$ is true.

(ii) By the Alexander–Hirschowitz theorem [Alexander and Hirschowitz 95], we know that $T(0,n;1,d;n+1)$ is true. Then if $(0,n;1,d;s)$ is superabundant and if $s\ge n+1$, then $T(0,n;1,d;s)$ is true.

Theorem 2.7. *Let $m=m'+m''+1$ and let $s=s'+s''$. If $(m',n;1,d;s';s''+t)$ and $(m'',n;1,d;s'';s'+t)$ are subabundant (respectively superabundant, equiabundant) and if $S(m',n;1,d;s';s''+t)$ and $S(m'',n;1,d;s'';s'+t)$ are true, then $(m,n;1,d;s;t)$ is subabundant (respectively superabundant, equiabundant) and $S(m,n;1,d;s;t)$ is true.*

Proof: Here we prove the theorem only in the case that $(m',n;1,d;s';s''+t)$ and $(m'',n;1,d;s'';s'+t)$ are both subabundant, because the remaining cases can be proved in a similar manner. Let V' and V'' be subspaces of V of dimensions $m'+1$ and $m''+1$ respectively. Suppose that V is the direct sum of V' and V''. Let $p = [u \otimes v^d] \in X_{m,n}$. If $u \in V'$, then we have

$$\begin{aligned} T_p(X_{m,n}) &= V \otimes v^d + u \otimes v^{d-1} W \\ &= (V' \otimes v^d + u \otimes v^{d-1}W) \oplus (V'' \otimes v^d) \\ &= T_p(X_{m',n}) \oplus Y_{p''}(X_{m'',n}) \end{aligned}$$

for some $p'' \in X_{m'',n}$ (p'' must be of the form $[u'' \otimes v^d]$ with $u'' \in V''$). Similarly, one can prove that if $u \in V''$, then $T_p(X_{m,n}) = Y_{p'}(X_{m',n}) \oplus T_p(X_{m'',n})$ for some $p' \in X_{m',n}$.

Let $q = [u' \otimes v'^d] \in X_{m,n}$. Then there exist $q' \in X_{m',n}$ and $q'' \in X_{m'',n}$ such that

$$\begin{aligned} Y_q(X_{m,n}) &= V \otimes v'^d \\ &= (V' \otimes v'^d) \oplus (V'' \otimes v'^d) \\ &= Y_{q'}(X_{m',n}) \oplus Y_{q''}(X_{m'',n}). \end{aligned}$$

Thus one can conclude that

$$U_{m,n}(s,t) \simeq U_{m',n}(s', s''+t) \oplus U_{m'',n}(s'', s'+t).$$

By assumption,

$$\dim U_{m',n}(s', s''+t) = s'(m'+n+1) + (s''+t)(m'+1)$$

and

$$\dim U_{m'',n}(s'', s'+t) = s''(m''+n+1) + (s'+t)(m''+1).$$

Thus

$$\begin{aligned} &\dim U_{m,n}(s,t) \\ &\quad = \dim U_{m',n}(s', s''+t) + \dim U_{m'',n}(s'', s'+t) \\ &\quad = s(m+n+1) + t(m+1) \\ &\quad \le (m'+1)\binom{n+d}{d} + (m''+1)\binom{n+d}{d} \\ &\quad = (m+1)\binom{n+d}{d}, \end{aligned}$$

and hence $(m,n;1,d;s,t)$ is subabundant and $S(m,n;1,d;s;t)$ is true. □

Below, we will discuss three examples to illustrate how to use Theorem 2.7. These examples will be used in later sections.

Example 2.8. In this example, we apply Theorem 2.7 to prove that $T(2,2;1,2;s)$ is true for every $s \le 3$. Note that $(2,2;1,2;s)$ is subabundant for $s \le 3$. Thus it suffices to show that $T(2,2;1,2;3)$ is true. Taking $m'=1$, $m''=0$ and $s'=2$, $s''=1$, one can reduce $T(2,2;1,2;3)$ to $S(1,2;1,2;2;1)$ and $S(0,2;1,2;1;2)$. Indeed, $(1,2;1,2;2;1)$ and $(0,2;1,2;1;2)$ are both subabundant. The statement $S(1,2;1,2;2;1)$ can be reduced again to twice $S(0,2;1,2;1;2)$ by taking

$$m' = m'' = 0 \quad \text{and} \quad s' = s'' = 1.$$

This means that $T(2,2;1,2;3)$ is reduced to the triple $S(0,2;1,2;1;2)$. Clearly $S(0,2;1,2;1;0)$ is true, and so is $S(0,2;1,2;1;2)$ by Remark 2.6(i). Hence we have completed the proof.

Example 2.9. We prove that $T(m,1;1,2;3)$ is true for any m. The proof is by induction. It has already been proved that $T(1,1;1,2;3)$ is true (see [Catalisano et al. 05]). Suppose now that $T(m-1,1;1,2;3)$ is true for some m. Note that $(m,1;1,2;3)$ is superabundant. Since $(m-1,1;1,2;3;0)$ and $(0,1;1,2;0;3)$ are also superabundant, we can reduce $T(m,1;1,2;3)$ to $T(m-1,1;1,2;3)$ and $S(0,1;1,2;0;3)$. Clearly, $S(0,1;1,2;0;3)$ is true, by Remark 2.6(i). Since $T(m-1,1;1,2;3)$ is true by the induction hypothesis, $T(m,1;1,2;3)$ is also true.

Example 2.10. Here we prove that $T(n+1,n;1,2;s)$ is true for any $s \le \lfloor \frac{n+1}{2} \rfloor + 1$ and any $n \ge 1$. Note that $(n+1,n;1,2;s)$ is subabundant for such an s. Thus it is sufficient to prove that $T(n+1,n;1,2;s)$ is true if $s = \lfloor \frac{n+1}{2} \rfloor + 1$.

First suppose that n is even, i.e., $n = 2k$ for some integer $k \ge 1$. Then $s = k+1$. Since $(2k,2k;1,2;k;1)$ and $(0,2k;1,2;1;k)$ are both subabundant, it follows that $T(2k+1,2k;1,2;k+1)$ can be reduced to $S(2k,2k;1,2;k;1)$ and $S(0,2k;1,2;1;k)$. Analogously, $S(2k,2k;1,2;k;1)$ can be reduced to $S(2k-1,2k;1,2;k-1;2)$ and $S(0,2k;1,2;1;k)$. This means that $T(2k+1,2k;1,2;k+1)$ is now reduced to $S(2k-1,2k;1,2;k-1;2)$ and twice $S(0,2k;1,2;1;k)$ (we will denote it by $2 * S(0,2k;1,2;1;k)$).

We can repeat the same process $(k-2)$ times to reduce $T(2k+1,2k;1,2;k+1)$ to $S(k,2k;1,2;0;k+1)$ and $(k+1) * S(0,2k;1,2;1;k)$. Indeed, we have only to check that $(2k+1-h,2k;1,2;k+1-h;h)$ is subabundant for any $1 \le h \le k+1$, which is true. Now the statement $S(k,2k;1,2;0;k+1)$ can be reduced to $S(k-1,2k;1,2;0;k+1)$ and $S(0,2k;1,2;0;k+1)$, since both $(k,2k;1,2;0;k+1)$ and $(k-1,2k;1,2;0;k+1)$ are subabundant.

Analogously $S(k-1,2k;1,2;0;k+1)$ can be reduced to $S(k-2,2k;1,2;0;k+1)$ and $S(0,2k;1,2;0;k+1)$.

Repeating the same process $k-2$ times, we can reduce $S(k, 2k; 1, 2; 0; k+1)$ to $(k+1) * S(0, 2k; 1, 2; 0; k+1)$. Recall that $(0, 2k; 1, 2; 1; k)$ and $(0, 2k; 1, 2; 0; k+1)$ are subabundant. Thus $S(0, 2k; 1, 2; 1; k)$ and $S(0, 2k; 1, 2; 0; k+1)$ are true, because $T(0, 2k; 1, 2; 1)$ and $T(0, 2k; 1, 2; 0)$ are true and by Remark 2.6(i). This implies that $T(2k+1, 2k; 1, 2; k+1)$ is true.

In the same way, we can also prove that $T(n+1, n; 1, 2; s)$ is true when n is odd. Indeed, $T(2k+2, 2k+1; 1, 2; k+2)$ can be reduced to

$$(k+2) * S(0, 2k+1; 1, 2; 1; k+1)$$

and

$$(k+1) * S(0, 2k+1; 1, 2; 0; k+2).$$

Since $S(0, 2k+1; 1, 2; 1; k+1)$ and $S(0, 2k+1; 1, 2; 0; k+2)$ are true, so is $T(2k+2, 2k+1; 1, 2; k+2)$.

As immediate consequences of Theorem 2.7, we can prove the following two propositions:

Proposition 2.11. *$T(m, n; 1, 2; s)$ is true if $s \leq m+1$ and $m \leq \binom{n+1}{2}$ or if $s \geq (m+1)(n+1)$.*

Proof: We first prove that if $m \leq \binom{n+1}{2}$, then $T(m, n; 1, 2; s)$ is true for any $s \leq m+1$. Since $(m, n; 1, 2; s)$ is subabundant for any $s \leq m+1$, it is enough to prove that $T(m, n; 1, 2; m+1)$ is true. Applying Theorem 2.7 $m+1$ times, we can reduce to $(m+1) * S(0, n; 1, 2; 1; m)$. Indeed, $(0, n; 1, 2; 1; m)$ is subabundant, since from the assumption $m \leq \binom{n+1}{2}$ it follows that

$$(n+1) + m \leq \binom{n+2}{2}.$$

It also follows that $(m-h, n; 1, 2; m+1-h; h)$ is subabundant for any $1 \leq h \leq m-1$. Since $S(0, n; 1, 2; 1; 0)$ is true, so is $S(0, n; 1, 2; 1; m)$, which implies that $T(m, n; 1, 2; m+1)$ is true.

To show that $T(m, n; 1, 2; s)$ is true for any $s \geq (m+1)(n+1)$, it is enough to prove that $T(m, n; 1, 2; (m+1)(n+1))$ is true, since $(m, n; 1, 2; (m+1)(n+1))$ is superabundant. In the same way as in the previous case the statement can be reduced to $(m+1) * S(0, n; 1, 2; m+1; (m+1)n)$. Since $(0, n; 1, 2; m+1)$ is superabundant and $T(0, n; 1, 2; m+1)$ is true, it follows that $(0, n; 1, 2; m+1; (m+1)n)$ is superabundant and $S(0, n; 1, 2; m+1; (m+1)n)$ is true. Thus $T(m, n; 1, 2; (m+1)(n+1))$ is true. □

Remark 2.12. In Sections 3 and 4, we will use different techniques to improve the bounds given in Proposition 2.11.

Proposition 2.13. *Suppose that $m \geq 1$ and $d \geq 3$. Let*

$$\ell = \left\lfloor \frac{\binom{n+d}{d}}{m+n+1} \right\rfloor \quad \text{and} \quad h = \left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil.$$

Then

(i) *$T(m, n; 1, d; s)$ is true for any $s \leq \ell(m+1)$.*

(ii) *If $(n, d) \neq (2, 4)$, $(3, 4)$, $(4, 3)$, $(4, 4)$ and if $s \geq h(m+1)$, then $T(m, n; 1, d; s)$ is true.*

(iii) *If $(n, d) = (2, 4)$, $(3, 4)$, $(4, 3)$, or $(4, 4)$, then $T(m, n; 1, d; s)$ is true for any $s \geq (h+1)(m+1)$.*

Proof: (i) Suppose that $s = \ell(m+1)$. Since

$$\begin{aligned}\ell(n+1) + \ell m &= \ell(m+n+1) \\ &\leq \frac{\binom{n+d}{d}}{m+n+1}(m+n+1)\binom{n+d}{d},\end{aligned}$$

then $(0, n; 1, d; \ell; \ell m)$ is subabundant (this implies that $(h, n; 1, d; \ell + h\ell; \ell(m-h))$ is also subabundant for all $1 \leq h \leq m$). Then $T(m, n; 1, d; \ell(m+1))$ can be reduced to $(m+1) * S(0, n; 1, d; \ell; \ell m)$. Furthermore, since $\ell < \left\lfloor \binom{n+d}{d}/n+1 \right\rfloor$, then $S(0, n; 1, d; \ell; 0)$ is true by the Alexander–Hirschowitz theorem. Thus $S(0, n; 1, d; \ell; \ell m)$ is true by Remark 2.6(i). This implies, by Theorem 2.7, that $T(m, n; 1, d; \ell(m+1))$ is true.

(ii) Let $s = h(m+1)$. Then $(m, n; 1, d; s)$ is clearly superabundant. The statement $T(m, n; 1, d; s)$ can be reduced to $(m+1) * S(0, n; 1, d; h; hm)$. Suppose that $n \neq 3, 4$. Then the Alexander–Hirschowitz theorem says that $T(0, n; 1, d; h)$ is true, and so is $S(0, n; 1, d; h; hm)$. Hence by Theorem 2.7 it follows that $T(m, n; 1, d; h(m+1))$ is true.

(iii) Suppose that $(n, d) = (2, 4)$, $(3, 4)$, $(4, 3)$ or $(4, 4)$. Then $T(0, n; 1, d; h+1)$ is true by the Alexander–Hirschowitz theorem, and thus $S(0, n; 1, d; h; (h+1)m)$ is also true. Therefore the same argument as in (ii) proves that $T(m, n; 1, d; (h+1)(m+1))$ is true. □

3. SEGRE–VERONESE VARIETIES $\mathbb{P}^m \times \mathbb{P}^n$ EMBEDDED BY $\mathcal{O}(1, 2)$: SUBABUNDANT CASE

Let V be an $(m+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{e_0, \ldots, e_m\}$ and let W be an $(n+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{f_0, \ldots, f_n\}$. As in the previous section, $X_{m,n}$ denotes $X^{1,2}_{m,n}$. Let U_L be a two-codimensional subspace of W and let $L = \mathbb{P}(V) \times \mathbb{P}(U_L)$. Note that if p is a point of $\nu_{1,2}(L)$, then the affine cone $T_p(X_{m,n})$ over the tangent space to $X_{m,n}$ at p modulo $V \otimes S_2(U_L)$ has dimension $(m+n+1)-(m+n-2+1) = 2$.

Definition 3.1. Let $k = \lfloor n/2 \rfloor$ and let

$$\underline{s}(m,n) = \begin{cases} (m+1)k - \frac{(m-2)(m+1)}{2}, & n \text{ even;} \\ (m+1)k - \frac{(m-3)(m+1)}{2}, & m,\, n \text{ odd;} \\ (m+1)k - \frac{(m-3)(m+1)+1}{2}, & m \text{ even, } n \text{ odd.} \end{cases}$$

Note that $\underline{s}(m, m-2) = 0$. We will sometimes drop the parameters m, n when they are clear from the context.

The goal of this section is to prove that if $m \leq n+2$, then $T(m,n;1,2;s)$ is true for any $s \leq \underline{s}(m,n)$. Since $(m,n;1,2;s)$ is subabundant, it is sufficient to prove that $T(m,n;1,2;\underline{s}(m,n))$ is true. The key point is to restrict to subspaces of codimension 2 and to use two-step induction on n.

It is obvious that $T(m,m-2;1,2;0)$ is true. It also follows from Example 2.10 that

$$T(m,m-1;1,2;\underline{s}(m,m-1))$$

is true. Thus it remains only to show that if $T(m,n-2;1,2;\underline{s}(m,n-2))$ is true, then so is $T(m,n;1,2;\underline{s}(m,n))$. To do this, we need to introduce the auxiliary statements $\underline{R}(m,n)$ and $Q(m,n)$ (see Definitions 3.2 and 3.6) and use double induction on m and n to prove such auxiliary statements.

Definition 3.2. Let k and $\underline{s} = \underline{s}(m,n)$ be as given in Definition 3.1. Note that

$$\underline{s}(m,n-2) = \underline{s} - (m+1).$$

Let $p_1, \ldots, p_{\underline{s}-(m+1)}$ be general points of L, let $q_1, \ldots, q_{m+1}$ be general points of $\mathbb{P}^{m,n} \setminus L$, and let $\underline{V}_{m,n}$ be the vector space

$$\left\langle V \otimes S_2(U_L), \sum_{i=1}^{\underline{s}-(m+1)} T_{p_i}(X_{m,n}), \sum_{i=1}^{m+1} T_{q_i}(X_{m,n}) \right\rangle.$$

Note that the following inequality holds:

$$\begin{aligned} \dim \underline{V}_{m,n} &\leq (m+1)\binom{n}{2} + 2[\underline{s} - (m+1)] \\ &\quad + (m+1)(m+n+1) \\ &= \begin{cases} (m+1)\binom{n+2}{2}, & n \text{ even, or } m,\, n \text{ odd;} \\ (m+1)\binom{n+2}{2} - 1, & m \text{ even, } n \text{ odd.} \end{cases} \end{aligned}$$

We say that $\underline{R}(m,n)$ *is true* if equality holds.

Remark 2.3 implies that $\underline{R}(m,n)$ is true if and only if

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z\cup L}(1,2)) = \begin{cases} 0, & n \text{ even or } m,\, n \text{ odd;} \\ 1, & m \text{ even, } n \text{ odd,} \end{cases}$$

where $Z = \{p_1^2, \ldots, p^2_{\underline{s}-(m+1)}, q_1^2, \ldots, q^2_{m+1}\}$.

Proposition 3.3. *Let k and $\underline{s} = \underline{s}(m,n)$ be as given in Definition 3.1. If $\underline{R}(m,n)$ is true and if $T(m,n-2;1,2;\underline{s}-(m+1))$ is true, then $T(m,n;1,2;\underline{s})$ is true.*

Proof: Let $p_1, \ldots, p_{\underline{s}} \in \mathbb{P}^{m,n}$ and $Z = \{p_1^2, \ldots, p_{\underline{s}}^2\}$. Then it is easy to check that

$$\begin{aligned} &\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \\ &\quad \geq \begin{cases} \frac{m^3-m}{2}, & n \text{ even or } m,\, n \text{ odd;} \\ k+1+\frac{m^3}{2}, & m \text{ even, } n \text{ odd.} \end{cases} \end{aligned}$$

Suppose that

$$p_1, \ldots, p_{\underline{s}-(m+1)} \in L$$

and that

$$p_{\underline{s}-m}, \ldots, p_{\underline{s}} \in \mathbb{P}^{m,n} \setminus L.$$

Let $Z = \{p_1^2, \ldots, p_{\underline{s}}^2\}$ and

$$Z' = Z \cap L = \{p_1^2, \ldots, p^2_{\underline{s}-(m+1)}\}.$$

Then we have the following short exact sequence:

$$0 \to \mathcal{I}_{Z\cup L}(1,2) \to \mathcal{I}_Z(1,2) \to \mathcal{I}_{Z',L}(1,2) \to 0.$$

Taking cohomology, we have

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z\cup L}(1,2)) \to H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \\ &\to H^0(L, \mathcal{I}_{Z'}(1,2)). \end{aligned}$$

Thus we must have

$$\begin{aligned} \dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) &\leq \dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z\cup L}(1,2)) \\ &\quad + \dim H^0(L, \mathcal{I}_{Z'}(1,2)). \end{aligned}$$

Since $\underline{R}(m,n)$ and $T(m,n-2;1,2;\underline{s}-(m+1))$ are true, we have

$$\dim H^0(\mathbb{P}^{m,n},\mathcal{I}_Z(1,2)) \leq \begin{cases} \dim H^0(L,\mathcal{I}_{Z'}(1,2))+1, & m \text{ even, } n \text{ odd;} \\ \dim H^0(L,\mathcal{I}_{Z'}(1,2)), & \text{otherwise,} \end{cases}$$

from which the proposition follows. $\square$

To prove that $T(m,n;1,2;\underline{s}(m,n))$ is true, it is therefore enough to show that $\underline{R}(m,n)$ is true if $m \leq n$. The proof is again by two-step induction on n. To be more precise, we first prove that $\underline{R}(m,m)$ and $\underline{R}(m,m+1)$ are true. Then we show that if $\underline{R}(m,n-2)$ is true, then $\underline{R}(m,n)$ is also true.

Proposition 3.4. *$\underline{R}(m,m)$ is true for any $m \geq 1$.*

Proof: Without loss of generality, we may assume that $U_L = \langle f_2,\dots,f_{m+1}\rangle$. Let $p_0,\dots,p_m \in \mathbb{P}^{m,m} \setminus L$. For each $i \in \{0,\dots,m\}$, we have $p_i = [u_i \otimes v_i^2]$, where $u_i \in V$ and $v_i \in W \setminus U_L$. Recall that

$$T_{p_i}(X_{m,m}) = V \otimes v_i^2 + u_i \otimes v_i W.$$

To prove the proposition, we will find explicit vectors u_i and v_i such that

$$V \otimes S_2(W) \equiv \sum_{i=0}^{m} T_{p_i}(X_{m,m}) \pmod{V \otimes S_2(U_L)}.$$

Let $u_i = e_i$ for each $i \in \{0,\dots,m\}$ and let

$$v_i = \begin{cases} f_i & \text{for } i=0,1, \\ if_0+f_1+f_i & \text{for } 2 \leq i \leq m. \end{cases}$$

Then we have

$$T_{p_i}(X_{m,m}) = \begin{cases} \left\langle e_0\otimes f_0^2,\dots,e_m\otimes f_0^2, e_0\otimes f_0f_1,\dots,e_0\otimes f_0f_m\right\rangle, \\ \qquad \text{if } i=0; \\ \left\langle e_0\otimes f_1^2,\dots,e_m\otimes f_1^2, e_1\otimes f_0f_1,\dots,e_1\otimes f_1f_m\right\rangle, \\ \qquad \text{if } i=1; \\ \big\langle e_0\otimes (if_0+f_1+f_i)^2,\dots,e_m\otimes (if_0+f_1+f_i)^2, \\ \qquad e_i\otimes (if_0+f_1+f_i)f_0,\dots, \\ \qquad\quad e_i\otimes (if_0+f_1+f_i)f_m\big\rangle, \\ \qquad \text{if } i\geq 2. \end{cases}$$

Now we prove that every monomial in

$$\{e_i\otimes f_jf_k \mid 0\leq i,j\leq 1, j\leq k\leq m\}$$

lies in $\langle V\otimes S_2(U_L), \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$.

For each $i\in\{2,\dots,m\}$, we have

$$\begin{aligned} &e_0\otimes(if_0+f_1+f_i)^2 \\ &\quad\equiv e_0\otimes(i^2f_0^2+f_1^2+f_i^2+2if_0f_1+2if_0f_i+2f_1f_i) \\ &\quad\equiv e_0\otimes 2f_1f_i \\ &\qquad \pmod{\langle V\otimes S_2(U_L), T_{p_1}(X_{m,m}), T_{p_2}(X_{m,m})\rangle}. \end{aligned}$$

Indeed, $e_0\otimes f_0^2, e_0\otimes f_0f_1$ and $e_0\otimes f_0f_i$ are in $T_{p_1}(X_{m,m})$, $e_0\otimes f_1^2$ is in $T_{p_2}(X_{m,m})$, $e_0\otimes f_i^2$ is in $V\otimes S_2(U_L)$. Similarly, one can prove that

$$e_1\otimes(if_0+f_1+f_i)^2 \equiv e_1\otimes 2if_0f_i \pmod{\langle V\otimes S_2(U_L), T_{p_1}(X_{m,m}), T_{p_2}(X_{m,m})\rangle}$$

for each $i\in\{2,\dots,m\}$. So we have proved that

$$e_i\otimes f_jf_k \in \sum_{i=0}^m T_{p_i}(X_{m,m})$$

if $i,j\in\{0,1\}$ and $k\in\{0,\dots,m\}$.

Note that for each $i\in\{2,\dots,m\}$,

$$\begin{aligned} e_i\otimes(if_0+f_1+f_i)f_0 &\equiv ie_i\otimes f_0f_1+e_i\otimes f_0f_i; \\ e_i\otimes(if_0+f_1+f_i)^2 &\equiv 2ie_i\otimes f_0f_1+2ie_i\otimes f_0f_i \\ &\qquad + 2e_i\otimes f_1f_i; \\ e_i\otimes(if_0+f_1+f_i)f_1 &\equiv ie_i\otimes f_0f_1+e_i\otimes f_1f_i \end{aligned}$$

modulo $\langle V\otimes S_2(U_L), \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$. Thus

$$\begin{aligned} &e_i\otimes(if_0+f_1+f_i)^2-2e_i\otimes(if_0+f_1+f_i)f_0 \\ &\quad -(2-2/i)e_i\otimes(if_0+f_1+f_i)f_1 \end{aligned}$$

is congruent to $(2/i)e_i\otimes f_1f_i$ modulo

$$\left\langle V\otimes S_2(U_L), \sum_{i=0}^m T_{p_i}(X_{m,m})\right\rangle.$$

Thus $e_i\otimes f_1f_i$ and hence $e_i\otimes f_0f_1$ and $e_i\otimes f_0f_i$ as well are in $\langle V\otimes S_2(U_L), \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$.

For every integer j such that $i\neq j$ and $j\geq 2$, we have

$$\begin{aligned} e_i\otimes(if_0+f_1+f_i)f_j &\equiv ie_i\otimes f_0f_j+e_i\otimes f_1f_j; \\ e_i\otimes(jf_0+f_1+f_j)^2 &\equiv 2je_i\otimes f_0f_j+2e_i\otimes f_1f_j \end{aligned}$$

modulo $\langle V\otimes S_2(U_L), \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$. Hence

$$\begin{aligned} &e_i\otimes(jf_0+f_1+f_j)^2-(2j/i)e_i\otimes(if_0+f_1+f_i)f_j \\ &\quad\equiv (2-2j/i)e_i\otimes f_1f_j. \end{aligned}$$

This implies that $e_i\otimes f_1f_j$, and hence $e_i\otimes f_0f_j$, is contained in $\langle V\otimes S_2(U_L), \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$, which completes the proof. $\square$

Proposition 3.5. $\underline{R}(m, m+1)$ *is true for any* $m \geq 1$.

Proof: We prove the statement only for m even, since the other case can be proved in the same way.

If m is even, then $\underline{s}(m, m+1) = 3m/2 + 1$. Let $p_1, \dots, p_{m/2} \in L$ and $q_1, \dots, q_{m+1} \in \mathbb{P}^{m,m+1} \setminus L$. Choose a subvariety $\mathbb{P}^{m,m} = \mathbb{P}(V) \times \mathbb{P}(W') \subset \mathbb{P}^{m,m+1}$ in such a way that the intersection of $\mathbb{P}^{m,m}$ with L is $\mathbb{P}^{m,m-2}$. We denote it by H. Specialize $q_1, \dots, q_{m+1}$ on $H \setminus L$. Suppose that $p_1, \dots, p_{m/2} \notin H$. Let $Z = \{p_1^2, \dots, p_{m/2}^2, q_1^2, \dots, q_{m+1}^2\}$. Then we have an exact sequence

$$0 \to \mathcal{I}_{Z \cup L \cup H}(1,2) \to \mathcal{I}_{Z \cup L}(1,2) \to \mathcal{I}_{(Z \cup L) \cap H, H}(1,2) \to 0.$$

By Proposition 3.4, statement $\underline{R}(m,m)$ is true. Thus $\dim H^0(\mathcal{I}_{(Z \cup L) \cap H, H}(1,2)) = 0$. So we have

$$\dim H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{Z \cup L \cup H}(1,2)) = \dim H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{Z \cup L}(1,2)).$$

Thus we need to prove that

$$\dim H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{Z \cup L \cup H}(1,2)) = 1.$$

Let $\widetilde{Z}$ be the residual of $Z \cup L$ by H. Then

$$H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{Z \cup L \cup H}(1,2)) \simeq H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{\widetilde{Z}}(1,1)).$$

Note that $\widetilde{Z}$ consists of L, $m+1$ simple points $q_1, \dots, q_{m+1}$, and $m/2$ double points $p_1^2, \dots, p_{m/2}^2$.

We denote by $X'_{m,m+1}$ the Segre variety $X^{1,1}_{m,m+1}$ obtained from embedding $\mathbb{P}^{m,m+1}$ by the morphism given by $\mathcal{O}(1,1)$. The condition that

$$\dim H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{\widetilde{Z}}(1,1)) = 1,$$

i.e.,

$$h_{\mathbb{P}^{m,m+1}}(\widetilde{Z}, (1,1)) = (m+1)(m+2) - 1,$$

is equivalent to the condition that the following subspace of $V \otimes W$ has dimension $(m+1)(m+2) - 1$:

$$\left\langle V \otimes U_L, \sum_{i=1}^{m/2} T_{p_i}(X'_{m,m+1}), \sum_{i=1}^{m+1} \langle u'_i \otimes v'_i \rangle \right\rangle,$$

where $q_i = [u'_i \otimes v'_i]$. Without loss of generality, we may assume that $U_L = \langle f_0, \dots, f_{m-1} \rangle$ and that $W' = \langle f_1, \dots, f_{m+1} \rangle$. Since $p_i \in L$ for each $i \in \{1, \dots, m/2\}$, there are $u_i \in V$ and $v_i \in U_L$ such that $p_i = [u_i \otimes v_i]$. Recall that $T_{p_i}(X'_{m,m+1}) = V \otimes v_i + u_i \otimes W$. So we have

$$T_{p_i}(X'_{m,m+1}) \equiv u_i \otimes \langle f_m, f_{m+1} \rangle \pmod{V \otimes U_L},$$

which implies that

$$\langle V \otimes U_L, T_{p_i}(X'_{m,m+1}) \rangle = (V \otimes f_0) \oplus \left\langle V \otimes (U_L \cap W'), \sum_{i=1}^{m/2} u_i \otimes \langle f_m, f_{m+1} \rangle \right\rangle.$$

Thus

$$\left\langle V \otimes U_L, \sum_{i=1}^{m/2} T_{p_i}(X'_{m,m+1}), \sum_{i=1}^{m+1} \langle u'_i \otimes v'_i \rangle \right\rangle = (V \otimes f_0) \oplus \left\langle V \otimes (U_L \cap W'), \sum_{i=1}^{m/2} u_i \otimes \langle f_m, f_{m+1} \rangle, \sum_{i=1}^{m+1} \langle u'_i \otimes v'_i \rangle \right\rangle.$$

Note that

$$T_1 = \{e_i \otimes f_0 \mid 0 \leq i \leq m\}$$

and

$$T_2 = \{e_i \otimes f_j \mid 0 \leq i \leq m, 1 \leq j \leq m-1\}$$

are bases for $V \otimes f_0$ and $V \otimes (U_L \cap W')$ respectively. Let $u_i = e_{i-1}$ for every $i \in \{1, \dots, m/2\}$. Then

$$T_3 = \{e_i \otimes f_j | 0 \leq i \leq m/2 - 1, m \leq j \leq m+1\}.$$

Let T_4 be the set of vectors of the standard basis for $V \otimes W$ not included in the set $T_1 \cup T_2 \cup T_3$. Then T_4 consists of $m+2$ distinct nonzero vectors. Choose $m+1$ distinct elements of T_4 as $u'_i \otimes v'_i$'s. Then $\bigcup_{i=1}^4 T_i$ spans a vector space of dimension $(m+1)(m+2) - 1$. Thus we have completed the proof. □

Let U_M be another two-codimensional subspace of W and let M be the subvariety of $\mathbb{P}^{m,n}$ of the form $\mathbb{P}(V) \times \mathbb{P}(U_M)$. If L and M are general, then we have

$$\begin{aligned} \dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{L \cup M}(1,2)) &= (m+1)\left[\binom{n+2}{2} - 2\binom{n}{2} + \binom{n-2}{2}\right] \\ &= 4(m+1). \end{aligned}$$

This is equivalent to the condition that the subspace of $V \otimes W$ spanned by $V \otimes U_L$ and $V \otimes U_M$ has codimension $4(m+1)$.

Definition 3.6. Let $p_1, \dots, p_{m+1}$ be general points of L and let $q_1, \dots, q_{m+1}$ be general points of M. We denote by $W_{m,n}$ the subspace of $V \otimes S_2(W)$ spanned by $V \otimes S_2(U_L)$, $V \otimes S_2(U_M)$, $\sum_{i=1}^{m+1} T_{p_i}(X_{m,n})$, and $\sum_{i=1}^{m+1} T_{q_i}(X_{m,n})$. Then $\dim W_{m,n}$ is expected to be

$(m+1)\binom{n+2}{2}$. We say that $Q(m,n)$ *is true* if $W_{m,n}$ has the expected dimension.

Remark 3.7. Keeping the same notation as in the previous definition, we denote by Z the zero-dimensional subscheme $\{p_1^2,\dots,p_{m+1}^2,q_1^2,\dots,q_{m+1}^2\}$. Then $Q(m,n)$ is true if and only if $\dim H^0(\mathbb{P}^{m,n},\mathcal{I}_{Z\cup L\cup M}(1,2))=0$.

Proposition 3.8. *If $Q(m,n)$ and $\underline{R}(m,n-2)$ are true, then $\underline{R}(m,n)$ is also true.*

Proof: Let $\underline{s}=\underline{s}(m,n)$, $p_1,\dots,p_{\underline{s}-(m+1)}\in L$, and $q_1,\dots,q_{m+1}\in\mathbb{P}^{m,n}\setminus L$. Suppose that $p_1,\dots,p_{\underline{s}-2(m+1)}\in L\cap M$, $p_{\underline{s}-2m-1}\dots,p_{\underline{s}-(m+1)}\in L\setminus(L\cap M)$, and $q_1,\dots,q_{m+1}\in M$. Let $Z'=Z\cap M$. Then we have the exact sequence

$$0\to\mathcal{I}_{Z\cup L\cup M}(1,2)\to\mathcal{I}_{Z\cup L}(1,2)\to\mathcal{I}_{Z'\cup(L\cap M),M}(1,2)\to 0.$$

Taking cohomology gives rise to the following exact sequence:

$$0\to H^0(\mathbb{P}^{m,n},\mathcal{I}_{Z\cup L\cup M}(1,2))\to H^0(\mathbb{P}^{m,n},\mathcal{I}_{Z\cup L}(1,2))\to H^0(M,\mathcal{I}_{Z'\cup(L\cap M),M}(1,2)).$$

By the assumption that $Q(m,n)$ is true, we have $\dim H^0(\mathbb{P}^{m,n},\mathcal{I}_{Z\cup L\cup M}(1,2))=0$. Thus the following inequality holds:

$$\dim H^0(\mathbb{P}^{m,n},\mathcal{I}_{Z\cup L}(1,2))\le\dim H^0(M,\mathcal{I}_{Z'\cup(L\cap M),M}(1,2)).$$

Hence if $\underline{R}(m,n-2)$ is true, then so is $\underline{R}(m,n)$. $\square$

Lemma 3.9. *If $Q(m-2,n)$ and $Q(1,n)$ are true, then $Q(m,n)$ is also true.*

Proof: Let V' be an $(m-1)$-dimensional subspace of V and let V'' be a two-dimensional subspace of V. Suppose that V can be written as the direct sum of V' and V''. Let $U=\langle V\otimes S_2(U_L),V\otimes S_2(U_M),T_p(X_{m,n})\rangle$. Suppose that $p=([u],[v])\in\mathbb{P}^{m-2,n}=\mathbb{P}(V')\times\mathbb{P}(U_L)$. Then $V\otimes v^2\subset V\otimes S_2(W)$. Thus

$$T_p(X_{m,n})\equiv T_p(X_{m-2,n})\pmod{V\otimes S_2(U_L)}.$$

Similarly, one can prove that

$$T_q(X_{m,n})\equiv T_q(X_{1,n})\pmod{V\otimes S_2(U_L)}$$

if $q=([u],[v])\in\mathbb{P}(V'')\times\mathbb{P}(W)$.

This means that if $p_1,\dots,p_{m+1}\in\mathbb{P}(V')\times\mathbb{P}(W)$ and if $q_1,\dots,q_{m+1}\in\mathbb{P}(V'')\times\mathbb{P}(W)$, then

$$\begin{aligned}\Big\langle V\otimes S_2(U_L),\sum_{i=1}^{m+1}T_{p_i}(X_{m,n}),\sum_{i=1}^{m+1}T_{q_i}(X_{m,n})\Big\rangle &=\Big\langle V'\otimes S_2(U_L),\sum_{i=1}^{m+1}T_{p_i}(X_{m-2,n})\Big\rangle\\ &\oplus\Big\langle V''\otimes S_2(U_L),\sum_{i=1}^{m+1}T_{q_i}(X_{1,n})\Big\rangle.\end{aligned}$$

In other words, $W_{m,n}\simeq W_{m-2,n}\oplus W_{1,n}$. Therefore, if $Q(m-2,n)$ and $Q(1,n)$ are true, so is $Q(m,n)$. $\square$

Lemma 3.10. *Let $n\ge 3$. Then $Q(1,n)$ and $Q(2,n)$ are true.*

Proof: Here we prove that $Q(1,n)$ is true only for $n\ge3$, because the proof of the remaining case follows the same path.

Without loss of generality, we may assume that $U_L=\langle f_0,\dots,f_{n-2}\rangle$ and $U_M=\langle f_2,\dots,f_n\rangle$. Let $U_K=\langle f_0,f_1,f_{n-1},f_n\rangle$ and let $K=\mathbb{P}(V)\times\mathbb{P}(U_K)$. Note that $S_2(W)=\langle S_2(U_L),S_2(U_M),S_2(U_K)\rangle$, and so

$$V\otimes S_2(W)=\langle V\otimes S_2(U_L),V\otimes S_2(U_M),V\otimes S_2(U_K)\rangle.$$

In other words, $H^0(\mathbb{P}^{1,n},\mathcal{I}_{L\cup M\cup K}(1,2))=0$. Specializing p_1 and p_2 on $K\cap L$ and q_1 and q_2 on $K\cap M$ yields the following short exact sequence, where $Z=\{p_1^2,p_2^2,q_1^2,q_1^2\}$:

$$0\to\mathcal{I}_{Z\cup L\cup M\cup K}(1,2)\to\mathcal{I}_{Z\cup L\cup M}(1,2)\to\mathcal{I}_{(Z\cup L\cup M)\cap K,K}(1,2)\to0.$$

Since $H^0(\mathbb{P}^{1,n},\mathcal{I}_{L\cup M\cup K}(1,2))$ vanishes, so does $H^0(\mathbb{P}^{1,n},\mathcal{I}_{Z\cup L\cup M\cup K}(1,2))$. Thus, in order to show that $Q(1,n)$ is true, it is enough to prove that $Q(1,3)$ is true.

Let p_1 and p_2 be general points of L and let q_1 and q_2 be general points of M. To prove that $Q(1,3)$ is true, we show directly that

$$W_{1,3}=\big\langle V\otimes S_2(U_L),V\otimes S_2(U_M),T_{p_1}(X_{1,3}),T_{p_2}(X_{1,3}),T_{q_1}(X_{1,3}),T_{q_2}(X_{1,3})\big\rangle. \tag{3–1}$$

Recall that $T_p(X_{1,3})$ for $p=[u\otimes v^2]$ is isomorphic to $V\otimes v^2+u\otimes vW$. Thus we can check equality (3–1) as follows. Let $S=\mathbb{C}[e_0,e_1,f_0,\dots,f_3]$. Choose randomly $u_1,\dots,u_4\in V$, $v_1,v_2\in U_L$, and $v_3,v_4\in U_M$. For each $i\in\{1,\dots,4\}$, let T_i be the ideal of S generated by

$u_i \otimes v_i^2, e_1 \otimes v_i^2, u_i \otimes v_i f_0, \ldots, u_i \otimes v_i f_3$. Let I_L and I_M be the ideals of S generated by $V \otimes S_2(U_L)$ and $V \otimes S_2(U_M)$ respectively and let $I = \sum_{i=1}^4 T_i + I_L + I_M$. The minimal set of generators for I can be computed in Macaulay2, and we checked that the members of the minimal generating set form a basis for $V \otimes S_2(W)$. □

Proposition 3.11. *Let $n \geq 3$. Then $Q(m,n)$ is true for any m.*

Proof: The proof is by two-step induction on m. Since we have already proved this proposition for $m = 1$ and 2, we may assume that $m \geq 3$. By the induction hypothesis, $Q(m-2,n)$ is true. Since $Q(1,n)$ is true by Lemma 3.10, it immediately follows from Lemma 3.9 that $Q(m,n)$ is true. □

As we have already mentioned, the following is an immediate consequence of Proposition 3.11:

Corollary 3.12. *Let $m \leq n$. Then $\underline{R}(m,n)$ is true.*

Proof: The proof is by induction on n. By Proposition 3.4, $\underline{R}(m,m)$ is true. The statement $\underline{R}(m,m+1)$ is also true by Proposition 3.5. Assume that $\underline{R}(m,n-2)$ is true for some $n \geq m$. We may also assume that $n \geq 3$. From Proposition 3.8 and Proposition 3.11 it therefore follows that $\underline{R}(m,n)$ is true. Thus we have completed the proof. □

Theorem 3.13. *Let k and $\underline{s} = \underline{s}(m,n)$ be as given in Definition 3.1 and suppose that $m \leq n+2$. Then $T(m,n;1,2;s)$ is true for any $s \leq \underline{s}$.*

Proof: Since $(m,n;1,2;s)$ is subabundant, it is enough to prove that $T(m,n;1,2;\underline{s})$ is true. The proof is by induction on n. If $n = m-2$, then $\underline{s}(m,m-2) = 0$. Thus $T(m,m-2;1,2;0)$ is clearly true. If $n = m-1$, then $\underline{s}(m,m-1) = \lfloor (m-1)/2 \rfloor + 1$. By Example 2.10, $T(m,m-1;1,2;s)$ is true for any $s \leq \lfloor m/2 \rfloor + 1$.

Now suppose that the statement is true for some $m \leq n$. By Proposition 3.3, $T(m,n;1,2;\underline{s})$ reduces to $T(m,n-2;1,2;\underline{s}-(m+1))$ and $\underline{R}(m,n)$. By Corollary 3.12, $\underline{R}(m,n)$ is true for any $m \leq n$. It follows therefore that $T(m,n;1,2;\underline{s})$ is true, which completes the proof. □

Define a function $r(m,n)$ as follows:

$$r(m,n) = \begin{cases} m^3 - 2m & \text{if } m \text{ is even and if } n \text{ is odd;} \\ \frac{(m-2)(m+1)^2}{2} & \text{otherwise.} \end{cases}$$

Corollary 3.14. *Suppose that $n > r(m,n)$. Then $T(m,n;1,2;s)$ is true if*

$$s \leq \left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor.$$

Proof: Since $(m,n;1,2;s)$ is subabundant, it suffices to show that $T(m,n;1,2;s)$ is true for $s = \left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor$. Note that

$$s = \begin{cases} (m+1)k - \frac{(m-2)(m+1)}{2} + \left\lfloor \frac{m^3-m}{2(m+n+1)} \right\rfloor, \\ \qquad n \text{ even;} \\ (m+1)k - \frac{(m-3)(m+1)}{2} + \left\lfloor \frac{m^3-m}{2(m+n+1)} \right\rfloor, \\ \qquad m,\ n \text{ odd;} \\ (m+1)k - \frac{(m-3)(m+1)+1}{2} + \left\lfloor \frac{n+m^3+2}{2(m+n+1)} \right\rfloor, \\ \qquad \text{otherwise.} \end{cases}$$

It is straightforward to show that if $n > r(m,n)$, then $s = \underline{s}(m,n)$. Thus it follows immediately from Theorem 3.13 that $T(m,n;1,2;s)$ is true. □

Remark 3.15. If $m = 1$, then $r(1,n) < 0$. Since $\underline{s}(1,n) = n+1$, it follows that $T(1,n;1,2;n+1)$ is true. Since $(1,n;1,2;n+1)$ is equiabundant, $T(1,n;1,2;s)$ is therefore true for any s.

4. SEGRE–VERONESE VARIETIES $\mathbb{P}^m \times \mathbb{P}^n$ EMBEDDED BY $\mathcal{O}(1,2)$: SUPERABUNDANT CASE

In this section, we keep the same notation as in Section 3. Let $k = \lfloor n/2 \rfloor$ and let

$$\bar{s}(m,n) = \begin{cases} (m+1)k+1 & \text{if } n \text{ is even;} \\ (m+1)k+3 & \text{otherwise.} \end{cases}$$

It is straightforward to show that $(m,n;1,2;\bar{s}(m,n))$ is superabundant. The main goal of this section is to prove that $T(m,n;1,2;\bar{s}(m,n))$ is true, which implies that $T(m,n;1,2;s)$ is true for any $s \geq \bar{s}(m,n)$.

Definition 4.1. Let $\bar{s} = \bar{s}(m,n)$, let $p_1, \ldots, p_{\bar{s}-(m+1)}$ be general points of L, let $q_1, \ldots, q_{m+1}$ be general points of $\mathbb{P}^{m,n} \setminus L$, and let $\overline{V}_{m,n}$ be the vector space

$$\left\langle V \otimes S_2(U_L), \sum_{i=1}^{\bar{s}-(m+1)} T_{p_i}(X_{m,n}), \sum_{i=1}^{m+1} T_{q_i}(X_{m,n}) \right\rangle.$$

Then the following inequality holds:

$$\dim \overline{V}_{m,n} \leq (m+1)\binom{n+2}{2}.$$

We say that $\overline{R}(m,n)$ *is true* if equality holds.

Remark 4.2. In the same way as in the proofs of Propositions 3.3 and 3.8, one can prove the following:

(i) If $\overline{R}(m,n)$ and $T(m,n-2;1,2;\bar{s}(m,n-2))$ are true, then $T(m,n;1,2;\bar{s}(m,n))$ is true.

(ii) If $Q(m,n)$ and $\overline{R}(m,n-2)$ are true, then $\overline{R}(m,n)$ is true. In particular, if $\overline{R}(m,n-2)$ is true, then $\overline{R}(m,n)$ is true, because $Q(m,n)$ is true for $n \geq 3$ by Proposition 3.11.

Definition 4.3. Suppose that $(m,n) \neq (1,1)$. A 4-tuple $(m,n;1,d)$ is said to be *balanced* if

$$m \leq \binom{n+d}{d} - d.$$

Otherwise, we say that $(m,n;1,d)$ is *unbalanced*.

Remark 4.4. The notion of unbalanced was first introduced for Segre varieties (see, for example, [Catalisano et al. 02, Abo et al. 09b]). Then it was extended to Segre–Veronese varieties in [Catalisano et al. 08]. In the same paper it is also proved that if $(m,n;1,d)$ is unbalanced, then $T(m,n;1,d;s)$ fails if and only if

$$\binom{n+d}{d} - n < s < \min\left\{m+1, \binom{n+d}{d}\right\}. \qquad (4\text{–}1)$$

In particular, $T(m,2;1,2;m+1)$ is true if $m \geq 5$, and $T(m,3;1,2;m+1)$ is true if $m \geq 8$.

Here we would like briefly to explain why if s satisfies the above inequalities, then $\sigma_s(X_{m,n})$ is defective. Let $p_1,\ldots,p_s$ be generic points in $X_{m,n}$. By assumption, we have $s < n+1$. Thus there is a proper subvariety of $\mathbb{P}^{m,n}$ of type $\mathbb{P}^{s-1,n}$ that contains $p_1,\ldots,p_s$. Thus we have

$$\begin{aligned}\dim \sigma_s(X_{m,n}) &\leq s(\dim \mathbb{P}^{m,n} - \dim \mathbb{P}^{s-1,n}) + s\binom{n+d}{d} \\ &= s\left[\binom{n+d}{d} + m + 1 - s\right].\end{aligned}$$

It is straightforward to show that if s satisfies the inequalities (4–1), then

$$\begin{aligned}& s\left[\binom{n+d}{d} + m + 1 - s\right] \\ &\qquad < \min\left\{s(m+n+1), (m+1)\binom{n+d}{d}\right\}.\end{aligned}$$

Thus $\sigma_s(X_{m,n})$ is defective. This also says that for such an s, the expected dimension of $\sigma_s(X_{m,n})$ is

$$s\left[\binom{n+d}{d} + m + 1 - s\right].$$

Lemma 4.5.

(i) *If $m \geq 3$, then $\overline{R}(m,n)$ is true for any $n \geq 2$.*

(ii) *$\overline{R}(2,n)$ is true for any $n \geq 3$.*

Proof: We first prove (i) for $m \geq 8$. By Proposition 3.11 and Remark 4.2(ii), it suffices to show that $\overline{R}(m,2)$ and $\overline{R}(m,3)$ are true for any $m \geq 8$. Suppose that $n \in \{2,3\}$. If $m \geq 8$, then $(m,n;1,2)$ is unbalanced. Furthermore, $(m,n;1,2;m+1)$ is superabundant. Thus $\overline{R}(m,n)$ can be reduced to $T(m,n;1,2;m+1)$. By Remark 4.4, $T(m,n;1,2;m+1)$ is true for $n \in \{2,3\}$. Thus $\overline{R}(m,n)$ is also true for $m \geq 8$ and $n \in \{2,3\}$.

The remaining cases of (i) can be checked directly as follows: Let $S = \mathbb{C}[e_0,\ldots,e_m,f_0,\ldots,f_n]$ and let $\bar{s} = \bar{s}(m,n)$. Choose randomly $u_1,\ldots,u_{\bar{s}} \in V$, $v_1,\ldots,v_{\bar{s}-(m+1)} \in U_L$, and $v_{\bar{s}-m},\ldots,v_{\bar{s}} \in W$. For each $i \in \{1,\ldots,\bar{s}\}$, let T_i be the ideal of S generated by $e_0 \otimes v_i^2,\ldots,e_m \otimes v_i^2$, $u_i \otimes v_i f_0,\ldots,u_i \otimes v_i f_n$ and let I_L be the ideal generated by $V \otimes S_2(U_L)$. Let $I = \sum_{i=1}^{\bar{s}} T_i + I_L$. Computing the minimal generating set of I, we can check in Macaulay2 that the vector space spanned by homogeneous elements of I of multidegree $(1,2)$ coincides with $V \otimes S_2(W)$.

Claim (ii) can also be checked in the same way. □

Theorem 4.6. *$T(m,n;1,2;s)$ is true for any $s \geq \bar{s}(m,n)$.*

Proof: In Example 2.9, we showed that $T(m,1;1,2;3)$ is true for any m. One can directly check that $T(2,2;1,2;4)$ is true. So, since $\overline{R}(2,n)$ is true for any $n \geq 3$ by Proposition 4.5, it follows from Remark 4.2(i) that $T(2,n;1,2;\bar{s})$ is true for any $n \geq 1$.

Suppose now that $m \geq 3$. If $n = 0$, then $\bar{s}(m,0) = 1$, and obviously $T(m,0;1,2;1)$ is true. If $n = 1$, $T(m,1;1,2;3)$ is true. Moreover, by Proposition 4.5, we know that $\overline{R}(m,n)$ is true for any $n \geq 2$. Hence, from Remark 4.2(i) it follows that $T(m,n;1,2;\bar{s}(m,n))$ is true for any n and any $m \geq 3$. This concludes the proof. □

5. CONJECTURE

Let $X_{m,n}$ be the Segre–Veronese variety $\mathbb{P}^{m,n}$ embedded by the morphism given by $\mathcal{O}(1,2)$. The main purpose of this section is to give a conjecturally complete list of defective secant varieties of $X_{m,n}$.

Let V be an m-dimensional vector space over $\mathbb{C}$ with basis $\{e_0,\ldots,e_m\}$ and let W be an n-dimensional vector space over $\mathbb{C}$ with basis $\{f_0,\ldots,f_n\}$. As mentioned at the beginning of Section 2, for a given point

$p = [u \otimes v^2] \in X_{m,n}$, the affine cone $T_p(X_{m,n})$ over the tangent space to $X_{m,n}$ at p is isomorphic to $V \otimes v^2 + u \otimes vW$. Let $A(p)$ be the $(m+1) \times (m+1)\binom{n+2}{2}$ matrix whose ith row corresponds to $e_i \otimes v^2$, and let $B(p)$ be the $(n+1) \times (m+1)\binom{n+2}{2}$ matrix whose ith row corresponds to $u \otimes vf_i$. Then $T_p(X_{m,n})$ is represented by the $(m+n+2) \times (m+1)\binom{n+2}{2}$ matrix $C(p)$ obtained by stacking $A(p)$ and $B(p)$:

$$C(p) = (A(p)||B(p)).$$

For randomly chosen points $p_1, \ldots, p_s \in X_{m,n}$, let $T_s(X_{m,n}) = \sum_{i=1}^{s} T_{p_i}(X_{m,n})$. Then $T_s(X_{m,n})$ is represented by the $s(m+n+2) \times (m+1)\binom{n+2}{2}$ matrix $C(p_1, \ldots, p_s)$ defined by

$$C(p_1, \ldots, p_s) = (C(p_1)||C(p_2)|| \cdots ||C(p_s)).$$

Thus Remark 2.2 and semicontinuity imply that if

$$\begin{aligned}\operatorname{rank} C(p_1, \ldots, p_s)\\ &= \min\left\{ s(m+n+1), (m+1)\binom{n+2}{2} \right\},\end{aligned}$$

then $\sigma_s(X_{m,n})$ has the expected dimension.

We programmed this in Macaulay2 and computed the dimension of $\sigma_s(X_{m,n})$ for $m, n \le 10$ to detect "potential" defective secant varieties of $X_{m,n}$. This experiment shows that $X_{m,n}$ is nondefective except for

- $(m, n; 1, 2)$ unbalanced;
- $(m, n) = (2, n)$, where n is odd and $n \le 10$;
- $(m, n) = (4, 3)$.

Remark 5.1. The defective cases we found in the experiments are all well known. In Remark 4.4, we gave an explanation of why if $(m, n; 1, 2)$ is unbalanced, then $X_{m,n}$ is defective. Here we will discuss the remaining known defective cases.

It is classically known that $\sigma_5(X_{2,3})$ is defective (see [Carlini and Chipalkatti 03] and [Carlini and Catalisano 07] for modern proofs). Carlini and Chipalkatti proved in their work on Waring's problem for several algebraic forms [Carlini and Chipalkatti 03] that $T(2, 5; 1, 2; 8)$ is false. In [Ottaviani 06], Ottaviani then proved, as a generalization of the Strassen theorem [Strassen 83] on three-factor Segre varieties, that $T(2, n; 1, 2; s)$ fails if $(n, s) = (2k+1, 3k+2)$ for any $k \ge 1$. Here we sketch his proof for the defectivity of $X_{2,2k+1}$. Recall that $X_{2,2k+1}$ is the image of the Segre–Veronese embedding

$$\nu_{1,2} : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes S^2W),$$

where V and W have dimension 3 and $2k+2$ respectively. For every tensor $\phi \in V \otimes S^2W$, let $S_\phi : V \otimes W^\vee \to \wedge^2 V \otimes W \cong V^\vee \otimes W$ be the contraction operator induced by ϕ. If P, Q, and R are the three symmetric slices of ϕ, then S_ϕ can be written as a skew-symmetric matrix of order $3(2k+2)$ of the form

$$S_\phi = \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix}.$$

The rank of S_ϕ is $3(2k+2)$ for a general tensor $\phi \in V \otimes S^2W$. On the other hand, since the contraction operator corresponding to a decomposable tensor has rank 2, we have $\operatorname{rank} S_\phi \le 2s$ if ϕ is the sum of s decomposable tensors. Since the decomposable tensors correspond to the points of the Segre–Veronese variety, we can deduce that if $s = 3k+2$, then $\sigma_s(X_{2,2k+1})$ does not fill $\mathbb{P}^{3\binom{2k+3}{2}-1}$.

The defectivity of $\sigma_6(X_{4,3})$ can be proved by the existence of a certain rational normal curve in $X_{4,3}$ passing through six generic points of $X_{4,3}$. Let $\pi_1 : \mathbb{P}^{4,3} \to \mathbb{P}^4$ and $\pi_2 : \mathbb{P}^{4,3} \to \mathbb{P}^3$ be the canonical projections. Given generic points $p_1, \ldots, p_6 \in \mathbb{P}^{4,3}$, there is a unique twisted cubic $\nu_3 : \mathbb{P}^1 \to C_3 \subset \mathbb{P}^3$ that passes through $\pi_2(p_1), \ldots, \pi_2(p_6)$. Let $q_i = \nu_3^{-1}(\pi_2(p_i))$ for each $i \in \{1, \ldots, 6\}$. Since any ordered subset of six points in general position in $\mathbb{P}^4$ is projectively equivalent to the ordered set $\{\pi_1(p_1), \ldots, \pi_1(p_6)\}$, there is a rational quartic curve $\nu_4 : \mathbb{P}^1 \to C_4 \subset \mathbb{P}^4$ such that $\nu_4(q_i) = \pi_1(p_i)$ for all $i \in \{1, \ldots, 6\}$. Let $\nu = (\nu_4, \nu_3)$ and let $C = \nu(\mathbb{P}^1)$. Then C passes through $p_1, \ldots, p_6$. The image of C under the morphism given by $\mathcal{O}(1,2)$ is a rational normal curve of degree 10 $(= 4 \cdot 1 + 2 \cdot 3)$ in $\mathbb{P}^{10}$. Thus we have

$$\dim \sigma_6(X_{4,3}) \le 10 + 6(7-1) = 46 < 6(4+3+1) - 1 = 47,$$

and so $\sigma_6(X_{4,3})$ is defective. See [Carlini and Chipalkatti 03] for an alternative proof.

The experiments with our program and Remark 5.1 suggest the following conjecture:

Conjecture 5.2. *Let $X_{m,n}$ be the Segre–Veronese variety $\mathbb{P}^{m,n}$ embedded by the morphism given by $\mathcal{O}(1,2)$. Then $\sigma_s(X_{m,n})$ is defective if and only if (m, n, s) falls into one of the following cases:*

(a) $(m, n; 1, 2)$ *is unbalanced and*

$$\binom{n+2}{2} - n < s < \min\left\{ m+1, \binom{n+2}{2} \right\};$$

(b) $(m,n,s) = (2, 2k+1, 3k+2)$ *with* $k \geq 1$*;*

(c) $(m,n,s) = (4,3,6)$.

It is known that the conjecture is true for $m = 1$ (see [Carlini and Chipalkatti 03]). Here we prove that the conjecture is true for $m = 2$ as a consequence of Theorems 3.13 and 4.6.

Theorem 5.3. *$T(2,n;1,2;s)$ is true for every s except $(n,s) = (2k+1, 3k+2)$ with $k \geq 1$.*

Proof: Assume first that $n = 2k$ is even. Then we have $\overline{s} = \underline{s} = 3k+1$. Hence, from Theorems 3.13 and 4.6, it follows that $T(2,2k;1,2;s)$ is true for any s.

Suppose now that $n = 2k+1$ is odd. Then we have $\underline{s} = 3k+1$ and $\overline{s} = 3k+3$. Thus $T(2,n;1,2;s)$ is true for any $s \leq 3k+1$, by Theorem 3.13, and for any $s \geq 3k+3$, by Theorem 4.6.

If $n = 1$, then $\underline{s} = 1$ and $\overline{s} = 3$. So it remains only to prove that also $T(2,1;1,2;2)$ is true. But this has been already proved in Example 2.10. So we have completed the proof. □

In [Ottaviani 06] it is also claimed that $\sigma_{3k+2}(X_{2,2k+1})$ is a hypersurface if $k \geq 1$ and that this can be proved by modifying Strassen's argument in [Strassen 83]. Then it follows that the equation of $\sigma_{3k+2}(X_{2,2k+1})$ is given by the Pfaffian of S_ϕ, where S_ϕ is the skew-symmetric matrix introduced in Remark 5.1(i). We conclude this paper by giving an alternative proof of the fact that $\sigma_{3k+2}(X_{2,2k+1})$ is a hypersurface for $k \geq 1$.

Definition 5.4. Suppose that n is odd. Let $s = 3\lfloor n/2 \rfloor + 2$, let $p_1, \ldots, p_{s-3}$ be general points of L, let q_1, q_2, q_3 be general points of $\mathbb{P}^{2,n} \setminus L$, and let $V_{2,n}$ be the vector space

$$\Big\langle V \otimes S_2(U_L), \sum_{i=1}^{s-3} T_{p_i}(X_{2,n}), \sum_{i=1}^{3} T_{q_i}(X_{2,n}) \Big\rangle.$$

Then the following inequality holds:

$$\dim V_{2,n} \leq 3\binom{n+2}{2}.$$

We say that $R(2,n)$ *is true* if equality holds.

Lemma 5.5. *Let $n \geq 3$ be an odd integer. Then $R(2,n)$ is true.*

Proof: The proof is very similar to that of Proposition 3.5. One can easily prove that if $Q(2,n)$ is true and if $R(2,n-2)$ is true, then $R(2,n)$ is true. Since we have already proved that $Q(2,n)$ is true, it suffices to show that $R(2,3)$ is true.

Let $p_1, p_2 \in L$ and let $q_1, q_2, q_3 \in \mathbb{P}^{2,3}$. Choose a subvariety H of $\mathbb{P}^{2,3}$ of the form $\mathbb{P}^{2,2} = \mathbb{P}(V) \times \mathbb{P}(W')$ such that $\mathbb{P}^{2,2}$ intersects L in $\mathbb{P}^{2,0}$. Suppose that $p_1, p_2 \notin H$. Specializing q_1, q_2, and q_3 in $H \setminus L$, we obtain the exact sequence

$$0 \to \mathcal{I}_{Z \cup L \cup H}(1,2) \to \mathcal{I}_{Z \cup L}(1,2) \to \mathcal{I}_{(Z \cup L) \cap H, H}(1,2) \to 0,$$

where $Z = \{p_1^2, p_2^2, q_1^2, q_2^2, q_3^2\}$. Since we have already proved that $\underline{R}(2,2)$ is true, we can conclude that $\dim H^0(\mathcal{I}_{(Z \cup L) \cap H, H}(1,2)) = 0$. Thus it is enough to prove that $H^0(\mathcal{I}_{Z \cup L \cup H}(1,2)) = 0$ or $H^0(\mathcal{I}_{\widetilde{Z}}(1,1)) = 0$, where $\widetilde{Z}$ is the residual of $Z \cup L$ by H. Note that $\widetilde{Z}$ consists of two double points p_1^2, p_2^2, and three simple points q_1, q_2, q_3 in H and L. Let $X'_{2,3}$ be the Segre–Veronese variety $\mathbb{P}^{2,3}$ embedded by $\mathcal{O}(1,1)$. We want to prove that L, $\sum_{i=1}^{2} T_{p_i}(X'_{2,3})$, and $\sum_{i=1}^{3} T_{q_i}(X'_{2,3})$ span $V \otimes W$. Note that if $p = [u \otimes v]$, then $T_p(X'_{2,3}) = V \otimes v + u \otimes W$. Now assume the following:

- $U_L = \langle f_0, f_1 \rangle$ and $W' = \langle f_1, f_2, f_3 \rangle$;
- $p_1 = e_0 \otimes f_0, p_2 = e_1 \otimes f_1 \in V \otimes U_L$;
- $q_1 = e_2 \otimes f_2, q_2 = e_2 \otimes f_3 \in V \otimes W'$.

For any nonzero $q_3 \in V \otimes W'$, one can show that

$$V \otimes W = \Big\langle L, \sum_{i=1}^{2} T_{p_i}(X'_{2,3}), \sum_{i=1}^{3} T_{q_i}(X'_{2,3}) \Big\rangle.$$

Thus we have completed the proof. □

Proposition 5.6. *If $(n,s) = (2k+1, 3k+2)$ for $k \geq 1$, then $\dim \sigma_s(X_{2,n}) = 3\binom{n+2}{2} - 2$.*

Proof: The proof is by induction on k. It is well known that $\sigma_5(X_{2,3})$ is a hypersurface. Thus we may assume that $k \geq 2$. Let $p_1, \ldots, p_s \in \mathbb{P}^{2,n}$. Then there is a subvariety L of $\mathbb{P}^{2,n}$ of the form $\mathbb{P}^{2,n-2}$ such that $p_1, p_2, p_3 \in L$. Let us suppose that $p_4, \ldots, p_s \in \mathbb{P}^{2,n} \setminus L$. Then we have the exact sequence

$$0 \to \mathcal{I}_{Z \cup L}(1,2) \to \mathcal{I}_Z(1,2) \to \mathcal{I}_{Z \cap L, L}(1,2) \to 0.$$

Taking cohomology, we get

$$\dim H^0(\mathcal{I}_Z(1,2)) \leq \dim H^0(\mathcal{I}_{Z \cup L}(1,2)) + \dim H^0(\mathcal{I}_{Z \cap L, L}(1,2)).$$

By Lemma 5.5, $\dim H^0(\mathcal{I}_{Z\cup L}(1,2)) = 0$. Thus, by the induction hypothesis,

$$\dim H^0(\mathcal{I}_Z(1,2)) \le \dim H^0(\mathcal{I}_{Z\cap L,L}(1,2)) \le 1.$$

As already claimed, it is known that $T(2,n;1,2;s)$ does not hold, i.e., $\dim H^0(\mathcal{I}_Z(1,2)) \ge 1$. It follows that $\sigma_s(X_{2,n})$ is a hypersurface in the ambient space $\mathbb{P}^{3\binom{n+2}{2}-1}$. $\square$

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