Constructing CF Groups by Coclass

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CONTENTS

A pro-p-group is a *CF group* if all (except possibly the first) of its lower central series factors have order p. We describe a construction for a class of just-infinite CF pro-p-groups with fixed coclass. Based on that, we observe that the class of infinite CF pro-p-groups is comparatively large within the class of all just-infinite pro-p-groups of fixed coclass.

1. INTRODUCTION

Blackburn [Blackburn 58] introduced two special types of p-groups as generalizations of the p-groups of maximal class: a p-group G is a *CF group* if it satisfies $[\gamma_i(G) : \gamma_{i+1}(G)] = p$ for $2 \leq i \leq c$, where $G = \gamma_1(G) \geq \gamma_2(G) > \cdots > \gamma_{c+1}(G) = \{1\}$ denotes the lower central series of G, and a CF p-group is an *ECF group* if additionally $G/\gamma_2(G)$ is elementary abelian. The p-groups of maximal class are the ECF p-groups with $G/\gamma_2(G)$ of order p^2.

The p-groups of maximal class have been studied in many places; see [Leedham-Green and McKay 02] for references. As a result, it is known that the structure of p-groups of maximal class can be quite complex, and the p-groups of maximal class form an interesting class of groups. The CF and ECF p-groups are less well investigated in the literature [McKay 87, McKay 90], perhaps because of the fact that the p-groups of maximal class have already proved to be a challenge.

In [Leedham-Green and Newman 80] it was suggested that p-groups be studied using the coclass as primary invariant. If G has order p^n and class c, then its coclass is $\mathrm{cc}(G) = n - c$. This suggestion has led to a major research project yielding many new deep insights into the structure of p-groups; see [Leedham-Green and McKay 02] for details. Our general idea is to use the methods of coclass theory to study CF and ECF groups.

A first and central step in understanding the p-groups of a fixed coclass is a classification of the infinite pro-p-groups of this coclass. It is known that for every prime p and every coclass r there are only finitely many infinite pro-p-groups of coclass r. Our central aim here is to

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investigate the infinite CF and ECF pro-p-groups of a given coclass.

Recall that a pro-p-group is *just infinite* if it is infinite and every normal subgroup has finite index. It is well known that for every prime p there is exactly one infinite pro-p-group of maximal class and that this group is just infinite. The following theorem shows that the class of infinite CF pro-p-groups is significantly larger. (See Section 4 for a proof.)

Theorem 1.1. *Let p be a prime and $r \in \mathbb{N}$.*

(a) *Every infinite CF pro-p-group of coclass r is a subdirect product of a finite abelian group of order p^{r+1} and a just-infinite CF pro-p-group of coclass at most r.*

(b) *There are at least $p^{p^{r-1}-(r+2)}$ isomorphism types of just-infinite CF pro-p-groups of coclass r. (An explicit construction for these groups is outlined below.)*

As shown in [Eick 05b], there exists a nonnegative $\epsilon = \epsilon(p, r)$ such that ϵ tends to 0 if p or r tends to infinity and there are at most

$$rp^{(1+\epsilon)r^2p^{r-1}}$$

isomorphism types of just-infinite pro-p-groups of coclass r. Theorem 1.1 says that there are at least

$$p^{(1-\epsilon)p^{r-1}}$$

just-infinite CF pro-p-groups of coclass r.

The just-infinite CF pro-p-groups constructed for Theorem 1.1 all satisfy $G/\gamma_2(G) \cong C_{p^r} \times C_p$. Hence these groups are ECF if and only if $r = 1$. Infinite ECF pro-p-groups of arbitrary coclass r can be obtained as subdirect products of the finite elementary abelian group C_p^{r+1} and the infinite pro-p-group of coclass 1.

The results of this paper are based on extensive experiments using the computer algebra system GAP. [The GAP Group 08]. Based on our experiments, we state two conjectures on the structure of the just-infinite CF pro-p-groups in Section 5. These conjectures should lead to an approach to classifying the just-infinite CF pro-p-groups up to isomorphism.

2. COCLASS THEORY

In this section we recall briefly some background on the classification of p-groups by coclass with a view toward CF groups. See [Leedham-Green and McKay 02] for further information on coclass theory.

The finite p-groups of coclass r can be sorted into a graph $\mathcal{G}(p, r)$: the vertices of the graph correspond to the isomorphism types of groups of coclass r, and two vertices are joined by an edge if their corresponding groups G and H satisfy $G/\gamma_c(G) \cong H$, where c is the class of G.

The inverse limit of the groups on an infinite path in $\mathcal{G}(p, r)$ is an infinite pro-p-group of coclass r. Conversely, the lower central factors of an infinite pro-p-group of coclass r yield an infinite path in $\mathcal{G}(p, r)$. These two constructions induce a one-to-one correspondence between the maximal infinite paths in $\mathcal{G}(p, r)$ and the isomorphism types of infinite pro-p-groups of coclass r. Hence a classification of infinite pro-p-groups of coclass r is central to understanding the shape of $\mathcal{G}(p, r)$.

2.1 The Graph $\mathcal{C}(p, r)$

We define $\mathcal{C}(p, r)$ as the full subgraph of $\mathcal{G}(p, r)$ consisting of all vertices that are CF groups. The following lemma shows that an infinite path of $\mathcal{G}(p, r)$ is either completely contained in $\mathcal{C}(p, r)$ or intersects trivially with $\mathcal{C}(p, r)$. Further, it observes that the infinite paths of $\mathcal{C}(p, r)$ correspond one-to-one to the infinite pro-p-groups that are CF groups and have coclass r.

Lemma 2.1. *Let $G_0, G_1, \ldots$ denote the groups on an infinite path in $\mathcal{G}(p, r)$.*

(a) *If G_i is a CF group for some i, then all groups $G_0, G_1, \ldots$ are CF groups.*

(b) *If all groups $G_0, G_1, \ldots$ are CF groups, then their inverse limit G is a CF group.*

Proof: (a) Suppose that G_i is CF. Then $G_0, \ldots, G_{i-1}$ are CF groups, since factors of CF groups are CF. Further, for every $j \in \mathbb{N}$ it follows that G_{i+j} is a CF group if G_{i+j-1} is a CF group, since $G_{i+j}/\gamma_c(G_{i+j}) \cong G_{i+j-1}$ and $\gamma_c(G_{i+j})$ has order p, where c is the class of G_{i+j}. Statement (b) follows directly from (a). □

It is not difficult to observe that every graph $\mathcal{C}(p, r)$ contains at least one infinite path: the cyclic group C_{p^r} has a faithful representation in dimension $d = p^{r-1}(p - 1)$, and the corresponding split extension $\mathbb{Z}_p^d \rtimes C_{p^r}$ is an infinite CF pro-p-group of coclass r.

2.2 Uniserial p-adic Space Groups

A group G is a *uniserial p-adic space group* if it is an extension of a normal subgroup $T \cong \mathbb{Z}_p^d$ (its translation subgroup) by a finite p-group P (its point group) such that P acts faithfully and uniserially on T; that is, the series defined by $T_0 = T$ and $T_{i+1} = [T_i, P]$ satisfies $[T_i : T_{i+1}] = p$ for all $i \in \mathbb{N}_0$. It is not difficult to observe that every uniserial p-adic space group G is a just-infinite pro-p-group of finite coclass; see [Leedham-Green and McKay 02, Section 7.4].

The rank d of the translation subgroup is also called the *dimension* of G. It is known that (see [Leedham-Green and McKay 02, Theorem 7.4.12])

$$d = \begin{cases} p^{s-1}(p-1) \text{ for some } 1 \le s \le \mathrm{cc}(G) - 1, & p \text{ odd}, \\ 1 \le s \le \mathrm{cc}(G) + 1, & \text{otherwise.} \end{cases}$$

Every infinite pro-p-group of finite coclass is an extension of a finite p-group by a uniserial p-adic space group; see [Leedham-Green and McKay 02, Theorem 7.4.12]. Hence the uniserial p-adic space groups coincide with the just-infinite pro-p-groups of finite coclass, and they play a central role in the classification of all infinite pro-p-groups by coclass.

In the following we recall a method to determine all uniserial p-adic space groups of dimension $d = p^{s-1}(p-1)$ and coclass r with a given point group $P \le \mathrm{GL}(d, \mathbb{Z}_p)$. We assume that P acts uniserially on $\mathbb{Z}_p^d$. Then $V = T \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^d$ is irreducible as a P-module and $Z(P)$ is a nontrivial cyclic group. Let C be the subgroup of order p in $Z(P)$ and define $F = \mathrm{Fix}_C(V/T)$, the fixed points of C in V/T. It is well known that F is elementary abelian of rank $q = p^{s-1}$ by [Leedham-Green et al. 86, Theorem 3.3] and $\overline{P} = P/C$ acts uniserially on F. Let $F = F_q > \cdots > F_0 = \{0\}$ be the unique maximal $\overline{P}$-invariant series through F.

The embedding $F_i \le F$ induces a natural homomorphism $H^1(\overline{P}, F_i) \to H^1(\overline{P}, F)$, and we define $E_i(P,T)$ as the image of this homomorphism. This yields a filtration of $H^1(\overline{P}, F)$ of the form

$$\{0\} = E_0(P,T) \le E_1(P,T) \le \cdots \le E_q(P,T) = H^1(\overline{P}, F).$$

This allows us to construct space groups by coclass, as the following theorem shows; see [Leedham-Green et al. 86, Theorem 3.3, Proposition 3.8] for a proof.

Theorem 2.2. (Leedham-Green, McKay, Plesken.) *Let $P \le \mathrm{GL}(d, \mathbb{Z}_p)$ with $|P| = p^l$, $N = N_{\mathrm{GL}(d,\mathbb{Z}_p)}(P)$, $T = \mathbb{Z}_p^d$.*

(a) *There is a natural isomorphism $H^2(P,T) \cong H^1(\overline{P}, F)$ that is compatible with the natural action of N.*

(b) *The isomorphism types of extensions of T by P of coclass r correspond one-to-one to the N-orbits of elements in $E_{l-r}(P,T) \setminus E_{l-r-1}(P,T)$.*

3. THE STRUCTURE OF CF PRO-p-GROUPS

The next theorem reduces a classification of all infinite CF pro-p-groups to a classification of all *uniserial p-adic CF space groups*, that is, the uniserial p-adic space groups that are also CF groups. It also implies Theorem 1.1(a).

Theorem 3.1. *Every infinite CF pro-p-group of coclass r is a subdirect product of a finite abelian p-group of order p^{r+1} and a uniserial p-adic CF space group of coclass at most r.*

Proof: Let G be an infinite CF pro-p-group of coclass r. We define $A := G/\gamma_2(G)$ and $H := G/Z_\infty(G)$, where $Z_\infty(G)$ is the hypercenter of G. Since G is a CF group of coclass r, it follows that A is a finite p-group of order p^{r+1}. We refer to [Leedham-Green and McKay 02] for a proof that H is a uniserial p-adic space group of coclass $r - \log_p |Z_\infty(G)|$. Since quotients of CF groups are CF, it follows that H is a CF group. It remains to show that G is a subdirect product of A and H, that is, that $Z_\infty(G) \cap \gamma_2(G) = \{1\}$. We observe that

$$[\gamma_i(G) : \gamma_{i+1}(G)] = [\gamma_i(H) : \gamma_{i+1}(H)][\gamma_i(G) \cap Z_\infty(G) : \gamma_{i+1}(G) \cap Z_\infty(G)]$$

for all $i \in \mathbb{N}$. Since G and H are both CF groups, it follows that $[\gamma_i(G) : \gamma_{i+1}(G)] = [\gamma_i(H) : \gamma_{i+1}(H)] = p$ for all $i \ge 2$. Thus $\gamma_i(G) \cap Z_\infty(G) = \gamma_{i+1}(G) \cap Z_\infty(G)$ for all $i \ge 2$. Since $Z_\infty(G)$ is finite, there exists $j \in \mathbb{N}$ with $\gamma_j(G) \cap Z_\infty(G) = \{1\}$. Hence $\gamma_2(G) \cap Z_\infty(G) = \{1\}$, as desired. □

It remains to investigate the structure of the uniserial p-adic CF space groups. The following lemma provides a first investigation of their point groups.

Lemma 3.2. *If G is a uniserial p-adic CF space group, then its point group P is a finite CF group with $\mathrm{cc}(P) = \mathrm{cc}(G)$ or $\mathrm{cc}(P) = \mathrm{cc}(G) - 1$.*

Proof: Every factor of a CF group is CF, and thus also the point group of G is CF. Further, we note that $[G : \gamma_2(G)] = [P : \gamma_2(P)][T : T \cap \gamma_2(G)]$. By the uniseriality of G, we find that $T \cap \gamma_2(G) \ge T_1$ and thus $[T : T \cap \gamma_2(G)] \in \{1, p\}$. Since G and P are both CF, we can read off the coclasses of G and P from their first lower central factors, and hence we obtain that $\mathrm{cc}(G) = \mathrm{cc}(P)$ or $\mathrm{cc}(G) = \mathrm{cc}(P) + 1$, as desired. □

Next we discuss the extension structure of the uniserial p-adic CF space groups G as extensions of their translation subgroups T by their point groups P. Since the coclasses of P and G are very close to each other, it follows that the extension of T by P defining G has to be "highly nonsplit."

Theorem 3.3. *Let $P \leq \mathrm{GL}(d, \mathbb{Z}_p)$ be a uniserial CF point group of class c. Then an extension G of T by P via $\delta \in H^1(\overline{P}, F)$ is a CF group if and only if*

(1) $\delta \notin E_{c-1}(P, T)$

or

(2) $\delta \in E_{c-1}(P, T) \setminus E_{c-2}(P, T)$ *and* $\gamma_2(G) \cap T = [T, P]$.

Proof: By Theorem 2.2, the case $\mathrm{cc}(P) = \mathrm{cc}(G)$ of Lemma 3.2 is equivalent to (1), and the case $\mathrm{cc}(P) = \mathrm{cc}(G) - 1$ of Lemma 3.2 is equivalent to (2). □

As noted in [McKay 94], there is no known example of a uniserial p-adic space group G with $\mathrm{cc}(G) = \mathrm{cc}(P)$ for its point group P. It is conjectured that such uniserial p-adic space groups do not exist. This induces the following conjecture.

Conjecture 3.4. *If G is a uniserial p-adic CF space group, then its point group P is a finite CF group with $\mathrm{cc}(P) = \mathrm{cc}(G) - 1$, and G is defined by an element*

$$\delta \in E_{c-1}(P, T) \setminus E_{c-2}(P, T)$$

such that $\gamma_2(G) \cap T = [T, P]$.

4. A CONSTRUCTION FOR CF SPACE GROUPS

In this section we construct a class of uniserial p-adic CF space groups. The constructed groups have coclass r and dimension $d = q(p-1)$ for $q = p^{r-1}$. We first introduce the point groups of these space groups and then we discuss their extensions.

4.1 A Special Class of Point Groups

Let $W \in \mathrm{GL}(p-1, \mathbb{Z}_p)$ be the companion matrix of $x^{p-1} + \cdots + x + 1$ and let $I \in \mathrm{GL}(p-1, \mathbb{Z}_p)$ be the identity matrix. We define $m \in \mathrm{GL}(d, \mathbb{Z}_p)$ as the block permutation matrix permuting blocks of dimension $p-1$ via the permutation $(1, \ldots, q)$, and for $1 \leq i \leq q$ we define the element $a_i \in \mathrm{GL}(d, \mathbb{Z}_p)$ as the block-diagonal matrix $\mathrm{diag}(I, \ldots, I, W, I, \ldots, I)$, where W is the ith block. Thus m and a_i have the following form:

$$m = \begin{pmatrix} 0 & I & & & & & \\ & 0 & I & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & I & \\ & & & & & 0 & I \\ I & & & & & & 0 \end{pmatrix},$$

and

$$a_i = \begin{pmatrix} I & & & & & & \\ & \ddots & & & & & \\ & & I & & & & \\ & & & W & & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{pmatrix}.$$

Definition 4.1. We define $U = \langle m a_1 \rangle$ and $A = \langle a_1, \ldots, a_q \rangle$.

Then A is elementary abelian of rank q, and U is cyclic of order p^r. The group U acts uniserially on A by conjugation. Hence the series $A = A_q > A_{q-1} > \cdots > A_1 > \langle 1 \rangle$ defined by $A_i = [A_{i+1}, U]$ satisfies $[A_{i+1} : A_i] = p$ for $1 \leq i \leq q-1$.

Definition 4.2. We define $U_i = U A_i$ for $1 \leq i \leq q$.

This yields a series of subgroups $U = U_1 \leq \cdots \leq U_q < \mathrm{GL}(d, \mathbb{Z}_p)$. The smallest group U acts uniserially on T by construction and hence all of the groups $U_1, \ldots, U_q$ are uniserial point groups. The following lemma lists some further properties of these groups.

Lemma 4.3. *Let $1 \leq i \leq q$.*

(1) $\gamma_j(U_i) = A_{i-(j-1)}$ *for* $2 \leq j$ *and* $\mathrm{cl}(U_i) = i$.

(2) $|U_i| = p^{i+r-1}$ *and* $\mathrm{cc}(U_i) = r - 1$.

(3) *U_i is a CF group with $U_i/\gamma_2(U_i) \cong C_{p^{r-1}} \times C_p$ if $i > 1$ and $U_1 \cong C_{p^r}$.*

Proof: (1) Let $j > 2$ and assume that the statement is true for all $k < j$. Then

$$\begin{aligned}\gamma_j(U_i) &= [\gamma_{j-1}(U_i), U_i] = [A_{i-((j-1)-1)}, U_i] \\ &= [A_{i-((j-1)-1)}, A_i U] = [A_{i-(j-2)}, U] \\ &= A_{i-(j-1)}.\end{aligned}$$

(2) $|U| = p^r$ and $|A_i| = p^i$ and $|U \cap A_i| = p$. This yields that $|U_i| = |U||A_i|/|U \cap A_i| = p^{i+r-1}$ and $\mathrm{cc}(U_i) = i + r - 1 - \mathrm{cl}(U_i) = r - 1$.

(3) This follows directly from (1). The proof is complete. □

Let $T = T_0 > T_1 > \cdots$ denote the maximal U-invariant series through T. This series is invariant under U_i for every i, and the action of U_i on T_j yields a uniserial subgroup of $\mathrm{GL}(d, \mathbb{Z}_p)$. The following lemma shows that there is a bound on the possible indices j yielding nonisomorphic actions.

Lemma 4.4. *$T_j \cong T_{j+q}$ as $\mathbb{Z}_p U_i$-lattices for all j and i.*

Proof: The group A_1 is a central subgroup of order p in every U_i. Every generator c of A_1 satisfies $T_j(c-1) = T_{j+q}$. □

It follows from Lemma 4.4 that the $\mathrm{GL}(d, \mathbb{Q}_p)$-class of the point group U_i splits into at most q classes under the action of $\mathrm{GL}(d, \mathbb{Z}_p)$.

4.2 Extensions

In this section we investigate the extensions of $T^* := T_{q-1}$ by U_i. First, we show that every such extension of coclass r yields a CF group. Then we determine a lower bound for the number of these extensions.

By Theorem 2.2, the extensions of T^* by U_i of coclass r correspond one-to-one to the N_i-orbits of elements of $E_{i-1}(U_i, T^*) \setminus E_{i-2}(U_i, T^*)$, where N_i is the normalizer of U_i in $\mathrm{GL}(d, \mathbb{Z}_p)$. We explicitly determine this underlying set in the following lemma. For this purpose take $C = A_1$ as a central subgroup of order p in U_i and let $F^* = \mathrm{Fix}_C(V/T^*)$ for $V = T^* \otimes \mathbb{Q}_p$ with series $F^* = F^*_q > \cdots > F^*_0 = \{0\}$.

Lemma 4.5. *Let $P = U_i$ for some $i \in \{1, \ldots, q\}$ and $\overline{P} = P/C$.*

(1) $H^1(\overline{P}, F^*) \cong F^*_{i-1}$.

(2) $E_j(P, T^*) \cong F^*_j$ *for* $j \in \{0, \ldots, i-1\}$.

Proof: (2) follows directly from (1), and it remains to prove (1).

Set $u = ma_1$ so that $U = \langle u \rangle \leq P$. Choose $x_q \in A_q \setminus A_{q-1}$ and define $x_i = [x_{i+1}, u]$ for $q - 1 \geq i \geq 1$. Then $x_i \in A_i \setminus A_{i-1}$ and $A_i = \langle x_1, \ldots, x_i \rangle$. Further, it follows that $P = U_i = A_i U = \langle x_i, u \rangle$ and $\overline{P} = P/C$ is the semidirect product of $\overline{A}_i$ with $\overline{U}$.

The map $\alpha : Z^1(\overline{P}, F^*) \to F^* \times F^* : \delta \mapsto (\delta(\overline{x}_i), \delta(\overline{u}))$ is a monomorphism. Let $\beta : Z^1(\overline{P}, F^*) \to F^* : \delta \mapsto \delta(\overline{u})$ denote its corresponding projection on the second component and let K denote the kernel of β.

We show that K is a complement to $B^1(\overline{P}, F^*)$ in $Z^1(\overline{P}, F^*)$. For this purpose let $\delta \in Z^1(\overline{P}, F^*)$. Then $\delta|_{\overline{U}} \in Z^1(\overline{U}, F^*) = B^1(\overline{U}, F^*)$, since $\overline{U}$ is a cyclic group acting uniserially on F^*. Thus there exists $f \in F^*$ with $\delta(\overline{u}) = \delta_f(\overline{u})$, where $\delta_f : \overline{g} \mapsto f^{\overline{g}} - f \in B^1(\overline{P}, F^*)$. Hence $\delta - \delta_f \in K$ and $\delta \in K + B^1(\overline{P}, F^*)$. Let $\delta \in K \cap B^1(\overline{P}, F^*)$. Then $\delta = \delta_f$ for some $f \in F^*$. Thus $\delta(\overline{u}) = f^{\overline{u}} - f = 0$. Hence $f \in F^*_1$, since U acts uniserially on F^*. This implies that $\delta = 0$ and $K \cap B^1(\overline{P}, F^*) = \{0\}$.

Thus $H^1(\overline{P}, F^*) \cong K$ and it remains to determine K. An element $\delta \in K$ is determined by its value $\delta(\overline{x}_i)$. A short computation yields that $\delta(\overline{x}_i) = \delta(\overline{x}_{i+1})(\overline{u} - 1)$. This implies that $\delta(\overline{x_i}) \in F^*_{i-1}$. Conversely, for every $f \in F^*_{i-1}$ there exists a $\delta \in K$ with $\delta(\overline{x}_i) = f$. This follows from the fact that $\overline{P}$ is the split extension of $\overline{A_i}$ with $\overline{U}$. □

It follows immediately from Theorem 2.2 and Lemma 4.5 that there are extensions of T^* by U_i of coclass r, since the set $E_{i-1}(U_i, T^*) \setminus E_{i-2}(U_i, T^*)$ is not empty. Next we show that all these extensions are CF groups.

Theorem 4.6. *Let G be an extension of T^* by U_i defined by an element of the set difference*

$$E_{i-1}(U_i, T^*) \setminus E_{i-2}(U_i, T^*).$$

Then G is a CF group of coclass r.

Proof: By Theorems 2.2 and 3.3, the group G has coclass r, and it remains to show that $\gamma_2(G) \cap T^* = [T^*, U_i] = T_q$. For this purpose we explicitly construct G as an affine matrix group following the well-known approach for space groups; see, for example, [Zassenhaus 48].

Let $\delta \in Z^1(\overline{P}, F^*)$ be an element defining G and define u and $x_1, \ldots, x_i$ as in the proof of Lemma 4.5. As in the proof of Lemma 4.5, we can assume that $\delta(\overline{u}) = 0$. We set $f = \delta(\overline{x}_i)$. Then $f \in F^*_{i-1} \setminus F^*_{i-2}$, since $\delta + B^1(\overline{P}, F^*) \in E_{i-1}(U_i, T^*) \setminus E_{i-2}(U_i, T^*)$.

Further, every element $g \in U_i$ can be written as $g = u^e x_i^{e_i} \cdots x_1^{e_1}$. This implies that $\delta(\overline{g}) = e\delta(\overline{u}) + e_i\delta(\overline{x}_i) + \cdots + e_1\delta(\overline{x}_1) = e_i f + e_{i-1} f(u-1) + \cdots + e_2 f(u-1)^{i-2}$. In particular, the cocycle δ is defined by f only.

Let $t \in T_{q-i} \setminus T_{q-i+1}$ such that $t + T^* = f$ and define the mapping $\sigma : U_i \to T_{-1} : g \mapsto e_i t + e_{i-1} t(u-1) + \cdots + e_1 t(u-1)^{i-1}$ for $g = u^e x_i^{e_i} \cdots x_1^{e_1} \in U_i$. Then $\sigma(g) + T^* = \delta(\overline{g})$ for every $g \in U_i$. Using σ we can write $G = \{\tilde{g}, \tilde{t} \mid g \in U_i, t \in T^*\}$ with

$$\tilde{g} = \left(\begin{array}{c|c} & 0 \\ g & \vdots \\ & 0 \\ \hline \sigma(g) & 1 \end{array} \right)$$

and

$$\tilde{t} = \left(\begin{array}{c|c} & 0 \\ 1 & \vdots \\ & 0 \\ \hline t & 1 \end{array} \right).$$

This construction implies that $\tilde{u}^{-1} = \widetilde{u^{-1}}$. Further, since x_j acts trivially on T/T_q and $\sigma(x_j) \in T$, it follows that

$$\tilde{x}_j^{-1} \equiv \left(\begin{array}{c|c} & 0 \\ x_j^{-1} & \vdots \\ & 0 \\ \hline -\sigma(x_j) & 1 \end{array} \right) \bmod T_q.$$

A direct computation now yields that $[\tilde{x}_j, \tilde{x}_k] \in T_q$ and $[\tilde{x}_{j-1}, \tilde{u}] \equiv \tilde{x}_j \bmod T_q$. Note that $[T^*, U_i] = T_q$. This implies that

$$\gamma_2(G) = \langle \tilde{x}_j, \tilde{t} \mid 1 \leq j \leq i-1, t \in T_q \rangle.$$

We obtain that $\gamma_2(G) \cap T^* = T_q$, which completes the proof. □

We determine a lower bound for the number of CF extensions obtained in Theorem 4.6 in the following theorem. This also provides a proof for Theorem 1.1(b).

Theorem 4.7.

(a) *If $p > 2$, then there are at least $p^{p^{r-1}-(r+2)}$ isomorphism types of CF extensions with coclass r of T^* by U_q.*

(b) *If $p = 2$, then there are at least $2^{2^{r-1}-r}$ isomorphism types of CF extensions with coclass r of T^* by U_q.*

Proof: The proof follows from Lemma 4.5 and Theorem 4.6 with arguments similar to [Eick 05b, Lemma 16 and Corollary 17]. It is proved there that the normalizer N of U_q in $\mathrm{GL}(d, \mathbb{Z}_p)$ acts as a group of order dividing $(p-1)^3 p^{r-2}$ on $E_{q-1}(U_q, T^*) \setminus E_{q-2}(U_q, T^*)$ if $r > 1$ and p is odd or $r > 2$ and $p = 2$. Thus if p is odd and $r > 1$, then the number of N-orbits on this set is at least

$$\frac{p^{q-1} - p^{q-2}}{(p-1)^3 p^{r-2}} = \frac{p^{q-r}}{(p-1)^2} \geq p^{q-r-2}.$$

Since there is exactly one extension of T^* by U_q if $r = 1$, the result follows. A similar argument gives the result for $p = 2$. □

5. EXPERIMENTAL EVIDENCE AND CONJECTURES

In [Eick 05a] there is an algorithm described to construct all uniserial p-adic space groups of a given coclass for an odd prime p. We used its implementation in GAP [The GAP Group 08] to construct the uniserial p-adic CF space groups for some small primes p and small coclasses r. The results of these experiments have led us to the construction outlined in Section 4.

There are further observations that arise from our experiments that we have not been able to prove. We list the most important ones as conjectures in the following. The first conjecture observes that the point groups chosen in Section 4 are the only ones possible.

Conjecture 5.1. *Let p be an odd prime and let G be a uniserial p-adic CF space group of coclass r and dimension $d = p^{s-1}(p-1)$ with point group P. Then $r = s$ and P is conjugate in $\mathrm{GL}(d, \mathbb{Q}_p)$ to U_i for some i.*

If this conjecture is true, then all uniserial p-adic CF space groups arise as extensions of a lattice T_j by a point group U_i for certain i and j. The next conjecture gives some further insight into the situation.

Conjecture 5.2. *Let p be an odd prime, and let $0 \leq j \leq q-1$ and $1 \leq i \leq q$ for $q = p^{r-1}$. Then either all or none of the extensions of coclass r of T_j by U_i are CF groups.*

We note that there are lattices T_j with $j \neq q-1$ that allow CF extensions of coclass r with certain point groups U_i. However, our experimental evidence suggests that $T^* = T_{q-1}$ is the only lattice for which all point groups $U_1, \ldots, U_q$ have CF extensions.

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