Algorithms for Projectivity and Extremal Classes of a Smooth Toric Variety

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CONTENTS

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In this paper we present two algorithms: the first tests the projectivity of a smooth complete toric variety and the second determines the extremal classes of the Mori cone of a smooth projective toric variety. The crucial fact is that we are able to give a complete description of $\mathcal{N}_1(X)$, determining a basis B of $\mathcal{N}_1(X)$ and the coordinates with respect to B of any element of $\mathcal{N}_1(X)$. The computational condition testing the projectivity is obtained by Kleiman's criterion of ampleness, while the condition determining the extremality of a class comes directly from the definition of a nonextremal class. The algorithms are used to study the Mori cone of Fano toric n-folds with dimension $n \leq 4$ and Picard number $\rho \geq 3$, computing all extremal rays of the Mori cone. Moreover, we describe a toric almost Fano variety of dimension 3 and Picard number 35 together with its Mori cone.

1. INTRODUCTION

A toric variety X of dimension n is a normal complex algebraic variety containing an algebraic group T isomorphic to $(\mathbb{C}^*)^n$ (a torus), as a dense open subset, with an algebraic action $T \times X \to X$ of T on X that extends the natural action of T on itself (multiplication in T). Moreover, we will assume that X is smooth and complete.

In this article we present an algorithm testing the projectivity of X and an algorithm to determine the extremal classes of the Mori cone of X, when X is a projective variety. After the description of algorithms we give two applications. In the first we study the Mori cone of Fano toric n-folds with $n \leq 4$ and $\rho \geq 3$, determining all its extremal rays. In the second we give a description of a toric almost Fano variety of dimension 3 and maximal Picard number ($\rho = 35$).

The definition of toric variety given above is a theoretical definition and does not allow us to obtain a computational description of X. We need an equivalent description for the variety X. We observe that the definition of fan (see [Ewald 96, Fulton 93, Oda 98]) gives a combinatorial description of X and that every geometrical

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property of X corresponds to a combinatorial property of the fan. In addition, combinatorial properties are easy to translate into computational properties. We will translate these combinatorial properties in a computational way to obtain our goal.

Given X, we deal with the characterization of the group of 1-cycles in X modulo numerical equivalence, denoted by $\mathcal{N}_1(X)$, and the associated vector space $\mathcal{N}_1(X)_{\mathbb{Q}} = \mathcal{N}_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Its dimension is ρ, where ρ denotes the Picard number of X. In $\mathcal{N}_1(X)_{\mathbb{Q}}$ we consider the Mori cone $\mathrm{NE}(X)$, the convex cone generated by classes of effective curves. Let $\mathrm{NE}(X)_{\mathbb{Z}}$ be the intersection of $\mathrm{NE}(X)$ with $\mathcal{N}_1(X)$.

Reid [Reid 88] proves that $\mathrm{NE}(X)_{\mathbb{Z}}$ is generated as a semigroup by the set $\mathcal{I}$ of all classes of invariant curves. Since $\dim \mathrm{NE}(X) = \rho$, it follows that $\mathcal{I}$ generates $\mathcal{N}_1(X)$ as a group and it contains a basis B of $\mathcal{N}_1(X)$.

If X is also projective, we observe that there is a subset of $\mathcal{I}$ that generates $\mathrm{NE}(X)_{\mathbb{Z}}$ as a semigroup. It is the set of all contractible classes of X and is denoted by $\mathcal{C}$. Again, we can conclude that the set $\mathcal{C}$ generates $\mathcal{N}_1(X)$ as a group and hence $\mathcal{C}$ contains a basis B of $\mathcal{N}_1(X)$ (see [Casagrande 03a]).

The sets $\mathcal{I}$ and $\mathcal{C}$ are sets of linear forms in the generators of the fan Σ_X. If we suppose that the generators of the fan Σ_X are t in number, then we can associate to every element in $\mathcal{I}$ (or in $\mathcal{C}$) a polynomial in t variables. Then we observe that a relation among the classes in $\mathcal{I}$ (or in $\mathcal{C}$) determines a relation among the associated polynomials and conversely. Thus we consider the set of all syzygies among the polynomials corresponding to all classes in $\mathcal{I}$ (or in $\mathcal{C}$). We observe that it is an ideal and it can be completely described by a system of generators. So we compute a system of generators for the ideal. Starting from this system of generators, we write an algorithm computing a basis B of $\mathcal{N}_1(X)$ contained in $\mathcal{I}$ (or in $\mathcal{C}$). Then, using the syzygies, we determine the vector $\underline{w}_\gamma$ of the coordinates of every class γ in $\mathcal{I}$ (or in $\mathcal{C}$) with respect to B. Consequently, the Mori cone is seen in $\mathbb{Q}^\rho$ as the cone generated by the vectors $\underline{w}_\gamma$, for $\gamma \in \mathcal{I}$ (or in $\mathcal{C}$).

Under this identification, we use the characterization of projectivity given by Kleiman's criterion of ampleness [Kleiman 66] to obtain the computational condition to test the projectivity of X; see Proposition 4.1.

If X is projective, we define its extremal classes: they are primitive elements of the intersection of 1-dimensional faces of $\mathrm{NE}(X)$ with $\mathcal{N}_1(X)$. By Reid's results we know that the set $\mathcal{E}$ of all extremal classes is a subset of $\mathcal{C}$ and that it is computationally determined by the definition of nonextremal class (see Proposition 5.2).

We will see that in both cases, we solve the problem by considering a linear Diophantine equation and proving that a nontrivial solution using nonnegative numbers exists.

In this paper we present two applications. The computations obtained in the first application tell us that for every considered toric Fano n-fold X (with $n \leq 4$ and $\rho \geq 3$), the sets $\mathcal{C}$ and $\mathcal{E}$ coincide. Hence every contractible class γ is extremal, and its associated variety X_γ is projective. Moreover, when X has $\rho = 3$, its Mori cone is always simplicial: we find three extremal rays. Finally, we observe that the program gives its answers in a short time: the longest CPU time spent by Mathematica is 0.531 seconds. In the other application we introduce a toric almost Fano variety X. We obtain a positive answer when we check its projectivity. Then we subsequently determine the extremal rays of $\mathrm{NE}(X)$. It has dimension 35 and 54 edges. In this case, Mathematica spends quite a long time, because in order to test the projectivity it constructs a linear Diophantine equation with 102 unknowns, and to compute extremal classes it constructs 56 linear Diophantine equations with 55 unknowns.

This article is divided into six sections. In Section 2 we recall some definitions about toric varieties [Ewald 96, Fulton 93, Oda 98] and some known results about toric Mori theory [Oda 98, Reid 88]. Then, in Section 3, we present an algorithm that computes a basis B of $\mathcal{N}_1(X)$ given by classes of invariant curves (or contractible classes) and all vectors of coordinates of all classes of invariant curves (or contractible classes) with respect to B. In Section 4 we give the computational condition of projectivity. In Section 5 we present the computational translation of the definition of nonextremal class and we explain how one can determine the set of all extremal classes as a subset of $\mathcal{C}$. In the last two sections we present two examples that we studied using our algorithms. In Section 6 we enumerate the results about the study of the Mori cone of toric Fano varieties of dimension $n \leq 4$ and Picard number $\rho \geq 3$. Section 7 is devoted to the study of a toric almost Fano variety of dimension 3 and maximal Picard number ($\rho = 35$).

Remark 1.1. Our algorithms, implemented using the programming language Mathematica 5.0, are collected in "Toric Varieties."[1] The web site also contains the instructions for all programs of the package.

[1] Available at the author's web page: http://annascaramuzza.googlepages.com/studies.

All computations in the two applications were carried out on a Pentium(R) 4 HP Pavilion ze5400 with 2.66 GHz and 256 MB RAM under Microsoft Windows XP.

2. BASIC DEFINITIONS AND KNOWN RESULTS

In this section we recall some definitions from toric geometry. More detailed references can be found in [Ewald 96, Fulton 93, Oda 98]. For a definition of primitive collections and relations we refer to [Batyrev 91] and [Batyrev 99]. Toric Mori theory is introduced in [Oda 98, Reid 88, Wiśniewski 02].

Let X be a smooth complete toric variety of dimension n. We introduce X using the fan Σ_X.

Let $N = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$ be a lattice of rank n, and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. We also define the vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$, whose dual space $M_{\mathbb{Q}}$ is identified with $M \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote by $(\ , \ )$ the natural pairing on $M_{\mathbb{Q}} \times N_{\mathbb{Q}}$.

We define the following set as a *rational convex polyhedral cone* (or simply *cone*) generated by $\{x_1, \dots, x_k\} \subset N$:

$$\langle x_1, \dots, x_k \rangle = \left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; \lambda_i \geq 0,\ \lambda_i \in \mathbb{Q} \right\}.$$

Given a cone $\sigma \in N_{\mathbb{Q}}$ we define its dual cone as follows:

$$\sigma^\vee = \{\, y \in M_{\mathbb{Q}} \mid (y, x) \geq 0 \text{ for all } x \in \sigma \,\}.$$

If $y \neq 0$ is in M we define

$$H_y = \{\, x \in N_{\mathbb{Q}} \mid (y, x) = 0 \,\},$$
$$H_y^+ = \{\, x \in N_{\mathbb{Q}} \mid (y, x) \geq 0 \,\}.$$

Then the cone $\tau = H_y \cap \sigma$ is a face of the cone σ if $y \in M \backslash \{0\}$ and $\sigma \subset H_y^+$.

A *fan* is a finite collection of rational convex polyhedral cones in the vector space $N_{\mathbb{Q}}$ satisfying the following conditions:

- Each face of a cone is a cone in the fan.
- The intersection of two cones belonging to the fan is a face of each cone.

Since X is smooth and complete, every cone in the fan is generated by a part of a basis of N, and the support of the fan Σ_X is the whole vector space $N_{\mathbb{Q}}$. We define the dimension of the cone σ as the dimension of the smallest linear subspace $\mathrm{Span}(\sigma)$ containing σ. Moreover, we denote by $\mathrm{RelInt}(\sigma)$ the interior of σ in $\mathrm{Span}(\sigma)$. For every 1-dimensional cone σ in Σ_X we consider its primitive generator x_σ. Thus the set

$$\{\, x_\sigma \mid \sigma \text{ is a 1-dimensional cone} \,\}$$

is the set of all generators of Σ_X. It is denoted by $G(\Sigma_X)$.

We will introduce a smooth complete toric variety enumerating all generators of its fan and all maximal cones belonging to the fan.

If X is projective, then we can describe the fan using the language of primitive collections and primitive relations introduced by Batyrev [Batyrev 91, Batyrev 99]. It gives another combinatorial way to introduce the fan Σ_X, and we will see that it is very useful in studying the Mori cone of X.

A subset $P \subseteq G(\Sigma_X)$ is a *primitive collection* if it does not generate a cone in the fan Σ_X while every proper subset of P generates a cone in the fan. The symbol $\mathrm{PC}(X)$ will denote the set of all primitive collections of Σ_X. By definition, it follows that for any subset S of $G(\Sigma_X)$ either S generates a cone in Σ_X or S contains a primitive collection. Let $P = \{x_1, \dots, x_k\} \subseteq G(\Sigma_X)$ be a primitive collection. Since X is complete, there exists a unique cone $\sigma_P = \langle y_1, \dots, y_r \rangle$ in Σ_X such that $x_1 + \cdots + x_k \in \mathrm{RelInt}(\sigma_P)$. Hence, there exist unique numbers $a_1, \dots, a_r \in \mathbb{Z}_{>0}$ such that

$$x_1 + \cdots + x_k - (a_1 y_1 + \cdots + a_r y_r) = 0.$$

This is called the *primitive relation* associated with the primitive collection P. The cone σ_P is the *cone associated* with the primitive collection P.

For every toric variety we can see that there is a bijection between the cones of dimension k in the fan and the set of orbits of the torus T in X of dimension $n-k$. Throughout this article we will denote by $V(\sigma)$ the Zariski closure of the orbit corresponding to the cone σ in X and we will refer to $V(\sigma)$ as an invariant subvariety. In the case of a 1-dimensional cone $\langle x \rangle$, we will use the notation $V(x)$ for the divisor.

Let us consider the group $\mathcal{N}_1(X)$ of algebraic 1-cycles on X modulo numerical equivalence and define the vector space $\mathcal{N}_1(X)_{\mathbb{Q}} = \mathcal{N}_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. In $\mathcal{N}_1(X)_{\mathbb{Q}}$ we define the Mori cone; it is the convex cone generated by the classes of effective curves modulo numerical equivalence:

$$\mathrm{NE}(X) = \left\{ \gamma \in \mathcal{N}_1(X)_{\mathbb{Q}} \;\middle|\; \gamma = \left[\sum a_i C_i \right], \text{ with } a_i \in \mathbb{Q}_{\geq 0} \right\}.$$

There is the following exact sequence:

$$0 \longrightarrow \mathcal{N}_1(X) \xrightarrow{\ \phi\ } \mathbb{Z}^t \xrightarrow{\ \psi\ } N \longrightarrow 0,$$

where $t = \mathrm{card}(G(\Sigma_X))$ and the maps ϕ, ψ are respectively defined by $\gamma \mapsto (\gamma \cdot V(x))_{x \in G(\Sigma_X)}$ and $(a_x)_{x\in G(\Sigma_X)} \mapsto \sum_{x\in G(\Sigma_X)} a_x x$. Then ϕ allows us to identify the group $\mathcal{N}_1(X)$ with the group of integral relations among the generators of the fan Σ_X. Hence every class $\gamma \in \mathcal{N}_1(X)$ can be identified with the relation

$$\sum_{x\in G(\Sigma_X)} (\gamma \cdot V(x))x = 0. \tag{2–1}$$

Thus every class $\gamma \in \mathcal{N}_1(X)$ is identified with an integral relation among the generators of the fan. In this paper we will consider the ring of polynomials $\mathbb{Q}[y_x]_{x\in G(\Sigma_X)}$ and we will associate to the class γ the linear polynomial $\sum_{x\in G(\Sigma_X)}(\gamma \cdot V(x))y_x$.

An invariant curve C corresponds to an $(n-1)$-dimensional cone σ in the fan. Suppose that $\sigma = \langle x_1, \dots, x_{n-1}\rangle$. There are two maximal cones σ' and σ'' such that $\sigma' \cap \sigma'' = \sigma$:

$$\sigma' = \langle x_1, \dots, x_{n-1}, x_n\rangle,$$
$$\sigma'' = \langle x_1, \dots, x_{n-1}, x_{n+1}\rangle.$$

Since X is smooth,

$$x_1, \dots, x_{n-1}, x_n$$

and

$$x_1, \dots, x_{n-1}, x_{n+1}$$

are two bases of N; hence there exist uniquely determined integers $a_1, \dots, a_{n-1}$ such that

$$x_n + x_{n+1} + a_1x_1 + \cdots + a_{n-1}x_{n-1} = 0. \tag{2–2}$$

Relation (2–2) is the relation corresponding to the numerical class of the invariant curve C.

The numerical classes of invariant curves allow Reid to characterize $\mathrm{NE}(X)_{\mathbb{Z}}$:

Theorem 2.1. [Reid 88, Corollary 1.7] *Let X be a smooth complete toric variety of dimension n with fan Σ_X. Let $\mathcal{I}$ be the set of all numerical classes of invariant curves of X. Then*

$$\mathrm{NE}(X)_{\mathbb{Z}} = \sum_{\gamma\in\mathcal{I}} \mathbb{Z}_{\geq 0}\gamma,$$

that is, $\mathcal{I}$ generates $\mathrm{NE}(X)_{\mathbb{Z}}$ as a semigroup.

Let us consider the group $\mathcal{N}^1(X)$ of divisors in X modulo numerical equivalence and the vector space $\mathcal{N}^1(X)_{\mathbb{Q}} = \mathcal{N}^1(X) \otimes \mathbb{Q}$ obtained from $\mathcal{N}^1(X)$. Inside $\mathcal{N}^1(X)_{\mathbb{Q}}$, we define the cone of nef $\mathbb{Q}$-divisors. It is denoted by $\mathrm{Nef}(X)$. We observe that $\mathcal{N}^1(X)_{\mathbb{Q}}$ is dual to $\mathcal{N}_1(X)_{\mathbb{Q}}$ and that $\mathrm{Nef}(X)$ is the dual cone of $\mathrm{NE}(X)$. By Kleiman's criterion of ampleness [Kleiman 66], we know that a divisor D is ample if and only if its numerical class lies in the interior of $\mathrm{Nef}(X)$. Then we have

$$\begin{aligned} \text{X is projective} &\Longleftrightarrow \text{there exists an ample divisor} D \\ &\Longleftrightarrow \mathrm{Nef}(X) \text{ has nonempty interior} \\ &\Longleftrightarrow \dim \mathrm{Nef}(X) = \dim \mathcal{N}^1(X)_{\mathbb{Q}} = \rho. \end{aligned}$$

By properties of dual cones (see [Fulton 93]), this is equivalent to saying that

$$\begin{aligned} \text{X is projective} \Longleftrightarrow\ & \mathrm{NE}(X) \text{ is strongly convex} \\ & (\text{that is, } \mathrm{NE}(X) \cap -\mathrm{NE}(X) = \{0\}). \end{aligned}$$

When X is projective we can characterize the Mori cone using a special subset of numerical classes of curves contained in the set of primitive relations: contractible classes.

Definition 2.2. Let $\gamma \in \mathrm{NE}(X)_{\mathbb{Z}}$ be primitive in $\mathbb{Z}_{\geq 0}\gamma$ and such that there exists some irreducible curve in X having numerical class in $\mathbb{Z}_{\geq 0}\gamma$. We say that γ is *contractible* if there exist a toric variety X_γ and an equivariant morphism $\varphi_\gamma : X \to X_\gamma$, surjective and with connected fibers, such that for every irreducible curve $C \subset X$,

$$\varphi_\gamma(C) = \{pt\} \iff [C] \in \mathbb{Q}_{\geq 0}\gamma.$$

Hence contractible classes correspond to "elementary" toric morphisms with connected fibers with source X.

We observe that every primitive relation is a relation among the generators of Σ_X; hence it can be interpreted as an element of $\mathcal{N}_1(X)$. Moreover, we have the following:

Proposition 2.3. [Kresch 00, Proposition 2.1] *Let $\gamma \in \mathcal{N}_1(X)$ be given by the relation*

$$a_1x_1 + \cdots + a_kx_k - (b_1y_1 + \cdots + b_ry_r) = 0,$$

with $a_i, b_j \in \mathbb{Z}_{>0}$ for each i, j. If $\langle y_1, \dots, y_r\rangle \in \Sigma_X$, then $\gamma \in \mathrm{NE}(X)$.

Proposition 2.3 says that for every primitive collection $P \in \mathrm{PC}(X)$, the primitive relation $r(P)$ belongs to $\mathrm{NE}(X)_{\mathbb{Z}}$. Moreover, Theorem 2.2 in [Casagrande 03a] says that every contractible class is also a primitive relation and is always the class of some invariant curve. The primitive relation of a contractible class γ can be used to describe the morphism φ_γ and the variety X_γ associated with it (see [Casagrande 03a]). Hence, we have the following important subsets of elements of $\mathrm{NE}(X)_{\mathbb{Z}}$, which are all finite:

$$\begin{array}{c} \mathcal{I} = \{\text{classes of invariant curves}\}, \\ \cup \\ \mathcal{C} = \{\text{contractible classes}\}, \\ \cap \\ \mathrm{PR} = \{\text{primitive relations}\}. \end{array}$$

One can see that it is easy to determine the set of all contractible classes of X as a subset of the set of primitive relations. In fact, there is the following combinatorial criterion:

Proposition 2.4. [Sato 00, Theorem 4.10], [Casagrande 03a, Proposition 3.4] *Let $P = \{x_1, \dots, x_h\}$ be a primitive collection in Σ_X, with primitive relation*

$$r(P): \quad x_1 + \cdots + x_h - a_1 y_1 - \cdots - a_k y_k = 0.$$

Then $r(P)$ is contractible if and only if for every primitive collection Q of Σ_X such that $Q \cap P \neq \emptyset$ and $Q \neq P$, the set $(Q \setminus P) \cup \{y_1, \dots, y_k\}$ contains a primitive collection.

We call a 1-dimensional face R of $\mathrm{NE}(X)$ an *extremal ray*, and the primitive element of $R \cap \mathrm{NE}(X)_{\mathbb{Z}}$ an *extremal class*. Let $\mathcal{E}$ be the set of all extremal classes of X. Thus we can reformulate Reid's results in toric Mori theory as follows:

Theorem 2.5. [Reid 88, Theorem 1.5] *Let X be a projective smooth toric variety. Any extremal class is contractible.*

The difference between contractible and extremal classes is given by the following:

Proposition 2.6. [Bonavero 00, Lemma 1], [Casagrande 03a, Corollary 3.3] *Let X be a projective smooth toric variety. Let $\gamma \in \mathrm{NE}(X)$ be a contractible class and let $\varphi_\gamma : X \to X_\gamma$ be the associated morphism. Then γ is not extremal if and only if φ_γ is birational and the variety X_γ is not projective.*

Proposition 2.6 gives a theoretical distinction between contractible and extremal classes, but we cannot use it to build our algorithm.

Finally, we notice that by definition of extremal classes, we have

$$\mathrm{NE}(X) = \sum_{\gamma \in \mathcal{E}} \mathbb{Q}_{\geq 0} \gamma,$$

but it is not known whether the same holds over $\mathbb{Z}$, i.e., whether the set $\mathcal{E}$ generates $\mathrm{NE}(X)_{\mathbb{Z}}$ as a semigroup. If we consider all contractible classes, we have the following theorem:

Theorem 2.7. [Casagrande 03a] *Let X be a projective smooth toric variety. Let $\mathcal{C}$ be the set of all contractible classes. Then*

$$\mathrm{NE}(X)_{\mathbb{Z}} = \sum_{\gamma \in \mathcal{C}} \mathbb{Z}_{\geq 0} \gamma,$$

that is, $\mathcal{C}$ generates $\mathrm{NE}(X)_{\mathbb{Z}}$ as a semigroup.

3. COMPUTING COORDINATES OF A CLASS BELONGING TO $\mathrm{NE}(X)_{\mathbb{Z}}$

In this section we will describe a technique to determine a basis B of $\mathcal{N}_1(X)$ and the coordinates of a class of invariant curves with respect to the basis B using the properties of the Mori cone $\mathrm{NE}(X)$.

To reach our goal we need to have some linear relations among the classes generating $\mathrm{NE}(X)_{\mathbb{Z}}$. We observe that every class of invariant curves generating $\mathrm{NE}(X)_{\mathbb{Z}}$ can be related to a linear polynomial with integer coefficients. Then the sought relations among these classes will be determined by syzygies among the associated polynomials. In this way, the problem is reduced to giving a complete description of the set of all syzygies among a finite set of polynomials and can be solved using techniques of computer algebra related to Gröbner bases and elimination theory.

Using these relations, we were able to determine the basis B and the coordinates of other classes with respect to B.

Let X be a smooth complete toric variety with Picard number ρ. We assume that its fan Σ_X is generated by the set $G(\Sigma_X) = \{x_1, \dots, x_t\}$. Let $\mathcal{S} \subset \mathrm{NE}(X)_{\mathbb{Z}}$ be a set generating $\mathrm{NE}(X)_{\mathbb{Z}}$. Then we can choose

(1) $\mathcal{S} = \mathcal{I}$: this is the general case (see Theorem 2.1);

(2) $\mathcal{S} = \mathcal{C}$: in this case X has to be projective (see Theorem 2.7).

In every case, a class $\gamma \in \mathcal{S}$ is a class in $\mathcal{N}_1(X)$ and can be identified with the integral relation as (2–1). This fact induces us to identify the class γ with a linear polynomial with integer coefficients. In the following we will explain how one can do this.

Let $y_1, \dots, y_t$ be a collection of t variables. Let $\mathbb{Z}[y_1, \dots, y_t]_1$ be the ring of linear polynomials in the variables $y_1, \dots, y_t$ with integer coefficients. We define the following assignment:

$$\begin{aligned} \varphi : \mathcal{N}_1(X) &\to \mathbb{Z}[y_1, \dots, y_t]_1, \\ \gamma \quad &\mapsto p_\gamma = \sum_{i=1}^{t} (\gamma \cdot V(x_i)) y_i, \end{aligned}$$

where p_γ is a linear polynomial in $y_1, \dots, y_t$. By definition, $p_\gamma(x_1, \dots, x_t) = 0$ is the relation associated with γ.

The assignment φ gives two injective homomorphisms $\mathcal{N}_1(X) \hookrightarrow \mathbb{Z}[y_1, \dots, y_t]_1$ and $\mathcal{N}_1(X)_{\mathbb{Q}} \hookrightarrow \mathbb{Q}[y_1, \dots, y_t]_1$. Assume that $\mathcal{S} = \{\gamma_1, \dots, \gamma_s\}$. Then the assignment defines s polynomials denoted by $p_1, \dots, p_s$.

If we suppose that $p_j = b_1 p_1 + \cdots + b_\rho p_\rho$, with $j \in \{\rho + 1, \dots, s\}$, we also have that

$$p_j(x_1, \dots, x_t) = b_1 p_1(x_1, \dots, x_t) + \cdots + b_\rho p_\rho(x_1, \dots, x_t).$$

Hence we have the following relation among the classes γ_j, for $j \in \{\rho + 1, \dots, s\}$:

$$\gamma_j - b_1 \gamma_1 - \cdots - b_\rho \gamma_\rho = 0.$$

Conversely, a linear relation among the classes $\gamma_1, \dots, \gamma_s$ determines a relation among the polynomials $p_1, \dots, p_s$.

Next we will explain how to determine an integral relation among the classes in $\mathcal{S}$. The technique involves the theory of Gröbner bases and elimination theory.

We consider every polynomial $p_i \in \mathbb{Z}[y_1, \dots, y_t]_1$ as a polynomial in $\mathbb{Q}[y_1, \dots, y_t]_1$. Let $\mathbb{Q}[z_1, \dots, z_s]$ be the ring of polynomials in the variables $z_1, \dots, z_s$. We define the following ring homomorphism:

$$\begin{aligned} \phi : \mathbb{Q}[z_1, \dots, z_s] &\to \mathbb{Q}[y_1, \dots, y_t]_1, \\ z_i \qquad &\mapsto p_i \text{ for } i = 1, \dots, s. \end{aligned}$$

The kernel $\operatorname{Ker}(\phi)$ is an ideal in $\mathbb{Q}[z_1, \dots, z_s]$, and it is equal to the set of all syzygies over $\mathbb{Q}$ among the polynomials $p_1, \dots, p_s$, that is, $f \in \operatorname{Ker}(\phi)$ if and only if $f(p_1, \dots, p_s) = 0$. Hence to describe completely the set of all syzygies among the polynomials $p_1, \dots, p_s$ over $\mathbb{Q}$ it is enough to determine a system of generators of $\operatorname{Ker}(\phi)$.

We use the following result in elimination theory to describe $\operatorname{Ker}(\phi)$:

Theorem 3.1. [Adams and Loustaunau 94, Theorem 2.4.2] *Let $\mathbb{Q}[y_1, \dots, y_t]$ be the ring of polynomials with rational coefficients. Let $\{p_1, \dots, p_s\}$ be a set of polynomials in $\mathbb{Q}[y_1, \dots, y_t]$. Let ϕ be a ring homomorphism defined as above. Let $\mathcal{F}$ be the ideal generated by polynomials $z_1 - p_1, \dots, z_s - p_s$ in $\mathbb{Q}[z_1, \dots, z_s, y_1, \dots, y_t]$.*

Then

$$\mathcal{F} \cap \mathbb{Q}[z_1, \dots, z_s] = \operatorname{Ker}(\phi).$$

The proof of Theorem 3.1 gives an algorithm to describe $\operatorname{Ker}(\phi)$. We have to compute a Gröbner basis $\mathcal{G}$ for the ideal $\mathcal{F}$ with respect to an elimination order, i.e., a fixed term order in which variables $y_1, \dots, y_t$ are larger than $z_1, \dots, z_s$.

Hence we consider $\mathbb{Q}[z_1, \dots, z_s, y_1, \dots, y_t]$, we fix an elimination order, and we compute the Gröbner basis $\mathcal{G}$ with respect to this term order. By the definition of Gröbner basis, $\mathcal{G}$ generates $\mathcal{F}$. By [Adams and Loustaunau 94, Theorem 2.3.4], the polynomials in $\mathcal{G}$ without any variable y_i form a Gröbner basis for the ideal $\mathcal{F} \cap \mathbb{Q}[z_1, \dots, z_s]$. Then the set $\widetilde{\mathcal{G}} = \mathcal{G} \cap \mathbb{Q}[z_1, \dots, z_s]$ is a system of generators for $\operatorname{Ker}(\phi)$ in $\mathbb{Q}[z_1, \dots, z_s]$.

We observe that in $\widetilde{\mathcal{G}}$ there are always exactly $s - \rho$ polynomials, because $\mathcal{N}_1(X)_{\mathbb{Q}}$ has dimension ρ and $\mathcal{S}$ contains a basis of $\mathcal{N}_1(X)$. Moreover, we have that every polynomial in $\widetilde{\mathcal{G}}$ has degree one with respect to every variable z_j, for $j = 1, \dots, s$.

The polynomials in $\widetilde{\mathcal{G}}$ may not have integer coefficients, in which case we consider the set

$$\mathcal{H} = \{h \in \mathbb{Z}[z_1, \dots, z_s] \mid h = k_{\widetilde{g}} \widetilde{g},\ \widetilde{g} \in \widetilde{\mathcal{G}}\},$$

where $k_{\widetilde{g}}$ is the least common multiple of all denominators of all coefficients in $\widetilde{g}$.

By definition, $\mathcal{H}$ is a Gröbner basis for $\operatorname{Ker}(\phi)$, and it is a set of $s - \rho$ polynomials with integer coefficients. Every $h \in \mathcal{H}$ determines an integral relation among $p_1, \dots, p_s$: $h(p_1, \dots, p_s) = 0$. Hence we have $s - \rho$ integral relations among the classes $\gamma_1, \dots, \gamma_s$.

The fact that $\mathcal{S}$ contains a basis of $\mathcal{N}_1(X)$ is crucial: it means that a polynomial p_j can be isolated in every relation $h(p_1, \dots, p_s) = 0$. In other words, there is a coefficient of a term in the polynomial $h(z_1, \dots, z_s)$ that is equal to ± 1. We assume that $\mathcal{H} = \{h_1, \dots, h_{s-\rho}\}$. Then we can determine ρ polynomials in $\{p_1, \dots, p_s\}$ (respectively ρ classes in $\mathcal{S}$) that describe the remaining $s - \rho$ polynomials (respectively $s - \rho$ classes in $\mathcal{S}$) as a linear combination of them.

Up to reordering the variables in the polynomials $h_1, \dots, h_{s-\rho}$, we can assume that

$$h_1: \quad z_{\rho+1} - \sum_{i=1}^{\rho} b_i^1 z_i,$$
$$\vdots$$
$$h_{s-\rho}: \quad z_s - \sum_{i=1}^{\rho} b_i^{s-\rho} z_i.$$

By definition of syzygy we have

$$h_1(p_1, \dots, p_s) = 0: \quad p_{\rho+1} = \sum_{i=1}^{\rho} b_i^1 p_i,$$
$$\vdots$$
$$h_{s-\rho}(p_1, \dots, p_s) = 0: \quad p_s = \sum_{i=1}^{\rho} b_i^{s-\rho} p_i.$$

Then the following integral relations are determined:

$$\begin{aligned} \gamma_{\rho+1} &= \sum_{i=1}^{\rho} b_i^1 \gamma_i, \\ &\vdots \\ \gamma_s &= \sum_{i=1}^{\rho} b_i^{s-\rho} \gamma_i. \end{aligned} \tag{3–1}$$

This means that $B = \{\gamma_1, \dots, \gamma_\rho\}$ is a basis of $\mathcal{N}_1(X)$. Moreover, equations (3–1) give the coordinates of $\gamma_{\rho+1}, \dots, \gamma_s$ with respect to this basis. We identify the class $\gamma_j \in B$ with the vector e_j of the canonical basis of $\mathbb{Q}^\rho$ and replace γ_j with e_j in (3–1). Thus we obtain the vector $\underline{w}_\gamma$ of the coordinates of the class $\gamma \in \mathcal{S}$ with respect to B. In this way, the Mori cone is seen in $\mathbb{Q}^\rho$ as the cone generated by the vectors $\underline{w}_\gamma$, for $\gamma \in \mathcal{S}$.

Remark 3.2. The system of generators of $\mathrm{Ker}(\phi)$ can be computed using the Mathematica built-in function `Eliminate` [Wolfram 03]. This command applies the elimination theory for linear polynomials to the set of polynomials $z_1 - p_1, \dots, z_s - p_s$.

Elimination theory for linear polynomials says that the Gröbner basis of the ideal $\mathcal{F}$ can be computed by considering the matrix A of coefficients of polynomials $z_1 - p_1, \dots, z_s - p_s$. We fix an elimination order, and then we order the polynomials $z_1 - p_1, \dots, z_s - p_s$ with respect to this term order. We then define the matrix A: the first s entries of the jth row are the coefficients of the polynomial $z_j - p_j$ with respect to the variables $z_1, \dots, z_s$, and the last t entries of the jth row are the coefficients of the polynomial $z_j - p_j$ with respect to the variables $y_1, \dots, y_t$.

Applying Gauss elimination to the matrix A, we obtain a triangular matrix B. The matrix B is the matrix of the coefficients of the polynomials of a Gröbner basis of $\mathcal{F}$ with respect to the fixed elimination order. The rows that have zero in the last t entries determine the coefficients of the polynomials of a Gröbner basis of $\mathcal{F} \cap \mathbb{Q}[z_1, \dots, z_s]$ with respect to the elimination order.

The function `Eliminate` computes the set $\mathcal{H}$ as a Gröbner basis of the ideal $\mathcal{F} \cap \mathbb{Q}[z_1, \dots, z_s]$ directly.

4. PROJECTIVITY

The problem of verifying the projectivity of a smooth complete toric variety X is equivalent to showing that $\mathrm{NE}(X)$ is strongly convex. By definition, $\mathrm{NE}(X)$ is strongly convex if and only if $\mathrm{NE}(X) \cap -\mathrm{NE}(X) = \{0\}$. Theorem 2.1 says that $\mathrm{NE}(X)_{\mathbb{Z}}$ is generated by the set of numerical classes of invariant curves $\mathcal{I}$. Suppose that $\mathcal{I} = \{\gamma_1, \dots, \gamma_s\}$.

Using the algorithm introduced in Section 3, we compute a basis B of $\mathcal{N}_1(X)$, contained in $\mathcal{I}$, and then the coordinates of all γ_i with respect to B. Hence, we have the set

$$\mathcal{V} = \{\underline{w}_i \mid i = 1, \dots, s\},$$

which contains the canonical basis of $\mathbb{Q}^\rho$. We identify every class with the vector of its coordinates, so that $\mathrm{NE}(X)_{\mathbb{Z}} = \langle \underline{w}_1, \dots, \underline{w}_s \rangle \subset \mathbb{Q}^\rho$.

Proposition 4.1. *Let X be a smooth complete toric variety. Let $\mathcal{I} = \{\gamma_1, \dots, \gamma_s\}$ be the set of all numerical classes of invariant curves. Let $\mathcal{V} = \{\underline{w}_1, \dots, \underline{w}_s\}$ be the set of vectors of coordinates of all classes in $\mathcal{I}$ with respect to a fixed basis of $\mathcal{N}_1(X)$. Then X is projective if and only if the equation*

$$\sum_{i=1}^{s} v_i \underline{w}_i = 0 \tag{4–1}$$

has only the trivial solution in the set of integers and nonnegative numbers $\mathbb{Z}^s_{\geq 0}$.

Proof: Suppose that there exists $\underline{w} \in \mathrm{NE}(X) \cap -\mathrm{NE}(X)$. This means that

$$\underline{w} = \sum_{i=1}^{s} a_i \underline{w}_i = \sum_{i=1}^{s} b_i(-\underline{w}_i),$$

where $a_i, b_i \in \mathbb{Z}_{\geq 0}$. Hence

$$\sum_{i=1}^{s}(a_i + b_i)\underline{w}_i = 0.$$

Then $(a_1 + b_1, \dots, a_s + b_s)$ is a nontrivial solution of the equation, which is a contradiction.

Conversely, let $\underline{w}$ be an element in $\mathrm{NE}(X) \cap -\mathrm{NE}(X)$. We are going to prove that $\underline{w} = \underline{0}$. Since $\underline{w} \in \mathrm{NE}(X) \cap -\mathrm{NE}(X)$, we have

$$\underline{w} = \sum_{i=1}^{s} a_i \underline{w}_i = \sum_{i=1}^{s} b_i(-\underline{w}_i),$$

where $a_i, b_i \in \mathbb{Z}_{\geq 0}$. Then

$$\sum_{i=1}^{s}(a_i + b_i)\underline{w}_i = 0. \tag{4–2}$$

By hypothesis, equation (4–2) has only the trivial solution; hence $(a_i + b_i) = 0$ for all $i = 1, \dots, s$. This implies that $a_i = b_i = 0$ for all $i = 1, \dots, s$ and that $\underline{w} = \underline{0}$. □

Proposition 4.1 allows us to write the algorithm `ProjQ[X]`. This command verifies the nonprojectivity of a variety X. Proposition 4.1 tells us that this is equivalent to showing that equation (4–1) has a nontrivial solution.

Remark 4.2. In order to determine the projectivity, we use the Mathematica command `FullSimplify[Exists[-,-]]`. The unknowns of the equation must be entered in the first argument of the command, while the equation must be entered in the second. This command allows us to obtain an answer in a very short time [Wolfram 03].

5. EXTREMAL CLASSES

Let X be a smooth projective toric variety. We consider the problem of determining all extremal classes in $\mathrm{NE}(X)_{\mathbb{Z}}$. By results presented in Section 2, we know that $\mathcal{E}$ is contained in $\mathcal{C}$ and that $\mathcal{C}$ can be determined by the criterion introduced in Proposition 2.4. Here, we explain how one can determine $\mathcal{E}$ inside $\mathcal{C}$.

Let us recall the definition of a nonextremal class:

Definition 5.1. A contractible class γ is *nonextremal* if there exist $Z_1, Z_2 \in \mathrm{NE}(X)$ such that $Z_1 + Z_2 \in \mathbb{Q}_{\geq 0}\gamma$ and $Z_1 \notin \mathbb{Q}_{\geq 0}\gamma$.

Then we have the following result:

Proposition 5.2. *Let X be a projective variety. Let γ be a contractible class and let $\mathcal{C} = \{\gamma_1 = \gamma, \dots, \gamma_s\}$ be the set of all contractible classes in X. Then the following are equivalent:*

(i) *γ is not extremal;*

(ii) *there exist $m_2, \dots, m_s \in \mathbb{Q}_{\geq 0}$ such that*

$$\gamma = \sum_{i=2}^{s} m_i \gamma_i.$$

Proof: (ii) $\Longrightarrow$ (i): Since $\gamma \neq 0$, at least one m_i is nonzero; we may assume that $m_2 \neq 0$. We set

$$Z_1 = m_2\gamma_2 \quad \text{and} \quad Z_2 = \sum_{i=3}^{s} m_i \gamma_i.$$

Then $Z_1 + Z_2 \in \mathbb{Q}_{\geq 0}\gamma$.

Suppose that $Z_1 \in \mathbb{Q}_{\geq 0}\gamma$. Then there exists $\lambda \in \mathbb{Q}_{\geq 0}$ such that $m_2\gamma_2 = \lambda\gamma$. Since every contractible class is primitive, this implies that $\gamma = \gamma_2$, which is a contradiction.

(i) $\Longrightarrow$ (ii): Let γ be a nonextremal contractible class. Let $Z_1, Z_2 \in \mathrm{NE}(X)$ be such that $Z_1 + Z_2 \in \mathbb{Q}_{\geq 0}\gamma$ and $Z_1 \notin \mathbb{Q}_{>0}\gamma$. Then

$$Z_1 + Z_2 = \lambda\gamma,$$

with $\lambda \in \mathbb{Q}_{>0}$.

Since

$$\mathrm{NE}(X) = \sum_{\eta \in \mathcal{E}} \mathbb{Q}_{\geq 0}\eta$$

and γ is nonextremal, we have

$$Z_i = \sum_{j=2}^{s} a_j^i \gamma_j,$$

where $i = 1, 2$, $a_j^i \in \mathbb{Q}_{\geq 0}$. Then

$$Z_1 + Z_2 = \sum_{i=1}^{2}\sum_{j=2}^{s} a_j^i \gamma_j = \lambda\gamma.$$

Since $\lambda \neq 0$, it follows that

$$\gamma = \sum_{j=2}^{s}\left(\frac{a_j^1 + a_j^2}{\lambda}\right)\gamma_j,$$

and we have the statement of the proposition. □

We use Proposition 5.2 to build the algorithm `ExtremalClasses[X]`. Also in this case, we solve the problem by considering a linear Diophantine equation. Again we use the Mathematica command `FullSimplify[Exists[-, -]]`, to determine whether the equation has a nontrivial solution [Wolfram 03].

6. EXTREMAL RAYS OF TORIC FANO MANIFOLDS

In this section we compute the extremal rays (classes) of the Mori cone of a smooth toric Fano variety of dimension $n \leq 4$ and Picard number $\rho \geq 3$.

A smooth complete variety X of dimension n is Fano if its anticanonical bundle $-K_X$ is ample. By definition, X is a projective variety.

The Mori cone $\mathrm{NE}(X)$ of a smooth Fano variety X is a closed polyhedral cone of dimension ρ. If $\mathrm{NE}(X)$ is simplicial, we know that there are exactly ρ extremal rays. However, when $\rho \geq 3$, the cone $\mathrm{NE}(X)$ is not always simplicial, and there are no estimates on the number of its extremal rays either in the toric case or in general. We recall that polyhedral cones of dimension 1 or 2 are always simplicial.

In the toric case, there is a finite number of smooth toric Fano varieties for each dimension n. They are classified up to dimension 7.

In dimension 2 there are five smooth toric Fano varieties: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and S_i, the blowup of $\mathbb{P}^2$ in i general points for $i = 1, 2, 3$. They are also called toric Del Pezzo surfaces. In this case, the Mori cone is always simplicial except for the surface S_3. The Mori cone of S_3 has dimension 4 and six extremal rays (they correspond to the (-1)-curves of the surface).

There are 18 smooth toric Fano 3-folds, and among these varieties, 13 have $\rho \geq 3$. In this case Batyrev and K. and M. Watanabe separately obtain the same classification (see [Batyrev 82, Batyrev 99, Watanabe and Watanabe 82]).

Smooth toric Fano 4-folds are classified by Batyrev in [Batyrev 99]. In [Sato 00], Sato describes a toric Fano 4-fold that does not appear in the list given by Batyrev. Both authors use the language of primitive relations to describe them. After these classifications, there are 124 toric Fano 4-folds, of which 114 have $\rho \geq 3$.

Recently, Kreuzer and Nill [Kreuzer and Nill 07] classified toric Fano 5-folds. Øbro [Øbro 07] studied the cases of dimensions 6 and 7. Kreuzer and Nill listed 866 toric Fano 5-folds using a computer program and the database of the 4-dimensional reflexive polytopes [Kreuzer 07]. Øbro presented an algorithm to classify all smooth Fano polytopes and studied the problem combinatorially.

We have computed all contractible and extremal classes of every toric Fano 3-fold and 4-fold using the package command `ExtremalClasses` (see Section 5). Here we present our results.

Let X be a smooth toric Fano variety of dimension at most 4.

Variety	ρ	card(PR)	card($\mathcal{E}$)	CPU time
$\mathcal{C}_1, \ldots, \mathcal{C}_5$	3	3	3	time = 0 s
$\mathcal{D}_1, \mathcal{D}_2$	3	5	3	time = 0 s
$\mathcal{E}_1, \ldots, \mathcal{E}_5$	4	6	4	time = 0 s
$\mathcal{F}_1, \mathcal{F}_2$	5	10	7 (•)	time = 0.04 s

TABLE 1. Smooth toric Fano 3-folds with $\rho \geq 3$.

From our computations, we see that X has no contractible nonextremal classes. Hence the variety X_γ associated with a contractible class γ of X is always projective. So far, there are no examples of contractible nonextremal classes in a smooth toric Fano variety with dimension $n \leq 4$ (see [Casagrande 03b, Section 5] for related details).

When $\rho = 3$ in the Mori cone there are exactly three extremal rays.

In dimension 3 there are only two cases with a nonsimplicial Mori cone. These have maximal Picard number 5 and seven extremal rays. We summarize our results in Table 1.

For every 3-fold we list

1. the name (we use the notation of [Batyrev 99, Section 2]);
2. the Picard number ρ (which is ≥ 3);
3. the cardinality of the set of primitive relations PR;
4. the cardinality of the set of extremal classes $\mathcal{E}$;
5. the CPU time in seconds (s). This is the CPU time spent by Mathematica to compute the set of extremal classes of the variety.

The symbol (•) denotes the varieties with a nonsimplicial Mori cone.

Let us consider the case of dimension 4. We summarize the results in Table 2. In the table, d_1 is the number of toric Fano 4-folds with Picard number ρ, while d_2 is the number of the varieties with Picard number ρ whose Mori cone is not simplicial. We use the symbol $\mathrm{card}_{\max}(\mathcal{E})$

ρ	d_2/d_1	$\mathrm{card}_{\max}(\mathcal{E})$
4	3/47	6
5	9/27	10
6	9/10	20
7	1/1	9
8	1/1	12

TABLE 2. Extremal rays in toric Fano 4-folds with $\rho \geq 4$.

Variety	ρ	card(PR)	card($\mathcal{E}$)	CPU time
$D_i, i = 1, \dots, 19$	3	3	3	0 s $\leq$ time $\leq$ 0.011 s
$E_i, i = 1, 2, 3$	3	5	3	time = 0.01 s
$G_i, i = 1, \dots, 6$	3	5	3	0 s $\leq$ time $\leq$ 0.01 s
$L_i, i = 1, \dots, 13$	4	4	4	time = 0 s
$H_i, i = 1, \dots, 10$	4	6	4	0 s $\leq$ time $\leq$ 0.01 s
$I_i, i = 1, \dots, 15$	4	6	4	0 s $\leq$ time $\leq$ 0.01 s
$M_i, i = 1, \dots, 5$	4	7	4	time = 0 s
J_1	4	9	5 ($\bullet$)	time = 0.03 s
J_2	4	9	6 ($\bullet$)	time = 0.05 s
Z_1	4	10	4	time = 0 s
Z_2	4	10	5 ($\bullet$)	time = 0.03 s
$Q_i, i = 1, \dots, 17$	5	7	5	0 s $\leq$ time $\leq$ 0.01 s
n. 108	5	10	5	time = 0.01 s
$K_i, i = 1, \dots, 4$	5	10	7 ($\bullet$)	0.03 s $\leq$ time $\leq$ 0.19 s
$R_i, i = 1, \dots, 3$	5	11 ($\star$)	7 ($\bullet$)	time = 0.05 s
n. 117	5	14	10 ($\bullet$)	time = 0.14 s
HS ($\diamond$)	5	18	9 ($\bullet$)	time = 0.1 s
$S_2 \times S_2$	6	10	6	time = 0 s
$U_i, i = 1, \dots, 8$	6	11	8 ($\bullet$)	time = 0.04 s
n. 118	6	25	20 ($\bullet$)	time = 0.531 s
$S_2 \times S_3$	7	16	9 ($\bullet$)	time = 0.05 s
$S_3 \times S_3$	8	18	12 ($\bullet$)	time = 0.12 s

TABLE 3. Smooth toric Fano 4-folds with $\rho \geq 3$. ($\star$) indicates that the varieties R_1, R_2, R_3 have 11 primitive relations, but in [Batyrev 99] we find only 10. The missing relation is $v_4 + v_6 = v_0$ (see [Batyrev 99]). ($\diamond$): HS is the variety described by Sato in [Sato 00].

to denote the maximal number of extremal rays among all Fano 4-folds with Picard number ρ. In Table 2 we consider $\rho \geq 4$ because $\mathrm{NE}(X)$ is always simplicial when $\rho = 3$.

It is interesting to observe that the variety with the maximal number of extremal rays, 20, has $\rho = 6$, which is not the maximal Picard number. This variety is the Del Pezzo 4-fold (see [Ewald 88] and [Voskresenskiĭ and Klyachko 85]), denoted in Table 3 by n. 118.

We enumerate in Table 3 all results obtained for the 114 smooth Fano 4-folds with $\rho \geq 3$. For each we give the same information considered for the smooth toric Fano 3-folds.

Remark 6.1. The maximal CPU time observed is 0.531 s. It is necessary to compute the extremal rays of the Del Pezzo toric 4-fold. This proves that the Mathematica program is able to obtain results in a very short amount of time. In dimension 3, the longest CPU time spent by Mathematica is 0.04 s (see Table 1). In this time Mathematica calculates the extremal rays of the varieties with the maximal Picard number ($\rho = 5$). Observe that in most cases, the CPU time is equal to 0 s. This means that the command executes some elementary computations that do not involve the use of memory [Wolfram 03].

7. AN ALMOST FANO THREEFOLD WITH $\rho = 35$

Here we present a smooth projective and almost Fano toric variety X of dimension 3 and $\rho = 35$.

A smooth projective variety X is almost Fano if its anticanonical bundle $-K_X$ is nef and big. The almost Fano varieties are a generalization of Fano varieties. We know that smooth Fano 3-folds have been classified and that they have Picard number $\rho \leq 10$. On the other hand, there is no classification of almost Fano 3-folds, and we do not know which maximal Picard number they can assume (see [Casagrande et al. 06] and [Nill 05]).

In the toric case, we know that for every dimension n, toric almost Fano varieties are finite in number (see [Batyrev 94]). In this case, it is convenient to use the concept of polytope to introduce the variety. A polytope is the convex hull of finitely many points.

When we consider Gorenstein toric Fano varieties of dimension n, we see that they correspond bijectively to special polytopes: *reflexive polytopes* (the definition was introduced in [Batyrev 94]). A property of reflexive polytopes is that their vertices are integral (i.e., in $\mathbb{Z}^n$), and the origin is their unique interior integral point. An almost Fano variety of dimension n is obtained by a crepant refinement of a reflexive polytope (see [Batyrev 94, Nill 05]).

In dimension 3, Kreuzer and Skarke give a complete classification of all reflexive polytopes (see [Kreuzer and Skarke 98], [Kreuzer 98]). There are 4319 of them. After this classification, we know that any toric almost Fano threefold X has $\rho \leq 35$. We are now going to describe an explicit example of such an X with maximal Picard number $\rho = 35$ and its Mori cone together with the extremal rays. In [Kreuzer 98] we see that there are only two 3-dimensional reflexive polytopes containing 39 integral points, and they are both simplices. We consider one of them, the simplex $\mathcal{P} \subset \mathbb{Z}^3$ with vertices, as follows:

$$v_1 = (1,0,0), \qquad v_2 = (1,2,0),$$
$$v_3 = (1,2,6), \qquad v_4 = (-5,-4,-6).$$

In order to obtain our example X, we have to determine all integral points (i.e., in $\mathbb{Z}^3$) and to triangulate each facet of $\mathcal{P}$. The vertices of each triangle must be a basis of $\mathbb{Z}^3$. The fan of X is given by the cones over all these triangles. This means that X is obtained by a crepant resolution of a singular Fano variety Y with Picard number $\rho_Y = 1$. All 38 integral points on the facets $F_1, \ldots, F_4$ give the generators of the fan. They are as follows:

```
x[1] = (-5,-4,-6),   x[2] = (-4,-3,-5),
x[3] = (-4,-3,-4),   x[4] = (-3,-2,-4),
x[5] = (-3,-2,-3),   x[6] = (-3,-2,-2),
x[7] = (-2,-2,-3),   x[8] = (-2,-1,-3),
x[9] = (-2,-1,-2),   x[10] = (-2,-1,-1),
x[11] = (-2,-1,0),   x[12] = (-1,-1,-2),
x[13] = (-1,-1,-1),  x[14] = (-1,0,-2),
x[15] = (-1,0,-1),   x[16] = (-1,0,0),
x[17] = (-1,0,1),    x[18] = (-1,0,2),
x[19] = (0,0,-1),    x[20] = (0,0,1),
x[21] = (0,1,-1),    x[22] = (0,1,0),
x[23] = (0,1,1),     x[24] = (0,1,2),
x[25] = (0,1,3),     x[26] = (0,1,4),
x[27] = (1,0,0),     x[28] = (1,1,0),
x[29] = (1,1,1),     x[30] = (1,1,2),
x[31] = (1,1,3),     x[32] = (1,2,0),
x[33] = (1,2,1),     x[34] = (1,2,2),
x[35] = (1,2,3),     x[36] = (1,2,4),
x[37] = (1,2,5),     x[38] = (1,2,6).
```

The triangulations described in Figures 1 and 2 have been chosen to define the fan of X. Observe that Σ_X has 38 generators; hence $\rho_X = 35$. Moreover, $-K_X$ is nef and big, but a priori X does not need to be projective.

There are 102 classes of invariant curves, so to test the projectivity, we have to determine 102 points in the lattice $\mathcal{N}_1(X) \cong \mathbb{Z}^{35}$. Then the equation corresponding to the projectivity condition has 102 unknowns, and our algorithm has to test the existence of a nontrivial solution of the equation. The answer of the algorithm is `True`, and it requires 42.922 s of CPU time. The variety X is thus projective.

The fan having been defined, we compute all its primitive collections. There are 596 of them, all of cardinality 2, except one of cardinality 3. Then we compute the corresponding 596 primitive relations. Among them there are 56 contractible classes of curves. We can describe for every contractible class the associated morphism and variety, analyzing the corresponding primitive relation (see [Casagrande 03a]). There are the following classes:

1. 51 classes of type

   `x[i] + x[j] - x[h] - x[k] = 0,` (7–1)

   with $i, j, h, k \in \{1, \ldots, 38\}$. The morphisms associated with these classes are small contractions with exceptional locus a curve $C \cong \mathbb{P}^1$ with normal bundle $\mathcal{N}_{C/X} = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

2. Two classes of type

   `x[i] + x[j] -2 x[h] = 0,` (7–2)

   with $i, j, h \in \{1, \ldots, 38\}$. The morphisms associated with these classes are birational and send a divisor to a singular curve.

3. Two classes of type

   `x[i] + x[j] - x[h] = 0,` (7–3)

   with $i, j, h \in \{1, \ldots, 38\}$. The associated morphisms are two smooth blowups with exceptional divisor given respectively by $V($`x[h]`$)$.

4. The contractible class of type

   `x[2] + x[3] + x[7] -2 x[1] = 0.` (7–4)

   This class is the primitive relation corresponding to the primitive collection of cardinality 3. The associated morphism is birational and sends a divisor to a singular point.

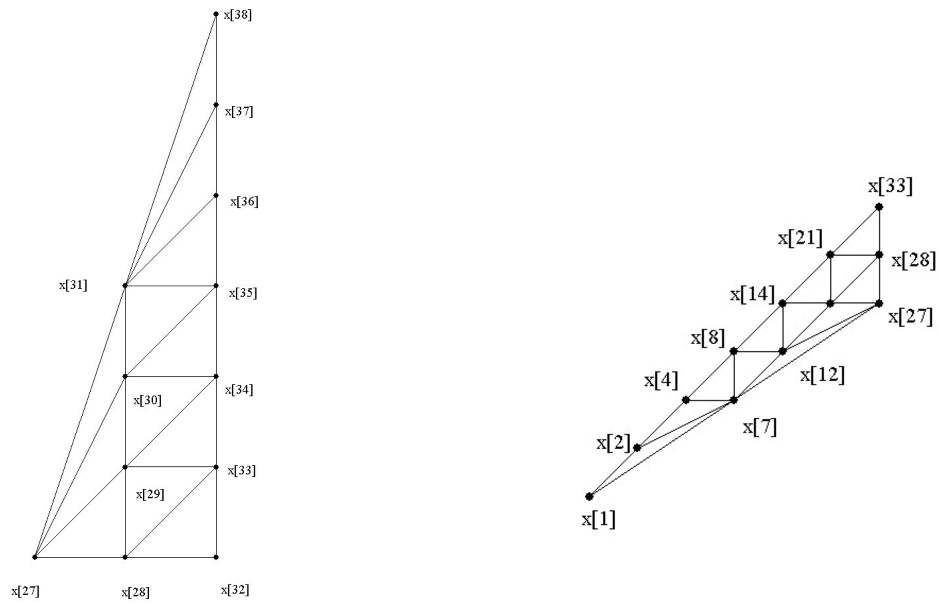

FIGURE 1. Triangulation of F_1 (on the left) and of F_2 (on the right).

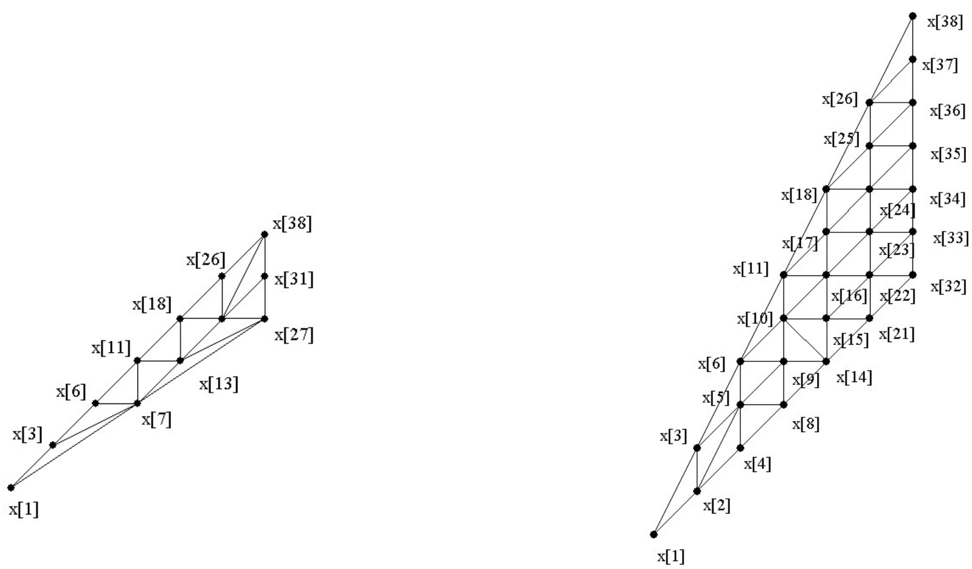

FIGURE 2. Triangulation of F_3 (on the left) and of F_4 (on the right).

Using the command `ExtremalClasses` we determine the extremal classes in $\mathrm{NE}(X)$. We see that all contractible classes are extremal except two: one of type (7–1) and one of type (7–3). Hence $\mathrm{NE}(X)$ is a cone of dimension 35 with 54 edges.

Let $H_{K_X} \subset \mathcal{N}_1(X)_{\mathbb{Q}}$ be the hyperplane of classes that have intersection zero with K_X. It cuts a facet of $\mathrm{NE}(X)$. We can see that 52 extremal classes lie on H_{K_X}:

- 50 classes of type (7–1);
- 2 classes of type (7–2).

The other two extremal classes have positive anticanonical degree and do not lie on H_{K_X}: one is a class of type (7–3), and the other is the class of type (7–4).

The two nonextremal classes have the following properties:

- the morphism associated with the first class sends a divisor $D \cong \mathbb{P}^2$ to a singular point;
- the morphism $\varphi : X \to Z$ associated with the second class is a smooth blowup of a nonprojective variety Z along a curve $C \cong \mathbb{P}^1$ and $\mathcal{N}_{C/Z} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, so that $-K_Z$ is not nef.

Observe that in this case, the toric almost Fano 3-fold has two contractible nonextremal classes, while every toric Fano 3-fold has no contractible nonextremal classes (see Table 1).

Note that the CPU time used by Mathematica to compute the extremal classes is 904.606 s. Here the algorithm has to test the existence of a nontrivial solution for 56 linear Diophantine equations.

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