For Which Triangles Is Pick's Formula Almost Correct?

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2000 AMS Subject Classification: 52C05, 57M25

Keywords: Pick's theorem, lattice points of a right-angled triangle, classification of two-bridge ribbon knots

We present an intriguing question about lattice points in triangles where Pick's formula is "almost correct." The question has its origin in knot theory, but its statement is purely combinatorial. After more than 30 years, the topological question was recently solved, but the lattice-point problem is still open.

1. ALMOST PICK'S FORMULA

Let p, q be positive integers and consider the triangle

$$\Delta = \Delta(p,q) := \operatorname{conv}\left\{ \big(0,0\big),\ \big(p,0\big),\ \left(p, \frac{q}{p}\right) \right\} \subset \mathbb{R}^2.$$

We count two types of lattice points in $\mathbb{Z}^2$:

$$\begin{aligned} &\operatorname{Pick}(\Delta) \\ &\quad := \#\{\text{interior lattice points, excluding boundaries}\} \\ &\qquad + \frac{1}{2}\#\{\text{boundary lattice points, excluding vertices}\} \\ &\qquad + \frac{1}{2}. \end{aligned}$$

If q/p is an integer, then Δ is a *lattice triangle*, and Pick's theorem says that

$$\operatorname{Area}(\Delta) = \operatorname{Pick}(\Delta).$$

This equality will no longer hold in general for $q/p \notin \mathbb{Z}$. Nevertheless, under favorable circumstances, Pick's formula can be *almost correct* in the following sense:

Definition 1.1. Let p, q be positive integers with q even, so that the area of our triangle Δ is $\frac{1}{2}q \in \mathbb{Z}$. We say that Pick's formula is *almost correct* for Δ if $\operatorname{Area}(\Delta) = \lfloor \operatorname{Pick}(\Delta) \rfloor$, where $\lfloor x \rfloor$ designates the integer part of $x \in \mathbb{R}$.

Notice that our counting formula defines $\operatorname{Pick}(\Delta)$ to be an integer or a half-integer. This means that Pick's formula is almost correct if and only if $\operatorname{Pick}(\Delta)$ equals either $\operatorname{Area}(\Delta)$ or $\operatorname{Area}(\Delta) + \frac{1}{2}$. Here are two typical examples:

Example 1.2. For $p = 5$, $q = 18$ we have $\text{Area}(\Delta) = 9$ and $\text{Pick}(\Delta) = 9$.

Example 1.3. For $p = 5$, $q = 4$ we have $\text{Area}(\Delta) = 2$ and $\text{Pick}(\Delta) = 2 + \frac{1}{2}$.

Starting with Δ we can consider the magnified triangles $r\Delta$ with $r \in \mathbb{N}$:

$$r\Delta := \text{conv}\left\{ (0,0),\ (pr,0),\ \left(pr, \frac{qr}{p}\right) \right\} \subset \mathbb{R}^2.$$

Of course, $p\Delta$ is a lattice triangle. We can now ask for the stronger condition that Pick's formula be almost correct for all $r\Delta$ with $r \in \mathbb{N}$. Notice that Pick's formula is almost correct for all $r \in \mathbb{N}$ if and only if it is almost correct for all $r = 1, 2, \ldots, p-1$.

Remark 1.4. It is a frequently studied question to bound the error between the area and a lattice-point count; see for instance the chapter "A Lattice-Point Problem" in [Hardy 40] and the literature cited there. Our setting can be seen as the inverse problem: we prescribe very strict error bounds and ask which triangles satisfy them.

2. THE CASSON–GORDON FAMILIES

For positive integers $p, q' \in \mathbb{N}$ with p odd and q' even, one has $q' = 2kp^2 \pm q$ for some $k \in \mathbb{N}$ and $1 < q < p^2$. Moreover,

$$\text{Pick}\left(\Delta(p, 2kp^2 + q)\right) = kp^2 + \text{Pick}\left(\Delta(p,q)\right),$$
$$\text{Pick}\left(\Delta(p, 2kp^2 - q)\right) = kp^2 - \text{Pick}\left(\Delta(p,q)\right) + \frac{1}{2}.$$

This shows that Pick's formula is almost correct for (p, q') if and only if it is almost correct for (p, q). It is thus natural to restrict attention to q with $1 < q < p^2$.

Theorem 2.1. **[**Casson and Gordon 86**]** *Let $p, q \in \mathbb{Z}$ be coprime integers with $1 < q < p^2$, p odd, q even. Suppose that p and q satisfy one of the following conditions:*

(i) $q = np \pm 1$ *for some* $n \in \mathbb{N}$ *with* $\gcd(n,p) = 1$;

(ii) $q = n(p \pm 1)$ *for some* $n \in \mathbb{N}$ *with* $n \mid 2p \mp 1$;

(iii) $q = n(p \pm 1)$ *for some* $n \in \mathbb{N}$ *with* $n \mid p \pm 1$, n *odd;*

(iv) $q = n(2p \pm 1)$ *for some* $n \in \mathbb{N}$ *with* $(p \mp 1)/n$ *odd.*

Then Pick's formula is almost correct for all triangles $r\Delta(p,q)$ with $r \in \mathbb{N}$, in other words, $\text{Area}(r\Delta) = \lfloor \text{Pick}(r\Delta) \rfloor$ *for $\Delta = \Delta(p,q)$ and all $r \in \mathbb{N}$.*

Remark 2.2. In the presentation given above, the four Casson–Gordon families may seem rather complicated at first sight. They can be reformulated in a more pleasant and symmetric fashion: each p^2/q has a continued fraction representation of one of the following three types: $[a_1, a_2, \ldots, a_k, \pm 1, -a_k, \ldots, -a_2, -a_1]$ with $a_i > 0$, $[2a, 2, 2b, -2, -2a, 2b]$, $[2a, 2, 2b, 2a, 2, 2b]$ with $a, b \neq 0$ (to obtain all examples we also allow negation and reversal of these continued fractions). See [Kanenobu 86, Theorem 6] for a hint on how to prove this for the first family and use direct calculations for the others.

3. KNOT-THEORETIC BACKGROUND

The only known proof of Theorem 2.1 is intricate and highly indirect, but its story is worth telling. Since the first version of the present note appeared, in February 2006, we have been questioned about the knot-theoretic background, and so we feel that we should summarize the proof here and give a brief account of its long-winded history. Even though it is not immediately relevant to the combinatorial question toward which we are heading, we thus take a detour in order to sketch the argument. We hope that this will serve to better situate the result and motivate the question that ensues.

Topological proof of Theorem 2.1: The proof is a byproduct of a profound topological investigation by Casson and Gordon in their seminal work [Casson and Gordon 86]. They apply the Atiyah–Singer G-signature theorem in dimension 4 in order to establish necessary conditions for a knot $K \subset \mathbb{R}^3$ to bound a ribbon disk $D \subset \mathbb{R}^3$, $\partial D = K$. As a corollary [Casson and Gordon 86, p. 188], they show that whenever the two-bridge knot represented by the fraction q/p^2 is a ribbon knot, then Pick's formula is almost correct for all triangles $r\Delta(p,q)$ with $r \in \mathbb{N}$. This obstruction allows them to exclude many two-bridge knots, by showing that they cannot bound any ribbon disk.

On the other hand, we have the four families displayed above, which have already been stated in [Casson and Gordon 86], alas without proof. Siebenmann [Siebenmann 75] proved for two of the Casson–Gordon families that the knot q/p^2 is a ribbon knot by explicitly constructing a ribbon disk. While pursuing a different approach, Lamm [Lamm 05] re-proved and extended Siebenmann's result by giving a unified construction showing that all four Casson–Gordon families indeed yield ribbon knots. Together with the fundamental result of Casson and Gordon this implies that for the above families Pick's formula must be almost correct, as stated in the theorem. □

Remark 3.1. As a historical note, we mention that Siebenmann's contribution [Siebenmann 75] has not been readily available, and thus the details of the constructive part have been completed in published form only recently, in [Lamm 05] and [Lisca 07]. The fundamental results of [Casson and Gordon 86] have circulated for more than ten years only in preprint form. Fortunately they have been saved from this fate and preserved for posterity in the book [Guillou and Marin 86].

Question 3.2. The proof via knot theory in dimensions 3 and 4 may seem far-fetched for a purely combinatorial statement that does not even mention knots or topology in any way. Is there a more direct (combinatorial) proof of Theorem 2.1?

Of course, for a fixed pair (p, q) the theorem can easily be verified by a (computer) count of lattice points. It is, however, not obvious how to prove the assertion in general. Is there some more satisfactory (number-theoretic) explanation?

4. IS THE LIST COMPLETE?

Having set the scene, we now come to the main point of the present note and formulate the delicate inverse question. Empirical evidence lets us conjecture that the list stated in Theorem 2.1 is complete. More explicitly, we make the following conjecture.

Conjecture 4.1. *If $p, q \in \mathbb{Z}$ are coprime integers with $1 < q < p^2$, p odd, q even, and Pick's formula is almost correct for all triangles $r\Delta(p, q)$ with $r \in \mathbb{N}$, then the pair (p, q) belongs to one of the four Casson–Gordon families stated above.*

This conjecture is already implicit in [Casson and Gordon 86], where the authors verified it for $p \leq 105$ on a computer. Although the question has been studied by knot theorists ever since the preprint of Casson and Gordon appeared in 1974, the above lattice-point conjecture remains unsolved after more than 30 years.

Remark 4.2. The topological problem, sketched above, of classifying two-bridge ribbon knots has recently been solved by Lisca [Lisca 07], using an independent topological approach avoiding the combinatorial problem. Apart from its own geometric appeal, an affirmative answer to Conjecture 4.1 would have an interesting application in knot theory, as indicated in the preceding proof: it would re-prove the result of Lisca, by showing that the Casson–Gordon families exhaust all possibilities.

Remark 4.3. We have verified the conjecture for $p < 5000$ using the straightforward counting method. On an Athlon processor running at 2 GHz this took less than two days. Notice, however, that in its naive form an exhaustive search takes time of order $O(n^5)$ and soon becomes too expensive, so certain optimizations are highly recommended.[1]

Remark 4.4. Following Casson and Gordon [Casson and Gordon 86, p. 187], in a modified formulation taken from Siebenmann [Siebenmann 75], we write $\sigma(p^2, q, r) := 4\,(\mathrm{Area}(r\Delta) - \mathrm{Pick}(r\Delta)) + 1$ and have

$$\sigma(p^2, q, r) = -\frac{2}{p^2} \sum_{s=1}^{p^2-1} \cot\left(\frac{\pi s}{p^2}\right) \cot\left(\frac{\pi q s}{p^2}\right) \sin^2\left(\frac{\pi q r s}{p}\right). \tag{4–1}$$

The computation can be sped up with the help of the continued fraction for p^2/q ("Eisenstein method"): define the numbers $a_i, q_i > 0$ by $q_0 = p^2$, $q_1 = q$ and $q_{i-1} = a_i q_i + q_{i+1}$. For $x \in \mathbb{R}$ define the function $\{x\}$ as (fractional part of x) $- \frac{1}{2}$ (this is not the standard notation) and the function $(\!(x)\!)$ as $\{x\}$ if x is not an integer, and 0 otherwise. According to [Siebenmann 75], we have (modulo a global sign)

$$\begin{aligned}\sigma(p^2, q, r) = \frac{1}{2} \sum_{i=1}^{k} (-1)^i a_i \left(1 - 4\left\{\frac{q_i r}{p}\right\}^2\right) \qquad (4\text{–}2)\\ - \sum_{i=1}^{k} (-1)^i \left(1 - 4\left(\!\!\left(\frac{q_i r}{p}\right)\!\!\right)\left(\!\!\left(\frac{q_{i-1} r}{p}\right)\!\!\right)\right).\end{aligned}$$

It would be a welcome complement to the existing literature to elucidate and further develop this ansatz. For instance, from (4–1) we easily obtain the symmetry $\sigma(p^2, q, r) = \sigma(p^2, q, p - r)$. Furthermore, for $qq' \equiv 1 \pmod{p^2}$ we have $\sigma(p^2, q, r) = \sigma(p^2, q', qr)$, and for $qq' \equiv -1 \pmod{p^2}$, we have similarly $\sigma(p^2, q, r) = -\sigma(p^2, q', qr)$. Equation (4–2) allows fast computations and is thus well suited for empirical exploration. Perhaps it can also provide some hints on how to attack Conjecture 4.1.

[1] If you want to check or further optimize our implementation, you can download it at http://www-fourier.ujf-grenoble.fr/~eiserm/software.html#pick.

Remark 4.5. Considering the average of $\sigma(p^2,q,r)$ over $r=1,\dots,p-1$, Sikora [Sikora 07] found a relationship with the classical Dedekind sum $s(q,p)$. By Theorem 2.1, if (p,q) belongs to one of the Casson–Gordon families, then $\sigma(p^2,q,r)=\pm 1$ for $r=1,\dots,p-1$. In particular,

$$\left|\frac{1}{p-1}\sum_{r=1}^{p-1}\sigma(p^2,q,r)\right| \le 1.$$

He shows that

$$\sum_{r=1}^{p-1}\sigma(p^2,q,r) = 4\cdot s(q,p) - 4p\cdot s(q,p^2)$$

in the following way:

Note that in equation (4–1) the variable r occurs only in the $\sin^2$ term. Therefore for summing $\sigma(p^2,q,r)$ over $r=1,\dots,p-1$ we need $\sum_{r=1}^{p-1}\sin^2(\frac{\pi qrs}{p})$ for given p, q, s. Because $\sum_{r=1}^{p-1}\sin^2(\frac{\pi qrs}{p}) = \frac{p}{2}$ if $p\nmid s$ (and the sum vanishes if $p \mid s$) we obtain

$$\begin{aligned}\sum_{r=1}^{p-1}\sigma(p^2,q,r) &= -\frac{2}{p^2}\sum_{\substack{0<s<p^2\\ p\nmid s}} \cot\left(\frac{\pi s}{p^2}\right)\cot\left(\frac{\pi q s}{p^2}\right)\frac{p}{2}\\ &= -4p\cdot s(q,p^2) + 4\cdot s(q,p).\end{aligned}$$

Hence if (p,q) belongs to one of the Casson–Gordon families, then

$$\frac{4}{p-1}\left|s(q,p) - p\cdot s(q,p^2)\right| \le 1. \tag{4–3}$$

Which (p,q), with p odd and q even, satisfy equation (4–3)? For example, for $p=9$ we obtain the solutions $q=22,56,68,70$ besides the Casson–Gordon families.

Remark 4.6. Writing the (mirrored) triangle Δ in the form $qx+p^2y \le pq$ with $x,y\ge 0$, we can apply the lattice-point counting formula of Beck and Robins [Beck and Robins 07, Theorem 2.10]. With

$$c_{p,q} := \frac{1}{4}\left(1+\frac{1}{p^2}+\frac{1}{q}\right) + \frac{1}{12}\left(\frac{p^2}{q}+\frac{q}{p^2}+\frac{1}{p^2q}\right),$$

the number of lattice points in the triangle $t\Delta$ is thus

$$\begin{aligned}L(t) = \frac{1}{2}qt^2 + \frac{1}{2}t\left(p+\frac{q}{p}+\frac{1}{p}\right) + c_{p,q} + s_{-tpq}(q,1;p^2)\\ + s_{-tpq}(p^2,1;q),\end{aligned}$$

where the last two terms denote the Fourier–Dedekind sums defined in [Beck and Robins 07]. (In order to stay as close as possible to the notation in the book, we denote the magnifying factor by t.)

Because q divides $-tpq$, we have $s_{-tpq}(p^2,1;q) = s_0(p^2,1;q)$, and we can use reciprocity to obtain

$$s_0(p^2,1;q) = -s_0(q,1;p^2) - c_{p,q} + 1.$$

Therefore

$$\begin{aligned}L(t) = \frac{1}{2}qt^2 + \frac{1}{2}t\left(p+\frac{q}{p}+\frac{1}{p}\right) + s_{-tpq}(q,1;p^2)\\ - s_0(q,1;p^2) + 1.\end{aligned}$$

In order to compute $\text{Pick}(t\Delta)$, we count the lattice points on the legs (tp and $\lfloor tq/p \rfloor$) and on the hypotenuse ($\lfloor t/p \rfloor$ because q and p^2 are coprime). Using the notation $\{x\} = x - \lfloor x \rfloor$, the result of subtracting half of the lattice points on the boundary from $L(t)$ (and taking care of the vertices in the way we specified) is

$$\begin{aligned}\text{Pick}(t\Delta) = \frac{1}{2}qt^2 + \frac{1}{2}\left\{\frac{t}{p}\right\} + \frac{1}{2}\left\{\frac{tq}{p}\right\} + s_{-tpq}(q,1;p^2)\\ - s_0(q,1;p^2).\end{aligned} \tag{4–4}$$

This shows that the t-variation of $\text{Area}(t\Delta) - \text{Pick}(t\Delta)$ depends mostly on $s_{-tpq}(q,1;p^2)$. The other terms do not contain t or are small.

For example, for $p=11, q=46$ and $t=1,2$ (illustrated in [Lamm 05, p. 8]), we have

$$\begin{aligned}\text{Pick}(\Delta) &= 23 + \frac{1}{22} + \frac{1}{11} + \frac{6}{11} - \frac{2}{11} = 23.5,\\ \text{Pick}(2\Delta) &= 92 + \frac{1}{11} + \frac{2}{11} - \frac{1}{11} - \frac{2}{11} = 92.\end{aligned}$$

Formula (4–4) can also be expressed in the form of Dedekind–Rademacher sums $r_n(q,p^2)$; see [Beck and Robins 07, Exercise 8.10]. Analyzing Formula (4–4), Beck and Pfeifle [Beck and Pfeifle 06] obtained partial results concerning Conjecture 4.1.

Remark 4.7. Extensions of Conjecture 4.1 are possible: with the definition

$$I(p,q) := \left\{\sigma(p^2,q,r) \mid r=1,\dots,p-1\right\},$$

Conjecture 4.1 now reads that (p,q) belongs to one of the Casson–Gordon families if and only if $I(p,q)=\{1\}$, $\{-1\}$, or $\{-1,1\}$.

For $I(p,q) = \{-3,-1\}$ we obtain the following families (with positive parameter a):

$$C_1(a) = [2a, -8, -2a, 2],$$
$$C_2(a) = [2, 2a, -2, 2, -2a, -6],$$
$$C_3(a) = [6, 2a, -2, 2, 2a, -2] \quad \text{(negative reversed fraction of } C_2\text{)},$$
$$C_4(a) = [2a, 2, -2, 2, -2, 2, -2, 2, -2a-2, 2],$$
$$C_5(a) = [2a, 2, -2, 2, -2, 2a+2, -2, 2, -2, 2],$$

as well as the sporadic case $[6, -4, -2, 2]$.

The case $I(p,q) = \{-3,-1\}$ and a similar set of families for $I(p,q) = \{1,3\}$ seem to be the only ones besides $\{-1,1\}$ for which exactly two values are attained, meaning that we do not find such families for $I(p,q) = \{3,5\}$, for example.

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Received June 13, 2008; accepted August 20, 2008.