Some Conjectures and Open Problems on Partition Hook Lengths

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CONTENTS

We present some conjectures and open problems on partition hook lengths motivated by known results on the subject. The conjectures were suggested by extensive experimental calculations using a computer algebra system. The first conjecture unifies two classical results on the number of standard Young tableaux and the number of pairs of standard Young tableaux of the same shape. The second unifies the classical hook formula and the marked hook formula. The third includes the longstanding Lehmer conjecture, which says that the Ramanujan tau function never assumes the value zero. The fourth is a more precise version of the third in the case of 3-cores. We also list some open problems on partition hook lengths.

1. INTRODUCTION

The hook lengths of partitions are widely studied in the theory of partitions, algebraic combinatorics, and group representation theory. In this paper we present some conjectures and open problems on partition hook lengths that are motivated by known results on the subject. The conjectures were suggested by extensive experimental calculations using a computer algebra system.

The basic notions needed here can be found in [Macdonald 95, p. 1], [Stanley 99, p. 287], [Lascoux 01, p. 1], [Knuth 98, p. 59], and [Andrews 76, p. 1].

A *partition* λ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The integers $(\lambda_i)_{i=1,2,\ldots,\ell}$ are called the *parts* of λ, the number ℓ of parts being the *length* of λ, denoted by $\ell(\lambda)$. The sum of its parts $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is denoted by $|\lambda|$.

Let n be an integer. A partition λ is said to be a partition of n if $|\lambda| = n$. We write $\lambda \vdash n$. The set of all partitions of n is denoted by $\mathcal{P}(n)$. The set of all partitions is denoted by $\mathcal{P}$, so that

$$\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}(n).$$

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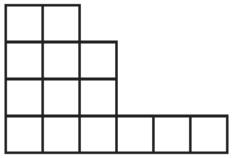

FIGURE 1. Partition.

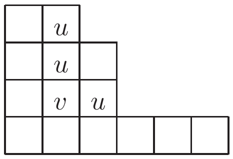

FIGURE 2. Hook length.

Each partition can be represented by its Ferrers diagram. For example, $\lambda = (6, 3, 3, 2)$ is a partition, and its Ferrers diagram is reproduced in Figure 1.

For each box v in the Ferrers diagram of a partition λ, or for each box v in λ, we define the *hook length* of v, denoted by $h_v(\lambda)$ or h_v, to be the number of boxes u such that $u = v$ or u lies in the same column as v and above v or in the same row as v and to the right of v (see Figure 2).

The *hook length multiset* of λ, denoted by $\mathcal{H}(\lambda)$, is the multiset of all hook lengths of λ. In Figure 3, the hook lengths of all boxes for the partition $\lambda = (6, 3, 3, 2)$ have been written in each box. We have

$$\mathcal{H}(\lambda) = \{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}.$$

Let t be a positive integer. Recall that a partition λ is a t-core if the hook length multiset of λ does not contain the integer t. It is known that the hook length multiset of each t-core does not contain any multiple of t [Knuth 98, pp. 69, 612], [Stanley 99, p. 468], [James and Kerber 81, p. 75].

Our first conjecture, stated in Section 2, unifies two classical results on the number of standard Young tableaux and the number of pairs of standard Young tableaux of the same shape. Our second conjecture unifies the classical hook formula and the marked hook formula (see Section 3). The third conjecture, presented in Section 4, includes the longstanding Lehmer conjecture, which says that the Ramanujan tau function never assumes the value zero. The fourth conjecture is a more precise version of the third one in the case of 3-cores (see Section 5). Finally, we list some open problems on partition hook lengths in Section 6.

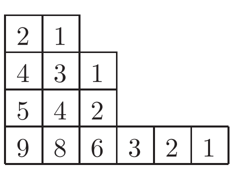

FIGURE 3. Hook lengths.

2. FIRST CONJECTURE

The hook length plays an important role in algebraic combinatorics thanks to the famous hook formula due to Frame, Robinson, and Thrall [Frame et al. 54]:

$$f_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}, \tag{2–1}$$

where f_λ is the number of standard Young tableaux of shape λ (see [Stanley 99, p. 376], [Knuth 98, p. 59], [Greene et al. 79, Gessel and Viennot 85, Zeilberger 84, Novelli et al. 97, Krattenthaler 99]).

Recall that the Robinson–Schensted–Knuth correspondence is a bijection between the set of ordered pairs of standard Young tableaux of $\{1, 2, \ldots, n\}$ of the same shape and the set of permutations of order n [Knuth 70] (see also [Knuth 98, pp. 49–59], [Stanley 99, p. 324]). It provides a combinatorial proof of the following identity:

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!\,. \tag{2–2}$$

Using (2–1), identity (2–2) can be written in the following generating function form:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x. \tag{2–3}$$

The Robinson–Schensted–Knuth correspondence also proves the fact that the number of standard Young tableaux of $\{1, 2, \ldots, n\}$ is equal to the number of involutions of order n (see [Knuth 98, p. 47]). In the generating function form, this means that

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x + x^2/2}. \tag{2–4}$$

Our first conjecture may be regarded as a hook-length formula that interpolates formulas (2–3) and (2–4), which hold for permutations and involutions, respectively. It was suggested by the hook-length expansion technique developed in [Han 08a].

Conjecture 2.1. (First conjecture.) *We have*

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \rho(z;h) = e^{x+zx^2/2}, \tag{2–5}$$

where the weight function $\rho(z;n)$ *is defined by*

$$\rho(z;n) = \frac{\sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} z^k}{n\sum_{k=0}^{\lfloor (n-1)/2\rfloor} \binom{n}{2k+1} z^k}. \tag{2–6}$$

The first values of the weight function $\rho(z,n)$ are listed below:

$$\begin{aligned}
\rho(z;1) &= 1,\\
\rho(z;2) &= \frac{1+z}{4},\\
\rho(z;3) &= \frac{3z+1}{9+3z},\\
\rho(z;4) &= \frac{z^2+6z+1}{16+16z},\\
\rho(z;5) &= \frac{5z^2+10z+1}{5z^2+50z+25},\\
\rho(z;6) &= \frac{z^3+15z^2+15z+1}{120z+36z^2+36},\\
\rho(z;7) &= \frac{7z^3+35z^2+21z+1}{7z^3+147z^2+245z+49}.
\end{aligned}$$

In fact, formula (2–6) has been verified up to $n\le 20$.

Using the real part $\Re$ and imaginary part $\Im$ operators for complex numbers, Conjecture 2.1 can be rewritten in the following equivalent form.

Conjecture 2.2. *We have*

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \frac{z\Re(1+iz)^h}{h\Im(1+iz)^h} = e^{x-z^2x^2/2}. \tag{2–7}$$

In the rest of this section we discuss some specializations of Conjecture 2.1. When $z=1$, then $\rho(1;n)=1/n$; we recover identity (2–4). When $z=0$, then $\rho(0;n)=1/n^2$; we recover identity (2–3). However, we cannot prove any other special cases of Conjecture 2.1 other than the above two values.

Now select the coefficients of $[zx^n]$ on both sides of (2–5). Since

$$\begin{aligned}
\rho(z;n) &= \frac{1+\binom{n}{2}z+O(z^2)}{n^2+n\binom{n}{3}z+O(z^2)} \\
&= \frac{1}{n^2}\left(1+\frac{n^2-1}{3}z\right)+O(z^2),
\end{aligned} \tag{2–8}$$

the coefficient of $[zx^n]$ on the left-hand side of (2–5) is

$$\begin{aligned}
&[z]\sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2}\left(1+\frac{h^2-1}{3}z\right)\\
&= \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}(\lambda)} \left(\frac{h^2-1}{3}\right)\\
&= \frac{1}{3}\sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}(\lambda)} h^2 - \frac{n}{3}\sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2}\\
&= \frac{1}{3}\sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}(\lambda)} h^2 - \frac{n}{3n!}.
\end{aligned} \tag{2–9}$$

The coefficient of $[zx^n]$ on the right-hand side of (2–5) is

$$\begin{aligned}
[zx^n]e^{x+zx^2/2} &= [zx^n]\sum_{k\ge 1}\frac{(x+zx^2/2)^k}{k!}\\
&= [zx^n]\sum_{k\ge 1}\frac{kx^{k-1}(zx^2/2)}{k!}\\
&= [x^n]\sum_{k\ge 1}\frac{x^{k-1}(x^2)}{2(k-1)!}\\
&= \frac{1}{2(n-2)!}.
\end{aligned} \tag{2–10}$$

By comparing (2–9) and (2–10), we obtain the next marked hook formula, which has been proved in [Han 08b, Han 08c].

Theorem 2.3. (Marked hook formula.) *We have*

$$\sum_{\lambda\vdash n} f_\lambda^2 \sum_{h\in\mathcal{H}(\lambda)} h^2 = \frac{n(3n-1)}{2}n!\,. \tag{2–11}$$

We can also select the coefficients of $[z^2x^n]$ on both sides of (2–5). Since

$$\begin{aligned}
\rho(z;n) &= \frac{1+\binom{n}{2}z+\cdots}{n^2+n\binom{n}{3}z+\cdots}\\
&= \frac{1}{n^2}\left(1+\frac{n^2-1}{3}z-\frac{(n^2-1)(n^2-4)}{45}z^2\right)+O(z^3),
\end{aligned}$$

the coefficient of $[zx^n]$ on the left-hand side of (2–5) is

$$\begin{aligned}
&[z^2]\sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2}\left(1+\frac{h^2-1}{3}z-\frac{(h^2-1)(h^2-4)}{45}z^2\right)\\
&= A+B,
\end{aligned}$$

with

$$A = \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{v\in\lambda} \frac{-(h_v^2-1)(h_v^2-4)}{45}$$

and

$$B = \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\substack{\{u,v\}\\ u\neq v}} \left(\frac{h_u^2-1}{3}\frac{h_v^2-1}{3}\right),$$

where the second sum in B ranges over all unordered pairs $\{u,v\}$ such that $u,v\in\lambda$ and $u\neq v$. Let us evaluate the two quantities A and B. We have

$$\begin{aligned} A &= -\frac{1}{45} \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}(\lambda)} (h^4-5h^2+4) \\ &= -\frac{1}{45}(A_1+A_2+A_3), \end{aligned}$$

with

$$A_1 = \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}(\lambda)} h^4, \tag{2–12}$$

$$A_2 = \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}(\lambda)} (-5h^2) = -\frac{5n(3n-1)}{2n!}$$

by (2–11),

$$A_3 = \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}(\lambda)} 4 = \frac{4n}{n!}.$$

We also have

$$\begin{aligned} B &= \frac{1}{9} \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u,v\}} \left(h_u^2h_v^2-(h_v^2+h_u^2)+1\right) \\ &= \frac{1}{9}(B_1+B_2+B_3), \end{aligned}$$

with

$$\begin{aligned} B_1 &= \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u,v\}} h_u^2h_v^2 \\ &= \frac{n(n-1)(27n^2-67n+74)}{24n!} \end{aligned}$$

by [Han 08b, Proposition 6.11],

$$\begin{aligned} B_2 &= \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u,v\}} \left(-(h_v^2+h_u^2)\right) \\ &= (n-1) \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{v} (-h_v^2) \\ &= -(n-1)\frac{n(3n-1)}{2n!}, \end{aligned}$$

$$B_3 = \sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u,v\}} 1 = \frac{1}{n!}\binom{n}{2}.$$

On the other hand, the coefficient of $[z^2x^n]$ on the right-hand side of (2–5) is

$$\begin{aligned} [z^2x^n]e^{x+zx^2/2} &= [z^2x^n]\sum_{k\geq 1}\frac{(x+zx^2/2)^k}{k!} \\ &= [z^2x^n]\sum_{k\geq 2}\frac{k(k-1)/2\times x^{k-2}(zx^2/2)^2}{k!} \\ &= [x^n]\sum_{k\geq 2}\frac{k(k-1)/2\times x^{k-2}(x^2/2)^2}{k!} \\ &= [x^n]\sum_{k\geq 2}\frac{x^{k+2}}{8(k-2)!} \\ &= \frac{1}{8(n-4)!}\,. \end{aligned} \tag{2–13}$$

By Conjecture 2.1 and (2–13) we have

$$\frac{1}{8(n-4)!} = -\frac{1}{45}(A_1+A_2+A_3)+\frac{1}{9}(B_1+B_2+B_3).$$

With the values of A_2, A_3, B_1, B_2, B_3 explicitly calculated, the expression for A_1 shown in (2–12) leads to the following conjecture.

Conjecture 2.4. *We have*

$$\sum_{\lambda\in\mathcal{P}(n)} \prod_{h\in\mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h\in\mathcal{H}(\lambda)} h^4 = \frac{n(40n^2-75n+41)}{6n!}. \tag{2–14}$$

3. SECOND CONJECTURE

Our next conjecture was suggested by the fact that formulas (2–2), (2–11), and (2–14) have the same form.

Conjecture 3.1. (Second conjecture.) *Let k be a positive integer. Then*

$$P_k(n) = (k+1)!(n-1)!\sum_{\lambda\vdash n}\Bigl(\prod_{v\in\lambda}\frac{1}{h_v^2}\Bigr)\Bigl(\sum_{u\in\lambda}h_u^{2k}\Bigr)$$

is a polynomial in n of degree k with integral coefficients.

Notice that the classical hook formula (2–2), the marked hook formula (2–11), and Conjecture 2.4 are all consequences of Conjecture 3.1 (the cases $k=0,1,2$), because if we know that $P_k(n)$ is a polynomial in n of degree k, we can determine the polynomial $P_k(n)$ by taking $(k+1)$ numerical values of $P_k(n)$ using the Lagrange interpolation formula. Let us go one more step by looking at the case $k=3$.

$$
\begin{aligned}
P_0(n) &= 1\\
P_1(n) &= 3\,n - 1\\
P_2(n) &= 40\,n^2 - 75\,n + 41\\
P_3(n) &= 1050\,n^3 - 4060\,n^2 + 5586\,n - 2552\\
P_4(n) &= 42336\,n^4 - 265860\,n^3 + 654360\,n^2 - 721800\,n + 291084\\
P_5(n) &= 2328480\,n^5 - 20956320\,n^4 + 77962500\,n^3 - 146671800\,n^2 + 136808100\,n\\
&\quad - 49470240\\
P_6(n) &= 163088640\,n^6 - 1941619680\,n^5 + 9851665824\,n^4 - 26869883040\,n^3\\
&\quad + 41020980000\,n^2 - 32822800920\,n + 10598574216\\
P_7(n) &= 13913499600\,n^7 - 206918712000\,n^6 + 1332526235040\,n^5 - 4753759570560\,n^4\\
&\quad + 10023914300400\,n^3 - 12352918032000\,n^2 + 8158628953440\,n - 2215386633600\\
P_8(n) &= 1401656256000\,n^8 - 24914439950400\,n^7 + 192568162026240\,n^6\\
&\quad - 830326365348480\,n^5 + 2134506603220992\,n^4 - 3232434128152320\,n^3\\
&\quad + 2628227513681280\,n^2 - 860196155051520\,n - 8832846318912.
\end{aligned}
$$

TABLE 1. The polynomials $P_k(n)$.

Conjecture 3.2. *We have*

$$\sum_{\lambda\vdash n} f_\lambda^2 \sum_{v\in\lambda} h_v^6 = \frac{n}{24}(1050n^3 - 4060n^2 + 5586n - 2552)\; n!$$

The first values of the polynomials $P_k(n)$ $(0 \le k \le 9)$, suggested by extensive experimental calculations using a computer algebra system, are shown in Table 1.

From Conjecture 2.4, we derive the following formula.

Conjecture 3.3. *Let n be an positive integer. We have*

$$\sum_{\lambda\vdash n}\Big(\prod_{v\in\lambda}\frac{1}{h_v^2}\Big)\Big(\sum_{u\in\lambda} h_u^2\Big)^2 = \frac{1}{12(n-1)!}(27n^3{-}14n^2{-}9n{+}8).$$

4. THIRD CONJECTURE

Let us state our third conjecture, followed by some specializations and remarks.

Conjecture 4.1. (Third conjecture.) *Let n, s, t be positive integers such that $t \ne 4, 10$ and $s \mid t$. Then the coefficient of x^n in*

$$\prod_{k\ge1}\frac{(1-x^{sk})^{t^2/s}}{1-x^k}$$

is equal to zero if and only if the coefficient of x^n in

$$\prod_{k\ge1}\frac{(1-x^{tk})^{t}}{1-x^k}$$

is also equal to zero.

Conjecture 4.1 has been verified by the author for all pairs (t, n) such that $t \le 13$ and $n \le 4000$.

Remark 4.2. Even if the conjecture is stated with the exceptions $t \ne 4, 10$, it is almost true in those cases. For example, up to $n = 4000$, there are only four exceptions $n = 53, 482, 1340, 2627$ for $s = 1, t = 4$; five exceptions $n = 35, 320, 890, 1745, 2885$ for $s = 2, t = 4$, and two exceptions $n = 24, 49$ for $s = 5, t = 10$. Ken Ono [Ono 08] has pointed out that there are infinitely many exceptions for $s = 1, t = 4$.

Let $\mathcal{P}(n;t)$ denote the set of all t-cores of n. The generating function for t-cores is given by the following formula:

$$\sum_{\lambda} x^{|\lambda|} = \prod_{k\ge1}\frac{(1-x^{tk})^t}{1-x^k}, \qquad (4\text{–}1)$$

where the sum ranges over all t-cores [Knuth 98, pp. 69, 612], [Stanley 99, p. 468], [Garvan et al. 90].

In [Han 08c, Corollary 5.3] we proved the following result.

Theorem 4.3. *We have*

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|}\prod_{v\in\lambda, s\mid h_v}\left(1-\frac{sz}{h_v^2}\right) = \prod_{k\ge1}\frac{(1-x^{sk})^z}{1-x^k}. \qquad (4\text{–}2)$$

Hence, Conjecture 4.1 can be rewritten using (4–1) and (4–2) as follows.

Conjecture 4.4. *Let n, s, t be positive integers such that $t \neq 4, 10$ and $s \mid t$. The expression*

$$\sum_{\lambda \in \mathcal{P}(n;t)} \prod_{v \in \lambda, s | h_v} \left(1 - \frac{t^2}{h_v^2}\right) \tag{4–3}$$

is equal to zero if and only if $\mathcal{P}(n;t) = \varnothing$.

Conjecture 4.1 is true for $s = 1$ and $t = 2$, thanks to the following two well-known formulas due respectively to Jacobi (see [Andrews 76, p. 21], [Knuth 98, p. 20]) and Gauss (see [Stanley 99, p. 518], [Andrews 76, p. 23]).

Theorem 4.5. (Jacobi.) *We have*

$$\prod_{m \geq 1} (1 - x^m)^3 = \sum_{m \geq 0} (-1)^m (2m+1) x^{m(m+1)/2}. \tag{4–4}$$

Theorem 4.6. (Gauss.) *We have*

$$\prod_{m \geq 1} \frac{(1 - x^{2m})^2}{1 - x^m} = \sum_{m \geq 0} x^{m(m+1)/2}. \tag{4–5}$$

Consider the specialization $s = 1$ and $t = 3$. Let $(a(n))$ be the coefficients in the expansion of the product

$$\begin{aligned}\prod_{m \geq 1} (1 - x^m)^8 &= \sum_{n \geq 0} a(n) x^n \\ &= 1 - 8x + 20x^2 - 70x^4 + 64x^5 + 56x^6 \\ &\quad - 125x^8 + \cdots - 20482x^{220} + 24050x^{224} \\ &\quad - 21624x^{225} + \cdots \end{aligned} \tag{4–6}$$

and $(b(n))$ the coefficients in the expansion of the product

$$\begin{aligned}\prod_{m \geq 1} \frac{(1 - x^{3m})^3}{1 - x^m} &= \sum_{b \geq 0} b(n) x^n \\ &= 1 + x + 2x^2 + 2x^4 + x^5 + 2x^6 + x^8 \\ &\quad + \cdots + 2x^{220} + 2x^{224} + 3x^{225} + \cdots . \end{aligned} \tag{4–7}$$

Notice that the coefficients $b(n)$ are rather small and $a(n)$ are rather large. Conjecture 4.1 may be restated as follows.

Conjecture 4.7. *Let n be a positive integer. Then $a(n) = 0$ if and only if $b(n) = 0$.*

Recall the following theorem due to Granville and Ono [Granville and Ono 96].

Theorem 4.8. *Let n, t be two positive integers such that $t \geq 4$. Then $\mathcal{P}(n;t) \neq \varnothing$.*

Hence Conjecture 4.1 can be rewritten in the following way.

Conjecture 4.9. *Let $t \geq 5, n, s$ be positive integers such that $s \mid t$ and $t \neq 10$. Then the coefficient of x^n in*

$$\prod_{k \geq 1} \frac{(1 - x^{sk})^{t^2/s}}{1 - x^k}$$

is not equal to zero.

In particular, when $s = 1$ and $t = 5$ in Conjecture 4.9, we recover the following longstanding Lehmer conjecture (see [Serre 70]). Recall that the Ramanujan τ-function is defined by (see [Serre 70, p. 156]):

$$\begin{aligned} x \prod_{m \geq 1} (1 - x^m)^{24} &= \sum_{n \geq 1} \tau(n) x^n \\ &= x - 24x^2 + 252x^3 - 1472x^4 \\ &\quad + 4830x^5 - 6048x^6 + \cdots . \end{aligned}$$

Conjecture 4.10. (Lehmer.) *For each n we have $\tau(n) \neq 0$.*

5. FOURTH CONJECTURE

Recall that $a(n)$ and $b(n)$ are defined by (4–6) and (4–7), respectively. The following conjectures characterize the integers n for which $a(n) = 0$ or $b(n) = 0$. They are suggested by Theorem 5.3, stated later in this section.

Conjecture 5.1. (Fourth conjecture.) *Let N be a positive integer.*

(i) *If there are integers $n \geq 0$, $m \geq 1$ such that*

$$N = 4^m n + (10 \cdot 4^{m-1} - 1)/3,$$

then $a(N) = 0$.

(ii) *If there are integers $n \geq 0, m \geq 1, k \geq 1$ with $m \not\equiv 2k - 1 \mod (6k-1)$ such that*

$$N = (6k-1)^2 n + (6k-1)m + 4k - 1,$$

then $a(N) = 0$.

(iii) *For all positive integers N we have $a(N) \neq 0$, except those in cases (i) and (ii).*

If the third conjecture is true, then Conjecture 5.1 is equivalent to the following conjecture for $b(n)$. It is known (see, e.g., [Garvan et al. 90]) that $b(n)$ is equal to the number of integer solutions of the Diophantine equation

$$3(x^2 + xy + y^2) + x + 2y = n.$$

Conjecture 5.2. *Let N be a positive integer.*

(i) *If there are integers $n \geq 0$, $m \geq 1$ such that*

$$N = 4^m n + \frac{10 \cdot 4^{m-1} - 1}{3},$$

then $b(N) = 0$.

(ii) *If there are integers $n \geq 0$, $m \geq 1$, $k \geq 1$ with $m \not\equiv 2k - 1 \bmod (6k - 1)$ such that*

$$N = (6k - 1)^2 n + (6k - 1)m + 4k - 1,$$

then $b(N) = 0$.

(iii) *For all positive integers N we have $b(N) \neq 0$, except those in cases (i) and (ii).*

Taking special values for m and k in Conjecture 5.1 yields the following relations.

Theorem 5.3. *We have*

$$\begin{aligned} a(4n+3) &= 0, \\ a(16n+13) &= 0, \\ a(25n+3) &= 0, \quad a(25n+13) = 0, \\ a(25n+18) &= 0, \quad a(25n+23) = 0, \\ a(64n+53) &= 0. \end{aligned}$$

Proof: In fact, the relations in Theorem 5.3 were discovered and automatically proved using a computer algebra program thanks to the next theorem, which asserts that a simple variation of the classical Macdonald identity [Macdonald 72] holds. For example, each term in identity (5.1) has two parameters k and m (or only one parameter k). To prove $a(4n+3) = 0$, we need to check $a(4n+3) = 0$ only for $k, m = 0, 1, 2, 3$, since the coefficient and the exponent in each term are both polynomials in k and m with integral coefficients. There are finitely many cases to verify. □

Theorem 5.4. *We have*

$$\begin{aligned} \prod_{k\geq 1}(1-q^k)^8 &= \sum_{k\geq 0}\Big((3k+1)^3 q^{3k^2+2k} - (3k+2)^3 q^{3k^2+4k+1}\Big) \\ &\quad + \sum_{k>m\geq 0}\Big((3k+1)(3m+1)(3k+3m+2) \\ &\qquad \times q^{k^2+k+m^2+m+km} \\ &\qquad - (3k+2)(3m+2)(3k+3m+4) \\ &\qquad \times q^{k^2+k+m^2+m+(k+1)(m+1)}\Big). \end{aligned} \tag{5–1}$$

In principle, any specialization of Conjecture 5.1 can be proved in the same way (if the computer is fast enough!). However, the general case requires a true mathematical investigation.

In the same manner, the following congruence properties were also discovered and automatically proved using a computer algebra program. However, we are not able to imagine a global formula similar to that of Conjecture 5.1.

Theorem 5.5. *We have*

$$\begin{aligned} a(2n+1) &\equiv 0 \mod 2, \\ a(4n+1) \equiv a(4n+2) &\equiv 0 \mod 4, \\ a(5n+2) \equiv a(5n+3) \equiv a(5n+4) &\equiv 0 \mod 5, \\ a(7n+3) \equiv a(7n+4) \equiv a(7n+6) &\equiv 0 \mod 7, \\ a(8n+1) \equiv a(8n+5) \equiv a(8n+6) &\equiv 0 \mod 8, \\ a(10n+2) \equiv a(10n+4) &\equiv 0 \mod 10, \\ a(11n+7) &\equiv 0 \mod 11, \\ a(14n+4) \equiv a(14n+6) \equiv a(14n+10) &\equiv 0 \mod 14. \end{aligned} \tag{5–2}$$

6. OPEN PROBLEMS

Is there a combinatorial proof of the marked hook formula (2–11) analogous to the Robinson–Schensted–Knuth correspondence for proving (2–2)? Let T be a standard Young tableau of shape λ (see [Knuth 98, p. 47]), u a box in λ, and m an integer such that $1 \leq m \leq h_u(\lambda)$. The triplet (T, u, m) is called a *marked Young tableau* of shape (λ, u). The number of marked Young tableaux of shape (λ, u) is then $f_\lambda h_u$. On the other hand, call each triplet (σ, j, k) such that $\sigma \in \mathfrak{S}_n$, $1 \leq j \leq n$, and $1 \leq k \leq n + j - 1$ a *marked permutation.*

$$
\begin{array}{|c|c|}\hline 5 & 9 \\ \hline 2_i & 8 \\ \hline \end{array}
\qquad\qquad
\begin{array}{|c|c|}\hline 8 & 9 \\ \hline 4_j & 5 \\ \hline \end{array}
$$
$$
\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline \end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline \end{array}
$$

FIGURE 4. Marked Young tableaux.

We say that the letter j within the permutation σ is marked k. The total number of marked permutations of order n is

$$\sum_{j=1}^{n}(n+j-1)n! = \frac{n(3n-1)}{2}n!\,.$$

Example 6.1. The sequence $6\ 4\ 9\ 5_k\ 7\ 1\ 2\ 8\ 3$ with $1 \le k \le 13$ is a marked permutation, whose letter 5 is marked k. The two diagrams in Figure 4 are marked Young tableaux of the same shape, where $1 \le i, j \le 3$.

Problem 6.2. Find a marked Robinson–Schensted–Knuth correspondence between pairs of marked Young tableaux and marked permutations that yields a direct proof of the marked hook formula, Theorem 2.3.

Keeping in mind that the number of all standard Young tableaux on $\{1, 2, \ldots, n\}$ is equal to the number of involutions of order n, see (2–4), we are led to pose the following problem.

Problem 6.3. Find a formula for the number of all marked standard Young tableaux (which could be called *marked involutions*):

$$\sum_{\lambda\vdash n} f_\lambda \sum_{v\in\lambda} h_v = ?$$

More generally, is there a simple formula for

$$\sum_{\lambda\in\mathcal{P}} \prod_{v\in\lambda}\left(1+\frac{1}{h_v}\right) x = ?$$

Let $t = 2t' + 1$ be an odd positive integer. In [Han 08c] we have constructed a bijection $\phi_V : \lambda \mapsto (v_0, v_1, \ldots, v_{t-1})$, which maps each t-core onto a V-coding such that

$$|\lambda| = \frac{1}{2t}(v_0^2 + v_1^2 + \cdots + v_{t-1}^2) - \frac{t^2-1}{24} \qquad (6\text{–}1)$$

and

$$\prod_{v\in\lambda}\left(1-\frac{t^2}{h_v^2}\right) = \frac{(-1)^{t'}}{1!\cdot 2!\cdot 3!\cdots(t-1)!}\prod_{0\le i<j\le t-1}(v_i - v_j). \qquad (6\text{–}2)$$

The right-hand side of (6–2) appears in the Macdonald identities for type $A_\ell^{(a)}$ (see [Macdonald 72]). Notice that the parameter t on the right-hand side of (6–2) can take only positive integer values, because t is a vector length, whereas on the left-hand side, t can be any complex number. For that reason we call formula (6–2) an *indiscretization* analogue of the Macdonald identities for $A_\ell^{(a)}$. This indiscretization principle led us to the following Nekrasov–Okounkov formula [Nekrasov and Okounkov 06, Han 08a]:

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{z}{h^2}\right) = \prod_{k\ge1}(1-x^k)^{z-1}. \qquad (6\text{–}3)$$

Problem 6.4. Find the indiscretization analogue of the Macdonald identities for the other affine root systems (see [Macdonald 72]) and deduce other expansion formulas for the powers of the Euler product.

The answer to Problem 6.2 will produce a large number of identities for powers of the Euler product. For example,

$$
\begin{aligned}
\prod_{m\ge1}(1-x^m) &= \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k+1)/2} && \text{(by Euler);}\\
&= \sum_{\lambda\in\mathcal{P}}\prod_{v\in\lambda}\Big(1-\frac{2}{h_v^2}\Big)x && \text{(by type } A_l^{(a)}\text{);}\\
&= ? && \text{(by type } B_l\text{);}\\
&\ \ \ldots
\end{aligned}
$$

In general, it is not easy to convert one identity to another directly.

Taking $z = 4$ in (6–3) yields the following identity due to Jacobi:

$$\prod_{m\ge1}(1-x^m)^3 = \sum_{m\ge0}(-1)^m(2m+1)x^{m(m+1)/2}. \qquad (6\text{–}4)$$

In fact, the general form of the Jacobi triple product identity reads

$$\prod_{n\ge0}(1+ax^{n+1})\left(1+\frac{x^n}{a}\right)(1-x^{n+1}) = \sum_{n=-\infty}^{+\infty} a^n x^{n(n+1)/2}. \qquad (6\text{–}5)$$

Problem 6.5. Find an a-analogue of (6–3) that can be transformed to the Jacobi triple product identity (6–5) by specialization.

7. LATEST NEWS AND COMMENTS

Mihai Cipu has proved Conjecture 5.2 [Cipu 08]. Richard Stanley and Greta Panova have recently proved Conjecture 3.1 [Stanley 08, Panova 08]. Ken Ono and the author have proved Conjecture 4.7 [Han and Ono 08]. Kevin Carde et al. have proved Conjecture 2.1 [Carde et al. 08]. Another conjecture referred to as Conjecture 1.7 in [Han 08b] has the same nature as the conjectures presented in this paper. It is not reproduced here, since it has just been proved by the author [Han 08c].

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