Convergence Properties of Donaldson's T-Iterations on the Riemann Sphere

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CONTENTS

In [Donaldson 05b], Donaldson gives three operators on a space of Hermitian metrics on a complex projective manifold: T, T_ν, T_K. Iterations of these operators converge to *balanced* metrics, and these themselves approximate constant scalar curvature metrics. In this paper we investigate the convergence properties of these iterations by examining the case of the Riemann sphere as well as higher-dimensional $\mathbb{CP}^n$.

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1. INTRODUCTION

Let X be a compact complex manifold with a positive holomorphic line bundle L. A long-standing open problem in Kähler geometry, building on Yau's solution of the Calabi conjecture [Yau 78], is to find sufficient conditions for the existence of a constant scalar curvature Kähler metric in $c_1(L)$. Another is as follows: can such a metric be obtained naturally as a limit of algebraic metrics via embeddings of X into $\mathbb{P}\,\mathrm{H}^0(X, L^k)$?

This idea of approximating Kähler metrics by restricting Fubini–Study metrics, advocated by Yau over the years, has led to the development of a rich theory relating analysis and notions of stability in the sense of geometric invariant theory (see [Yau 93, Tian 90, Tian 97, Donaldson 02]). In a fundamental paper, Donaldson [Donaldson 01] showed that under an assumption on the space of automorphisms, the metrics induced from *balanced* embeddings [Zhang 96] of X into projective space by sections of L^k converge to the constant scalar curvature metric as $k \to \infty$. The balanced condition means that

$$\int_X \frac{Z_i \overline{Z_j}}{|Z|^2} d\mu = c\delta_{ij}$$

(where $d\mu$ is the volume form on X induced by the Fubini–Study metric and c is a constant depending on

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the data (X, L^k) and not on the particular embedding), and this is equivalent to the Chow stability of the embedding [Zhang 96, Luo 98, Phong and Sturm 03].

Recently, Donaldson [Donaldson 05a, Donaldson 05b] has devised iterative procedures on the space of Hermitian metrics on $\mathrm{H}^0(X, L^k)$ to find approximations to these balanced metrics. For sufficiently large k, these approximations are close to a constant scalar curvature metric.

Explicit numerical computations, focused on the Calabi–Yau case, where there are possible applications to string theory, have been obtained in [Donaldson 05b, Douglas et al. 06a, Douglas et al. 06b]. See also [Headrick and Wiseman 06, Keller 07b, Rubinstein 07], where different methods are used.

Donaldson's three iterative maps T, T_ν, and T_K, described below, are interesting in their own right. Indeed, as pointed out in [Donaldson 05b], it is likely that these maps can be viewed as discrete approximations to the Ricci and Calabi flows.

Instead of pursuing general questions of existence, in this paper we pick a simple compact complex manifold, the Riemann sphere, and investigate the convergence properties of each of T, T_ν, and T_K on the space of Hermitian metrics induced from Veronese embeddings into $\mathbb{CP}^n$. In Section 5, we briefly investigate the case $\mathbb{CP}^n$ when $n > 1$.

There is a natural notion of distance on the space of Hermitian metrics $\mathrm{GL}(n+1, \mathbb{C}) / \mathrm{U}(n+1)$, and indeed, as k increases, this distance function is expected [Phong and Sturm 06] to approximate that on the infinite-dimensional space of Kähler metrics [Mabuchi 87, Semmes 92, Donaldson 99, Chen 00].

A natural question one might ask is whether any of the T, T_ν, and T_K iterations are distance-reducing on the space of metrics. In Section 3.3, we show that the T operator does not satisfy this property.

One goal of this study was to find an effective bound on the distance between the rth iteration of a metric under T, T_ν, or T_K and the limiting balanced metric. One such bound is proposed in Section 3.3. In Section 3.2, we list the observed asymptotic behavior of each of these iterations. In Section 4, we give some examples. In Section 5, we investigate the case for higher-dimensional projective space.

It has recently come to the author's attention that on Julien Keller's web site [Keller 07a], one can find a program to compute a Ricci flat metric on a particular $K3$ surface using the techniques of Donaldson on which this paper is based. More information can be found there.

All computations and all graphs in this paper were done using Maple 9.

2. THE T, T_ν, AND T_K OPERATORS

Let X be an n-dimensional complex projective manifold, and $L \to X$ an ample line bundle. In [Donaldson 05b], Donaldson examines three different actions on the space of Hermitian metrics on $\mathrm{H}^0(X, L^k)$: T, T_ν, T_K. We briefly recall how he defines each.

Given a Hermitian metric G on $\mathrm{H}^0(X, L^k)$ and an orthonormal basis $\{s_i\}$ with respect to G, one defines the Fubini–Study metric $h = \mathrm{FS}(G)$ on the line bundle L^k by the requirement that $\sum_i |s_i|_h^2 = 1$. The result is independent of the orthonormal basis chosen. Now given this metric h on L^k, we define a new Hermitian metric on $\mathrm{H}^0(X, L^k)$, denoted by $\mathrm{Hilb}(h)$, by

$$\|s\|_{\mathrm{Hilb}}^2 = R \int_X |s|_h^2 \omega_h^n / n!,$$

where ω_h is the Kähler form $-\sqrt{-1}\, \partial \overline{\partial} \log h$ and R is the constant

$$R = \frac{\dim \mathrm{H}^0(X, L^k)}{\mathrm{Vol}(X, \omega_h^n / n!)}.$$

This defines the T map: $T(G) = \mathrm{Hilb}(\mathrm{FS}(G))$.

The T_ν map is defined analogously, but instead of the volume form $\omega_h^n / n!$, we fix a volume form ν of our choosing. As above, we set

$$\|s\|_{\mathrm{Hilb}_\nu} = R_\nu \int_X |s|_h^2 \nu,$$

where

$$R_\nu = \frac{\dim \mathrm{H}^0(X, L^k)}{\mathrm{Vol}(X, \nu)}.$$

Then we define $T_\nu(G) = \mathrm{Hilb}_\nu(\mathrm{FS}(G))$.

The T_K function is defined in case $L^k = K^{-p}$, where K is the canonical bundle. Again we modify only the volume form, this time choosing

$$\omega_{G,K} = \left(\sum s_i \otimes \overline{s}_i \right)^{-1/p}.$$

The resulting metric on $\mathrm{H}^0(X, L^k) = \mathrm{H}^0(X, K^{-p})$ is given as above:

$$\|s\|_{\mathrm{Hilb}_K} = R_K \int_X |s|_h^2 \omega_{G,K},$$

where

$$R_K = \frac{\dim \mathrm{H}^0(X, L^k)}{\mathrm{Vol}(X, \omega_{G,K})}.$$

As before, set $T_K(G) = \mathrm{Hilb}_K(\mathrm{FS}(G))$.

A Hermitian metric G is *balanced with respect to* T (respectively T_ν, T_K) if $T(G) = G$ (respectively $T_\nu(G) = G, T_K(G) = G$). The basic philosophy is that if $F = T, T_\nu, T_K$ and if there exists some balanced metric, then starting with any Hermitian metric G the iterations $F^{(r)}(G)$ should tend to a balanced metric as r tends to infinity (see [Donaldson 05b] and also [Sano 06]). In this paper we will concern ourselves only with a very simple case and study in some detail the properties of this convergence.

Specifically, we take as our manifold the Riemann sphere $X = \mathbb{CP}^1$ and line bundle $L = O_X(1)$. We note that the presence of the automorphism group $\mathrm{SL}(2,\mathbb{C})$ means that strictly speaking, some aspects of the theory may need to be developed further, in the manner of [Mabuchi 05], for example, but since we are focusing on numerical results here, we will not dwell on this issue. Fix a holomorphic coordinate $z \in \mathbb{C}$. Then $\mathrm{H}^0(X, L^k) = \mathrm{H}^0(\mathbb{CP}^1, O(k)) \cong \mathbb{C}^{k+1}$ has basis $1, z, z^2, \ldots, z^k$. Hermitian metrics can now be associated with $(k+1)\times(k+1)$ positive definite Hermitian matrices. For the T_ν function we fix our volume form ν as the standard Fubini–Study form

$$\nu = \sqrt{-1}\,\partial\,\overline{\partial}\log(1+|z|^2) = \frac{\sqrt{-1}}{(1+|z|^2)^2}\,\mathrm{d}z\wedge\mathrm{d}\overline{z}. \quad (2\text{–}1)$$

In the case of the T_K map, we note that $K = O(-2)$; hence $L^k = K^{-p}$ precisely when $k = 2p$.

We simplify further by considering only those metrics invariant under the S^1 action $z \mapsto e^{i\theta}z$ on the Riemann sphere. This restricts our attention to *diagonal* positive definite Hermitian $(k+1)\times(k+1)$ matrices G. We will suppose G has entries $a_0^{-1}, a_1^{-1}, \ldots, a_k^{-1}$ (taking inverses simplifies later computations), and we will use the notation

$$G = (a_0, a_1, \ldots, a_k)$$

to denote this metric. Each of T, T_ν, and T_K is a function of $(a_0, a_1, \ldots, a_k)$, and in the remainder of this section we write them down explicitly.

We begin with T. Taking G as above, we can choose the orthonormal basis $\{s_i = \sqrt{a_i}z^i,\ i = 0,\ldots,k\}$. Then

$$h = \mathrm{FS}(G) = \left(\sum a_i|z|^{2i}\right)^{-1},$$

and we calculate

$$\begin{aligned}\omega_h &= \sqrt{-1}\,\partial\,\overline{\partial}\log\left(\sum a_i|z|^{2i}\right)\\ &= \sqrt{-1}\frac{\sum_{i>j} a_i a_j (i-j)^2|z|^{2(i+j-1)}}{\left(\sum a_i|z|^{2i}\right)^2}\,\mathrm{d}z\wedge\mathrm{d}\overline{z}.\end{aligned}$$

Write $T(a_0,\ldots,a_k) = (\tilde{a}_0,\ldots,\tilde{a}_k)$. Then

$$\tilde{a}_q^{-1} = R\int_{\mathbb{C}} |z|^{2q} h\omega_h,$$

where $R = (k+1)/\operatorname{Vol}(X,\omega_h)$. Using polar coordinates $z = re^{i\theta}$ and setting $x = r^2$, we get

$$\tilde{a}_q = 1\Big/\left(2\pi R\int_0^\infty \frac{\sum_{i>j} a_i a_j (i-j)^2 x^{i+j-1}}{\left(\sum a_i x^i\right)^2} x^q\,\mathrm{d}x\right).$$

Thus after substituting for R, we obtain

$$T : a_q \mapsto \frac{\int_0^\infty \frac{\sum_{i>j} a_i a_j (i-j)^2 x^{i+j-1}}{(\sum a_i x^i)^2}\,\mathrm{d}x}{(k+1)\int_0^\infty \frac{\sum_{i>j} a_i a_j (i-j)^2 x^{i+j-1}}{(\sum a_i x^i)^3} x^q\,\mathrm{d}x}, \quad (2\text{–}2)$$

$q = 0, 1, \ldots, k$.

By a similar computation, noting that the T_ν map has the simpler volume form (2–1), we obtain

$$T_\nu : a_q \mapsto \left((k+1)\int_0^\infty \frac{x^q\,\mathrm{d}x}{(1+x)^2\sum a_i x^i}\right)^{-1}, \quad (2\text{–}3)$$

$q = 0, 1, \ldots, k$.

For the T_K map, the volume form is

$$\omega_{G,K} = \sqrt{-1}\left(\sum a_i|z|^{2i}\right)^{-1/p}\,\mathrm{d}z\wedge\mathrm{d}\overline{z},$$

and we calculate as above,

$$T_K : a_q \mapsto \frac{\int_0^\infty \left(\sum a_i x^i\right)^{-2/k}\,\mathrm{d}x}{(k+1)\int_0^\infty \left(\sum a_i x^i\right)^{-1-2/k} x^q\,\mathrm{d}x}, \quad (2\text{–}4)$$

$q = 0, 1, \ldots, k$.

Often, it is simpler still to work with (S^1-invariant) metrics invariant under the inversion $z \mapsto z^{-1}$. We call such metrics *palindromic*, since they are characterized as those metrics $(a_0, a_1, \ldots, a_k)$ that satisfy

$$a_0 = a_k,\ a_1 = a_{k-1}, \ldots,\ a_{\lfloor k/2\rfloor} = a_{\lceil k/2\rceil}.$$

Thus in the palindromic case, there are exactly $\lceil k/2\rceil$ real (positive) parameters, while in the nonpalindromic case, there are $k+1$. However, we note that for any of the operators $F = T, T_\nu, T_K$, and any starting metric $(a_0,\ldots,a_k)$, if we let $(\tilde{a}_0,\ldots,\tilde{a}_k)$ denote the metric after an application of F, then we have a relation

$$\sum_{i=0}^{k} \frac{a_i}{\tilde{a}_i} = k+1. \quad (2\text{–}5)$$

This is immediately verified by checking formulas (2–2), (2–3), (2–4).

3. FINDINGS

In investigating the behavior of the convergence of a sequence of Hermitian metrics, we need to decide what we mean when we say that two metrics are close. Let $M = \mathrm{GL}(k+1,\mathbb{C})/U(k+1)$ be the space of Hermitian metrics on $\mathbb{CP}^k$. The $\mathrm{GL}(k+1,\mathbb{C})$-invariant Kähler metric is given by the form $g_H(U,V) = \mathrm{tr}(H^{-2}UV)$, where U, V are in the tangent space to H on M. Geodesics on M are given by the images of one-parameter subgroups, e.g.,

$$\begin{pmatrix} e^{\alpha_0 t} & & \\ & \ddots & \\ & & e^{\alpha_k t} \end{pmatrix}.$$

Let $A = (a_0, \dots, a_k)$ and $B = (b_0, \dots, b_k)$ be two metrics in M. Writing $a_i = e^{\alpha_i}$ and $b_i = e^{\beta_i}$ for $i = 0, \dots, k$, we find that the geodesic from A to B is given by $P(t)$, $0 \le t \le 1$, where $P(t)$ is the diagonal matrix with entries $e^{(\beta_i - \alpha_i)t + \alpha_i}$, $i = 0, \dots, k$. Now we can calculate the distance between A and B as $\int_0^1 \left|\frac{dP}{dt}\right|_P dt = \sqrt{\sum(\beta_i - \alpha_i)^2}$, or

$$\mathrm{dist}(A,B) = \sqrt{\sum_{i=0}^{k} \left(\log \frac{b_i}{a_i}\right)^2}. \qquad (3\text{–}1)$$

One goal is then to understand how well the rth iteration of $F = T, T_\nu, T_K$ applied to a Hermitian metric G approximates the limiting balanced metric $B := F^{(\infty)}(G)$. That is, we wish to understand the function

$$\mathrm{err}_{F,k}(G,r) = \mathrm{dist}\left(F^{(r)}(G), F^{(\infty)}(G)\right).$$

In particular, we would like to give an effective bound:

$$\mathrm{err}_{F,k}(G,r) < \mathrm{bnd}_{F,k}(d,r),$$

where $d = \mathrm{dist}(G,B)$. We propose such a bound in Section 3.3.

3.1 The Balanced Metrics

The metrics obtained by taking the coefficients of the polynomial $\alpha(1+cX)^k$, i.e., $a_q = \alpha c^q \binom{k}{q}$, for any $\alpha, c > 0$, are fixed for both the T and the T_K maps; it is not fixed for T_ν unless $c = 1$, in which case we get the *round metric*, the only palindromic balanced metrics for any k. This can be explained by the fact that both the T and T_K maps respect the induced action of $\mathrm{SL}(2,\mathbb{C})$ on the space of metrics, while T_ν does not.

Starting with arbitrary $G = (a_0, a_1, \dots, a_k)$, it is not entirely clear toward which balanced metric iterations of any of the operators T, T_ν, T_K will tend; all we can say is that the coefficients will be of the form $B = (b_0, \dots, b_k)$, where $b_q = \alpha c^q \binom{k}{q}$ for some $\alpha, c > 0$, and if G is palindromic or the operator is T_ν, then $c = 1$. We also note that when $k = 2$, we can calculate the value c as $c = \sqrt{a_2/a_0}$, and thus the balanced metric will be of the form $\alpha(a_0, 2\sqrt{a_0 a_2}, a_2)$ for some scalar $\alpha > 0$.

3.2 Asymptotic Behavior

In the long run, the behavior of the iterations of $F = T, T_\nu, T_K$ is predictable. For each function, the limiting ratio

$$\sigma_{F,k} := \lim_{r\to\infty} \frac{\mathrm{dist}(F^{(r+1)}(G),\ F^{(\infty)}(G))}{\mathrm{dist}(F^{(r)}(G),\ F^{(\infty)}(G))}$$

exists and converges to a simple limit. In [Donaldson 05b], Donaldson proves that in the case of the T_ν iteration and starting with a palindromic metric, this σ-value can be computed as

$$\sigma_{T_\nu,k} = \frac{(k-1)k}{(k+2)(k+3)} \quad \text{(if } G \text{ is palindromic).} \qquad (3\text{–}2)$$

By examining many examples, we also observed that if G is not palindromic, we get

$$\sigma_{T_\nu,k} = \frac{k}{k+2} \quad \text{(if } G \text{ is not palindromic),} \qquad (3\text{–}3)$$

while in the case of the T iteration, we have

$$\sigma_{T,k} = \frac{(k-1)(k+6)}{(k+2)(k+3)}, \qquad (3\text{–}4)$$

and for T_K we get

$$\sigma_{T_K,k} = \frac{k-1}{k+3}. \qquad (3\text{–}5)$$

In neither of these latter two cases does it matter whether we start with a palindromic metric.

We see that when $k = 2$, we have

$$\sigma_{T_\nu,2}\ \text{(not palindromic)} > \sigma_{T,2} > \sigma_{T_K,2} > \sigma_{T_\nu,2}\ \text{(palindromic)},$$

while for $k \ge 3$ we have

$$\sigma_{T,k} \ge \sigma_{T_\nu,k}\ \text{(not palindromic)} > \sigma_{T_K,2} > \sigma_{T_\nu,2}\ \text{(palindromic)},$$

with strict inequalities for every $k > 3$. So in general, if we start with a palindromic metric G, we expect that the T_ν iterations will converge the most quickly, followed by T_K and then by T. Starting with a nonpalindromic G, the T_ν iterations will slow down, and we find that T_K will converge fastest. Here T is still slowest to converge.

r	a_0	a_1	a_2	dist$(-, B)$	bnd
0	0.8826	15.0043	31.7738	0.2848	1.0180
1	0.9738	12.6377	35.0561	0.0640	0.3027
2	0.9946	12.1292	35.8067	0.0131	0.0683
3	0.9989	12.0259	35.9612	0.0026	0.0140
4	0.9998	12.0052	35.9922	0.0005	0.0028
5	1.0000	12.0010	35.9984	0.0001	0.0006

TABLE 1. Example iteration with $k = 2$.

3.3 The Effect on Distance

Despite this simple long-term behavior of the T, T_ν, and T_K iterations, the early behavior is still somewhat mysterious. Perhaps one surprising fact along these lines is that in general, the T operator is not distance-reducing on the space of Hermitian metrics on $\mathrm{H}^0\left(\mathbb{CP}^1, O(k)\right)$. An example when $k = 6$ is given in Section 4. This is the smallest value of k for which the author has found such an example.

In [Calabi and Chen 02], the authors show that the Calabi flow is, in a certain sense, distance-reducing. Hence it might be surprising that T is not distance-reducing, given the expectation that it can be viewed as a discrete version of such a flow.

While it can happen that $T(G)$ is farther from the balanced metric than G is, it does not appear to be the case that it can be arbitrarily farther. Indeed, for each of the operators T, T_ν, T_K, the amount by which it can "magnify" the distance from the balanced metric appears to be simply bounded by a slow function of k. This leads us to conjecture a bound for how far the rth iteration of any of the operators can be from the balanced metric.

Let $F = T, T_\nu, T_K$, let G be any metric, and set $B = F^{(\infty)}(G)$ to be the balanced metric to which the dynamical system $\{F^{(r)}(G),\ r = 0, 1, 2, \ldots\}$ converges. Recall that we define

$$\mathrm{err}_{F,k}(G, r) = \mathrm{dist}\left(F^{(r)}(G), B\right).$$

Let d denote the initial distance from G to B in the space of Hermitian metrics. Then we propose that in fact,

$$\mathrm{err}_{F,k}(G, r) < \log\left(1 + e^{kd}\sigma^r_{F,k}\right) \qquad (3\text{–}6)$$

for every $k > 1$. We do not expect this bound to be sharp.

4. EXAMPLES

In this section we illustrate the findings from Section 3 with some examples. We will always scale all metrics uniformly so that the limiting balanced metric begins with a one. Note that each of the operators T, T_ν, T_K respects scaling.

We begin with $k = 2$ and a nonpalindromic metric proportional to $G = (1, 17, 36)$, and consider the T_K iterations. According to Section 3.1, the limiting balanced metric will be, after scaling, $B = (1, 12, 36)$. The results are displayed in Table 1. The first column gives the iteration r; the next three, the entries of the metric; the second to last gives the distance from the balanced metric, or $\mathrm{err}_{T_K,k}(r, G)$; and the last column gives the bound $\mathrm{bnd}_{T_K,k}(d, r) = \log(1+e^{kd}\sigma^r_{T_K,k})$ proposed in Section 3.3.

We consider another nonpalindromic metric, proportional to $G = (1, 25, 0.07, 13)$, with $k = 3$. We use the T_ν operator and list the results of the first few iterations $T_\nu^{(r)}(G)$ in Table 2. We note that the limiting metric is $B = (1, 3, 3, 1)$.

We give one more table, Table 3, this time beginning with a metric that moves away from the limiting metric after the first application of the operator T. We choose the palindromic

$$G = (1,\ 6000,\ 150000,\ 20000000000,\ 150000,\ 6000,\ 1),$$

with $k = 6$. Each iterate will be of the form $(a_0, a_1, a_2, a_3, a_2, a_1, a_0)$, so we keep track only of a_0, a_1, a_2, a_3. Again we uniformly scale so that the limiting metric is exactly $B = (1, 6, 15, 20, 15, 6, 1)$.

We finish the Riemann sphere case with a visual example of Donaldson's T-iterations. We choose a palindromic metric that we can realize as induced from an embedding

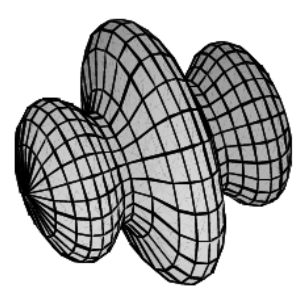

FIGURE 1. $\mathbb{CP}^1$ with metric induced from $G = (1, 300, 300, 300, 1)$.

r	a_0	a_1	a_2	a_3	dist$(-, B)$	bnd
0	0.20720	5.18011	0.01450	2.69366	5.67338	17.02014
1	0.57206	2.68260	3.45522	1.58209	0.74488	16.50932
2	0.73295	2.72858	3.31411	1.32528	0.44129	15.99849
3	0.83372	2.82894	3.18320	1.18836	0.26423	15.48766
4	0.89777	2.89557	3.10812	1.11040	0.15845	14.97684
5	0.93773	2.93684	3.06435	1.06526	0.09505	14.46601
10	0.99505	2.99505	3.00496	1.00497	0.00739	11.91189
15	0.99961	2.99962	3.00039	1.00039	0.00057	9.35784
20	0.99997	2.99997	3.00003	1.00003	0.00004	6.80474

TABLE 2. Example iteration with $k = 3$.

r	a_0	a_1	a_2	a_3	err	bnd
0	0.00010	0.58903	14.72580	1963439.38600	17.69856	106.19139
1	0.00010	0.48814	1073.02459	733382.16850	18.10011	106.00906
2	0.00011	0.60722	1196.93120	414634.58830	17.67812	105.82674
3	0.00013	0.72695	1195.91914	257759.72070	17.21170	105.64441
4	0.00016	0.84269	1147.31003	167930.51810	16.72422	105.46208
5	0.00020	0.95726	1076.08572	112611.11230	16.22342	105.27976
10	0.00068	1.58083	669.18359	18910.93755	13.62571	104.36813
20	0.01002	3.32601	190.00391	970.58975	8.42894	102.54488
30	0.11205	5.07732	52.17933	117.34474	3.98456	100.72162
40	0.51092	5.88292	22.24884	34.20518	1.22538	98.89836
50	0.87358	5.99470	16.26035	22.28184	0.24744	97.07511
60	0.97741	5.99984	15.20684	20.36883	0.04187	95.25185
70	0.99629	6.00000	15.03350	20.05958	0.00682	93.42860
80	0.99940	6.00000	15.00541	20.00962	0.00110	91.60534
90	0.99990	6.00000	15.00088	20.00156	0.00018	89.78209
100	0.99998	6.00000	15.00014	20.00025	0.00003	87.95883

TABLE 3. Non-distance reducing example ($k = 6$).

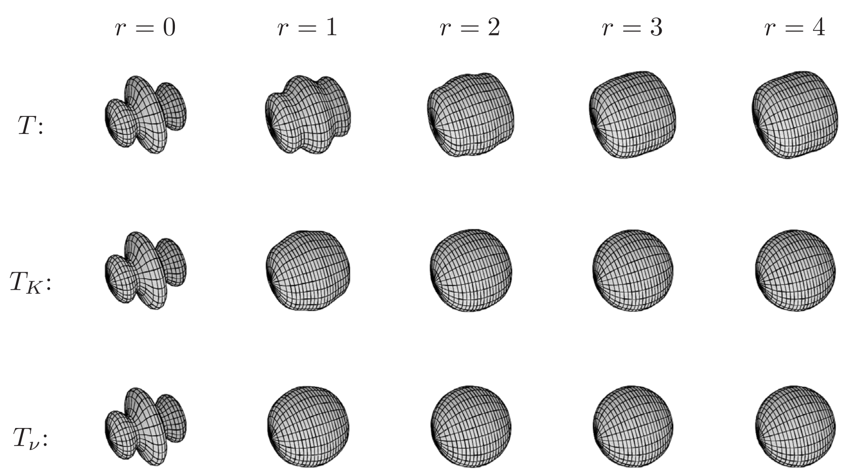

FIGURE 2. The first four iterations.

of $\mathbb{CP}^1$ into $\mathbb{R}^3$. In particular, we pick

$$G = (1, 300, 300, 300, 1)$$

on $\mathrm{H}^0(\mathbb{CP}^1, O(4))$, which is a metric obtained if one were to pinch the sphere around two latitudes, giving it two narrow necks. See Figure 1.

Now, in Figure 2, we plot the evolution of the metric G under the iterations of T, T_K, and T_ν, respectively.

Clearly, the T iterations are much slower in converging to a round sphere. Not until the third iteration does it become convex. At the other extreme lie the T_ν iterations, where the first iteration is already almost indistinguishable from a round sphere. Intermediate between the two are the T_K iterations. The figure visually presents the observations in Section 3.2, where rates of convergence were compared using asymptotic behavior.

5. HIGHER-DIMENSIONAL PROJECTIVE SPACE

Let us now investigate the complex projective space $X = \mathbb{CP}^n$, where $n > 1$. We will consider exclusively the T_ν iteration. Let $z_1, \ldots, z_n$ be local coordinates on $X = \mathbb{CP}^n$. Let us fix once and for all a volume form ν on X using that induced by the normalized Fubini–Study metric. That is, if

$$\begin{aligned}\omega &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(1+\sum|z_k|^2\right)\\ &= \frac{\sqrt{-1}}{2\pi}\,\frac{\sum_{i,j}\left[\left(1+\sum_k|z_k|^2\right)\delta_{ij}-z_j\overline{z}_i\right]}{\left(1+\sum_k|z_k|^2\right)^2}dz_i\wedge d\overline{z}_j\end{aligned}$$

is the normalized Fubini–Study metric in local coordinates, then we set

$$\nu = \omega^n = n!\left(\frac{\sqrt{-1}}{2\pi}\right)^n\frac{dz_1\wedge d\overline{z}_1\wedge\cdots\wedge dz_n\wedge d\overline{z}_n}{\left(1+|z_1|^2+\cdots+|z_n|^2\right)^{n+1}}.$$

It is not hard to check that with this choice of volume form we get

$$\mathrm{Vol}(\mathbb{CP}^n) = \int_{\mathbb{CP}^n}\nu = 1.$$

Again we set $L = O(1)$ and fix a $k > 0$. Note that a basis of $H^0(X, L^k)$ is given by the set of monomials in the z_i of total degree $\le k$. Denote these by $w_1, \ldots, w_N$, where $N = \binom{n+k}{k}$. In this setup we are studying embeddings

$$X = \mathbb{CP}^n \hookrightarrow \mathbb{P}\left(H^0(\mathbb{CP}^n, O(k))\right) \cong \mathbb{CP}^{N-1}.$$

As above, we take h to be the metric on $L^k = O(k)$ defined by

$$h = \left(\sum_{i=1}^N|w_i|^2\right)^{-1}.$$

Now if G is a (positive definite Hermitian) matrix on $H^0(X, L^k)$, then $T_\nu(G)$ is the matrix giving rise to the norm

$$\|s\|_{\mathrm{Hilb}_\nu} = R_\nu\int_X|s|_h^2\nu,$$

where

$$R_\nu = \frac{\dim H^0(X, L^k)}{\mathrm{Vol}(X,\nu)} = N.$$

The matrix G has rows and columns indexed by the terms w_i, $i = 1, \ldots, N$. Let us take a diagonal matrix with terms a_i^{-1}. Such a matrix corresponds to an (algebraic) metric invariant under the torus $\Lambda_n = (S^1)^n$ action $z_l \mapsto e^{i\theta_l}z_l$, $l = 1, \ldots, n$.

An orthonormal basis, according to G, is given by

$$\{\sqrt{a_i}w_i,\ i = 1, \ldots, N\}.$$

Then in terms of the a_i's, the matrix $T_\nu(G)$ will have diagonal entries $\tilde{a}_i^{-1}$ equal to

$$\begin{aligned}\|w_i\|_{\mathrm{Hilb}_\nu} &= Nn!\left(\frac{\sqrt{-1}}{2\pi}\right)^n\\ &\quad\times\int_{\mathbb{C}^n}\frac{|w_i|^2dz_1\wedge d\overline{z}_1\wedge\cdots\wedge dz\wedge d\overline{z}_n}{\left(\sum_{p=1}^N a_p|w_p|^2\right)\left(1+\sum_{q=1}^n|z_q|^2\right)^{n+1}}.\end{aligned}$$

Changing to polar coordinates $z_j = r_j\exp\left(\sqrt{-1}\,\theta_j\right)$ and substituting $x_j = r_j^2$, we get

$$\begin{aligned}T(G)_{ii} &= \tilde{a}_i^{-1}\\ &= Nn!\int_0^\infty\!\!\cdots\int_0^\infty\frac{w_i(x)dx_1\cdots dx_n}{\left(\sum_{p=1}^N a_pw_p(x)\right)\left(1+\sum_{q=1}^n x_q\right)^{n+1}},\end{aligned}\tag{5–1}$$

where $w(x)$ denotes the monomial w with the substitutions $x_k = z_k$, $k = 1, \ldots, n$.

5.1 Asymptotic Behavior in Higher Dimensions

Let us consider the asymptotic behavior of T_ν. Recall (see Section 3.2) that in the case of $n = 1$, i.e., when $X = \mathbb{CP}^1$, we defined

$$\sigma_{T_\nu,k} := \lim_{r\to\infty}\frac{\mathrm{dist}(T_\nu^{(r+1)}(G),\ T_\nu^{(\infty)}(G))}{\mathrm{dist}(T_\nu^{(r)}(G),\ T_\nu^{(\infty)}(G))}.$$

This value depends on whether the initial metric G is invariant under the inversion map $z \mapsto z^{-1}$, or equivalently in homogeneous coordinates, $Z_0 \leftrightarrow Z_1$. In [Donaldson 05b], Donaldson computes these values theoretically, and our investigations corroborate his result:

$$\sigma_{T_\nu,k} = \begin{cases}\frac{(k-1)k}{(k+2)(k+3)} & \text{if } G \text{ is invariant under } Z_0 \leftrightarrow Z_1,\\ \frac{k}{(k+2)} & \text{otherwise.}\end{cases}\tag{5–2}$$

Our goal is to show evidence for a simple extension of this formula valid on $X = \mathbb{CP}^n$, $n \ge 1$.

When $n > 1$ there are many possible ways to extend the notion of a "palindromic" metric (as we defined in Section 2): for the Riemann sphere we have those metrics invariant under $Z_0 \leftrightarrow Z_1$, but in general, there are many permutations of the homogeneous coordinates $Z_0, \ldots, Z_n$, and it is trivial to check that if G is a metric invariant under such a symmetry, then so is $T_\nu(G)$.

We might then expect that there can be distinct values for σ depending on various symmetries under which the metric G could be invariant. Thus we may find a different value for each (conjugacy class of) subgroup of $\mathrm{Sym}(n+1)$ (the symmetric group on $n+1$ characters) corresponding to metrics G invariant under the automorphisms $Z_i \mapsto Z_{\pi(i)}$, $i = 0, 1, \ldots, n$, for π ranging over the subgroup.

We present here some numerical findings in the cases of $n = 2$ and $n = 3$. The iterated integrals (5–1) grow in computational complexity quickly with increasing n.

We start with a metric G that is torus-invariant, but otherwise "random" in the sense that it is not invariant under any permutation of the homogeneous coordinates. We tabulate approximate numerical values for the asymptotic constant σ here, all computed starting with "random" (but torus-invariant) metrics:

σ	$k=2$	3	4	5
$n=2$	0.40	0.50	0.57	0.63
3	0.33	0.43	0.50	0.56

For the moment, let us just note that the above values apparently follow the pattern

$$\sigma = \frac{k}{k+n+1}. \tag{5–3}$$

When $n = 1$, the fundamental case that we considered, this formula specializes to (5–2).

In the nongeneric case, in which G might be invariant under a permutation of the homogeneous coordinate variables, we find simple behavior:

- If there is no fixed-point-free permutation of the homogeneous coordinate variables under which G is invariant, then σ is the same as computed in the asymmetric case.
- Otherwise, suppose G is invariant under some fixed-point-free permutation of the homogeneous coordinate variables. Then we get new values for σ, as tabulated in the following table:

σ	$k=2$	3	4	5
$n=2$	0.07	0.14	0.21	0.28
3	0.05	0.11	0.17	0.22

One can check that approximate fractional equivalents to these numbers follow the pattern

$$\sigma = \frac{(k-1)k}{(k+n+2)(k+n+3)}. \tag{5–4}$$

We should stress that when $n = 1$, equation (5–4), together with (5–3), specializes to (5–2). This together with various experimental evidence leads the author to pose the following question:

Question 5.1. Let G be a torus-invariant metric arising from a matrix on $H^0(\mathbb{CP}^n, O_{\mathbb{CP}^n}(k))$, and let $B = T_\nu^\infty(G)$ be the limiting balanced metric under the T_ν iteration. Define

$$\sigma_G(n,k) := \lim_{r\to\infty} \frac{\mathrm{dist}(T_\nu^{(r+1)}(G),\ B)}{\mathrm{dist}(T_\nu^{(r)}(G),\ B)}.$$

Let us say that G is *generally symmetric* if it is invariant under some fixed-point-free permutation of the homogeneous coordinates. Then is

$$\sigma_G(n,k) = \begin{cases} \frac{(k-1)k}{(k+n+1)(k+n+2)} & \text{if } G \text{ is generally symmetric,} \\ \frac{k}{(k+2)} & \text{otherwise,} \end{cases} \tag{5–5}$$

a general formula?

5.2 Example Computation

To illustrate a typical computation leading to some of the numbers above, take $n = 3, k = 4$. Then

$$N = \dim H^0\left(\mathbb{CP}^3, O(4)\right) = \binom{3+4}{4} = 35,$$

and a basis of $H^0(\mathbb{CP}^3, O(4))$ is (in local coordinates)

$$\{w_1, \ldots, w_{35}\} = \{1, z_1, z_2, z_3, z_1^2, \ldots, z_2 z_3^3, z_3^4\}. \tag{5–6}$$

We choose a G that is invariant under *every* permutation of the homogeneous coordinates Z_i, $i = 0, \ldots, 3$ (where $z_i = \frac{Z_i}{Z_0}$). Taking into account these symmetries, there are only five distinct basis elements:

$$1,\ z_1,\ z_1^2,\ z_1 z_2,\ z_1 z_2 z_3.$$

In the order in which the basis elements are listed in (5–6), first by degree, then lexicographically, these are the first, second, fifth, sixth, and fifteenth elements. In the notation used at the beginning of this section we pick diagonal entries of G: a_i^{-1} in the row and column determined by the basis element w_i. Due to the symmetries we will have five parameters:

$$G : a_1, a_2, a_5, a_6, a_{15}.$$

The iterations of T_ν on these parameters, denote them by $a_{i,r}$, $r = 0, 1, \ldots, \infty$, will (after uniform scaling) tend toward the values $1, 4, 6, 12, 24$ for $i = 1, 2, 5, 6, 15$ respectively. This can readily be checked by noting that the

$T_\nu^{(r)}$	a_2	a_5	a_6	a_{15}	$\tilde{\sigma}_r$
0	20.0000000	30.0000000	40.0000000	50.0000000	0.0000
1	4.3071170	6.5967335	13.0915039	25.9850356	0.0192
2	4.0344368	6.0688663	12.1588436	24.3600437	0.1121
3	4.0052604	6.0105224	12.0258597	24.0613530	0.1528
4	4.0008611	6.0017223	12.0042908	24.0102741	0.1637
5	4.0001430	6.0002860	12.0007145	24.0017140	0.1661
6	4.0000238	6.0000476	12.0001191	24.0002857	0.1665
7	4.0000040	6.0000079	12.0000198	24.0000476	0.1666
8	4.0000007	6.0000013	12.0000033	24.0000079	0.1667

TABLE 4. Example iteration for $x = \mathbb{CP}^3$.

Fubini–Study metric is the balanced metric B. At this point we should recall that the a_i coefficients are actually entries in the inverse matrix G^{-1}; hence the entries of G will tend to $1, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}$. However, we can compute the approximate σ values via

$$\lim_{r\to\infty} \frac{\operatorname{dist}(T_\nu^{(r+1)}(G),\ B)}{\operatorname{dist}(T_\nu^{(r)}(G),\ B)} = \lim_{r\to\infty} \frac{\frac{1}{a_{2,r+1}} - \frac{1}{4}}{\frac{1}{a_{2,r}} - \frac{1}{4}} = \lim_{r\to\infty} \frac{a_{2,r+1} - 4}{a_{2,r} - 4},$$

say (note that the last equality follows because the $a_{i,r}$ are convergent). Denote this last quotient, within the limit, by $\tilde{\sigma}_{r+1}$. Its value should tend to the σ value determining the asymptotic behavior of the T_ν iterations on this metric.

Let us take $(a_1, a_2, a_5, a_6, a_{15}) = (1, 20, 30, 40, 50)$. The limiting balanced metric will have corresponding coordinates proportional to $(1, 4, 6, 12, 24)$, as noted above. However, instead of uniformly scaling all metrics so the result is exactly this metric, we will this time scale each metric so that its first coordinate (the a_1) is equal to one. There is no loss of information: relation (2–5) has the obvious adaptation to this situation; namely, $\sum_{i=1}^{35} a_i/\tilde{a}_i = 35$. Using this, one can iteratively obtain the original numbers. The advantage of doing this is that we no longer need to keep track of the first coordinate a_1.

With this convention, we get Table 4, showing the first eight iterations as well as the approximate σ values.

We note that the apparent limiting value, $0.1\overline{6} = \frac{1}{6}$, matches the value in equation (5–5).

6. FURTHER QUESTIONS

The case of a nondiagonal matrix (thus corresponding to a metric not invariant under $z \mapsto e^{i\theta}z$) was not treated in this paper. In investigating this direction, one might see whether the asymptotic values (see (3–4) and (3–5) or (5–5)) remain valid, and whether the bound (3–6) still holds. If the bound does still hold, then it would be interesting to work toward a sharp bound.

In another direction, one might ask whether the operators T, T_ν, T_K are distance-decreasing *after* the first iteration; or put another way, is the square of each of these operators distance-reducing? No counterexample to this was found.

The next step is to look beyond $\mathbb{CP}^n$, perhaps to toric varieties (see, for example, [Bunch and Donaldson 08]), $K3$ surfaces, Calabi–Yau 3-folds, etc., and work out the same convergence properties of these dynamical systems. It would also be interesting to compare the convergence properties of the T-iterations to those of PDE methods for finding canonical metrics, such as the Ricci flow. All these questions the author hopes to examine later.

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