A Note on the Rational Points of $X_0^+(N)$

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CONTENTS

Let C be the image of a canonical embedding ϕ of the Atkin–Lehner quotient $X_0^+(N)$ associated with the Fricke involution w_N. In this note we exhibit some relations among the rational points of C. For each $g = 3$ (respectively the first $g = 4$) curve C we found that there are one or more lines (respectively planes) in $\mathbb{P}^{g-1}$ whose intersection with C consists entirely of rational Heegner points or the cusp point, where N is prime. We also discuss an explanation of the first nonhyperelliptic exceptional rational point.

1. INTRODUCTION

Fix an integer $N > 1$ and let $X_0(N)$ be the moduli space of (ordered) pairs (E, E') of generalized elliptic curves E and E' linked by a cyclic isogeny $\varphi\colon E \longrightarrow E'$ of degree N. Consider the Atkin–Lehner quotient curve $X_0^+(N)$ defined by the involution w_N of $X_0(N)$ induced by mapping an isogeny $\varphi\colon E \longrightarrow E'$ to its dual $\hat{\varphi}\colon E' \longrightarrow E$. The quotient curve $X_0^+(N)$ has been studied by Galbraith [Galbraith 99], Mazur [Mazur 78], and Momose [Momose 87], among others. Galbraith [Galbraith 96] studied the rational points of a canonical image $C \subset \mathbb{P}^{g_N^+-1}$ of $X_0^+(N)$ of genera $g_N^+ = 3$, 4, and 5, for many prime levels N. In each case he locates the cusp and rational Heegner points, and moreover, for $N = 137$ and 311 he exhibits a rational point that is neither a cusp point nor a Heegner point. In this note we exhibit an explicit set of hyperplanes $\{H_1, \dots, H_s\}$ in $\mathbb{P}^{g_N^+-1}$ such that the intersection of each H_i with C (over a fixed algebraic closure $\mathbb{Q}^{\mathrm{al}}$ of $\mathbb{Q}$) consists entirely of rational points of C, for each prime level N such that $g_N^+ = 3$, i.e., $N = 97$, 109, 113, 127, 139, 149, 151, 179, and 239, and the first prime level N such that $g_N^+ = 4$, i.e., $N = 137$. For the latter case we found a further plane defined by three different rational Heegner points that also contains the exceptional point found by Galbraith [Galbraith 96, p. 88].

The material is organized as follows. Section 2 introduces our basic notation as well as some background

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material. The collinearity relations are discussed in Section 3, while the coplanarity relations are discussed in Section 4.

2. PRELIMINARIES

Let X be an algebraic curve defined over a field k and let $\Omega^1(X)$ be the k-vector space of its holomorphic differentials. Also let $\{\omega_1, \dots \omega_g\}$ be a basis of $\Omega^1(X)$. The integer g is called the *genus* of X. The *canonical map* ϕ of X in projective space $\mathbb{P}^{g-1}$ is the morphism

$$X \xrightarrow{\phi} \mathbb{P}^{g-1}, \quad P \mapsto (\omega_1(P) : \cdots : \omega_g(P)).$$

It is well known that the map ϕ is an embedding if the genus g exceeds 2 and X is not hyperelliptic. Now fix an integer $N > 1$ and recall that $X_0(N) = \Gamma_0(N) \setminus \mathcal{H}^*$, where $\mathcal{H}^* = \{\tau \in \mathbb{C} \colon \Im(\tau) > 0\} \cup \mathbb{P}^1(\mathbb{Q})$, and $\Gamma_0(N)$ consists of the matrices $\mu = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \equiv 0 \pmod{N}$, and each μ acts as usual as a fractional linear transformation $\tau \mapsto \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$.

Now let $S_2(\Gamma_0(N))$ be the $\mathbb{C}$-vector space of modular forms $f \colon \mathcal{H} \longrightarrow \mathbb{C}$ of weight 2 and trivial character with respect to $\Gamma_0(N)$. It is well known that $f \mapsto f\, d\tau$ defines an isomorphism from $S_2(\Gamma_0(N))$ onto $\Omega^1(X_0(N))$. Also, the involution w_N is induced by the action of the matrix $W_N = \left(\begin{smallmatrix} 0 & N \\ -1 & 0 \end{smallmatrix}\right)$ on $\mathcal{H}$. So the canonical map of $X_0^+(N)$ into projective space is given by a basis

$$\{f_1, \dots, f_{g_N^+}\}$$

of the $+1$-eigenspace $S_2^+(\Gamma_0(N))$ of $S_2(\Gamma_0(N))$ with respect to the action of the matrix W_N on modular forms.

It is a classical fact that the above basis may be chosen such that each Fourier expansion $f_k(\tau) = \sum_{n=1}^{\infty} a(n)q^n$ has only rational integer coefficients, where as usual $q = e^{2\pi i\tau}$ and $\tau \in \mathcal{H}$. Moreover, it is possible to use the Fourier expansions of certain sets of products $f_{k_1} f_{k_2} \cdots f_{k_s}$ to compute projective equations

$$\begin{gathered} F_1(f_1, \dots, f_{g_N^+}) = 0, \\ \vdots \\ F_r(f_1, \dots, f_{g_N^+}) = 0, \end{gathered}$$

for the image C of the canonical map ϕ. In particular, the coefficients of each polynomial F_k are rational integers. Elkies [Elkies 98] and Galbraith [Galbraith 96] have gathered experimental evidence that suggests that the basis $\{f_1, \dots, f_{g_N^+}\}$ may be chosen such that the coefficients of each F_k are of size $O(\log N)$. For example, Elkies [Elkies 98, p. 63] obtained (in a slightly different notation) the affine equation[1]

$$y^3 - (x^2 - x + 2)y^2 + (x^3 + x^2 - x + 3)y + x^2 + x - 1 = 0,$$

for an image C of the genus-three curve $X_0^+(239)$ in $\mathbb{P}^2$, where

$$\begin{aligned} x &= \frac{f_1}{f_3} = q^{-1} + q^5 + q^6 + \cdots, \\ y &= \frac{f_2}{f_3} = -q + q^3 + q^4 - q^6 - 2q^7 - q^8 + q^9 + 2q^{10} + \cdots, \end{aligned}$$

and each $f_j \in S_2^+(N)$ is determined by

$$\begin{aligned} f_1 &= q - q^2 - q^5 - q^7 + q^8 - 2q^9 - q^{12} + \cdots, \\ f_2 &= -q^3 + q^4 + q^5 - q^8 - q^{10} + q^{11} + q^{12} + \cdots, \\ f_3 &= q^2 - q^3 - q^6 - 2q^8 + q^9 - q^{10} + q^{12} + \cdots. \end{aligned}$$

From now on, let us assume that N is prime. Using [Gross 87, Proposition 3.1] and the Riemann–Hurwitz formula, we see that

$$g_N^+ = \frac{1}{2}\left(g_N + 1 - H(N)\right),$$

where

$$H(N) = \begin{cases} \frac{1}{2}h(-4N), & \text{if } N \equiv 1 \pmod{4}, \\ \frac{1}{2}(h(-N) + h(-4N)), & \text{otherwise.} \end{cases}$$

Here $h(D)$ is the class number of the imaginary quadratic order of discriminant D, and g_N is the genus of $X_0(N)$, which is given by $g_N = \left\lfloor \frac{N+1}{12} \right\rfloor$, unless $N = 12q + 1$ in which case $g = q - 1$.

In particular, using explicit upper bounds on the class number $h(D)$ it may be found that the primes N such that $X_0^+(N)$ has genus 3 are indeed $N = 97$, 109, 113, 127, 139, 149, 151, 179, and 239. Similarly, it may be found that there are exactly five prime numbers N such that $X_0^+(N)$ has genus $g = 4$, namely $N = 137$, 173, 199, 251, and 311. From now on, we assume that the genus g_N^+ of $X_0^+(N)$ is either 3 or 4. For each of these 14 levels N, Galbraith [Galbraith 96] proved that the curve $X_0^+(N)$ is nonhyperelliptic, exhibited equations for C, and located the cusp as well as the rational points predicted by the theory of complex multiplication, i.e., the rational Heegner points.

Following Gross [Gross 84], Heegner points may be succinctly defined as points $(E, E') \in X_0^+(N)$ such that

[1] There is an equation for $X_0^+(239)$ with smaller coefficients than the one obtained by Elkies [Elkies 98, p. 63], as shown in Table 9.

$\mathcal{O}_D = \mathrm{End}(E) = \mathrm{End}(E') > \mathbb{Z}$, where $\mathcal{O}_D$ is the imaginary quadratic order of discriminant $D < 0$. In particular, using results of Gross [Gross 84], the fact that $N > 89$ implies that there are no rational Heegner points of discriminant $D = -Np$ on $X_0^+(N)$ if p is prime. Thus the discriminants of rational Heegner points that may arise are only $D = -3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163$.

3. GENUS THREE: COLLINEARITY RELATIONS

Let us assume that N is one of the nine primes N such that $X_0^+(N)$ has genus $g_N^+ = 3$. Let

$$\{f_1, f_2, f_3\}$$

be a basis of $S_2^+(N)$ with rational integral Fourier coefficients. The associated canonical map defines an embedding ϕ of $X_0^+(N)$ into $\mathbb{P}^2$ defined over $\mathbb{Q}$. The image C of ϕ in $\mathbb{P}^2$ has degree 4. In other words, the lines L in the projective plane $\mathbb{P}^2$ will intersect the curve C in four points if we take into account intersection multiplicities.

Usually, we expect that the line defined by two rational points P_1 and P_2 of C (which we define to be the tangent line of C at say P_1 in case $P_1 = P_2$) will intersect C at two further points whose field of definition is a quadratic extension of $\mathbb{Q}$. However, we shall exhibit for each of the levels under consideration a nonempty set of lines

$$\mathcal{S} = \{L_1, \ldots, L_s\}$$

such that the intersection of each L_i with C (over a fixed algebraic closure $\mathbb{Q}^{\mathrm{al}}$ of $\mathbb{Q}$) consists entirely of rational points. We obtained $\mathcal{S}$ by computing $L \cap C$ for each line L defined by a pair P, Q of different rational points on C, and also for each tangent line L at a rational point on C. So $\mathcal{S}$ is maximal, i.e., it must contain all lines L such that $L \cap C \subset C(\mathbb{Q})$. We show each set of collinearity relations in a diagram (Figures 1 through 9 with associated Tables 1 through 9). of the real locus of the affine equation for C defined by $z = 1$ (depicting parts of the line $z = 0$ when necessary). The figures are self explanatory; the intersection multiplicity ∂ of L_i with C at a point $P \in L_i \cap C$ is indicated only when necessary. To ease notation we regard the cusp $i\infty$ as labeled by $D = 0$.

All rational Heegner points P on $X_0^+(N)$ may be identified as follows. Clearly, we may choose, among all bases $\{f_1, f_2, f_3\}$ as above, one such that, say, the coordinate function $x = f_1/f_3$ has no poles on $X_0^+(N)$ except at the cusp. This and the fact that the q-expansion of x is integral imply that $x(P)$ lies in $\mathbb{Z}$. Since the q-expansion of x converges at every $\tau \in \mathcal{H}$, it is possible to determine $x(P)$ by approximating it to within $\frac{1}{2}$. If the q-expansion of the other coordinate function does not converge at some Heegner rational point P, then we may repeat this process (keeping in mind the changes of coordinates between the different choices of basis $\{f_1, f_2, f_3\}$) until we pin down every P.

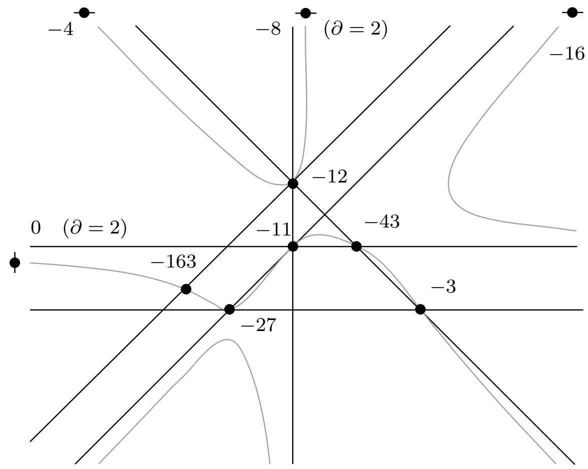

FIGURE 1. Relations for $N = 97$.

$$f_1 = q - q^2 - q^3 - 2q^5 - 2q^7 + 3q^{10} - 2q^{11} + q^{12} + \cdots$$
$$f_2 = q^3 - q^4 - 2q^5 - q^6 + q^7 + 4q^8 - 2q^9 + 3q^{10} + q^{12} + \cdots$$
$$f_3 = -q^2 + q^3 + 2q^4 - q^5 + q^6 + q^7 - q^8 - 3q^9 + q^{10} + \cdots$$

$$yx^3 - (y+1)x^2 - (y^3 + y - 1)x + y^2 - y = 0$$

0	$(1:0:0)$	−3	$(2:-1:1)$	−4	$(1:-1:0)$
−8	$(0:1:0)$	−11	$(0:0:1)$	−12	$(0:1:1)$
−16	$(1:1:0)$	−27	$(1:1:-1)$	−43	$(1:0:1)$
−163	$(5:2:-3)$				

TABLE 1. Level $N = 97$.

$$f_1 = q - q^2 - q^3 - q^5 - 2q^7 - q^9 + 2q^{10} - 5q^{11} + 2q^{12} + \cdots$$
$$f_2 = q^4 - 2q^5 - q^6 + 3q^7 - 2q^8 + q^9 + q^{10} + q^{11} + \cdots$$
$$f_3 = q^2 - q^3 - q^4 - q^5 - q^6 + 2q^7 - q^8 + 3q^9 - 3q^{10} + \cdots$$

$$yx^3 + yx^2 - (y^3 + y^2 - 2y + 1)x + (y^3 - y) = 0$$

0	$(1:0:0)$	−3	$(-2:2:3)$	−4	$(0:-1:1)$
−7	$(0:1:0)$	−12	$(0:0:1)$	−16	$(0:1:1)$
−27	$(-1:1:0)$	−28	$(-2:1:2)$	−43	$(1:1:0)$

TABLE 2. Level $N = 109$.

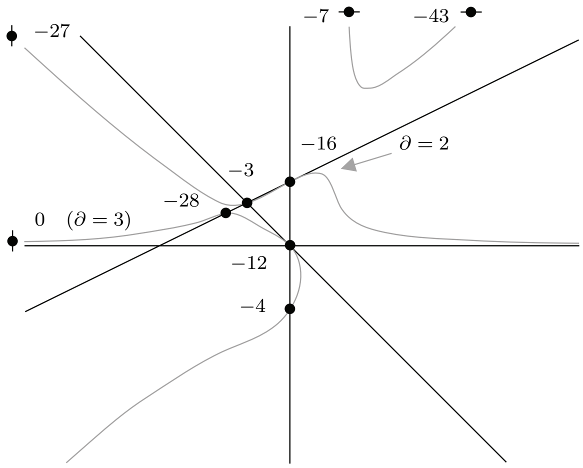

FIGURE 2. Relations for $N = 109$.

$$f_1 = q^2 - q^3 - q^4 - 2q^6 - q^8 + 4q^9 + q^{10} - q^{11} + 3q^{12} + \cdots$$
$$f_2 = q^3 - q^4 - 2q^5 + q^7 + 2q^8 - 3q^9 + 2q^{10} + 3q^{11} + \cdots$$
$$f_3 = q - q^2 - q^3 - q^5 + q^6 - 3q^7 - q^9 - q^{10} + 2q^{11} - q^{12} + \cdots$$

$$(y-1)x^3 - x^2 - (y^3 - y^2 - y)x + y^3 + y^2 + y = 0$$

0	(0 : 0 : 1)	−4	(1 : 1 : 0)	−7	(−1 : 0 : 1)
−8	(0 : 1 : 0)	−11	(1 : 0 : 0)	−16	(−1 : 1 : 0)
−28	(1 : −2 : 1)	−163	(2 : 1 : 2)		

TABLE 3. Level $N = 113$.

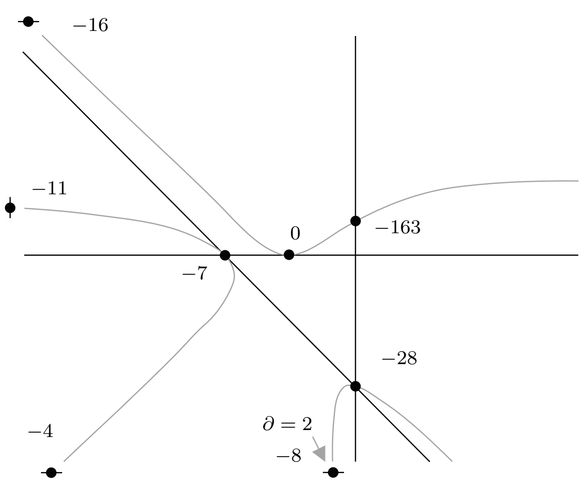

FIGURE 3. Relations for $N = 113$.

$$f_1 = -q^2 + 2q^4 + q^5 + 2q^6 + q^7 - 2q^8 - q^9 - 2q^{11} - 3q^{12} + \cdots$$
$$f_2 = -q^2 + q^3 + q^4 + q^6 + q^8 - 2q^9 + 2q^{10} - 3q^{11} - 2q^{12} + \cdots$$
$$f_3 = q - q^3 - q^4 - 3q^5 - 2q^6 - 2q^7 + q^9 + q^{10} + 2q^{11} + \cdots$$

$$x^4 - (y-1)x^3 + (2y^2 - y - 1)x^2 - (2y^3 + 2y^2 + 1)x + y^3 + y = 0$$

0	(0 : 0 : 1)	−3	(3 : 1 : −3)	−7	(−1 : 0 : 1)
−12	(1 : 1 : −1)	−27	(0 : 1 : 0)	−28	(1 : 0 : 1)
−43	(1 : 1 : 0)	−67	(2 : 1 : −3)		

TABLE 4. Level $N = 127$.

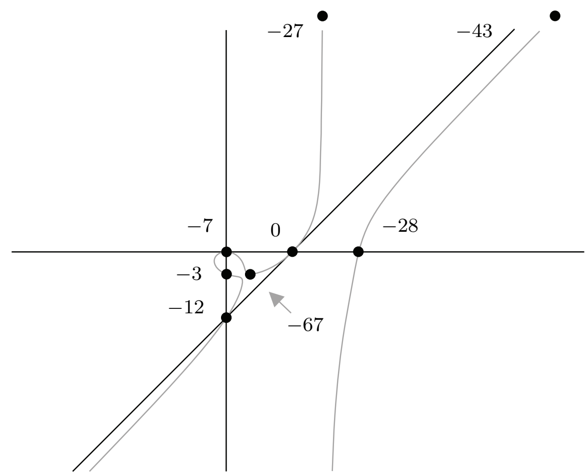

FIGURE 4. Relations for $N = 127$.

$$f_1 = -q^2 + q^3 + q^4 + q^6 - q^7 + q^8 - 2q^9 + 2q^{10} + q^{11} + \cdots$$
$$f_2 = q - q^3 - q^4 - 3q^5 - q^6 - q^8 - 2q^{11} + q^{12} + q^{14} + \cdots$$
$$f_3 = -q^2 + 2q^4 + q^5 + q^6 + q^7 - q^8 - q^9 + q^{10} - 2q^{11} + \cdots$$

$$x^3 + (y^2 - 2y - 2)x^2 + (y^3 - 2y^2 + 2)x - y^3 + y$$

0	(0 : 1 : 0)	−3	(3 : 1 : 2)	−8	(0 : 0 : 1)
−12	(−1 : 1 : 0)	−19	(1 : 0 : 0)	−27	(0 : −1 : 1)
−43	(0 : 1 : 1)				

TABLE 5. Level $N = 139$.

$$f_1 = q - q^3 - q^4 - 2q^5 - q^6 - 2q^7 - q^{10} + q^{14} + 2q^{15} + \cdots$$
$$f_2 = q^3 - q^4 - q^5 - q^7 + q^8 - 2q^9 + 2q^{10} + 2q^{11} + \cdots$$
$$f_3 = -q^2 + q^4 + 2q^5 + 2q^6 + q^8 - q^9 - q^{10} - 3q^{11} + \cdots$$

$$yx^3 - x^2 - (y^3 - 2y^2 - y + 1)x + y^3 + y^2 - y - 1 = 0$$

0	(1 : 0 : 0)	−7	(0 : −1 : 1)	−4	(−1 : 1 : 0)
−16	(1 : 1 : 0)	−19	(0 : 1 : 0)	−28	(0 : 1 : 1)
−67	(2 : 1 : −2)				

TABLE 6. Level $N = 149$.

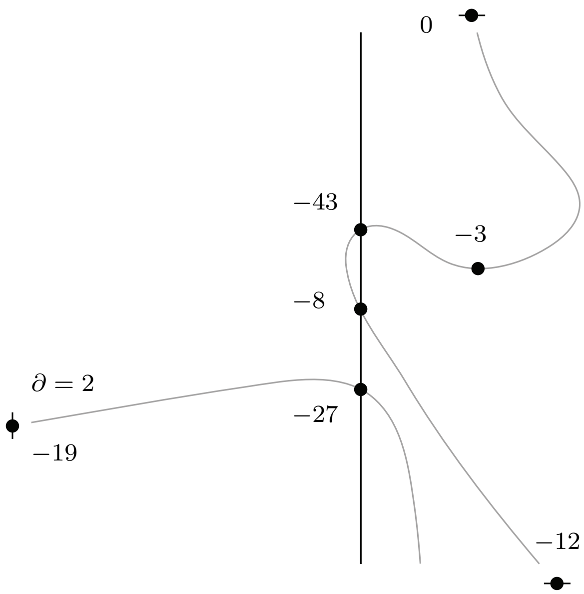

FIGURE 5. Relations for $N = 139$.

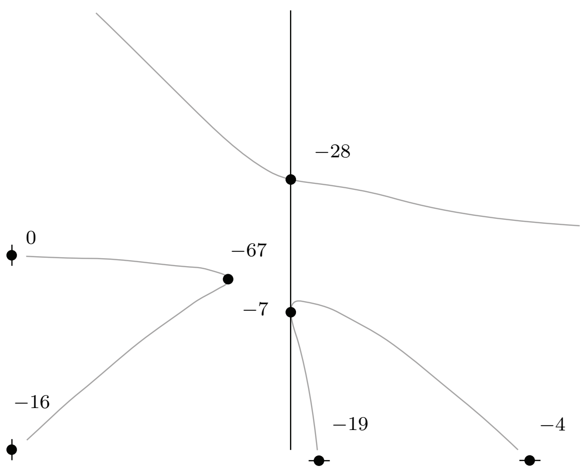

FIGURE 6. Relations for $N = 149$.

$$f_1 = q^4 - q^5 - q^6 - 2q^8 + q^9 + q^{10} + 2q^{11} + q^{12} + 3q^{13} + \cdots$$
$$f_2 = -q^2 + q^3 + q^4 + q^8 - q^9 + 3q^{10} - 2q^{11} - 2q^{13} + q^{14} + \cdots$$
$$f_3 = -q + q^3 + q^4 + 2q^5 + q^6 + q^7 + q^8 + q^9 + q^{11} + \cdots$$

$$x^4 + (2y+1)x^3 - (y-1)x^2 - (y^3 + y^2 + 2y - 1)x + y^3$$

0	$(0:0:1)$	-3	$(1:3:2)$	-7	$(0:1:0)$
-12	$(1:-1:0)$	-67	$(-1:-1:1)$	-27	$(-1:0:1)$
-28	$(-2:1:2)$	-163	$(-1:1:3)$		

TABLE 7. Level $N = 151$.

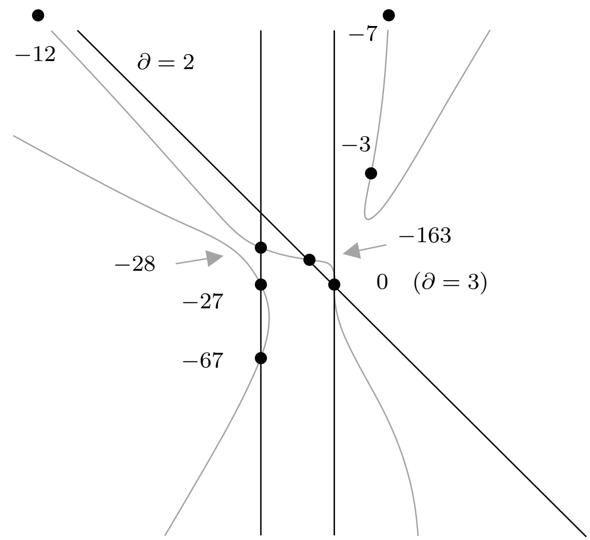

FIGURE 7. Relations for $N = 151$.

$$f_1 = q^4 - q^5 - q^6 - q^8 + q^9 + 2q^{11} - q^{13} + q^{14} + q^{15} - 3q^{16} + \cdots$$
$$f_2 = q - q^2 - q^4 - 2q^7 + 2q^8 - 3q^9 + q^{10} - 3q^{11} + q^{12} + \cdots$$
$$f_3 = q^2 - q^3 - q^5 - q^6 + q^7 - 2q^8 + 2q^9 - 2q^{10} + q^{11} + \cdots$$

$$(y-1)x^3 + (y^2 - y + 1)x^2 + (y^3 + y^2 + y)x - y-$$

0	$(0:1:0)$	-7	$(1:0:0)$	-8	$(0:-1:1)$
-11	$(1:-1:1)$	-28	$(1:-2:2)$	-163	$(1:-5:3)$

TABLE 8. Level $N = 179$.

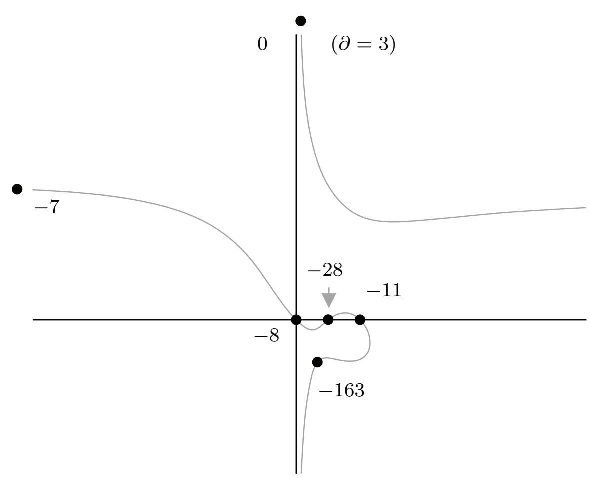

FIGURE 8. Relations for $N = 179$.

$$f_1 = -q^3 + q^4 + q^5 - q^8 - q^{10} + q^{11} + q^{12} + q^{13} + 2q^{15} + \cdots$$
$$f_2 = -q^2 + q^4 + q^5 + q^6 + q^8 - q^9 + q^{11} + q^{13} + q^{14} + \cdots$$
$$f_3 = q - q^4 - 2q^5 - q^6 - q^7 - q^9 - q^{11} - q^{12} - 3q^{13} - q^{15} + \cdots$$

$$x^4 - (y-1)x^3 + x^2 + (y^3 - y^2 + 1)x - y^4 + y^3 + y^2 = 0$$

0	$(0:0:1)$	-7	$(1:1:0)$	-19	$(1:1:-1)$
-28	$(1:-1:0)$	-43	$(-1:0:1)$		

TABLE 9. Level $N = 239$.

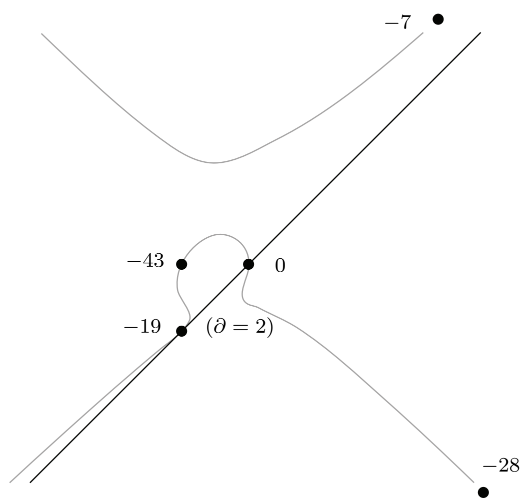

FIGURE 9. Relations for $N = 239$.

4. GENUS FOUR: COPLANARITY RELATIONS

We study the rational points of the first genus-four Atkin–Lehner quotient curve $X_0^+(N)$, i.e., level $N = 137$. It is known that $X_0^+(N)$ is not hyperelliptic. So C may be expressed as the intersection of a quadric and a cubic surface in $\mathbb{P}^3$. (See [Galbraith 96, p. 12] and the reference contained therein, i.e., [Hartshorne 77, Example IV.5.2.2].) Given a basis $\{f_1, f_2, f_3, f_4\}$ of $S_2^+(\Gamma_0(N))$ (with integer coefficients) as above, these surfaces may be found by computing a basis $\mathcal{B}_d$ for the $\mathbb{Q}$-vector space V_d of forms $\alpha(w, x, y, z)$ of degree d (and the form 0) such that $\alpha(f_1, f_2, f_3, f_4) = 0$, for $d = 2$ and 3. (The quadric may be found by picking the only element of $\mathcal{B}_2$, say F. Then the cubic surface may be obtained by picking an element $G \in \mathcal{B}_3$ such that the intersection of $F = 0$ and $G = 0$ has dimension one.) Table 10 shows the first few terms of the q-expansion of f_1,f_2, f_3, and f_4 and the affine equation corresponding to $z = 1$ for each of these surfaces. The table also identifies the cusp, the rational Heegner points, and an exceptional rational point, in terms of projective coordinates $(w : x : y : z)$.

Let $\Pi(P_1, P_2, P_3)$ be the plane defined by given noncollinear rational points P_1, P_2, and P_3 of C. Since the degree of C is 6, in general, the set-theoretic intersection $C \cap \Pi(P_1, P_2, P_3)$ will contain three further points, whose field of definition is expected to be a cubic extension of $\mathbb{Q}$. However, it turns out that by computing the set-theoretic intersection $C \cap \Pi(P_1, P_2, P_3)$ and the intersection multiplicity

$$(C, \Pi(P_1, P_2, P_3))_P$$

of each point $P \in C \cap \Pi(P_1, P_2, P_3)$, for each relevant set of rational points of the form $\{P_1, P_2, P_3\}$, it is possible to find four different planes Π_1, Π_2, Π_3, and Π_4 in $\mathbb{P}^3$ defined by

$$\Pi_1 : x - y = 0,$$
$$\Pi_2 : x + z - 2y = 0,$$
$$\Pi_3 : x + z + w - y = 0,$$
$$\Pi_4 : x = 0,$$

such that each of these planes intersects C at exactly six rational points with multiplicities given by Table 10.

We claim that if a plane Π is such that the intersection $\Pi \cap C$ consists entirely of rational points, then Π must be one of the above four planes. To see this, it suffices to consider the plane determined by the tangent line T_P at P and a point Q not in T_P and also the osculating plane O_R at R, for each P, Q, and R in $C(\mathbb{Q})$. From [Willmore 59, p. 16] we may see that on each affine open subset U of $\mathbb{P}^3$ we have

$$T_P \cap U = \{v(P)t + P \in U : t \in \mathbb{Q}^{\mathrm{al}}\},$$

where, all in standard notation,

$$v = \nabla(F) \times \nabla(G)$$

and

$$O_R \cap U = \{S \in U : S \cdot (b(R)) = 0\} + R,$$

where

$$b = (v\nabla(v)) \times v.$$

Our claim follows by straightforward computations on each relevant U. Further calculations show that the intersection of $L_{i,j} = \Pi_i \cap \Pi_j$ with C has degree 3, and also, $L_{i,j} \cap C \subset C(\mathbb{Q})$, for $(i, j) = (1, 2)$, $(2, 3)$, $(3, 4)$, and $(4, 1)$, where the local intersection numbers are

$$(C, L_{i,j})_P = 2$$

for $(i, j, P) = (4, 1, 0)$ and $(C, L_{i,j})_P = 1$ otherwise.

$$\begin{aligned} f_1 &= -q + q^2 + q^4 + q^5 + 3q^7 - 2q^8 + 2q^9 - 2q^{10} + 3q^{11} + \cdots \\ f_2 &= -q^4 + q^5 + q^6 + q^7 + q^8 - 2q^9 - 5q^{11} + q^{12} - q^{13} + \cdots \\ f_3 &= q^3 - q^4 - q^5 - q^6 + 2q^8 - 2q^9 + 3q^{10} - q^{11} + 2q^{12} + \cdots \\ f_4 &= q - 2q^4 - 3q^5 - 3q^6 - 3q^7 + 2q^8 - q^9 + 4q^{10} + 4q^{12} + \cdots \end{aligned}$$

$$\begin{aligned} wx - yw + y^2 - y &= 0 \\ x^3 - 2(y-1)x^2 - (y^2+y-2)x + w^2 + (y+2)w + 1 &= 0 \end{aligned}$$

0	$(-1:0:0:1)$	-4	$(0:-1:1:1)$
-7	$(-1:0:1:2)$	-8	$(1:0:0:0)$
-11	$(-1:1:1:1)$	-16	$(-2:1:1:1)$
-19	$(0:-1:0:1)$	-28	$(1:0:1:0)$
(exceptional point)	$(-15:8:12:17)$		

	0	-4	-7	-8	-11	-16	-19	-28
Π_1	2			2	1	1		
Π_2			1	1	2	1	1	
Π_3	1		2		1		1	1
Π_4	3		1	1				1

TABLE 10. Level $N = 137$.

Note that the Heegner rational point of discriminant $D = -4$ is not contained in any of the planes Π_i. However, the plane defined by $x + 2z + 2w - y = 0$ contains this point, the cusp, the rational Heegner point with $D = -11$, the exceptional point, and two further points defined over the real quadratic field $\mathbb{Q}(\sqrt{2})$.

5. CONCLUDING REMARKS

It is not hard to see that for most of the levels N we have discussed, the exhibited relations among rational points of $X_0^+(N)$ may be heuristically explained by the fact that the naive heights of the rational points of C are rather small. So it seems worthwhile to extend the above list of examples to higher levels, hoping that more extensive experimental evidence will help us to grasp the nature of this phenomenon. This might shed some light on the nature of $X_0^+(N)(\mathbb{Q})$ for prime levels N, which is extremely interesting, as expressed in [Mazur 78].

Via the Gross–Kohnen–Zagier theorem [Gross et al. 87], all these relations translate into relations among coefficients of a suitable weight-$\frac{3}{2}$ modular form of level $4N$.

So one open question that we would like to raise now is whether the collinearity/coplanarity relations discussed here are telling us something meaningful about the Fourier expansion of certain modular forms of half-integral weight.

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