Motivic Proof of a Character Formula for SL(2)

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CONTENTS

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This paper provides a proof of a p-adic character formula by means of motivic integration. We use motivic integration to produce virtual Chow motives that control the values of the characters of all depth-zero supercuspidal representations on all topologically unipotent elements of p-adic $\mathrm{SL}(2)$; likewise, we find motives for the values of the Fourier transform of all regular elliptic orbital integrals having minimal nonnegative depth in their own Cartan subalgebra, on all topologically nilpotent elements of p-adic $\mathfrak{sl}(2)$. We then find identities in the ring of virtual Chow motives over $\mathbb{Q}$ that relate these two classes of motives. These identities provide explicit expressions for the values of characters of all depth-zero supercuspidal representations of p-adic $\mathrm{SL}(2)$ as linear combinations of Fourier transforms of semisimple orbital integrals, thus providing a proof of a p-adic character formula.

1. INTRODUCTION

The representation theory of groups over finite fields is stated entirely in terms of algebraic geometry through the work of Deligne and Lusztig [Deligne and Lusztig 76, Lusztig 85]. Naturally, it is desirable to "lift" the geometric constructions to the p-adic groups. Motivic integration offers an approach to this problem that is very different from all previous ones; although it is not aiming at a complete understanding of the geometry underlying various objects of representation theory, it could in principle provide an algorithm for computing them in each individual case. This paper is essentially an experiment illustrating this point.

In this paper we work with the group $\mathrm{SL}(2)$ over a p-adic field for the simple reason that the geometric objects that arise as a result of computing the "motivic" volumes turn out to be easily computable by hand; in fact, all the ones that are nontrivial turn out to be conics. For bigger groups, both the algorithms and the results are of course much more complicated, but we believe that still there are computable geometric objects responsible for the character values in general, and this will be the subject of a future article.

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There is already substantial evidence that the objects arising in harmonic analysis on reductive p-adic groups are "motivic" (in particular, computable). The most spectacular result in this direction appears in a very recent article [Cluckers et al. 07] that shows that the orbital integrals arising in the fundamental lemma are motivic. Earlier papers on this topic include [Cunningham and Hales 04], which provides an explicit description of a certain class of semisimple orbital integrals, and [Gordon 04], where it is proved that the values of characters of depth-zero representations are motivic near the identity.

The virtual Chow motives we calculate appear in the context of the following expansion for the character of any depth-zero supercuspidal representation of p-adic SL(2). (The precise statement is Theorem 2.5.) Let p be an odd prime, let $\mathbb{K}$ be a p-adic field with residue field $\mathbb{F}_q$ (so q is a power of p), and let $G = \mathrm{SL}(2, \mathbb{K})$. We use the modified Cayley transform $\mathrm{cay}(Y) = (1+Y/2)(1-Y/2)^{-1}$ to pass between the topologically nilpotent elements in the Lie algebra and the topologically unipotent elements in the group. We show that there is a finite set of regular elliptic orbits represented by the elements X_z in the Lie algebra $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{K})$, each having minimal nonnegative depth in its Cartan subalgebra, with the following property. Let π be an arbitrary depth-zero supercuspidal representation of G and let Θ_π be the distribution character of π in the sense of Harish-Chandra. Then there are rational numbers $c_z(\pi)$ such that

$$\Theta_\pi\left(\mathrm{cay}(Y)\right) = \sum_z c_z(\pi)\ \hat{\mu}_{X_z}(Y), \qquad (1\text{–}1)$$

for all regular topologically nilpotent elements Y of $\mathfrak{g}$. (Here $\hat{\mu}_{X_z}$ denotes the generalized function on $\mathfrak{g}$ corresponding to the Fourier transform of the orbital integral at X_z.) The coefficients $c_z(\pi)$ are given in Table 4, from which one sees that up to sign (given by a quadratic character at -1), $c_z(\pi)$ is in fact a rational function in q (the cardinality of the residue field of $\mathbb{K}$) with integer coefficients that are independent of q; of course, this is a meaningful observation only if one clarifies in what sense the left- and right-hand sides of equation (1–1) may be viewed as functions in q, as we shall do later in the paper. We shall refer to equation (1–1) as a *semisimple character expansion*. The question of uniqueness of the coefficients is discussed in Section 6.

It is fair to say that the existence of a semisimple character expansion is well known in one form or another, as is the fact that it extends, mutatis mutandis, to a much larger class of representations and groups. The novelty of this paper lies in our proof. We use motivic integration to "separate" the character and each of the orbital integrals into a linear combination of the values of the corresponding function defined over the residue field, with coefficients that are virtual Chow motives. Then we can see directly that on each side of the semisimple character expansion we have the combination of the same values with the same coefficients. This approach clearly shows two ways in which algebraic geometry appears in the values of the characters and orbital integrals: the finite field function is in fact the characteristic function of a character sheaf; and the "motivic" coefficients come from the process of inflation and induction that connects our representation with the representation of the group over the finite field.

As a consequence of this perspective, we find much more than the coefficients $c_z(\pi)$ promised above. In fact, we produce expressions for the values of the characters of all depth-zero supercuspidal representations on all topologically unipotent elements of G. Likewise, we produce expressions for the values of the Fourier transforms of all regular elliptic orbital integrals having minimal nonnegative depth in their own Cartan subalgebra, on all topologically nilpotent elements of $\mathfrak{g}$. Comparing these leads to Table 4 and the proof of equation (1–1). The character tables for $\mathrm{SL}(2,\mathbb{K})$ were computed by Sally and Shalika in [Sally and Shalika 68]. Since they use a different construction of the supercuspidal representations of SL(2) from that of induction from a compact subgroup used here, it is difficult to match our calculations with their classical calculation before we see the result. However, once we have the character values, it becomes easy to find them in the character tables computed by Sally and Shalika. This can, in fact, be used to match the representations of depth zero obtained by the modern construction with the equivalent representations appearing in those classical tables.

2. BASIC NOTIONS

Throughout this paper, $\mathbb{K}$ denotes a p-adic field; by this we mean that $\mathbb{K}$ is a field equipped with a nonarchimedean valuation such that it is complete and locally compact with respect to the topology determined by the norm, and such that the residue field (which is necessarily finite) has characteristic p. Notice that we do not put any condition on the characteristic of $\mathbb{K}$. As is well known, any such $\mathbb{K}$ is a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p(\!(t)\!)$, where t is transcendental over $\mathbb{F}_p$. The ring of integers of $\mathbb{K}$ will be denoted by $\mathcal{O}_\mathbb{K}$, and the maximal ideal in $\mathcal{O}_\mathbb{K}$ will be denoted by $\mathfrak{p}_\mathbb{K}$. The residue field will be denoted by $\mathbb{F}_q$. We reserve q for the cardinality of $\mathbb{F}_q$, and p for the characteristic of $\mathbb{F}_q$, so that q is a power of p.

Next, we fix a prime ℓ different from p and an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. We will shortly assume $p \neq 2$. Henceforth, by "representation" we mean a representation in a vector space over $\bar{\mathbb{Q}}_\ell$.

In Sections 2.1 and 4.1 we make brief reference to the Bruhat–Tits building and Moy–Prasad filtrations for G (see [Moy and Prasad 94]). In those sections of the paper, we will use the notation of Moy–Prasad. In particular, for each pair (x, r), where x is a point and r is a nonnegative real number as above, $G_{x,r}$ is a compact open subgroup of G; when $r = 0$, this is often abbreviated to G_x. For each pair (x, r) as above, let $\underline{G}_{x,r}$ be the smooth integral model for $G_{\mathbb{K}}$ introduced in [Yu 02]; the set of integral points in $\underline{G}_{x,r}$ coincides with $G_{x,r}$. We write $\bar{G}_{x,r}$ for the reductive quotient of the special fiber of $\underline{G}_{x,r}$. Analogous notions apply to the Lie algebra $\mathfrak{g}$: for any point x and any real number r, $\mathfrak{g}_{x,r}$ denotes the Moy–Prasad lattice in $\mathfrak{g}$, and $\underline{\mathfrak{g}}_{x,r}$ denotes the corresponding integral model for $\mathfrak{g}_{\mathbb{K}}$, while $\bar{\mathfrak{g}}_{x,r}$ denotes the corresponding Lie algebra scheme over $\mathbb{F}_q$. When we use $\mathfrak{g}_{x,r}$ and $G_{x,r}$, we will often write these down explicitly, with the hope that readers unfamiliar with Moy–Prasad filtrations will have little difficulty with the essential features of this paper.

Note that we have not yet chosen a uniformizer for $\mathbb{K}$, and that none of the constructions above require such a choice.

2.1 Depth-Zero Representations

In this section we remind the reader how to construct all depth-zero supercuspidal irreducible representations of G, up to equivalence.

Let (π, V) be a representation of G. For each point x in the Bruhat–Tits building for G, let V_x denote the subspace consisting of those $v \in V$ such that $\pi(k)v = v$ for each $k \in G_{x,0^+}$. Then $G_{x,0}$ acts on V_x, and the resulting representation is denoted by π_x. By definition, the representation π has *depth zero* if V_x is nontrivial for some point x in the Bruhat–Tits building for G.

The canonical map $\mathcal{O}_{\mathbb{K}} \to \mathbb{F}_q$ will be denoted by $x \mapsto \bar{x}$, and with x and r as above, we write $\rho_{x,r} : \mathfrak{g}_{x,r} \to \bar{\mathfrak{g}}_{x,r}$ for the quotient map with kernel equal to the pronilpotent radical $\mathfrak{g}_{x,r^+}$ of $\mathfrak{g}_{x,r}$.

For each depth-zero π and each point x in the Bruhat–Tits building for G, the representation π_x factors through $\rho_{x,0} : G_{x,0} \to \bar{G}_{x,0}$. Let $\bar{\pi}_x$ denote the representation of $\bar{G}_{x,0}$ on V_x such that $\pi_x(k) = (\bar{\pi}_x \circ \rho_{x,0})(k)$ for all $k \in G_x$; then $\bar{\pi}_x$ is called the representation of $\bar{G}_{x,0}$ obtained by compact restriction and is also denoted by $\mathrm{cRes}^G_{G_{x,0}}(\pi, V)$.

From the other direction, let x be a point in the Bruhat–Tits building for G and let (σ, W) be a representation of $\bar{G}_x$, and let $(\rho^*_{x,0}\sigma, W)$ be the representation of G_x defined by $(\rho^*_{x,0}\sigma)(k) = \sigma(\rho_{x,0}(k))$, for each $k \in G_x$. The right-regular representation of G on the space of compactly supported functions $f : G \to W$ such that $f(kg) = (\rho^*_{x,0}\sigma)(k)f(g)$ for all $g \in G$ and all $k \in G_x$ is called the representation of G obtained from (σ, W) (or $(\rho^*_{x,0}\sigma, W)$) by compact induction and denoted by $\mathrm{cInd}^G_{G_x}(\sigma, W)$.

Although $\left(\mathrm{cRes}^G_{G_x}, \mathrm{cInd}^G_{G_x}\right)$ is an adjoint pair of functors (see [Vigneras 03]), one must be careful: even if σ is a cuspidal irreducible representation of $\bar{G}_x$, it does not follow in general that $\mathrm{cInd}^G_{G_x}\sigma$ is an admissible representation of G, let alone a supercuspidal representation. To clarify matters somewhat, we have the following result, which we paraphrase from independent results by Lawrence Morris and Moy–Prasad. For each point x in the Bruhat–Tits building for G, the compact restriction functor $\mathrm{cRes}^G_{G_x}$ restricts to a surjection from supercuspidal irreducible representations of G to cuspidal irreducible representations of $\bar{G}_x$. Moreover, if π is a depth-zero supercuspidal representation of G, then there is a vertex x in the Bruhat–Tits building for G such that $\mathrm{cInd}^G_{G_x}(\mathrm{cRes}^G_{G_x}\pi)$ is equivalent to π.

There are exactly two G-orbits of vertices in the Bruhat–Tits building for G; let (0) and (1) be the adjacent vertices corresponding to the following maximal compact subgroups:

$$\begin{aligned} G_{(0)} &:= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \,\middle|\, a, b, c, d \in \mathcal{O}_{\mathbb{K}};\ ad - bc = 1 \right\}, \\ G_{(1)} &:= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \,\middle|\, a, d \in \mathcal{O}_{\mathbb{K}}, b \in \mathfrak{p}_{\mathbb{K}}^{-1}, c \in \mathfrak{p}_{\mathbb{K}}^{1};\ ad - bc = 1 \right\}. \end{aligned} \tag{2–1}$$

The special fibers of the integral models $\underline{G}_{(0)}$ and $\underline{G}_{(1)}$ are both $\mathrm{SL}(2)_{\mathbb{F}_q}$; it follows that $\bar{G}_{(0)}$ and $\bar{G}_{(1)}$ are both $\mathrm{SL}(2, \mathbb{F}_q)$. Therefore, up to equivalence, all depth-zero supercuspidal irreducible representations of G are produced by compact induction from cuspidal irreducible representations of $\mathrm{SL}(2, \mathbb{F}_q)$. Accordingly, to enumerate all depth-zero supercuspidal irreducible representations of G, it is necessary to recall the construction of cuspidal irreducible representations of $\mathrm{SL}(2, \mathbb{F}_q)$. This we will do in Section 2.3. In the meantime, we record the result: each depth-zero supercuspidal irreducible representation of G is equivalent to one in Table 1, where all terms are defined in Section 2.3.

$\pi(0,\theta) := \mathrm{cInd}_{G_{(0)}}^{G}(\sigma_\theta)$	$\pi(1,\theta) := \mathrm{cInd}_{G_{(1)}}^{G}(\sigma_\theta)$
$\pi(0,+) := \mathrm{cInd}_{G_{(0)}}^{G}(\sigma_+)$	$\pi(1,+) := \mathrm{cInd}_{G_{(1)}}^{G}(\sigma_+)$
$\pi(0,-) := \mathrm{cInd}_{G_{(0)}}^{G}(\sigma_-)$	$\pi(1,-) := \mathrm{cInd}_{G_{(1)}}^{G}(\sigma_-)$

TABLE 1. The representations π appearing in Theorem 2.5.

2.2 Gauss Sums

Before moving on to a review of the cuspidal irreducible representations of $\mathrm{SL}(2,\mathbb{F}_q)$, we say a few words about Gauss sums, which are the prototype for much that follows. Let $\mathrm{sgn} : \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell$ be the quadratic character of $\mathbb{F}_q^\times$ extended by zero to $\mathbb{F}_q$. *Arbitrarily but irrevocably*, we fix an additive character $\bar\psi : \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell$. Let $\widehat{\mathrm{sgn}}$ denote the Fourier transform of sgn with respect to $\bar\psi$ in the following sense:

$$\widehat{\mathrm{sgn}}(a) = \sum_{x\in\mathbb{A}^1(\mathbb{F}_q)} \mathrm{sgn}(x)\bar\psi(ax).$$

(Note that this Fourier transform is not unitary.) Consider the Gauss sums $\gamma_\pm : \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell$ defined by

$$\begin{aligned}\gamma_+(a) &:= \sum_{\{x\in\mathbb{A}^1(\mathbb{F}_q)\,|\,\mathrm{sgn}(x)=+1\}} \bar\psi(xa),\\ \gamma_-(a) &:= \sum_{\{x\in\mathbb{A}^1(\mathbb{F}_q)\,|\,\mathrm{sgn}(x)=-1\}} \bar\psi(xa).\end{aligned} \tag{2–2}$$

Then $\gamma_+ - \gamma_- = \widehat{\mathrm{sgn}}$. Elementary arguments show that $\gamma_+ + \gamma_- = -1$ and $(\gamma_+ - \gamma_-)^2 = q\,\mathrm{sgn}(-1)$. Fix a square root $\sqrt{q}$ of q in $\bar{\mathbb{Q}}_\ell$. Then there is a unique square root $\zeta \in \bar{\mathbb{Q}}_\ell$ of $\mathrm{sgn}(-1)$ (determined by $\bar\psi$) such that

$$\widehat{\mathrm{sgn}} = \sqrt{q}\zeta\,\mathrm{sgn}. \tag{2–3}$$

Observe that sgn is an eigenvector for the Fourier transform with eigenvalue $\sqrt{q}\zeta$.

Remark 2.1. We wish to emphasize here that $\zeta \in \bar{\mathbb{Q}}_\ell$ is determined by two choices: the square root $\sqrt{q}$ of q in $\bar{\mathbb{Q}}_\ell$ and the additive character $\bar\psi : \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell$. Also, although ζ is a fourth root of unity, it is not necessarily a primitive fourth root of unity; indeed, $\zeta^2 = \mathrm{sgn}(-1)$.

2.3 Cuspidal Representations of $\mathrm{SL}(2,\mathbb{F}_q)$

Let T denote the anisotropic torus of $\mathrm{SL}(2)_{\mathbb{F}_q}$ with $\mathbb{F}_q$-rational points

$$T(\mathbb{F}_q) = \left\{ \begin{bmatrix} x & y\\ \epsilon y & x\end{bmatrix} \in \mathrm{SL}(2,\mathbb{F}_q) \,|\, x^2 - y^2\epsilon = 1 \right\}, \tag{2–4}$$

where ϵ is a nonsquare in $\mathbb{F}_q^\times$.

Let R_T^G denote the Deligne–Lusztig induction functor of [Deligne and Lusztig 76]; this takes virtual representations of $T(\mathbb{F}_q)$ to virtual representations of $\mathrm{SL}(2,\mathbb{F}_q)$. If θ is nontrivial and the order of θ is not 2 (so θ is in "general position") then $(-1)R_T^G(\theta)$ is an irreducible cuspidal representation of $\mathrm{SL}(2,\mathbb{F}_q)$, which we henceforth denote by σ_θ; in other words, as virtual representations,

$$R_T^G(\theta) = -\sigma_\theta. \tag{2–5}$$

Let Q_T denote the restriction of trace $R_T^G(\theta)$ to the set of unipotent elements of $\mathrm{SL}(2,\mathbb{F}_q)$, where θ is in general position; as the notation suggests, Q_T is independent of θ. This is the (classical) Green's polynomial for $\mathrm{SL}(2,\mathbb{F}_q)$. For future reference,

$$Q_T(g) = \begin{cases} 1, & g \neq \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right] \wedge \mathrm{trace}\, g = 2,\\ 1-q, & g = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right],\\ 0, & \text{otherwise.}\end{cases} \tag{2–6}$$

(See, for example, the appendix to [Digne and Michel 91].)

If θ_0 is the quadratic character of $T(\mathbb{F}_q)$, then the Deligne–Lusztig virtual representation $R_T^G(\theta_0)$ contains two irreducible cuspidal representations, which form the Lusztig series for (T,θ_0). It is well known that the difference between the characters of these two representations is supported by the set of regular unipotent elements of $\mathrm{SL}(2,\mathbb{F}_q)$ (see Remark 2.3). We label the representations in the Lusztig series for (T,θ_0) by σ_+ and σ_- and define

$$Q_G := \mathrm{trace}\,\sigma_+ - \mathrm{trace}\,\sigma_- \tag{2–7}$$

in such a way that

$$Q_G\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) = \sqrt{q}\zeta^3. \tag{2–8}$$

Then,

$$Q_G\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = \begin{cases}\sqrt{q}\zeta^3\mathrm{sgn}(b), & a+d=2 \wedge b\neq 0,\\ \sqrt{q}\zeta\mathrm{sgn}(c), & a+d=2\wedge c\neq 0,\\ 0, & \text{otherwise.}\end{cases} \tag{2–9}$$

In particular, Q_G is supported by regular unipotent elements. (For this calculation we refer readers to the lovely table from the appendix to [Digne and Michel 91] with the caveat that what they denote by $\sqrt{q\,\mathrm{sgn}(-1)}$ is here denoted by $\sqrt{q}\zeta^3$.)

For a variety of reasons, it is the characters σ_+ and σ_- that are the most interesting.

$a_Q(\sigma)$	$Q = Q_T$	$Q = Q_G$
$\sigma = \sigma_\theta$	-1	0
$\sigma = \sigma_\pm$	$-\frac{1}{2}$	$\mp\frac{1}{2}$

TABLE 2. $\chi_\sigma = \sum_Q a_Q(\sigma)Q$.

Since Q_G and Q_T are supported by unipotent elements, for all primes p except 2, we can and shall often abuse notation by considering these as functions on the nilpotent elements of $\mathfrak{sl}(2,\mathbb{F}_q)$ by composing them with the modified Cayley transform $\text{cay}(X) = (1+(X/2))(1-(X/2))^{-1}$. We then have

$$\begin{aligned} Q_G\left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right) &= \sqrt{q}\zeta^3, \\ Q_G\left(\left[\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right]\right) &= \sqrt{q}\zeta, \\ Q_G\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) &= 0, \end{aligned}$$

and

$$\begin{aligned} Q_T\left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right) &= 1, \\ Q_T\left(\left[\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right]\right) &= 1, \\ Q_T\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) &= 1-q. \end{aligned}$$

Remark 2.2. We wish to emphasize that the choice of $\sqrt{q}$ and $\bar{\psi}$ determined ζ, and that we labeled the representations in the Lusztig series $\{\sigma_+, \sigma_-\}$ for (T, θ_0) precisely so that equation (2–8) would be true.

We close this section by recalling one small consequence of Lusztig's celebrated work on character sheaves and representations of finite groups of Lie type (see [Lusztig 85]). Let σ be any cuspidal representation of $\mathrm{SL}(2,\mathbb{F}_q)$ and let χ_σ denote the restriction of trace σ to unipotent elements. Then there are unique $a_Q(\sigma) \in \bar{\mathbb{Q}}_\ell$ such that

$$\chi_\sigma = \sum_Q a_Q(\sigma)Q,$$

where the sum is taken over the set $\{Q_T, Q_G\}$ of (generalized) Green's polynomials. The values of $a_Q(\sigma)$ are given in Table 2; they will play a role in the calculations for the coefficients $c_z(\pi)$ appearing in Table 4.

Remark 2.3. The characters of the representations σ_θ and σ_+ and σ_- introduced above are perhaps best understood in terms of characteristic functions of character sheaves. Let θ be any character of $T(\mathbb{F}_q)$ and let $\mathcal{L}_\theta$ be the corresponding Frobenius-stable Kummer local system on the étale site of $T_{\bar{\mathbb{F}}_q}$; in this case, the characteristic function $\chi_{\mathcal{L}_\theta} : T(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$ of $\mathcal{L}_\theta$ coincides with θ (see [Lusztig 85, 8.4] for the definition of "characteristic function"). Let ind_T^G denote the cohomological induction functor of [Lusztig 85]; this takes Frobenius-stable character sheaves for $T_{\bar{\mathbb{F}}_q}$ to Frobenius-stable perverse sheaves for $G_{\bar{\mathbb{F}}_q}$. If θ is nontrivial and not quadratic (i.e., in general position for SL(2)), then $\mathrm{ind}_T^G \mathcal{L}_\theta[1]$ is a Frobenius-stable character sheaf and the characteristic function of this perverse sheaf is the character of $R_T^G(\theta)$; in other words,

$$\text{trace}\, R_T^G(\theta) = \chi_{\mathrm{ind}_T^G \mathcal{L}_\theta[1]}$$

when θ is in general position. (Note that $\mathrm{ind}_T^G \mathcal{L}_\theta[1]$ is not a *cuspidal* character sheaf, even though trace $R_T^G(\theta)$ is a cuspidal function.) On the other hand, $\mathrm{ind}_T^G \mathcal{L}_{\theta_0}[1]$ is not a character sheaf; rather, it is a direct sum of character sheaves. The algebraic group $G_{\bar{\mathbb{F}}_q}$ admits two cuspidal character sheaves: one, denoted by C_0, is unipotent (i.e., supported by the unipotent cone in $\mathrm{SL}(2,\bar{\mathbb{F}}_q)$), while the other, denoted by C_1, is supported by -1 times the unipotent cone. Comparing with the notation above, we have $Q_G = \chi_{C_0}$. In this paper we are chiefly concerned with the characters of cuspidal representations of $\mathrm{SL}(2,\mathbb{F}_q)$ when restricted to unipotent elements in $\mathrm{SL}(2,\mathbb{F}_q)$; the vector space spanned by characters of cuspidal representations restricted to unipotent elements is two-dimensional, and a basis for this space is given by Q_G (the restriction of χ_{C_0} to the unipotent cone) and Q_T (the restriction of χ_{C_θ} to the unipotent cone), where θ is any fixed character in general position.

2.4 The Elements X_z

Before specifying the orbits appearing in equation (1–1) we say a few words about Cartan subalgebras of $\mathfrak{g}$, or equivalently, about conjugacy classes of forms of $\mathrm{GL}(1)_\mathbb{K}$ in $\mathrm{SL}(2)_\mathbb{K}$. (These are the proper twisted-Levi subgroups of G.) This is precisely the kernel $\mathfrak{C}(\mathbb{K})$ of the map in Galois cohomology $\alpha_\mathbb{K} : H^1(\mathbb{K}, N) \to H^1(\mathbb{K}, G)$ that is induced by the inclusion $N \to G$, where N denotes the normalizer of the split torus of diagonal matrices in G. To study this kernel, it is useful to first pass to the unramified closure $\mathbb{K}_{nr}$ of $\mathbb{K}$ in $\bar{\mathbb{K}}$ and observe that $\mathfrak{C}(\mathbb{K}_{nr}) = \ker \alpha_{\mathbb{K}_{nr}}$ contains exactly one nontrivial element and this element corresponds to the conjugacy class of forms that split over a (totally ramified) quadratic extension of $\mathbb{K}_{nr}$. Let τ_0 be the trivial element in $\mathfrak{C}(\mathbb{K}_{nr})$ and let τ_1 be the nontrivial element in $\mathfrak{C}(\mathbb{K}_{nr})$. The inclusion $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K}_{nr}) \to \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$ induces a surjection in Galois cohomology, which in turn restricts to a surjection

$\mathfrak{C}(\mathbb{K}) \to \mathfrak{C}(\mathbb{K}_{nr})$, as pictured below:

$$\begin{array}{ccc} H^1(\mathbb{K}, G) & \twoheadrightarrow & H^1(\mathbb{K}_{nr}, G) \\ \uparrow{\scriptstyle \alpha_{\mathbb{K}}} & & \uparrow{\scriptstyle \alpha_{\mathbb{K}_{nr}}} \\ H^1(\mathbb{K}, N) & \twoheadrightarrow & H^1(\mathbb{K}_{nr}, N) \\ \uparrow & & \uparrow \\ \mathfrak{C}(\mathbb{K}) & \longrightarrow & \mathfrak{C}(\mathbb{K}_{nr}) \\ \uparrow & & \uparrow \\ 1 & \longrightarrow & 1 \end{array}$$

The fiber above τ_0 consists of three classes in $H^1(\mathbb{K}, N)$, and the fiber above τ_1 consists of four classes in $H^1(\mathbb{K}, N)$ if $\mathrm{sgn}(-1) = 1$ and two classes in $H^1(\mathbb{K}, N)$ if $\mathrm{sgn}(-1) = -1$, where $\mathrm{sgn} : \mathbb{K} \to \bar{\mathbb{Q}}_\ell$ is the quadratic character of $\mathcal{O}_{\mathbb{K}}^*$ extended by zero to all $\mathbb{K}$.

We now return to the orbits appearing in equation (1–1). In order to define these orbits, we now pick a uniformizer ϖ for $\mathbb{K}$ and a nonsquare unit ε in $\mathcal{O}_{\mathbb{K}}$. For reasons that will be apparent later, we also now introduce a new parameter, v, taking values in $\mathcal{O}_{\mathbb{K}}^*$ and define $X_z(v)$ in Table 3. Representatives for the orbits appearing in equation (1–1) are given by $X_z := X_z(1)$. Notice that each $X_z(v) \in \mathfrak{g}$ above is *good*, in the sense of [Adler 98, Section 2.2].

Each of the $X_z(v)$ determines a compact Cartan subalgebra $\mathfrak{g}$ (independent of $v \in \mathcal{O}_{\mathbb{K}}^*$), and therefore a *cocycle* representing a class in $\ker \alpha_{\mathbb{K}}$. Thus, the summation set in equation (1–1) is the set of cocycles $\{s_1, s_2, t_0, t_1, t_2, t_3\} \subset Z^1(\mathbb{K}, N)$. However, some of these cocycles represent the same cohomology class in $H^1(\mathbb{K}, N)$, depending on the sign of -1 in $\mathbb{K}$; specifically, if $\mathrm{sgn}(-1) = -1$, then t_0 and t_1 represent the same cohomology class, and t_2 and t_3 also represent the same class in $H^1(\mathbb{K}, N)$.

Remark 2.4. Observe that picking ϖ and ε is exactly equivalent to picking representative cocycles for the cohomology classes in $\ker \alpha_{\mathbb{K}}$. In fact, our choice for s_1 corresponds to our choice of ε, and our choice for t_0 corresponds to our choice of ϖ.

2.5 Normalization of the Measures

We choose the Haar measure on $\mathrm{SL}(2, \mathbb{K})$ that coincides with the Serre–Oesterlé measure on $\mathrm{SL}(2, \mathcal{O}_{\mathbb{K}})$; that is, our Haar measure for G is normalized so that the volume of the maximal compact subgroup $\mathrm{SL}(2, \mathcal{O}_{\mathbb{K}})$ is the cardinality of $\mathrm{SL}(2, \mathbb{F}_q)$, which is $q(q^2 - 1)$. In this setting, all the fibers of the projection $\mathrm{SL}(2, \mathcal{O}_{\mathbb{K}}) \to \mathrm{SL}(2, \mathbb{F}_q)$ have volume 1. We denote this measure on G by m. This choice determines a Haar measure on $\mathfrak{g}$ such that the volume of of the kernel of the map $\mathfrak{sl}(2,\mathcal{O}_{\mathbb{K}}) \to \mathfrak{sl}(2,\mathbb{F}_q)$ is also 1; i.e., the volume of $\mathfrak{sl}(2,\mathcal{O}_{\mathbb{K}})$ equals the cardinality of $\mathfrak{sl}(2,\mathbb{F}_q)$. We denote this measure on $\mathfrak{g}$ by vol. Observe that the volume of $\mathfrak{sl}(2,\mathcal{O}_{\mathbb{K}})$ is $\frac{q^2}{q^2-1}$ times that of $\mathrm{SL}(2, \mathcal{O}_{\mathbb{K}})$.

With these choices, the formal degree of a representation π produced by compact induction from a cuspidal irreducible representation σ on the finite reductive quotient of a maximal parahoric subgroup of G coincides with the dimension $\mathrm{trace}\,\sigma(1)$ of σ.

2.6 Statement of the Character Formula

We now state the character formula appearing in the introduction more carefully. Fix measures for G and $\mathfrak{g}$ as specified in Section 2.5.

For any smooth representation π of G, let Θ_π denote the generalized function corresponding to the distribution character π with respect to the Haar measure on G above. For any regular semisimple X in $\mathfrak{g}$ and any locally constant compactly supported $f : \mathfrak{g} \to \bar{\mathbb{Q}}_\ell$, let $\mu_X(f)$ denote the orbital integral of f at X. Fix an additive character ψ of $\mathbb{K}$ with conductor $\mathcal{O}_{\mathbb{K}}$ such that the induced additive character of the residue field $\mathbb{F}_q$ is $\bar{\psi}$ (see Section 2.2). For any f as above, let $\hat{f}$ denote the Fourier transform of f taken with respect to the Killing form $\langle Y, Z \rangle := \mathrm{trace}\,(YZ)$, character ψ, and Haar measure on $\mathfrak{g}$ specified in Section 2.5; thus

$$\hat{f}(Y) = \int_{\mathfrak{g}} f(Z)\, \psi(\langle Y, Z \rangle)\, dZ.$$

Let $\hat{\mu}_X(f)$ denote the orbital integral of $\hat{f}$ at X.

Theorem 2.5. *Let $\mathbb{K}$ be a p-adic field with $p \neq 2$. For each depth-zero supercuspidal representation π of G and for each cocycle $z \in \{s_1, s_2, t_0, t_1, t_2, t_3\}$ (see Section 2.4), there is a regular elliptic $X_z \in \mathfrak{g}$ and $c_z(\pi) \in \mathbb{Q}$ such that*

$$\Theta_\pi(\mathrm{cay}^* f) = \sum_{z \in \{s_1, s_2, t_0, t_1, t_2, t_3\}} c_z(\pi) \hat{\mu}_{X_z}(f) \qquad (2\text{–}10)$$

for all Schwartz functions f supported by topologically nilpotent elements in $\mathfrak{g}$. The coefficients $c_z(\pi)$ are all rational numbers. The coefficients $c_z(\pi)$ are given in Table 4.

This paper is devoted to understanding Theorem 2.5 from the motivic perspective. To that end, we present a brief review of motivic integration in Section 2.7.

z	s_1	s_2	t_0	t_1	t_2	t_3
$X_z(v)$	$\begin{bmatrix} 0 & v \\ \varepsilon v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^{-1}v \\ \varepsilon\varpi v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & v \\ \varpi v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon v \\ \varpi\varepsilon^{-1}v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & v \\ \varpi\varepsilon v & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon v \\ \varpi v & 0 \end{bmatrix}$

TABLE 3. Elements $X_z(v)$. Orbits $X_z := X_z(1)$ appear in Theorem 2.5.

$c_z(\pi)$	$z=s_1$	$z=s_2$	$z=t_0$	$z=t_1$	$z=t_2$	$z=t_3$
$\pi=\pi(0,\theta)$	$q-1$	0	0	0	0	0
$\pi=\pi(1,\theta)$	0	$q-1$	0	0	0	0
$\pi=\pi(0,+)$	$\frac{q-1}{2}$	0	$-\frac{q^2-1}{2^3q}\zeta^2$	$+\frac{q^2-1}{2^3q}\zeta^2$	$-\frac{q^2-1}{2^3q}\zeta^2$	$+\frac{q^2-1}{2^3q}\zeta^2$
$\pi=\pi(0,-)$	$\frac{q-1}{2}$	0	$+\frac{q^2-1}{2^3q}\zeta^2$	$-\frac{q^2-1}{2^3q}\zeta^2$	$+\frac{q^2-1}{2^3q}\zeta^2$	$-\frac{q^2-1}{2^3q}\zeta^2$
$\pi=\pi(1,+)$	0	$\frac{q-1}{2}$	$-\frac{q^2-1}{2^3q}\zeta^2$	$+\frac{q^2-1}{2^3q}\zeta^2$	$+\frac{q^2-1}{2^3q}\zeta^2$	$-\frac{q^2-1}{2^3q}\zeta^2$
$\pi=\pi(1,-)$	0	$\frac{q-1}{2}$	$+\frac{q^2-1}{2^3q}\zeta^2$	$-\frac{q^2-1}{2^3q}\zeta^2$	$-\frac{q^2-1}{2^3q}\zeta^2$	$+\frac{q^2-1}{2^3q}\zeta^2$

TABLE 4. The coefficients $c_z(\pi)$ appearing in Theorem 2.5. (Here we write ζ^2 for $\operatorname{sgn}(-1)$ in order to save space.)

Our proof of Theorem 2.5 is sketched in Section 2.8 and executed in Section 5; we will calculate the coefficients $c_z(\pi)$ for each depth-zero supercuspidal π, up to equivalence.

Remark 2.6. More properties of the coefficients $c_z(\pi)$ are given in Section 6. We show there that each coefficient may be interpreted as a virtual Chow motive. The issue of uniqueness of the coefficients is also addressed in Section 6.

2.7 Motivic Integration

The goal of arithmetic motivic integration is essentially to reduce the calculation of p-adic volumes to the computation of the number of points on varieties over finite fields. In particular, suppose we are talking about subsets of an affine space defined over a p-adic field. Consider the measurable subsets that can be described in a language (of logic) that depends neither on p nor on the choice of the uniformizer of the valuation (one such language is the language of Denef–Pas that is described in the next subsection). Once a set is described by a logical formula in such a language, it is possible to associate a geometric object defined over $\mathbb{Q}$ with it, in such a way that for almost all primes p, the volume of our set can be recovered from the number of points of the reduction of this object over $\mathbb{F}_q$, where $q = p^r$ is the cardinality of the residue field of the given local field.

Motivic integration is based on the algorithm of elimination of quantifiers (there is also a more recent version [Cluckers and Loeser 08], where in some parts quantifier elimination is replaced with cell decomposition, also an algorithmic procedure). Since at present, motivic integration as an algorithm is not implemented, it has not previously been used for computing examples; nevertheless, it can be used to prove general existence results for the required geometric objects (which already has implications, as explored in [Hales 05a, Cunningham and Hales 04, Gordon 04]).

We refer to the original papers [Denef and Loeser 01, Cluckers and Loeser 08] and to the beautiful exposition [Hales 05b] for the detailed description the concept of motivic integration and the statements of "comparison theorems" that relate the classical p-adic volumes with the motivic volumes. In the next few subsections we just give a list of the techniques we use.

2.7.1 The Language of Rings and the Language of Denef–Pas.

The language of rings is the language of logic such that its formulas can be interpreted in any ring with identity; thus, any such ring is a structure for this language. The language of rings has symbols for 0 and 1, symbols for countably many variables, and the symbols for the operations of addition $+$ and multiplication $\times$. A formula is a syntactically correct expression built from finitely many of these symbols, and also parentheses ( ); quantifiers $\exists$, $\forall$; and symbols of conjunction $\wedge$, disjunction $\vee$, and negation $\neg$. We will usually use this language to work with the residue field of our valued field.

There is also a first-order language that is perfectly suited for defining subsets of nonarchimedean-valued fields: it is the language of Denef–Pas. Formulas in this language have variables of three *sorts*: the valued field sort, the residue field sort, and the value sort. There also are symbols for the function $\mathrm{ord}(\cdot)$, which takes variables of the valued field sort to the variables of the value sort, and the function $\mathrm{ac}(\cdot)$ that takes the variables of the valued field sort to the variables of the residue field sort. The formulas are built using algebraic operations on variables of the same sort; the symbols $\mathrm{ord}(\cdot)$ and $\mathrm{ac}(\cdot)$ (which can be applied to variables of the valued field sort); quantifiers; and the symbols for the logical operations of conjunction, disjunction, and negation.

Every valued field with a choice of a uniformizer $(\mathbb{K}, \varpi)$ is a structure for the language of Denef–Pas; then the functions ord and ac match the usual valuation and "angular component" maps, where the "angular component" $\mathrm{ac}(x)$ equals the first nonzero coefficient of the p-adic expansion of x, i.e., if x is a unit, $\mathrm{ac}(x) = x \mathrm{mod}\ (\varpi)$; if x is not a unit, $\mathrm{ac}(x) = x\varpi^{-\mathrm{ord}(x)} \mathrm{mod}\ (\varpi)$.

A subset of an affine space over a local field $\mathbb{K}$ is called *definable* if it can be defined by a formula in the language of Denef–Pas. For a formula Φ with m free variables of the residue field sort and no other free variables, given a local field with the ring of integers $\mathcal{O}_{\mathbb{K}}$ and a choice of the uniformizer, we will denote by $Z(\Phi, \mathcal{O}_{\mathbb{K}})$ the subset of ${\mathcal{O}_{\mathbb{K}}}^{\oplus m}$ defined by the formula Φ.

2.7.2 The Ring of Virtual Chow Motives.

In all versions of motivic integration, the motivic volume takes values in some ring built from the Grothendieck ring of the category of algebraic varieties over the base field, which is $\mathbb{Q}$ in our case.

Strictly speaking, the so-called ring of virtual Chow motives is a natural choice of the ring of values for the arithmetic motivic volume, for reasons described in [Hales 05b]. We will not define Chow motives here (see [Scholl 94] for a good introduction). As a first approximation, it is possible to think just of formal linear combinations of isomorphism classes of varieties with rational coefficients.

The motivic volume takes values in the ring Mot, which we will now define. Let $\mathcal{M}ot_{\mathbb{Q}}$ be the category of Chow motives over $\mathbb{Q}$. We take its Grothendieck ring $K_0(\mathcal{M}ot_{\mathbb{Q}})$, i.e., the ring of formal linear combinations of the isomorphism classes of the Chow motives, with natural relations (see [Scholl 94]), and the product coming from tensor product. This ring has a unit 1. For every smooth projective variety V there is a Chow motive that corresponds to V in a natural way. It is a deep theorem that this map from varieties to motives extends to all (not just smooth projective) varieties, and induces a homomorphism from the Grothendieck ring of the category of varieties $K_0(Var_{\mathbb{Q}})$ to $K_0(\mathcal{M}ot_{\mathbb{Q}})$. Let us denote the image of this map by $K_0(\mathcal{M}ot)^v$. The image of the class of the affine line under this map is usually denoted by $\mathbb{L}$ (called the Lefschetz motive); the point maps to 1. Note that naturally, in $K_0(\mathcal{M}ot_{\mathbb{Q}})$ we have $\mathbb{L}+1 = [\mathbb{P}^1]$, where $[\mathbb{P}^1]$ is the class of the Chow motive corresponding to the projective line.

The ring Mot is obtained from $K_0(\mathcal{M}ot)^v \otimes_{\mathbb{Z}} \mathbb{Q}$ by localizing at $\mathbb{L}$ followed by localizing further at the set $\{\mathbb{L}^{-i} - 1 \mid i > 0\}$ (i.e., all these elements can be formally inverted). Note that the localization at $\{\mathbb{L}^{-i} - 1 \mid i > 0\}$ (which is equivalent to adding in the sums of geometric series with quotient $\mathbb{L}^i$) replaces the completion that was done in the original version [Denef and Loeser 01].

Let us also adopt the following convention: for a polynomial $F(q) \in \mathbb{Z}[q]$, we will denote by $[F(\mathbb{L})]$ the class of $F(\mathbb{L})$ in the ring Mot.

For every prime power q, there is an action of TrFrob_q on the elements of Mot that comes from the Frobenius action on the Chow motives. It can be thought of as a generalization of counting points on a variety over $\mathbb{F}_q$. The trace of the Frobenius operator on a Chow motive is the alternating sum of the traces of Frobenius acting on its ℓ-adic cohomology groups, and it is a priori an element of $\bar{\mathbb{Q}}_\ell$ (see [Denef and Loeser 01, Section 3.3]), but in fact, this number lies in $\mathbb{Q}$ for all elements of Mot that arise from the motivic volumes (in particular, for us the choice of ℓ doesn't matter). It is the trace of the Frobenius action that allows us to relate the values of the motivic measure (elements of Mot) to the usual p-adic volumes (rational numbers).

2.7.3 The Motivic Volume.

We cannot describe the construction of the motivic volume here, but we need only very simple examples in this paper, so we list just a few main principles and introduce the notation. Following the pattern of [Hales 05b], we start by declaring that there is a map from formulas in the language of rings to the ring Mot defined above. We denote the image of a formula ϕ under this map by $[\phi]$. We will use two properties of this map:

- The motivic volume of a formula in the language of rings with no quantifiers that defines a smooth algebraic variety is the class of that variety.

- If ϕ_1 and ϕ_2 are equivalent ring formulas, then they have equal motivic volumes. If ϕ_1 is an n-fold cover of ϕ_2 (both the equivalence and the term "cover" are understood in the sense of [Hales 05b]), then

$$[\phi_2] = \frac{1}{n}[\phi_1].$$

Now let us say a few words about the construction of the motivic volume for definable sets that is compatible with the classical volume of p-adic sets. First, we need to talk about sets in a way that does not depend on the field—and that is done by considering *definable* sets, as described above. In order to pass back and forth between sets and formulas defining them, it is very convenient to use the notion of a *definable subassignment* introduced in [Denef and Loeser 01].

Let $h_{\mathbb{A}^m}$ be the functor of points of the affine space that takes fields to sets ($K \mapsto \mathbb{A}^m(K)$). A *subassignment* of $h_{\mathbb{A}^m}$ is simply a collection of subsets of $\mathbb{A}^m(K)$, one for each field K. A formula Φ in the language of Denef–Pas with m free variables naturally defines a subassignment of $h_{\mathbb{A}^m}$ on a suitable category of local fields (e.g., the category of all nonarchimedean completions of a given global field): for each local field K, take the subset of $\mathbb{A}^m(K)$ on which the formula takes the value "true." Such subassignments are called *definable*.

The *motivic volume* is defined on definable subassignments and takes values in the ring Mot of virtual Chow motives. We denote the motivic volume by μ, so that $\mu(\Phi)$ denotes the motivic volume of the subassignment defined by the formula Φ. Sometimes, if we have a definable set H (defined by some formula Φ), we will write $\mu(H)$, meaning $\mu(\Phi)$.

The strategy of motivic integration is to replace the formula that defines our subassignment with an equivalent formula that has no quantifiers ranging over the valued field (the language of Denef–Pas admits quantifier elimination). The next step is to "approximate" the new formula by the formulas in the language of rings whose variables range only over the finite field. Finally, ring formulas can be mapped to the ring Mot defined above, as mentioned in the beginning of this subsection.

The main result that makes motivic integration relevant for us is the comparison theorem [Denef and Loeser 01, Theorems 8.3.1, 8.3.2]. Let Φ be a formula in the language of Denef–Pas. The comparison theorems (one of which deals with the fields of characteristic zero, and the other one with function fields) state that for all but finitely many primes p, for any local field $\mathbb{K}$ with the ring of integers $\mathcal{O}_{\mathbb{K}}$ and the residue field $\mathbb{F}_q$ of cardinality $q = p^r$, the volume of the set $Z(\Phi, \mathcal{O}_{\mathbb{K}})$ equals the trace of the Frobenius action on $\mu(\Phi)$ (which is an element of Mot).

In fact, it is possible to some extent to keep track of the "bad primes" where the comparison theorem fails, and we will do this as we compute the motivic volumes in our examples.

2.7.4 Motivic Integration with Parameters. Here we quote one more technical notation from [Cluckers and Loeser 08] that will be used only in the proof of Lemma 3.3. Cluckers and Loeser [Cluckers and Loeser 08] have a more general construction of the motivic volume: they define motivic volume on definable subassignments of the functor of points of $\mathbb{A}_K^m \times \mathbb{A}_k^n \times \mathbb{Z}^r$, which is denoted by $h[m, n, r]$ (here K stands for the valued field, and k for its residue field), which corresponds to considering families of definable subassignments of $h_{\mathbb{A}_K^m}$ with parameters in k and in $\mathbb{Z}^r$.

2.8 Outline of the Proof of Theorem 2.5

Since the rest of the paper is devoted to the proof of Theorem 2.5, we describe the strategy before we get into the technicalities.

We need to prove an equality of two distributions represented by locally constant locally integrable functions on the set of regular topologically unipotent elements in the group G. Each of these functions takes countably many values. We prove the equality by partitioning the domain into sets on which both sides are constant, and showing equality on each of these sets. Moreover, the problem is in fact not just countable, it is in some sense finite. Each of the two functions is obtained by a Frobenius-like formula from a function that takes finitely many values (explicitly, three distinct values for the function on the left-hand side and five distinct values for the function on the right-hand side). Motivic integration allows us to isolate these values and the coefficients at each one of them that are acquired in the Frobenius formula. Those latter coefficients also turn out to be computable.

It is in this sense that the proof is better suited for a computer than for a human, and that is in fact the point of this paper.

In order to be more precise, we must first say a few words about regular topologically unipotent elements of G. To do this, we start by studying regular topologically nilpotent elements in the Lie algebra $\mathfrak{g}$; we will then invoke our assumption that p is odd, in which case these correspond exactly, via the modified Cayley transform $\operatorname{cay}(X) = (1 + (X/2))(1 - (X/2))^{-1}$, to regular topologically unipotent elements of G. (In fact, cay establishes a bijection from topologically nilpotent elements of $\mathfrak{g}$ to topologically unipotent elements of G.)

What we need, ideally, is a partition of the set of regular topologically unipotent elements into *definable* sets such that all the functions involved (the characters and Fourier transforms of the orbital integrals) will be constant on them. We will not explicitly use any local constancy results (except in Section 6); instead, we will first make the partition, and then see that it is the right one.

Recall that at the moment, ε (a nonsquare unit) and ϖ (the uniformizer of the valuation) are fixed. One of the points of the motivic integration approach is to do the calculation independently of these choices, but that requires some tricks (discussed below), since they are both very deeply ingrained in all the definitions: ε is linked with the cocycles z, and ϖ plays a role in the construction of our representations. For now, we write down explicitly seven Cartan subalgebras of $\mathfrak{g}$ (one noncompact and six compact):

$$\mathfrak{h}_{s_0} = \left\{ \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \;\middle|\; x \in \mathbb{K} \right\},$$

$$\mathfrak{h}_{s_1} = \left\{ \begin{bmatrix} 0 & x \\ x\varepsilon & 0 \end{bmatrix} \;\middle|\; x \in \mathbb{K} \right\},$$

$$\mathfrak{h}_{s_2} = \left\{ \begin{bmatrix} 0 & x\varpi^{-1} \\ x\varepsilon\varpi & 0 \end{bmatrix} \;\middle|\; x \in \mathbb{K} \right\},,$$

$$\mathfrak{h}_{t_0} = \left\{ \begin{bmatrix} 0 & x \\ x\varpi & 0 \end{bmatrix} \;\middle|\; x \in \mathbb{K} \right\},,$$

$$\mathfrak{h}_{t_1} = \left\{ \begin{bmatrix} 0 & x\varepsilon \\ x\varpi\varepsilon^{-1} & 0 \end{bmatrix} \;\middle|\; x \in \mathbb{K} \right\},$$

$$\mathfrak{h}_{t_2} = \left\{ \begin{bmatrix} 0 & x \\ x\varpi\varepsilon & 0 \end{bmatrix} \;\middle|\; x \in \mathbb{K} \right\},$$

$$\mathfrak{h}_{t_3} = \left\{ \begin{bmatrix} 0 & x\varepsilon \\ x\varpi & 0 \end{bmatrix} \;\middle|\; x \in \mathbb{K} \right\}.$$

We have labeled our Cartan subalgebras by cocycles $z \in \{s_0, s_1, s_2, t_0, t_1, t_2, t_3\}$, one from each cohomology class in $\mathfrak{C}(\mathbb{K})$ (see Section 2.4) in the case $\operatorname{sgn}(-1) = 1$. In the case $\operatorname{sgn}(-1) = -1$, the Cartan subalgebras labeled by t_0 and t_1 are conjugate; and those labeled by t_2 and t_3 are conjugate (which means that the corresponding cocycles represent the same cohomology class). We will always consider each one of these subalgebras separately, which means that we are doing one-third more work than necessary half the time.

Our Cartan subalgebras are filtered by an index n that is closely related to the depth. To write the filtration lattices explicitly, in Table 5 we define, for each $z \in \{s_0, s_1, s_2, t_0, t_1, t_2, t_3\}$, integer n, and unit u in $\mathcal{O}_{\mathbb{K}}$, an element $Y_{z,n}(u) \in \mathfrak{h}_z$. Let $\mathfrak{h}_{z,n} := \{Y_{z,n}(u) \mid u \in \mathcal{O}_{\mathbb{K}}^*\}$. If z is s_0, s_1, or s_2, then $\mathfrak{h}_{z,n}$ is the set of elements of $\mathfrak{h}_z$ with depth n; however, if z is t_0, t_1, t_2, or t_3 then $\mathfrak{h}_{z,n}$ is the set of elements of $\mathfrak{h}_z$ with depth $\frac{1}{2} + n$.

In this paper we are particularly interested in understanding our distributions on topologically nilpotent elements of $\mathfrak{g}$. In anticipation of the correct partition of regular topologically nilpotent elements of $\mathfrak{g}$, we now define

$$\mathfrak{h}_{z,n,+} := \{Y_{z,n}(u) \mid u \in \mathcal{O}_{\mathbb{K}}^* \wedge \operatorname{sgn}(u) = +1\},$$

and

$$\mathfrak{h}_{z,n,-} := \{Y_{z,n}(u) \mid u \in \mathcal{O}_{\mathbb{K}}^* \wedge \operatorname{sgn}(u) = -1\},$$

and denote by ${}^G\mathfrak{h}_{z,n,\pm}$ the corresponding G-invariant sets. If $z = s_0$, s_1, or s_2, then ${}^G\mathfrak{h}_{z,n,\pm}$ consists of topologically nilpotent elements if and only if $n \geq 1$; if $z = t_0$, t_1, t_2, or t_3 then ${}^G\mathfrak{h}_{z,n,\pm}$ consists of topologically nilpotent elements if and only if $n \geq 0$.

Further, let ${}^G\mathfrak{h}_{z,n}$ (respectively ${}^G\mathfrak{h}_{z,n,+}$, ${}^G\mathfrak{h}_{z,n,-}$) denote the smallest G-invariant set containing $\mathfrak{h}_{z,n}$, where G acts on $\mathfrak{g}$ by adjoint action. The disjoint union of all the sets ${}^G\mathfrak{h}_{z,n}$ (as z ranges over the fixed set of representatives for $\mathfrak{C}(\mathbb{K})$ and n ranges over all nonnegative integers, with the exception of $n = 0$ in ${}^G\mathfrak{h}_{s_0,n}$, ${}^G\mathfrak{h}_{s_1,n}$, and ${}^G\mathfrak{h}_{s_1,n}$) coincides with the set of regular topologically nilpotent elements in $\mathfrak{g}$.

In fact, conjugation by G removes the ambiguity caused by the choice of specific cocycles (and therefore the choice of ε): different representatives of the same cohomology class correspond to different, but conjugate, Cartan subalgebras. Hence, it would be more appropriate to label the sets ${}^G\mathfrak{h}_{z,n}$ by the cohomology classes, and call them ${}^G\mathfrak{h}_{\tau,n}$, where τ is the cohomology class represented by z. We would like to stress that this produces a partition of the topologically nilpotent set that depends only on $\operatorname{sgn}(-1)$ (because the number of the distinct cohomology classes depends on this sign), and not on any choices. However, we do need the specific representatives

z	s_0	s_1	s_2
$Y_{z,n}(u)$	$\begin{bmatrix} \varpi^n u & 0 \\ 0 & -\varpi^n u \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^n u \\ \varepsilon\varpi^n u & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^{n-1} u \\ \varepsilon\varpi^{n+1} u & 0 \end{bmatrix}$

z	t_0	t_1	t_2	t_3
$Y_{z,n}(u)$	$\begin{bmatrix} 0 & \varpi^n u \\ \varpi^{n+1} u & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon\varpi^n u \\ \varepsilon^{-1}\varpi^{n+1} u & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varpi^n u \\ \varepsilon\varpi^{n+1} u & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \varepsilon\varpi^n u \\ \varpi^{n+1} u & 0 \end{bmatrix}$

TABLE 5. Elements $Y_{z,n}(u)$.

$Y_{z,n}(u)$ of each one of the sets ${}^G\mathfrak{h}_{\tau,n}$ in order to carry out our calculations. Our choice of these representatives depends on ε and ϖ. The strategy is to carry out all the explicit calculations first, get to the level of logical formulas, and at that stage ϖ would disappear entirely. To remove the dependence of ε, in Section 3.3 we turn it into a parameter ranging over the residue field, and then average over it, also showing that in fact we are averaging a constant function.

Note that with the way we defined our subalgebras, in the case $\mathrm{sgn}(-1) = -1$ when two pairs of them become conjugate, this conjugation switches the $\pm$ sign: ${}^G\mathfrak{h}_{t_0,n,+}$ corresponds to ${}^G\mathfrak{h}_{t_1,n,-}$, and ${}^G\mathfrak{h}_{t_3,n,+}$ corresponds to ${}^G\mathfrak{h}_{t_2,n,-}$ (we will see that this is reflected in our character table).

On the group side, we write $\Gamma_{z,n}$ (respectively $\Gamma_{z,n,+}$, $\Gamma_{z,n,-}$) for the image of $\mathfrak{h}_{z,n}$ (respectively $\mathfrak{h}_{z,n,+}$, $\mathfrak{h}_{z,n,-}$) under the modified Cayley transform cay, when defined; we denote the corresponding G-invariant sets by ${}^G\Gamma_{z,n}$ (respectively ${}^G\Gamma_{z,n,+}$, ${}^G\Gamma_{z,n,-}$). The characteristic functions of the sets ${}^G\Gamma_{z,n,\pm}$ will be denoted by $f_{z,n,\pm}$, and the characteristic functions of the sets ${}^G\mathfrak{h}_{z,n,\pm}$ will be denoted by $\tilde{f}_{z,n,\pm}$ throughout the paper.

Given a depth-zero representation π induced from a maximal compact subgroup G_x, in Section 3 we associate three virtual Chow motives (denoted by $M^{x,0}_{z,n,\pm}$, $M^{x,1}_{z,n,\pm}$, and $M^{x,\varepsilon}_{z,n,\pm}$) with each triple $(z, n, \pm)$ as above, so that the value of the distribution character of π at $f_{z,n,\pm}$ can be recovered from this triple of virtual Chow motives for almost all residual characteristics p (in fact, for all $p \neq 2$). Moreover, we will see that the only way in which these motives depend on π is through the compact subgroup on which π has nontrivial compact restriction (i.e., through the choice of the vertex in the building $x = (0)$ or $x = (1)$; see Section 2.1).

In Section 4, we also associate with each $(z, n, \pm)$ five virtual Chow motives, so that all orbital integrals that appear on the right-hand side of the semisimple character expansion can be recovered from these motives.

In Section 4.4, we put together all the results about groups over the finite fields that are crucial for our understanding of the p-adic "lifts."

Since both sides of the semisimple character expansion are invariant under conjugation by G, all we have to do is check the equality on each of the sets $\Gamma_{z,n}$. This is done in Section 5. Due to the very mechanical nature of this proof, we do not include all the details. In all proofs, we include all the details in one "unramified" case $z = s_1$ and one "ramified" case $z = t_2$; in the other cases we just indicate the differences and summarize the results in the tables.

3. MOTIVES CORRESPONDING TO OUR CHARACTERS

Throughout Section 3, x denotes either the standard vertex (0) or the vertex (1) in the Bruhat–Tits building for G (see equation (2–1)); we also reserve the symbol y for the barycenter of the facet with boundary $\{(0), (1)\}$.

3.1 Consequences of the Frobenius Formula for the Character

This section follows the method of expressing the character as a sum over conjugacy classes in the reductive quotient that was used in [Gordon 04].

Let π be a supercuspidal depth-zero representation of G and let Θ_π be its distribution character in the sense of Harish-Chandra. Let f be a test function supported on some compact subset H of the set of regular topologically unipotent elements in G. We can assume that $f = f_H$ is the characteristic function of such a set without loss of generality. Let χ be the character of the representation of the finite group $\bar{G}_x$ that gave rise to π (see Section 2.1). Since G_x is a maximal compact subgroup of G, the Frobenius formula gives the following expression for the character:

$$\begin{aligned}\Theta_\pi(f_H) &= \int_G \int_G f_H(ghg^{-1})\chi_{x,0}(h)\,dh\,dg\\ &= \int_G \int_H \chi_{x,0}(g^{-1}hg)\,dh\,dg\\ &= \int_{G/G_x}\int_{G_x}\int_H \chi_{x,0}(k^{-1}g^{-1}hgk)\,dh\,dk\,dg\\ &= \mathrm{m}(G_x)\int_{G/G_x}\int_H \chi_{x,0}(g^{-1}hg)\,dh\,dg.\end{aligned} \tag{3–1}$$

(Recall from Section 2.5 that the formal degree $d(\pi)$ of π is exactly $\chi_{x,0}(1)$.) The real work is to prove that these integrals converge. Of course, Harish-Chandra did this long ago when the characteristic of $\mathbb{K}$ is 0. This is also done in the proof of [Bushnell and Henniart 96, Theorem A.14(ii)]. We will see the convergence when we do the calculations by hand.

The next step is to rewrite the outside integral in equation (3–1) as a sum using the Cartan decomposition $G = G_x A G_x$, where A is the set of elements of the form $a_\lambda := \mathrm{diag}(\varpi^\lambda, \varpi^{-\lambda})$ with λ a nonnegative integer. This is also done in [Bushnell and Henniart 96, Theorem A.14(ii)]; see equation (3–3) below. We will also take this formula one step further by collecting the terms corresponding to each value of the character of $\mathrm{SL}_2(\mathbb{F}_q)$ at unipotent elements.

Before we can carry out this plan, we need to introduce some more notation. Recall that there are three unipotent conjugacy classes in $\mathrm{SL}(2,\mathbb{F}_q)$: the class U_0 of the identity (one element), the class U_1 of the element $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$, and the class U_ϵ of the element $\left[\begin{smallmatrix}1&\epsilon\\0&1\end{smallmatrix}\right]$, where ϵ is a nonsquare in $\mathbb{F}_q$. Suppose U is a unipotent conjugacy class in $\mathrm{SL}(2,\mathbb{F}_q)$, λ is a nonnegative integer, and x is a vertex in the Bruhat–Tits building for G. For any regular topologically unipotent element h of G, let $N^x_{U,\lambda}(h)$ denote the number of right G_x-cosets gG_x inside the double coset $G_x a_\lambda G_x$ that satisfy the following condition:

$$g^{-1}hg \in G_x \wedge \rho_{x,0}(g^{-1}hg) \in U.$$

Proposition 3.1. *Let π be a supercuspidal depth-zero representation of G and let Θ_π be its distribution character in the sense of Harish-Chandra. Let H be a compact set of regular topologically unipotent elements in G and let f_H be the characteristic function of H. Let χ be the character of the representation of the finite group $\bar{G}_x$ that gave rise to π (see Section 2.1). Then*

$$\Theta_\pi(f_H) = \sum_\lambda \sum_U \chi(U)\int_H N^x_{U,\lambda}(h)\,dh, \tag{3–2}$$

where U runs over unipotent conjugacy classes in $\mathrm{SL}(2,\mathbb{F}_q)$ and λ runs over nonnegative integers. The sum over λ has only finitely many nonzero terms.

Proof: Consider a fixed double coset $G_x a_\lambda G_x$. We observe that if $g_1, g_2 \in G_x$ and $g_1 a_\lambda G_x = g_2 a_\lambda G_x$, then $\chi_{x,0}(g_1^{-1}hg_1) = \chi_{x,0}(g_2^{-1}hg_2)$. Hence, using Cartan decomposition, the double integral from the formula (3–1) can be rewritten as a double sum, and we obtain the formula (cf. [Bushnell and Henniart 96, Theorem A.14(ii)])

$$\Theta_\pi(f_H) = \mathrm{m}(G_x)\sum_{a\in G_x\backslash G/G_x}\ \sum_{g\in G_x a G_x/G_x}\int_H \chi_{x,0}(g^{-1}hg)\,dh, \tag{3–3}$$

where the summation index in the outside sum runs over the set of representatives of the double cosets, that is, over A, and the summation index in the inner sum runs over the set of representatives of the left G_x-cosets inside the given double coset. We refer to [Bushnell and Henniart 96, Theorem A.14(ii)] for the proof that there are only finitely many values of λ that give nonzero summands.

Now consider the contribution of the double coset

$$G_x \begin{bmatrix}\varpi^\lambda & 0\\ 0 & \varpi^{-\lambda}\end{bmatrix} G_x$$

to the sum on the right-hand side of equation (3–3). The following two observations make the calculation of the character fairly simple. First, the function $\chi_{x,0}$ vanishes outside the maximal compact subgroup G_x, while on G_x, the value $\chi_{x,0}(g)$ depends only on $\rho_{x,0}(g)$. Second, the set H is contained in the set of topologically unipotent elements, so for every h in H, the element $\rho_{x,0}(g^{-1}hg)$ is a unipotent element in $\bar{G}_x$, provided that $g^{-1}hg$ lies in G_x. Using these observations, it is possible to rewrite the formula for the character (equation (3–3)) in the following form:

$$\Theta_\pi(f_H) = \sum_\lambda \sum_U \chi(U)\int_H N^x_{U,\lambda}(h)\,dh. \tag{3–4}$$

That completes the proof. □

In the next section, we will see that the numbers $N^x_{U,\lambda}(h)$ give rise to geometric objects. Note that these numbers essentially depend only on the group; they depend on the representation only through the choice of the vertex x that indexes the maximal compact subgroup from which our representation is induced.

3.2 Almost Definable Sets $W^x_{U,\lambda}(h)$

In this section we begin the process of finding Chow motives related to the numbers $N^x_{U,\lambda}(h)$ introduced in Section 3.1 by expressing these numbers as p-adic volumes of some sets $W^x_{U,\lambda}(h)$; we will then use the comparison theorem of Denef and Loeser (see Section 2.7.3) to recover these p-adic volumes from the motivic volumes. But first we need to introduce yet more notation.

Suppose U is a unipotent conjugacy class in $\bar{G}_x$, λ is a nonnegative integer, and x is a vertex in the Bruhat–Tits building for G as before. For any regular topologically unipotent element h of G, define

$$W^x_{U,\lambda}(h) := \{y \in G_x \mid a_\lambda^{-1} y^{-1} h y a_\lambda \in G_x \wedge \rho_{x,0}((ya_\lambda)^{-1} h(ya\lambda)) \in U\}. \tag{3–5}$$

Lemma 3.2. *With notation as above,*

$$N^x_{U,\lambda}(h) = (q+1)q^{2\lambda-1}\frac{\mathrm{m}(W^x_{U,\lambda}(h))}{\mathrm{m}(G_x)}, \tag{3–6}$$

for all regular topologically unipotent $h \in G$.

Proof: Let a_λ denote the element $\mathrm{diag}(\varpi^\lambda, \varpi^{-\lambda})$, as before. For each $\lambda \in \mathbb{N} \cup \{0\}$, we say that two elements y_1 and y_2 in G_x are λ-equivalent if $y_1 a_\lambda$ and $y_2 a_\lambda$ are in the same left G_x-coset, that is, if $a_\lambda^{-1} y_1^{-1} y_2 a_\lambda \in G_x$. The λ-equivalence class of an element y is denoted by $[y]_\lambda$. Each set $W^x_{U,\lambda}(h)$ is a finite disjoint union of λ-equivalence classes (see [Gordon 04]), and if it contains an element y, it contains the whole class $[y]_\lambda$. With this notation, the number $N^x_{U,\lambda}(h)$ equals the number of λ-equivalence classes in the set $W^x_{U,\lambda}(h)$.

Now, for $\lambda > 0$, the cardinality of $G_x a_\lambda G_x / G_x$ is $(q+1)q^{2\lambda-1}$, as can be shown using the affine Bruhat decomposition for G (see [Bruhat and Tits 96], for example). Since all equivalence classes have equal volumes (see [Gordon 04, Lemma 4]), it follows that

$$N^x_{U,\lambda}(h) = (q+1)q^{2\lambda-1}\frac{\mathrm{m}(W^x_{U,\lambda}(h))}{\mathrm{m}(G_x)}, \tag{3–7}$$

as claimed. □

The sets $W^x_{U,\lambda}(h)$ are almost definable, but they depend on the parameter h, which cannot be specified within the language. Still, it is possible to use motivic integration to calculate their volumes, using the version of motivic integration that allows parameters.

3.3 Motivic Volumes of the Sets $W^x_{U,\lambda}(h)$

In this section, we write down the formulas defining the sets $W^x_{U,\lambda}(h)$ of the previous section, and then calculate their motivic volumes. This is done case by case in z. We start with the most interesting case $z = s_1$. We include the details only for $x = (0)$. Before we start the calculation, two remarks are due.

First, recall that we had to fix a nonsquare unit ε and a uniformizer ϖ, as discussed in Section 2.8. In this section, we introduce a variable δ that will be allowed to range over nonsquare units. As discussed in Section 2.7.1, a p-adic field together with the choice of the uniformizer is a structure for the language of Denef–Pas (i.e., such a choice provides an interpretation of all the formulas). If we choose a nonsquare unit ε in the given field, carry out all the constructions that have appeared so far, and then plug in the value of ε for δ in our formulas, we will get the corresponding sets $W^x_{U,\lambda}(h)$. We will see, however, that their motivic volumes (and therefore, also p-adic volumes) are independent of the choice of ε.

Second, the comparison theorem relates the motivic volumes to p-adic volumes for all but finitely many primes p. There are two sources of "bad" primes in this statement: singularities of the varieties that come up, and quantifier elimination. Since we are doing all the calculations by hand, and our cases are very simple, we will see that the only prime that needs to be excluded is 2.

3.3.1 The Logical Formulas for $W^{(0)}_{U,\lambda}(h)$ for $h \in \Gamma_{s_1,n}$.

Recall that all the elements of the set $\Gamma_{s_1,n}$ have the form

$$h = (1+*)\begin{bmatrix} 1+* & u\varpi^n \\ \varepsilon u \varpi^n & 1+* \end{bmatrix}, \tag{3–8}$$

where $u \in \mathcal{O}^*_{\mathbb{K}}$, and $*$ stands for elements of order at least $2n$. We write $y = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G_{(0)}$ and use the entries a, b, c, d as free variables in our formulas. We begin by fixing U and h as above and finding explicit conditions on a, b, c, d and λ that ensure that $y \in W^{(0)}_{U,\lambda}(h)$. A direct computation gives

$$y^{-1}hy = (1+*)\begin{bmatrix} 1+\text{h.o.t} & u(d^2-b^2\varepsilon)\varpi^n+\text{h.o.t} \\ u(a^2\varepsilon-c^2)\varpi^n+\text{h.o.t} & 1+\text{h.o.t} \end{bmatrix}, \tag{3–9}$$

where h.o.t means "higher-order terms," which refers to the terms with valuation greater than that of the leading term.

As we see from equation (3–5), we must now conjugate this element by a_λ. Conjugation by a_λ induces multiplication of the entry in the upper right-hand corner by $\varpi^{-2\lambda}$, and multiplication of the entry in the lower left-hand corner by $\varpi^{2\lambda}$. It follows that $a_\lambda^{-1} y^{-1} h y a_\lambda \in G_{(0)}$

if and only if

$$\begin{aligned} &-2\lambda + \mathrm{ord}(u(d^2 - b^2\varepsilon)\varpi^n)) \geq 0 \\ &\quad \wedge\, 2\lambda + \mathrm{ord}(u(a^2\varepsilon - c^2)\varpi^n)) \geq 0. \end{aligned} \tag{3–10}$$

Consequently, we have the following list of cases:

Case 1. $2\lambda < n$. In this case, for any $y \in G_{(0)}$ the element $a_\lambda^{-1} y^{-1} h y a_\lambda$ lies in $G_{(0)}$ and projects to the identity under the reduction mod ϖ.

Case 2. $2\lambda > n$. We observe that $\mathrm{ord}(d^2 - b^2\varepsilon) = 0$: indeed, either both a and d or b and c have to be units because $ad - bc = 1$, and $\rho_{(0),0}(d^2 - b^2\varepsilon) \neq 0$, since ε is a nonsquare. Similarly, $\mathrm{ord}(a^2\varepsilon - c^2) = 0$. This implies that if $2\lambda > n$, the set of y satisfying the conditions is empty.

Case 3. $2\lambda = n$. (This case corresponds to the interesting situations.) In this case the image of $a_\lambda^{-1} y^{-1} h y a_\lambda$ under $\rho_{(0),0}$ is an element of the form $\left[\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right]$, where $\beta = \mathrm{ac}(d^2 - b^2\varepsilon)\mathrm{ac}(u) \in \mathbb{F}_q$. Then for $h \in \Gamma_{s_1,n,+}$, the reduction $\rho_{(0),0}(a_\lambda^{-1} y^{-1} h y a_\lambda)$ falls into U_1 if $\mathrm{ac}(d^2 - b^2\varepsilon)$ is a square and into U_ε otherwise.

Notice that when λ is fixed and h is fixed, the condition $y \in W^{(0)}_{U,\lambda}(h)$ does not depend on u, as long as h is confined to one of the sets $\Gamma_{s_1,n,+}$ and $\Gamma_{s_1,n,-}$.

We now reformulate what we have found concerning the character in terms of formulas in Pas's language. We start by observing that the maximal compact subgroup $G_{(0)}$ is defined by the formula

$$\begin{aligned} &ad - bc = 1 \wedge \mathrm{ord}(a) \geq 0 \wedge \mathrm{ord}(b) \geq 0 \wedge \mathrm{ord}(c) \geq 0 \\ &\quad \wedge \mathrm{ord}(d) \geq 0. \end{aligned}$$

For the cases $2\lambda < n$ and $2\lambda > n$, no other calculations are needed. Indeed, if $2\lambda < n$, then we are within Case 1, and therefore both sets $W^{(0)}_{U_{1,\varepsilon},\lambda}(h)$ are empty, and $W^{(0)}_{U_0,\lambda}(h) = G_{(0)}$. If $2\lambda > n$, then we are within Case 2, and the set of y satisfying the conditions is also empty.

Let us now consider the case n even, and $2\lambda = n$. Let $\psi(b, d, \delta)$ be the formula (in Pas's language)

$$\psi(b, d, \delta) = \exists x(d^2 - b^2\delta = x^2).$$

Since we will often pass back and forth between the valued field and the finite field, let us introduce an abbreviation for the "reduction mod ϖ" map: let

$$\bar{x} = \begin{cases} \mathrm{ac}(x), & \mathrm{ord}(x) = 0, \\ 0, & \mathrm{ord}(x) > 0. \end{cases}$$

If we know that δ is a nonsquare unit, then by Hensel's lemma, to check whether the triple (b, d, δ) with $(\bar{b}, \bar{d}) \neq (0, 0)$ satisfies the formula ψ, it is enough to know whether its reduction mod (ϖ) satisfies the same formula (where now the variables are interpreted as residue-field variables). Hensel's lemma is applicable because $\bar{d}^2 - \bar{b}^2\bar{\delta}$ cannot be 0 when we assume that $\bar{\delta}$ is a nonsquare and $\bar{b}, \bar{d}$ are not simultaneously zero.

Let us consider the family of formulas depending on a parameter η that ranges over the set of nonsquares in $\mathbb{F}_q$ (note that this is a definable set):

$$\phi_\eta(a, b, c, d) = \left[ad - bc = 1 \wedge \exists\xi(\bar{b}^2 - \bar{d}^2\eta = \xi^2)\right]. \tag{3–11}$$

For every value of $\eta \in {\mathbb{F}_q}^* \setminus {\mathbb{F}_q}^{*2}$, if we let all the variables range over $\mathcal{O}_\mathbb{K}$, the formula $\phi_\eta(a, b, c, d)$ defines the set $W^{(0)}_{U_1,\lambda}(h)$ for any $h \in \Gamma_{s_1,n,+}$ if the unit ε that was fixed in order to define the sets $\Gamma_{s_1,n,\pm}$ has the property $\bar{\varepsilon} = \eta$. The same formula also defines the set $W^{(0)}_{U_\varepsilon,\lambda}(h)$ with $h \in \Gamma_{s_1,n,-}$. In the next subsection we find the motivic volumes of these sets.

3.3.2 The Motivic Volumes of $W^{(0)}_{U,\lambda}(h)$, $h \in \Gamma_{s_1,n,\pm}$.

The calculation of these volumes is very simple. Recall that our Haar measure on G is normalized in such a way that the fibers of the projection $G \to G(\mathbb{F}_q)$ have volume 1. The formula ϕ_η imposes a condition on the variables b, d that apparently depends only on $\bar{b}, \bar{d}$. This means that either a whole fiber over a point $(\bar{a}, \bar{b}, \bar{c}, \bar{d}) \in \mathrm{SL}(2, \mathbb{F}_q)$ satisfies this condition, or the whole fiber does not satisfy it. Hence, to calculate the volume of the set $W^{(0)}_{U_1,\lambda}(h)$, all we need to do is count the number of points in $\left[\begin{smallmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{F}_q)$ that satisfy the condition $\exists\xi : \bar{b}^2 - \bar{d}^2\eta = \xi^2$, where η is a parameter that is a quadratic residue in $\mathbb{F}_q$. This calculation is carried out carefully in [Gordon 09]. Here we state the result as a lemma.

In fact, this is the only place in the present paper where interesting geometric objects arise from p-adic volumes. Indeed, as we see from the argument above, the p-adic volumes of the sets $W^{(0)}_{U,\lambda}(h)$ with the "borderline" value of $\lambda = n/2$ (for n even) are connected with the number of points of a conic over the finite field.

Lemma 3.3. *Suppose $h \in \Gamma_{s_1,n}$. If n is even and $\lambda = n/2$, then the motivic volumes of the sets $W^{(0)}_{U_1,\lambda}(h)$ and $W^{(0)}_{U_\varepsilon,\lambda}(h)$ both equal $\frac{1}{2}\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)$, which is half of the motivic volume of the maximal compact subgroup $G_{(0)}$; otherwise, the motivic volumes of the sets $W^{(0)}_{U_1,\lambda}(h)$ and $W^{(0)}_{U_\varepsilon,\lambda}(h)$ both equal 0.*

Proof: The statement follows from the calculation of the motivic volume of the formula

$$\begin{aligned}&\Phi(a,b,c,d,\eta)\\&=\left[ad-bc=1\wedge\exists\xi\neq 0(\bar{d}^2-\bar{b}^2\eta=\xi^2)\wedge\nexists\beta(\eta=\beta^2)\right],\end{aligned}\tag{3–12}$$

which is carried out in [Gordon 09]. Note that the calculation ultimately boils down to computing the class of the "hyperbola" $x^2-y^2=1$, which is $\mathbb{L}-1$. This is the reason that we get an answer that is polynomial in $\mathbb{L}$. □

3.3.3 The Motivic Volumes of $W_{U,\lambda}^{(0)}(h)$ for $h\in\Gamma_{s_2,n}$.

In the case $z=s_2$, the answer is essentially the same as in the case $z=s_1$, but the calculation is slightly more complicated. Here we sketch the calculation of the motivic volume of the sets $W_{U,\lambda}^{(0)}(h)$ for $h\in\Gamma_{s_2,n,\pm}$, indicating the differences with the case $z=s_1$.

Exactly as before, we consider the element $a_\lambda^{-1}y^{-1}\gamma_{s_2,n}ya_\lambda$, and use the entries of the matrix $y=\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in G_{(0)}$ as free variables in our logical formulas.

The conditions (3–10) are now replaced with

$$\begin{aligned}-2\lambda+\operatorname{ord}(d^2u\varpi^{n-1}-b^2u\varepsilon\varpi^{n+1}))&\geq 0,\\2\lambda+\operatorname{ord}(a^2u\varepsilon\varpi^{n+1}-c^2u\varpi^{n-1})&\geq 0.\end{aligned}\tag{3–13}$$

The second condition is satisfied automatically if $\lambda\geq 0$, so we need to focus only on the first one. Similarly to the case $z=s_1$, this condition implies that when $\lambda>\frac{n+1}{2}$, the element $a_\lambda^{-1}y^{-1}hya_\lambda$ is outside $G_{(0)}$; if $\lambda<\frac{n-1}{2}$, this element is in $G_{(0)}$ and projects to U_0.

Suppose for now that n is odd. Unlike the case $z=s_1$, now there are two interesting cases: $\lambda=\frac{n-1}{2}$ and $\lambda=\frac{n+1}{2}$. If $\lambda=\frac{n-1}{2}$ and $\operatorname{ord}(d)=0$, then $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda)\in U_1$ if $\operatorname{sgn}(u)=1$ and $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda)\in U_\varepsilon$ if $\operatorname{sgn}(u)=1$. If $\operatorname{ord}(d)>0$, then $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda)\in U_0$. If $\lambda=\frac{n+1}{2}$, then $a_\lambda^{-1}y^{-1}hya_\lambda$ is in $G_{(0)}$ only if $\operatorname{ord}(d)>0$. In this case, the projection of the element $a_\lambda^{-1}y^{-1}hya_\lambda$ depends on the sign of $\operatorname{ac}(d^2-b^2\varepsilon)$ and on $\operatorname{sgn}(u)$, which is similar to the case $z=s_1$. The only difference is that here there is an additional condition $\operatorname{ord}(b)=0$ (if $\operatorname{ord}(d)>0$ then the determinant condition forces $\operatorname{ord}(b)=0$).

One finds that the motivic volume of the subset of G defined by the formula $\exists\xi\neq 0(\operatorname{ac}(d^2-b^2\varepsilon)=\xi^2)\wedge(\operatorname{ord}(b)=0)\wedge(\operatorname{ord}(d)>0)$ equals $\frac{1}{2}(\mathbb{L}-1)^2$. The motivic volume of its complement, i.e., the set defined by the formula $\nexists\xi\neq 0(\operatorname{ac}(d^2-b^2\varepsilon)=\xi^2)\wedge(\operatorname{ord}(b)=0)\wedge(\operatorname{ord}(d)>0)$, equals $\mathbb{L}(\mathbb{L}^2-1)-\frac{1}{2}(\mathbb{L}-1)^2=\frac{1}{2}\mathbb{L}^2-\frac{1}{2}$. The last two sentences are of course an abbreviation. In truth, we have to replace ε with a variable δ and do everything exactly the same way as was done in the previous case.

For simplicity, suppose that $\operatorname{sgn}(u)=1$. Putting all these calculations together, we get

$$\begin{aligned}\mu\left(W_{\frac{n+1}{2},U_\varepsilon}\right)&=\frac{1}{2}\mathbb{L}^2-\frac{1}{2},\\\mu\left(W_{\frac{n-1}{2},U_\varepsilon}\right)&=0,\\\mu\left(W_{\frac{n-1}{2},U_1}\right)&=\mathbb{L}^2(\mathbb{L}-1),\\\mu\left(W_{\frac{n+1}{2},U_1}\right)&=\frac{1}{2}(\mathbb{L}-1)^2.\end{aligned}\tag{3–14}$$

The case n even and the calculation of $\mu(W_{\lambda,U_0})^{(0)}(h)$ are similar to the case $z=s_1$.

3.3.4 The Ramified Cases.

We have the following lemma.

Lemma 3.4. *Let x be a vertex in the Bruhat–Tits building for G. Suppose $h\in\Gamma_{z,n,\pm}$, with $z\in\{t_0,t_1,t_2,t_3\}$. Then the sets $W_{U,\lambda}^x(h)$ are definable for all nonnegative integers λ; their motivic volumes are independent of h and the choice of ε, and can be explicitly computed.*

Proof: We prove this lemma only for $x=(0)$. The proof in the case $x=(1)$ is very similar; the results of these calculations become part of the expressions for $M_{z,n,\pm}^{(1),0}$, $M_{z,n,\pm}^{(1),1}$, and $M_{z,n,\pm}^{(1),\varepsilon}$ appearing in Tables 8 and 9. So everywhere in this proof $x=(0)$, and we drop the superscript x from the notation $W_{U,\lambda}^x(h)$.

The argument is very similar to that in the unramified case; the only difference is that the actual calculation of the motivic volumes of the corresponding sets $W_{U,\lambda}(h)$ is simpler. Here we carry out the proof for the case $z=t_2$. The other three ramified cases are almost identical to it. First, as in the previous subsection, we write the elements h of the set $\Gamma_{t_2,n}$ explicitly as

$$h=(1+*)\begin{bmatrix}1+*&u\varpi^n\\\varepsilon u\varpi^{n+1}&1+*\end{bmatrix},\tag{3–15}$$

where $u\in\mathcal{O}_{\mathbb{K}}^*$ and $*$ denotes the terms of order at least $2n$, as before.

As in the proof of the previous lemma, we let $y=\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$ be a variable running over $G_{(0)}$ (so that the symbols for its entries a,b,c,d will become the formal variables in Pas's language formulas defining the sets $W_{U,\lambda}^{(0)}(h)$).

As before, we compute $y^{-1}hy$, which leads to the following conditions on y,λ for the element $a_\lambda^{-1}y^{-1}hya_\lambda$ to be in $G_{(0)}$:

$$\begin{aligned}-2\lambda+\operatorname{ord}(ud^2\varpi^n-b^2\varepsilon u\varpi^{n+1})&\geq 0,\\2\lambda+\operatorname{ord}(ua^2\varepsilon\varpi^{n+1}-c^2u\varpi^n)&\geq 0.\end{aligned}\tag{3–16}$$

As before, it is convenient to consider the cases n even and n odd separately.

Suppose n is even. Looking at the left-hand side of the inequalities (3–16), we see that if $\lambda < n/2$, then the set $W_{U_0,\lambda}(h)$ coincides with the whole of $G_{(0)}$, and the sets $W_{U_1,\lambda}(h)$, $W_{U_\varepsilon,\lambda}(h)$ are empty.

If $\lambda > n/2$, then, since n is even and λ is an integer, 2λ is at least $n+2$, which forces the element $a_\lambda^{-1}y^{-1}hya_\lambda$ outside $G_{(0)}$, and all three sets are empty.

Finally, in the case $\lambda = n/2$, the outcome depends on the entry d: If $\mathrm{ord}(d) = 0$, then the term $d^2u\varpi^n$ in the expression $d^2u\varpi^n - b^2\varepsilon u\varpi^{n+1}$ dominates, and therefore $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_1$ if $\mathrm{sgn}(u) = 1$, and $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_\varepsilon$ if $\mathrm{sgn}(u) = -1$. If $\mathrm{ord}(d) > 0$, then $\rho_{(0),0}(a_\lambda^{-1}y^{-1}hya_\lambda) \in U_0$.

Hence, for $h \in \Gamma_{t_2,n,\pm}$, the set $W_{U_0,n/2}(h)$ is defined by the formula $\mathrm{ord}(d) > 0$ in conjunction with the formulas defining $G_{(0)}$. Note that the volume of this set is the same as that of $G_{(01)}$, i.e., equals $\mathbb{L}(\mathbb{L}-1)$. The sets $W_{U_1,n/2}(h)$, $W_{U_\varepsilon,n/2}(h)$ are defined by the formula $\mathrm{ord}(d) = 0$ in conjunction with the formulas defining $G_{(0)}$ for $h \in \Gamma_{t_2,n,+}$ and $h \in \Gamma_{t_2,n,-}$, respectively, and are respectively empty for $h \in \Gamma_{t_2,n,-}$ and $h \in \Gamma_{t_2,n,+}$.

The case n odd is very similar. If $\lambda = \frac{n+1}{2}$ (the most interesting case) and $\mathrm{ord}(d) = 0$, then the second of the conditions (3–16) is not satisfied, so all three sets $W_{U,\frac{n+1}{2}}(h)$ are empty. If $\mathrm{ord}(d) > 0$, then automatically $\mathrm{ord}(b) = 0$, and the leading term in the expression $d^2u\varpi^n - b^2\varepsilon u\varpi^{n+1}$ is $-b^2u\varepsilon\varpi^{n+1}$, which has sign opposite to $\mathrm{sgn}(-1)\mathrm{sgn}(u)$.

We obtain that if $\lambda \le (n-1)/2$, then

$$W_{U_0,\lambda} = G_{(0)} \quad \text{and} \quad W_{U_1,\lambda}(h) = W_{U_\varepsilon,\lambda}(h) = \varnothing,$$

for any $h \in \Gamma_{t_2,n}$; if $\lambda > (n+1)/2$, then all three sets are empty; if $\lambda = (n+1)/2$, we have

$$\begin{aligned} &W_{U_1,\frac{n+1}{2}}(h) = G_{(0)} \cap \{\mathrm{ord}(d) > 0\}, W_{U_\varepsilon,\frac{n+1}{2}}(h) = \varnothing, \\ &\qquad \text{if } h \in \Gamma_{t_2,n,\mathrm{sgn}(-1)}; \\ &W_{U_\varepsilon,\frac{n+1}{2}}(h) = G_{(0)} \cap \{\mathrm{ord}(d) > 0\}, W_{U_1,\frac{n+1}{2}}(h) = \varnothing, \\ &\qquad \text{if } h \in \Gamma_{t_2,n,-\mathrm{sgn}(-1)}. \end{aligned} \tag{3–17}$$

It follows that $\mu(W_{U_1,\frac{n+1}{2}}(h))$ equals 0 or equals $\mu(G_{(01)}) = \mathbb{L}(\mathbb{L}-1)$ depending on whether $h \in \Gamma_{t_2,n,\mathrm{sgn}(-1)}$ or $h \in \Gamma_{t_2,n,-\mathrm{sgn}(-1)}$; the same is true for $\mu(W_{U_\varepsilon,\frac{n+1}{2}}(h))$.

As we see from this proof, a different choice of ε could not have affected the motivic volumes of these sets; also, clearly no “bad” primes were acquired. □

3.3.5 The Case $x = (1)$. In all the proofs in the present section we have been assuming that $x = (0)$. The only major difference in the case $x = (1)$ is that the element $y = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ now belongs to $G_{(1)}$, not $G_{(0)}$. The requirement is that the element $a_\lambda^{-1}y^{-1}hya_\lambda$ belong to $G_{(1)}$ and that the reduction map applied to this element be $\rho_{(1),0}$ instead of $\rho_{(0),0}$. Therefore, all the cases look slightly different, but no new varieties appear in the calculations of the motivic volumes of the new sets.

3.4 Motives Corresponding to the Distribution Characters

Proposition 3.5. *The Harish-Chandra character of each depth-zero supercuspidal representation is constant on each set ${}^G\Gamma_{z,n,\pm}$. Moreover, there exist virtual motives $M^{x,0}_{z,n,\nu}$, $M^{x,1}_{z,n,\nu}$, and $M^{x,\varepsilon}_{z,n,\nu}$ (where z is any cocycle defined in Section 2.8, n is a positive integer for $z \in \{s_0, s_1, s_2\}$ and a nonnegative integer for $z \in \{t_0, t_1, t_2, t_3\}$, the sign ν is $\pm$, and x is a vertex (0) or (1)) such that*

$$\begin{aligned} &\frac{1}{m({}^G\Gamma_{z,n,\nu})}\Theta_\pi(f_{z,n,\nu}) \\ &\quad = \chi|_{U_0}\mathrm{TrFrob}M^{x,0}_{z,n,\nu} + \chi|_{U_1}\mathrm{TrFrob}M^{x,1}_{z,n,\nu} \\ &\qquad + \chi|_{U_\varepsilon}\mathrm{TrFrob}M^{x,\varepsilon}_{z,n,\nu}. \end{aligned} \tag{3–18}$$

The virtual motives $M^{x,0}_{z,n,\nu}$, $M^{x,1}_{z,n,\nu}$, and $M^{x,\varepsilon}_{z,n,\nu}$ are explicitly given in Tables 6 and 7 in the case $x = (0)$, and in Tables 8 and 9 in the case $x = (1)$.

Proof: We will show the details of the proof of this proposition in the case $x = (0)$ only. The results of the similar calculations in the case $x = (1)$ are summarized in the tables.

Let us apply Proposition 3.1 to the test functions $f_{z,n,\nu}$ (so that the set H in that Proposition is ${}^G\Gamma_{z,n,\nu}$). Note that in G, for all z but s_0, $\mathfrak{h}_z$ is elliptic, i.e., it is a compact Cartan subalgebra. We consider the elliptic cases first. As we will explicitly see below, the sum over λ that appears in the expression (3–2) has only finitely many terms in these cases, so we can permute the two sums, and obtain

$$\Theta_\pi(f_{z,n,\nu}) = \sum_U \chi(U) \sum_\lambda \int_{{}^G\Gamma_{z,n,\nu}} N^x_{U,\lambda}(h)\, dh, \tag{3–19}$$

where the index U runs over U_0, U_1, and U_ε. That is, the character is already expressed as a linear combination of the values $\chi|_{U_0}$, $\chi|_{U_1}$, and $\chi|_{U_\varepsilon}$. All we need to do is “evaluate” the coefficients $\sum_\lambda \int_{{}^G\Gamma_{z,n,\nu}} N^x_{U,\lambda}(h)$. We

z	$M^{(0),0}_{z,n,\nu}$	$M^{(0),1}_{z,n,\nu}$	$M^{(0),\varepsilon}_{z,n,\nu}$
s_0	$\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}$	$\mathbb{L}^n$	$\mathbb{L}^n$
s_1	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$, n odd $\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}$, n even	0, n odd $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}$, n even	0, n odd $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}$, n even
s_2	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$, n odd $\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}$, n even	0, n odd $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}$, n even	0, n odd $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}$, n even

TABLE 6. Virtual motives for the characters of $\pi(0,\theta)$, $\pi(0,+)$, and $\pi(0,-)$ at $Y_{z,n}(u)$ for $z \in \{s_0, s_1, s_2\}$.

z	$M^{(0),0}_{z,n,\nu}$	$M^{(0),1}_{z,n,\nu}$	$M^{(0),\varepsilon}_{z,n,\nu}$
t_0	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n$, $\zeta^{2n}=\nu$ 0, otherwise	0, $\zeta^{2n}=\nu$ $\mathbb{L}^n$, otherwise
t_1	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n$, $\zeta^{2n}=-\nu$ 0, otherwise	0, $\zeta^{2n}=-\nu$ $\mathbb{L}^n$, otherwise
t_2	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n$, $\zeta^{2n}=(-1)^n\nu$ 0, otherwise	0, $\zeta^{2n}=(-1)^n\nu$ $\mathbb{L}^n$, otherwise
t_3	$\frac{\mathbb{L}^n-1}{\mathbb{L}-1}$	$\mathbb{L}^n$, $\zeta^{2n}=(-1)^{n+1}\nu$ 0, otherwise	0, $\zeta^{2n}=(-1)^{n+1}\nu$ $\mathbb{L}^n$, otherwise

TABLE 7. Virtual motives for the characters of $\pi(0,\theta)$, $\pi(0,+)$, and $\pi(0,-)$ at $Y_{z,n}(u)$ for $z \in \{t_0, t_1, t_2, t_3\}$. (Recall that $\zeta^2 = \mathrm{sgn}(-1)$.)

recall Lemma 3.2, which relates the numbers $N^x_{U,\lambda}(h)$ to the volumes of the sets $W^x_{U,\lambda}(h)$. Then we evaluate their *motivic* volumes for h in each of the sets ${}^G\Gamma_{z,n,\nu}$ and sum them over λ with coefficients that come from Lemma 3.2.

We start with $z = s_1$. Let h be an element of $\Gamma_{s_1,n,\pm}$, and let λ be a nonnegative integer. By the comparison theorem, the equality (3–7) of Lemma 3.2 can be written in a "motivic" form: for all primes $p \neq 2$ (recall that q is a power of p), we have $N^x_{U,\lambda}(h) = \mathrm{TrFrob}_q M^x_{U,\lambda}$, where

$$M^x_{U,\lambda} = \left[(q+1)q^{2\lambda-1}\right]\frac{\mu(W^x_{U,\lambda})(h)}{\mu(G_x)} = (\mathbb{L}+1)\mathbb{L}^{2\lambda-1}\frac{\mu(W^x_{U,\lambda})(h)}{\mu(G_x)}.$$

For now, let $x = (0)$. Note that a priori, the right-hand side depends on the element h. However, by Lemma 3.3, for all $h \in \Gamma_{s_1,n,\pm}$, and for every U, the motivic volume of $W^x_{U,\lambda}(h)$ does not depend on h and equals

$$\begin{cases} \frac{1}{2}\mathbb{L}(\mathbb{L}^2-1), & \text{if } \lambda = n/2,\ n \text{ even},\ U = U_1 \text{ or } U_\varepsilon, \\ \mathbb{L}(\mathbb{L}^2-1), & \text{if } \lambda < n/2,\ U = U_0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us define the virtual Chow motive $M^{(0),0}_{s_1,n,\pm}$ that corresponds to the conjugacy class U_0 by the formulas

$$M^{(0),0}_{s_1,n,\pm} = \begin{cases} 1 + (\mathbb{L}+1)\sum_{\lambda=1}^{n/2-1}\mathbb{L}^{2\lambda-1} = \frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}, & n \text{ even}, \\ 1 + (\mathbb{L}+1)\sum_{\lambda=1}^{(n-1)/2}\mathbb{L}^{2\lambda-1} = \frac{\mathbb{L}^{n}-1}{\mathbb{L}-1}, & n \text{ odd}. \end{cases} \tag{3–20}$$

Also let

$$M^{(0),1}_{s_1,n,\pm} = M^{(0),\varepsilon}_{s_1,n,\pm} = \begin{cases} \frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{3–21}$$

z	$M^{(1),0}_{z,n,\nu}$	$M^{(1),1}_{z,n,\nu}$	$M^{(1),\varepsilon}_{z,n,\nu}$
s_0	$\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}$	$\mathbb{L}^n$	$\mathbb{L}^n$
s_1	$\frac{\mathbb{L}^{n}-1}{\mathbb{L}-1}$, n even $\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}$, n odd	0, n even $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}$, n odd	0, n even $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}$, n odd
s_2	$\frac{\mathbb{L}^{n}-1}{\mathbb{L}-1}$, n even $\frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}$, n odd	0, n even $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}$, n odd	0, n even $\frac{1}{2}(\mathbb{L}+1)\mathbb{L}^{n-1}$, n odd

TABLE 8. Virtual motives for the characters of $\pi(1,\theta)$, $\pi(1,+)$ and $\pi(1,-)$ at $Y_{z,n}(u)$ for $z \in \{s_0, s_1, s_2\}$.

z	$M^{(1),0}_{z,n,\nu}$	$M^{(1),1}_{z,n,\nu}$	$M^{(1),\varepsilon}_{z,n,\nu}$
t_0	$\frac{\mathbb{L}^{n}-1}{\mathbb{L}-1}$	$\mathbb{L}^n$, $\zeta^{2n+2}=\nu$ 0, otherwise	0, $\zeta^{2n+2}=\nu$ $\mathbb{L}^n$, otherwise
t_1	$\frac{\mathbb{L}^{n}-1}{\mathbb{L}-1}$	$\mathbb{L}^n$, $\zeta^{2n}=-\nu$ 0, otherwise	0, $\zeta^{2n}=-\nu$ $\mathbb{L}^n$, otherwise
t_2	$\frac{\mathbb{L}^{n}-1}{\mathbb{L}-1}$	$\mathbb{L}^n$, $\zeta^{2n+2}=(-1)^{n+1}\nu$ 0, otherwise	0, $\zeta^{2n+2}=(-1)^{n+1}\nu$ $\mathbb{L}^n$, otherwise
t_3	$\frac{\mathbb{L}^{n}-1}{\mathbb{L}-1}$	$\mathbb{L}^n$, $\zeta^{2n+2}=(-1)^{n}\nu$ 0, otherwise	0, $\zeta^{2n+2}=(-1)^{n}\nu$ $\mathbb{L}^n$, otherwise

TABLE 9. Virtual motives for the characters of $\pi(1,\theta)$, $\pi(1,+)$, and $\pi(1,-)$ at $Y_{z,n}(u)$ for $z \in \{t_0, t_1, t_2, t_3\}$.

Then, combining the equations above, we obtain, for $p \neq 2$,

$$\begin{aligned}\frac{\Theta_\pi(f_{s_1,n,\nu})}{m({}^G\Gamma_{s_1,n,\nu})} &= \Theta_\pi(h)\\ &= \chi|_{U_0}\,\mathrm{TrFrob}\,M^{(0),0}_{s_1,n,\nu} + \chi|_{U_1}\,\mathrm{TrFrob} M^{(0),1}_{s_1,n,\nu}\\ &\quad + \chi|_{U_\varepsilon}\,\mathrm{TrFrob} M^{(0),\varepsilon}_{s_1,n,\nu},\end{aligned} \tag{3–22}$$

where we write Θ_π both for the distribution character and for the locally integrable function on the regular set that represents it. Note that there is no difference in the formulas for $\Theta_\pi(f_{s_1,n,+})$ and $\Theta_\pi(f_{s_1,n,-})$, because $M^{(0),1}_{s_1,n,\pm}$ in any case coincides with $M^{(0),\varepsilon}_{s_1,n,\pm}$. The proposition in the case $z = s_1$ is proved.

In the case $z = s_2$, the calculation is very similar to the case $z = s_1$, except that when n is odd, the expressions for $M^{(0),1}_{s_2,n,\nu}$ and $M^{(0),\varepsilon}_{s_2,n,\nu}$ contain one or two terms depending on ν, yet the final answer is the same in both cases. Using equation (3–14), we get

$$\begin{aligned}M^{(0),1}_{s_2,n,+} &= M^{(0),\varepsilon}_{s_2,n,-}\\ &= \frac{1}{2}\frac{(\mathbb{L}-1)^2}{\mathbb{L}(\mathbb{L}^2-1)}\mathbb{L}^{2\frac{n+1}{2}-1}(\mathbb{L}+1)\\ &\quad + \frac{\mathbb{L}}{\mathbb{L}+1}\mathbb{L}^{2\frac{n-1}{2}-1}(\mathbb{L}+1)\\ &= \frac{1}{2}\mathbb{L}^{n-1}(\mathbb{L}+1),\\ M^{(0),1}_{s_2,n,-} &= M^{(0),\varepsilon}_{s_2,n,+} = \frac{1}{2}\frac{(\mathbb{L}^2-1)}{\mathbb{L}(\mathbb{L}^2-1)}\mathbb{L}^{2\frac{n+1}{2}-1}(\mathbb{L}+1)\\ &= \frac{1}{2}\mathbb{L}^{n-1}(\mathbb{L}+1).\end{aligned}$$

The values of $M^{(0),0}_{s_2,n,\nu}$ are computed similarly to the case $z = s_1$.

Let us now prove the proposition for the ramified elements. All ramified cases are very similar to each other. We show the details for the case $z = t_2$. The argument is exactly the same as in the case $z = s_1$, but we have to use Lemma 3.4 instead of Lemma 3.3. In the case n even, we get

$$\begin{aligned} M^{(0),0}_{t_2,n,\pm} &= 1 + \sum_{\lambda=1}^{n/2-1} \mathbb{L}^{2\lambda-1} + \frac{\mu(G_{(01)})}{\mu(G_{(0)})}\mathbb{L}^{2\frac{n}{2}-1}(\mathbb{L}+1) \\ &= \frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1} + \frac{1}{\mathbb{L}+1}\mathbb{L}^{2\frac{n}{2}-1}(\mathbb{L}+1) = \frac{\mathbb{L}^n-1}{\mathbb{L}-1}; \\ M^{(0),1}_{t_2,n,+} &= M^{(0),\varepsilon}_{t_2,n,-} = \frac{\mu(G_{(0)})-\mu(G_{(01)})}{\mu(G_{(0)})}\mathbb{L}^{2\frac{n}{2}-1}(\mathbb{L}+1) \\ &= \frac{\mathbb{L}}{\mathbb{L}+1}\mathbb{L}^{2\frac{n}{2}-1}(\mathbb{L}+1) = \mathbb{L}^n; \\ M^{(0),1}_{t_2,n,-} &= M^{(0),\varepsilon}_{t_2,n,+} = 0. \end{aligned} \tag{3–23}$$

If n is odd, a similar calculation yields

$$\begin{aligned} M^{(0),0}_{t_2,n,\pm} &= 1 + (\mathbb{L}+1)\sum_{\lambda=1}^{\frac{n-1}{2}} \mathbb{L}^{2\lambda-1}(\mathbb{L}+1) \\ &= \frac{\mathbb{L}^n-1}{\mathbb{L}-1}, \\ M^{(0),1}_{t_2,n,\mathrm{sgn}(-1)} &= M^{(0),\varepsilon}_{t_2,n,-\mathrm{sgn}(-1)} = \frac{\mathbb{L}}{\mathbb{L}+1}\mathbb{L}^{2\frac{n+1}{2}-1}(\mathbb{L}+1) \\ &= \mathbb{L}^n, \\ M^{(0),1}_{t_2,n,-\mathrm{sgn}(-1)} &= M^{(0),\varepsilon}_{t_2,n,\mathrm{sgn}(-1)} = 0. \end{aligned} \tag{3–24}$$

The Proposition for $z = t_2$ follows.

Finally, let us address the case $s = s_0$. The elements of the set $\Gamma_{s_0,n}$ have the form $h = \left[\begin{smallmatrix} 1+u\varpi^n & 0 \\ 0 & (1+u\varpi^n)^{-1} \end{smallmatrix}\right]$, where u is a unit. Following the pattern of the previous section, we take $y = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G_{(0)}$, and write down the conditions ensuring that the element $a_\lambda^{-1}y^{-1}hya_\lambda$ belongs to $G_{(0)}$ and projects to a given conjugacy class under the map $\rho_{(0),0}$. As before, we see that this depends on the entry in the right-hand corner of the matrix $a_\lambda^{-1}y^{-1}hya_\lambda$, which equals $\varpi^{-2\lambda}bd((1+u\varpi^n)-(1+u\varpi^n)^{-1})$. We have (when $p \neq 2$)

$$\begin{aligned} &\mathrm{ord}(\varpi^{-2\lambda}bd((1+u\varpi^n)-(1+u\varpi^n)^{-1})) \\ &\quad = -2\lambda + n + \mathrm{ord}(bd), \\ &\mathrm{ac}(\varpi^{-2\lambda}bd((1+u\varpi^n)-(1+u\varpi^n)^{-1})) \\ &\quad = 2\mathrm{ac}(bd)\mathrm{ac}(u). \end{aligned} \tag{3–25}$$

Here the situation is quite different from the elliptic cases, because the valuation of bd can be arbitrarily large, and therefore there are infinitely many values of λ such that $a_\lambda^{-1}y^{-1}hya_\lambda$ belongs to $G_{(0)}$. We know, of course, that the sum in equation (3–2) has to be finite anyway. Here we will see explicitly that it happens because both for $\chi = Q_T$ and $\chi = Q_G$ (and therefore, for any linear combination of these two functions as well), for large values of λ the sum of three terms $\sum_U \chi|_U N^{(0)}_{U,\lambda}(h)$ vanishes. Indeed, if $\chi = Q_G$, it is easy to see that this sum is always zero, because Q_G vanishes on the class U_0 and takes opposite values on U_1 and U_ε. Suppose $\chi = Q_T$. Then for each positive integer k, we will need the volumes of the subsets of $G_{(0)}$ defined by the formulas $\{\mathrm{ord}(b) \geq k\}$ and $\{\mathrm{ord}(d) \geq k\}$. Note that these sets are disjoint and have equal volumes. From the point of view of motivic integration, it is easy to see that these sets are stable at level k in the language of [Denef and Loeser 01], and their motivic volumes equal $(\mathbb{L}-1)\mathbb{L}^{-(k-2)}$. The only varieties appearing in this calculation are affine spaces, so we acquire no bad primes. Now it is easy to see that when $2\lambda > n$, the value $\chi|_{U_0}$ appears with the coefficient $(\mathbb{L}-1)\mathbb{L}^{-(k-2)}$, and the values $\chi|_{U_1}$, $\chi|_{U_\varepsilon}$ each appear with the coefficient $\frac{1}{2}(\mathbb{L}-1)^2\mathbb{L}^{-(k-1)}$, which leads to the cancellation in the case $\chi = Q_T$. Finally, we are again in a situation similar to all the previous cases, in which we need to sum only over all λ not exceeding $n/2$. We omit the details of getting the answers that appear in the first rows of Tables 6 and 8. This ends the proof of Proposition 3.5. □

4. MOTIVES FOR THE FOURIER TRANSFORMS OF OUR ORBITAL INTEGRALS

In this section we use the notation of [Cunningham and Hales 04].

4.1 The Fourier Transform of Good Orbital Integrals

For any rational number s, let $\mathfrak{g}_s$ denote the union of the Moy–Prasad lattices $\mathfrak{g}_{x,s}$ as x ranges over all points in the extended Bruhat–Tits building $I(G,\mathbb{K})$ for G (see [Moy and Prasad 94] for the definition of $\mathfrak{g}_{x,s}$). Let $\mathcal{H}(\mathfrak{g})$ denote the Hecke algebra of locally constant, compactly supported functions $f : \mathfrak{g} \to \bar{\mathbb{Q}}_\ell$. As in [Cunningham and Hales 04, Section 1.3], for any pair of rational numbers $s \leq r$, we write $\mathcal{H}(\mathfrak{g})^s_r$ for the $\bar{\mathbb{Q}}_\ell$-vector space of elements of $\mathcal{H}(\mathfrak{g})$ such that f is supported by $\mathfrak{g}_s$ and $\hat{f}$ is supported by $\mathfrak{g}_{-r}$. (Recall that the Fourier transform is taken with respect to an additive character of $\mathbb{K}$ with conductor $\mathcal{O}_\mathbb{K}$.) Then the Fourier transform defines an isomorphism of $\bar{\mathbb{Q}}_\ell$-vector spaces

$$\begin{aligned} \mathcal{H}(\mathfrak{g})^s_r &\to \mathcal{H}(\mathfrak{g})^{-r}_{-s}, \\ f &\mapsto \hat{f}. \end{aligned}$$

Again following [Cunningham and Hales 04, Section 1.3], we write $\mathcal{H}(\mathfrak{g})^s$ for the union of the spaces $\mathcal{H}(\mathfrak{g})^s_r$ with

$s \leq r$, and $\mathcal{H}(\mathfrak{g})_r$ for the union of the spaces $\mathcal{H}(\mathfrak{g})_r^s$ with $s \leq r$.

For any $\varphi : \bar{\mathfrak{g}}_{x,r} \to \bar{\mathbb{Q}}_\ell$, we write $\varphi_{x,r}$ for the element of $\mathcal{H}(\mathfrak{g})_r^r$ such that $\varphi_{x,r}(Y) = (\varphi \circ \rho_{x,r})(Y)$ if $Y \in \mathfrak{g}_{x,r}$ and $\varphi_{x,r}(Y) = 0$ otherwise. As explained in [Cunningham and Hales 04, Section 1], on the level of reductive quotients we have another Fourier transform taking functions on $\bar{\mathfrak{g}}_{x,r}$ to functions on $\bar{\mathfrak{g}}_{x,-r}$. With respect to these definitions we have

$$\widehat{\varphi_{x,r}} = \mathrm{vol}(\mathfrak{g}_{x,r})\hat{\varphi}_{x,-r}, \tag{4–1}$$

where vol refers to the measure on $\mathfrak{g}$. For elaboration and proofs, the reader is referred to [Cunningham and Hales 04, Section 1].

Before stating the next proposition, we remind the reader that if X is regular elliptic, then there is a unique point x in the Bruhat–Tits building for G corresponding to the centralizer X in G, since G has compact center. Moreover, in this case, the depth of X in $\mathfrak{g}$ is the unique real number r (rational number, actually) such that $X \in \mathfrak{g}_{x,r}$ and $X \notin \mathfrak{g}_{x,r^+}$.

Proposition 4.1. *Suppose X is a regular elliptic good element of $\mathfrak{g}$. Let x be the point in the Bruhat–Tits building for G corresponding to the centralizer of X in G and let r be the depth of X. Let $\bar{X}$ denote the image of X under $\rho_{x,r} : \mathfrak{g}_{x,r} \to \bar{\mathfrak{g}}_{x,r}$ and let $\varphi : \bar{\mathfrak{g}}_{x,r} \to \mathbb{Q}$ denote the characteristic function of the $\bar{G}_x$-orbit of $\bar{X} \in \bar{\mathfrak{g}}_{x,r}$ divided by the cardinality of that orbit. If $f \in \mathcal{H}(\mathfrak{g})^{-r}$ then*

$$\hat{\mu}_X(f) = \int_G \int_{\mathfrak{g}} f(\mathrm{Ad}(g)Y)\hat{\varphi}_{x,-r}(Y)\, dY\, dg. \tag{4–2}$$

Proof: Suppose $f \in \mathcal{H}(\mathfrak{g})^{-r}$. Then $f \in \mathcal{H}(\mathfrak{g})_{-r}^{-s}$ for some $-r \leq -s$. Thus, $\hat{f} \in \mathcal{H}(\mathfrak{g})_r^s$, so $\hat{f} \in \mathcal{H}(\mathfrak{g})_r$. Now, by [Cunningham and Hales 04, Proposition 1.22] and elementary properties of the Fourier transform,

$$\begin{aligned}
&\hat{\mu}_X(f)\\
&= \mu_X(\hat{f})\\
&= \mathrm{vol}(\mathfrak{g}_{x,r})^{-1} \int_G \int_{\mathfrak{g}} \hat{f}(\mathrm{Ad}(g)Y)\varphi_{x,r}(Y)\, dY\, dg \qquad (4\text{–}3)\\
&= \mathrm{vol}(\mathfrak{g}_{x,r})^{-1} \int_G \int_{\mathfrak{g}} f(\mathrm{Ad}(g)Y)\widehat{\varphi_{x,r}}(Y)\, dY\, dg.
\end{aligned}$$

By [Cunningham and Hales 04, Proposition 1.13],

$$\widehat{\varphi_{x,r}}(Y) = \mathrm{vol}(\mathfrak{g}_{x,r})\hat{\varphi}_{x,-r}(Y),$$

so

$$\hat{\mu}_X(f) = \int_G \int_{\mathfrak{g}} f(\mathrm{Ad}(g)Y)\hat{\varphi}_{x,-r}(Y)\, dY\, dg, \tag{4–4}$$

as claimed. □

Remark 4.2. We will sometimes write $\varphi_{\bar{X}}$ (respectively $\hat{\varphi}_{\bar{X}}$) for the function φ (respectively $\hat{\varphi}$) appearing in Proposition 4.1 above; in that case, $\varphi_{x,r}$ becomes $(\varphi_{\bar{X}})_{x,r}$ and $\hat{\varphi}_{x,-r}$ becomes $(\hat{\varphi}_{\bar{X}})_{x,-r}$.

4.2 Application to Our Orbital Integrals

In order to apply [Cunningham and Hales 04, Proposition 1.22] to the Lie algebra $\mathfrak{g}$ and the orbital integrals appearing in Theorem 2.5, we must find the function $(\hat{\varphi}_{\bar{X}_z})_{x_z,-r_z}$ for each X_z appearing in Theorem 2.5, where x_z is the point in the Bruhat–Tits building for X_z and r_z is the depth of X_z in $\mathfrak{g}$.

4.2.1 Case: $z = s_1$. Recall (from Section 2.4) that

$$X_{s_1}(v) := \begin{bmatrix} 0 & v \\ \varepsilon v & 0 \end{bmatrix}.$$

Here we will assume that v is a unit. The point x_{s_1} is the standard vertex of the Bruhat–Tits building for G and the depth r_{s_1} of $X_{s_1}(v)$ is 0; in other words, $x_{s_1} = (0)$ and $r_{s_1} = 0$. Thus,

$$\mathfrak{g}_{x_{s_1},r_{s_1}} = \mathfrak{g}_{(0),0} = \left\{ \begin{bmatrix} z & x \\ y & -z \end{bmatrix} \mid x, y, z \in \mathcal{O}_{\mathbb{K}} \right\}.$$

The reduction map $\rho_{x_{s_1},r_{s_1}}$ is given by

$$\begin{bmatrix} z & x \\ y & -z \end{bmatrix} \mapsto \begin{bmatrix} \bar{z} & \bar{x} \\ \bar{y} & -\bar{z} \end{bmatrix},$$

where $\bar{x}$, $\bar{y}$, and $\bar{z}$ denote the images of x, y, and z respectively under $\mathcal{O}_{\mathbb{K}} \to \mathbb{F}_q$. Let $\bar{X}_{s_1}(v)$ denote the image of $X_{s_1}(v)$ under $\rho_{x_{s_1},r_{s_1}}$. The $\bar{G}_{x_{s_1}}(\mathbb{F}_q)$-orbit of $\bar{X}_{s_1}(v)$ in $\bar{\mathfrak{g}}_{x_{s_1},r_{s_1}}$ is

$$\left\{ \begin{bmatrix} z & x \\ y & -z \end{bmatrix} \in \mathfrak{sl}(2,\mathbb{F}_q) \middle| xy + z^2 = \bar{\varepsilon}\bar{v}^2 \right\},$$

which has cardinality $q(q-1)$. Thus, $\varphi_{\bar{Y}_{s_1}(v)} : \bar{\mathfrak{g}}_{x_{s_1},r_{s_1}} \to \bar{\mathbb{Q}}_\ell$ is given by

$$\varphi_{\bar{X}_{s_1}(v)}\left(\begin{bmatrix} z & x \\ y & -z \end{bmatrix} \right) = \begin{cases} \frac{1}{q(q-1)} & xy + z^2 = \bar{\varepsilon}\bar{v}^2, \\ 0 & \text{otherwise.} \end{cases} \tag{4–5}$$

In order to find the (relative) Fourier transform of this function (in the sense of [Cunningham and Hales 04]) we observe that the Killing form $\langle X,Y\rangle := \mathrm{trace}\,(XY)$ gives a pairing between lattices

$$\mathfrak{g}_{x_{s_1},r_{s_1}} \times \mathfrak{g}_{x_{s_1},-r_{s_1}} \to \mathcal{O}_{\mathbb{K}},$$
$$\left(\begin{bmatrix} z & x \\ y & -z \end{bmatrix}, \begin{bmatrix} c & a \\ b & -c \end{bmatrix}\right) \mapsto xb + ya + 2zc,$$

which in turn gives a bilinear form $\bar{\mathfrak{g}}_{x_{s_1},r_{s_1}} \times \bar{\mathfrak{g}}_{x_{s_1},-r_{s_1}} \to \mathbb{F}_q$. The relative Fourier transform is taken with respect to this form. The image of the set of topologically nilpotent elements in $\mathfrak{g}_{x_{s_1},-r_{s_1}}$ under $\rho_{x_{s_1},-r_{s_1}}$ is the cone

$$\left\{ \begin{bmatrix} c & a \\ b & -c \end{bmatrix} \in \mathfrak{sl}(2,\mathbb{F}_q) \middle| ab + c^2 = 0 \right\}.$$

In Section 4.4 we will see that if $ab + c^2 = 0$, then

$$Q_T\left(\begin{bmatrix} c & a \\ b & -c \end{bmatrix}\right) = (1-q)\hat{\varphi}_{\bar{X}_{s_1}(v)}\left(\begin{bmatrix} c & a \\ b & -c \end{bmatrix}\right), \qquad (4\text{–}6)$$

where Q_T is given in equation (2–6). This completes our description of the relevant properties of $\hat{\varphi}_{\bar{X}_{s_1}(v)}$.

4.2.2 Case: $z = s_2$. Since this case is very similar to the case above, we only summarize the results here. Recall (from Section 2.4) that

$$X_{s_2}(v) := \begin{bmatrix} 0 & v\varpi \\ \varepsilon v \varpi^{-1} & 0 \end{bmatrix},$$

where v, as above, is assumed to be a unit. The point x_{s_2} is the vertex (1) of the Bruhat–Tits building for G (see Section 2.3), and the depth r_{s_2} of $X_{s_2}(v)$ is 0. Thus,

$$\mathfrak{g}_{x_{s_2},r_{s_2}} = \mathfrak{g}_{(1),0} = \left\{ \begin{bmatrix} z & x\varpi \\ y\varpi^{-1} & -z \end{bmatrix} \mid x,y,z \in \mathcal{O}_{\mathbb{K}} \right\}.$$

The reduction map $\rho_{x_{s_2},r_{s_2}}$ is given by

$$\begin{bmatrix} z & x\varpi \\ y\varpi^{-1} & -z \end{bmatrix} \mapsto \begin{bmatrix} \bar{z} & \bar{x} \\ \bar{y} & -\bar{z} \end{bmatrix}.$$

The function $\varphi_{\bar{X}_{s_2}(v)} : \bar{\mathfrak{g}}_{x_{s_2},r_{s_2}} \to \bar{\mathbb{Q}}_\ell$ is exactly as in the preceding case, so

$$Q_T\left(\begin{bmatrix} c & a \\ b & -c \end{bmatrix}\right) = (1-q)\hat{\varphi}_{\bar{X}_{s_2}(v)}\left(\begin{bmatrix} c & a \\ b & -c \end{bmatrix}\right), \qquad (4\text{–}7)$$

as above. This completes our description of the relevant properties of $\hat{\varphi}_{\bar{X}_{s_2}(v)}$.

4.2.3 Case: $z \in \{t_0, t_1, t_2, t_3\}$. Then the point x_z is (01), and the depth r_z is $\frac{1}{2}$. Thus,

$$\mathfrak{g}_{x_z,r_z} = \mathfrak{g}_{(01),\frac{1}{2}} = \left\{ \begin{bmatrix} z\varpi & x \\ y\varpi & -z\varpi \end{bmatrix} \mid x,y,z \in \mathcal{O}_{\mathbb{K}} \right\}.$$

Thus, $\bar{\mathfrak{g}}_{x_z,r_z} = \mathbb{A}^2(\mathbb{F}_q)$, and the reduction map $\rho_{x_z,r_z} : \mathfrak{g}_{x_z,r_z} \to \mathbb{A}^2(\mathbb{F}_q)$ is given by

$$\begin{bmatrix} z\varpi & x \\ y\varpi & -z\varpi \end{bmatrix} \quad\mapsto\quad (\bar{x},\bar{y}).$$

The reduction map on $G_{x_z} \to \mathrm{GL}(1,\mathbb{F}_q)$ is given by

$$\begin{bmatrix} a & b \\ \varpi c & d \end{bmatrix} \quad\mapsto\quad \bar{a}.$$

Thus, the action of $\bar{G}_{x_z}$ on $\bar{\mathfrak{g}}_{x_z,r_z}$ corresponds to the action of $\mathrm{GL}(1,\mathbb{F}_q)$ on $\mathbb{A}^2(\mathbb{F}_q)$ given by $t \cdot (x,y) := (t^2 x, t^{-2} y)$. It follows immediately from the definitions above that

$$\varphi_{\bar{X}_{t_0}(v)}(x,y) = \begin{cases} \frac{2}{q-1} & xy = \bar{v}^2 \wedge \mathrm{sgn}(x) = \mathrm{sgn}(v), \\ 0 & \text{otherwise}, \end{cases}$$
$$\varphi_{\bar{X}_{t_1}(v)}(x,y) = \begin{cases} \frac{2}{q-1} & xy = \bar{v}^2 \wedge \mathrm{sgn}(x) = \mathrm{sgn}(\varepsilon v), \\ 0 & \text{otherwise}, \end{cases}$$
$$\varphi_{\bar{X}_{t_2}(v)}(x,y) = \begin{cases} \frac{2}{q-1} & xy = \bar{\varepsilon}\bar{v}^2 \wedge \mathrm{sgn}(x) = \mathrm{sgn}(v), \\ 0 & \text{otherwise}, \end{cases}$$
$$\varphi_{\bar{X}_{t_3}(v)}(x,y) = \begin{cases} \frac{2}{q-1} & xy = \bar{\varepsilon}\bar{v}^2 \wedge \mathrm{sgn}(x) = \mathrm{sgn}(\varepsilon v), \\ 0 & \text{otherwise}. \end{cases}$$

4.3 Two Functions on $\mathbb{A}^2(\mathbb{F}_q)$

In this section we introduce two functions that play a crucial role in our proof of Theorem 2.5. Using notation from the preceding subsection, define

$$\begin{aligned} \varphi_{(0)} &:= \left(\varphi_{\bar{X}_{t_0}} - \varphi_{\bar{X}_{t_1}}\right) + \left(\varphi_{\bar{X}_{t_2}} - \varphi_{\bar{X}_{t_3}}\right), \\ \varphi_{(1)} &:= \left(\varphi_{\bar{X}_{t_0}} - \varphi_{\bar{X}_{t_1}}\right) - \left(\varphi_{\bar{X}_{t_2}} - \varphi_{\bar{X}_{t_3}}\right). \end{aligned} \qquad (4\text{–}8)$$

To find the Fourier transform of these functions (in the sense of [Cunningham and Hales 04]), observe that the Killing form $\langle X,Y\rangle := \mathrm{trace}\,(XY)$ gives a pairing between lattices

$$\begin{aligned} \mathfrak{g}_{x_z,r_z} \times \mathfrak{g}_{x_z,-r_z} &\to \mathcal{O}_{\mathbb{K}}, \\ \left(\begin{bmatrix} z\varpi & x \\ y\varpi & -z\varpi \end{bmatrix}, \begin{bmatrix} c & a\varpi^{-1} \\ b & -c \end{bmatrix}\right) &\mapsto xb + ya + 2\varpi zc, \end{aligned}$$

which in turn gives the bilinear form

$$\begin{aligned} \bar{\mathfrak{g}}_{x_z,r_z} \times \bar{\mathfrak{g}}_{x_z,-r_z} &\to \mathbb{F}_q, \\ ((x,y),(a,b)) &\mapsto xb + ya. \end{aligned}$$

The relative Fourier transform is taken with respect to this form. The image of the set of topologically nilpotent elements in $\mathfrak{g}_{x_z,-r_z}$ under $\rho_{x_z,-r_z}$ is the normal crossing

$$\left\{(a,b) \in \mathbb{A}^2(\mathbb{F}_q) \mid ab = 0\right\}.$$

In Section 4.4 we will see that

$$\begin{aligned}\hat{\varphi}_{(0)}(a,0) &= 0,\\ \hat{\varphi}_{(0)}(0,b) &= \frac{2^2}{q-1}\sqrt{q}\zeta^3\mathrm{sgn}(b),\\ \hat{\varphi}_{(1)}(a,0) &= \frac{2^2}{q-1}\sqrt{q}\zeta\mathrm{sgn}(a),\\ \hat{\varphi}_{(1)}(0,b) &= 0.\end{aligned} \tag{4–9}$$

4.4 Some Finite-Field Calculations

In this section we defend equations (4–6), (4–7), and (4–9).

Recall the definition of the function $\gamma_\pm : \mathbb{A}^2(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$ from equation (2–2). Recall also that we equip $\mathbb{A}^2(\mathbb{F}_q)$ with the bilinear form $\langle (x,y),(a,b)\rangle = xb + ya$, as explained in Section 4.2. Finally, recall the definition of $\varphi_{\bar{X}_z(v)}$ for $z \in \{t_0,t_1,t_2,t_3\}$. Then

$$\begin{aligned}\hat{\varphi}_{\bar{X}_{t_0}(v)}(a,b) &= \sum_{(x,y)\in\mathbb{A}^2(\mathbb{F}_q)} \bar{\psi}\langle (x,y),(a,b)\rangle\ \varphi_{\bar{X}_{t_0}(v)}(x,y),\\ &= \frac{2}{q-1}\sum_{\substack{xy=\bar{v}^2\\ \mathrm{sgn}(x)=\mathrm{sgn}(v)}} \bar{\psi}(xb)\bar{\psi}(ya),\\ &= \frac{2}{q-1}\sum_{\mathrm{sgn}(x)=\mathrm{sgn}(v)} \bar{\psi}(xb)\bar{\psi}(\bar{v}^2x^{-1}a).\end{aligned}$$

Thus,

$$\begin{aligned}\hat{\varphi}_{\bar{X}_{t_0}(v)}(a,0) &= \frac{2}{q-1}\sum_{\mathrm{sgn}(x)=\mathrm{sgn}(v)} \bar{\psi}(0)\bar{\psi}(\bar{v}^2x^{-1}a),\\ &= \frac{2}{q-1}\sum_{\mathrm{sgn}(t)=\mathrm{sgn}(v)} \bar{\psi}(ta),\\ &= \frac{2}{q-1}\gamma_{\mathrm{sgn}(v)}(a),\end{aligned}$$

and

$$\begin{aligned}\hat{\varphi}_{\bar{X}_{t_0}(v)}(0,b) &= \frac{2}{q-1}\sum_{\mathrm{sgn}(x)=\mathrm{sgn}(v)} \bar{\psi}(xb)\bar{\psi}(0),\\ &= \frac{2}{q-1}\gamma_{\mathrm{sgn}(v)}(b).\end{aligned}$$

Similar arguments show that

$$\begin{aligned}\hat{\varphi}_{\bar{X}_{t_1}(v)}(a,0) &= \frac{2}{q-1}\gamma_{\mathrm{sgn}(\varepsilon v)}(a),\\ \hat{\varphi}_{\bar{X}_{t_1}(v)}(0,b) &= \frac{2}{q-1}\gamma_{\mathrm{sgn}(\varepsilon v)}(b),\end{aligned}$$

and

$$\begin{aligned}\hat{\varphi}_{\bar{X}_{t_2}(v)}(a,0) &= \frac{2}{q-1}\gamma_{\mathrm{sgn}(\varepsilon v)}(a),\\ \hat{\varphi}_{\bar{X}_{t_2}(v)}(0,b) &= \frac{2}{q-1}\gamma_{\mathrm{sgn}(v)}(b),\end{aligned}$$

and

$$\begin{aligned}\hat{\varphi}_{\bar{X}_{t_3}(v)}(a,0) &= \frac{2}{q-1}\gamma_{\mathrm{sgn}(v)}(a),\\ \hat{\varphi}_{\bar{X}_{t_3}(v)}(0,b) &= \frac{2}{q-1}\gamma_{\mathrm{sgn}(\varepsilon v)}(b).\end{aligned}$$

Thus,

$$\begin{aligned}&\left(\hat{\varphi}_{\bar{X}_{t_0}(v)} - \hat{\varphi}_{\bar{X}_{t_1}(v)} + \hat{\varphi}_{\bar{X}_{t_2}(v)} - \hat{\varphi}_{\bar{X}_{t_3}(v)}\right)(a,0)\\ &\quad= \frac{2}{q-1}\left(\gamma_{\mathrm{sgn}(v)} - \gamma_{\mathrm{sgn}(\varepsilon v)} + \gamma_{\mathrm{sgn}(\varepsilon v)} - \gamma_{\mathrm{sgn}(v)}\right)(a)\\ &\quad= 0,\end{aligned}$$

and

$$\begin{aligned}&\left(\hat{\varphi}_{\bar{X}_{t_0}(v)} - \hat{\varphi}_{\bar{X}_{t_1}(v)} + \hat{\varphi}_{\bar{X}_{t_2}(v)} - \hat{\varphi}_{\bar{X}_{t_3}(v)}\right)(0,b)\\ &\quad= \frac{2}{q-1}\left(\gamma_{\mathrm{sgn}(v)} - \gamma_{\mathrm{sgn}(\varepsilon v)} + \gamma_{\mathrm{sgn}(v)} - \gamma_{\mathrm{sgn}(\varepsilon v)}\right)(b)\\ &\quad= \frac{2^2}{q-1}(\gamma_{\mathrm{sgn}(v)} - \gamma_{\mathrm{sgn}(\varepsilon v)})(b)\\ &\quad= \frac{2^2}{q-1}\mathrm{sgn}(v)\widehat{\mathrm{sgn}}(b)\\ &\quad= \frac{2^2}{q-1}\mathrm{sgn}(v)\sqrt{q}\zeta^3\mathrm{sgn}(b).\end{aligned}$$

Letting $v = 1$, we recover the first two parts of equation (4–9). Likewise,

$$\begin{aligned}&\left(\hat{\varphi}_{\bar{X}_{t_0}(v)} - \hat{\varphi}_{\bar{X}_{t_1}(v)} - \hat{\varphi}_{\bar{X}_{t_2}(v)} + \hat{\varphi}_{\bar{X}_{t_3}(v)}\right)(a,0)\\ &\quad= \frac{2}{q-1}\left(\gamma_{\mathrm{sgn}(v)} - \gamma_{\mathrm{sgn}(\varepsilon v)} - \gamma_{\mathrm{sgn}(\varepsilon v)} + \gamma_{\mathrm{sgn}(v)}\right)(a)\\ &\quad= \frac{2^2}{q-1}(\gamma_{\mathrm{sgn}(v)} - \gamma_{\mathrm{sgn}(\varepsilon v)})(a)\\ &\quad= \frac{2^2}{q-1}\mathrm{sgn}(v)\widehat{\mathrm{sgn}}(a)\\ &\quad= \frac{2^2}{q-1}\mathrm{sgn}(v)\sqrt{q}\zeta^1\mathrm{sgn}(a),\end{aligned}$$

and

$$\begin{aligned}&\left(\hat{\varphi}_{\bar{X}_{t_0}(v)} - \hat{\varphi}_{\bar{X}_{t_1}(v)} - \hat{\varphi}_{\bar{X}_{t_2}(v)} + \hat{\varphi}_{\bar{X}_{t_3}(v)}\right)(0,b)\\ &\quad= \frac{2}{q-1}\left(\gamma_{\mathrm{sgn}(v)} - \gamma_{\mathrm{sgn}(\varepsilon v)} - \gamma_{\mathrm{sgn}(v)} + \gamma_{\mathrm{sgn}(\varepsilon v)}\right)(b)\\ &\quad= 0.\end{aligned}$$

Letting $v = 1$, we recover the last two parts of equation (4–9).

Equation (4–6) (and therefore equation (4–7) also) is exactly Springer's hypothesis in our context. In fact, it is quite easy to verify this equality by direct calculation of the relevant Fourier transforms, but since this paper is already long, we omit the details.

4.5 Motives for the Fourier Transforms of Orbital Integrals

As we see from Section 4.2, for our purposes, there are exactly three important points in the Bruhat–Tits building: (0), (01), and (1). For the remainder of this section, the symbol x is reserved for the vertices (0) and (1), while $y = (01)$.

Let f be a test function supported on the topologically nilpotent elements in $\mathfrak{g}(\mathbb{K})$. All we need to know in order to find $\hat{\mu}_{X_z}(f)$ is the values of the corresponding function $\hat{\varphi}_{X_z}$ on the corresponding reductive quotient, and the information about the projections of the elements of the form $h^{-1}Yh$ for the elements Y in the support of f.

Recall the notation $\mathfrak{h}_{z,n} = \{Y_{z,n}(u) \mid u \in \mathcal{O}_{\mathbb{K}}^*\} \subset \mathfrak{g}$ and $\tilde{f}_{z,n,\pm} = \mathrm{cay}^* f_{z,n,\pm}$ (the characteristic function of the set ${}^G\mathfrak{h}_{z,n,\pm}$, where G acts by adjoint action).

Proposition 4.3. *Suppose $z \in \{s_0, s_1, s_2, t_0, t_1, t_2, t_3\}$. Let $\tilde{f}_{z,n,\pm}$ be the characteristic function of the set ${}^G\mathfrak{h}_{z,n,\pm}$ and let $\mathfrak{u}_0$, $\mathfrak{u}_1$, and $\mathfrak{u}_\varepsilon$ denote the nilpotent orbits in $\mathfrak{sl}(2,\mathbb{F}_q)$ corresponding to the unipotent conjugacy classes U_0, U_1, and U_ε respectively. Let κ be an arbitrary linear combination of the functions Q_T and Q_G viewed as functions on $\mathfrak{sl}(2,\mathbb{F}_q)$ (see Section 4.4 for their definition). Then*

$$\begin{aligned}\int_G\int_{\mathfrak{g}} &\tilde{f}_{z,n,\pm}(gYg^{-1})\kappa_{x,0}(Y)dYdg \\ &= \kappa|_{\mathfrak{u}_0}\mathrm{TrFrob}M^{x,0}_{z,n,\pm} + \kappa|_{\mathfrak{u}_1}\mathrm{TrFrob}M^{x,1}_{z,n,\pm} \\ &\quad + \kappa|_{\mathfrak{u}_\varepsilon}\mathrm{TrFrob}M^{x,\varepsilon}_{z,n,\pm},\end{aligned} \tag{4–10}$$

where $M^{x,0}_{z,n,\pm}$, $M^{x,1}_{z,n,\pm}$, and $M^{x,\varepsilon}_{z,n,\pm}$ are defined in Tables 6 and 7 in the case $x = (0)$ and Tables 8 and 9 in the case $x = (1)$.

Proof: This proposition essentially follows from Proposition 3.5. We have

$$\begin{aligned}\int_G\int_{\mathfrak{g}} &\tilde{f}_{z,n,\pm}(gYg^{-1})\kappa_{x,0}(Y)dYdg \\ &= \int_G\int_{\mathfrak{g}} \tilde{f}_{z,n,\pm}(Y)\kappa_{x,0}(g^{-1}Yg)dYdg \\ &= \int_G\int_{{}^G\Gamma_{z,n,\pm}} (\kappa_{x,0}\circ \mathrm{cay}^*)(g^{-1}\gamma g)d\gamma dg.\end{aligned}$$

Note that in the above integral, the function $\kappa_{x,0}\circ\mathrm{cay}^*$ is evaluated only at topologically unipotent elements. The modified Cayley transform cay is measure-preserving on this set, and that is why the integral can be rewritten as a double integral over the group G. This is exactly the expression that appears in equation (3–1), and therefore by Proposition 3.5 it has the required form. □

Corollary 4.4. *Suppose $z \in \{s_1, s_2\}$ and $z' \in \{s_0, s_1, s_2, t_0, t_1, t_2, t_3\}$. For each $n \in \mathbb{N}$, the Fourier transform $\hat{\mu}_{X_z}$ of the orbital integral at X_z is constant on the set ${}^G\mathfrak{h}_{z',n,\pm}$, and*

$$\begin{aligned}&\hat{\mu}_{X_z}(\tilde{f}_{z',n,\pm}) \\ &\quad = \hat{\varphi}_{\bar{X}_z}|_{\mathfrak{u}_0}\mathrm{TrFrob}M^{x,0}_{z',n,\pm} + \hat{\varphi}_{\bar{X}_z}|_{\mathfrak{u}_1}\mathrm{TrFrob}M^{x,1}_{z',n,\pm} \\ &\qquad + \hat{\varphi}_{\bar{X}_z}|_{\mathfrak{u}_\varepsilon}\mathrm{TrFrob}M^{x,\varepsilon}_{z',n,\pm},\end{aligned}$$

where $x = (0)$ if $z = s_1$ and $x = (1)$ if $z = s_2$. See Remark 4.2 for the definition of $\varphi_{\bar{X}_z}$.

Proof: From Section 4.2 we see that our elements X_z correspond to vertices x in the building, and their depths are all $r = 0$. First, recall that by Proposition 4.1, we have

$$\Phi(X_z, \tilde{f}_{z',n,\pm}) = \int_G\int_{\mathfrak{g}} \tilde{f}_{z',n,\pm}(gYg^{-1})(\hat{\varphi}_{\bar{X}_z})_{x,0}(Y).$$

Now we can plug in $\kappa = \hat{\varphi}_{\bar{X}_z}$ in Proposition 4.3. By Springer's hypothesis (see Section 4.4), the function $\hat{\varphi}_{\bar{X}_z}$ restricted to the set of nilpotent elements is a constant multiple of the Green's polynomial Q_T (thought of as a function on the Lie algebra), so the assumptions of the proposition are satisfied. □

Let us now consider the elements X_z of the ramified elliptic tori, that is, $z = t_0$, t_1, t_2, or t_3. Then (see Section 4.2) $y = (01)$ and $r = \frac{1}{2}$, and $\bar{\mathfrak{g}}_{y,-r}(\mathbb{F}_q) = \mathbb{A}^2(\mathbb{F}_q)$; in the rest of this section we write y for (01) and r for $\frac{1}{2}$. The image of the set of topologically nilpotent elements in $\mathfrak{g}_{y,-1/2}$ under $\rho_{y,-r}$ is contained in $\{(x,y) \in \mathbb{A}^2(\mathbb{F}_q) \mid xy = 0\}$. This set is the union of the following five orbits of the action of $\mathrm{GL}(1,\mathbb{F}_q)$ on $\mathbb{A}^2(\mathbb{F}_q)$:

$$\begin{aligned}V^0 &:= \{(0,0)\}, \\ V^{1,+} &:= \{(0,x) \mid \mathrm{sgn}(x) = 1\}, \\ V^{1,-} &:= \{(0,x) \mid \mathrm{sgn}(x) = -1\}, \\ V^{2,+} &:= \{(x,0) \mid \mathrm{sgn}(x) = 1\}, \\ V^{2,-} &:= \{(x,0) \mid \mathrm{sgn}(x) = -1\}.\end{aligned} \tag{4–11}$$

Proposition 4.5. *Let κ be a $\mathrm{GL}(1)$-invariant function defined on $\mathbb{A}^2(\mathbb{F}_q)$ with respect to the action defined in Section 4.4. Then, for each $z \in \{s_1, s_2, t_0, t_1, t_2, t_3\}$, each $\nu = \pm$, and each nonnegative integer n in the case $z \in \{t_0, t_1, t_2, t_3\}$, and each positive integer n in the case $z \in \{s_1, s_2\}$, there exist virtual Chow motives $\mathcal{N}^0_{z,n,\nu}$,*

$\mathcal{N}^{1,\pm}_{z,n,\nu}$, and $\mathcal{N}^{2,\pm}_{z,n,\nu}$ such that

$$\int_{G(\mathbb{K})}\int_{\mathfrak{g}(\mathbb{K})}\tilde{f}_{z,n,\nu}(Ad(g)Y)\kappa_{y,-r}(Y)\,dY\,dg$$
$$=\kappa|_{V_0}\mathrm{TrFrob}\mathcal{N}^0_{z,n,\nu}+\sum_{\alpha=\pm}\kappa|_{V^{1,\alpha}}\mathrm{TrFrob}\mathcal{N}^{1,\alpha}_{z,n,\nu}$$
$$+\sum_{\alpha=\pm}\kappa|_{V^{2,\alpha}}\mathrm{TrFrob}\mathcal{N}^{2,\alpha}_{z,n,\nu}.$$

Moreover, $\mathcal{N}^0_{z,n,\nu}$ and $\mathcal{N}^{1,\pm}_{z,n,\nu}$ and $\mathcal{N}^{2,\pm}_{z,n,\nu}$ are rational functions of $\mathbb{L}$. The virtual Chow motives $\mathcal{N}^{1,\pm}_{z,n,\nu}$ and $\mathcal{N}^{2,\pm}_{z,n,\nu}$ are given in Tables 10 and 11.

Proof: We rewrite the left-hand side as the sum over the nilpotent orbits in the reductive quotient, as we did with the character in Section 3.1. In order to do this, we make the following definition. Suppose V is one of the $\mathrm{GL}(1,\mathbb{F}_q)$-orbits in $\mathbb{A}^2(\mathbb{F}_q)$ appearing in equation (4–11). Let n be an integer and recall that y denotes the point (01) in the Bruhat–Tits building for G. For any regular topologically nilpotent element Y of $\mathfrak{g}$, let $N_{V,\lambda}(Y)$ denote the number of G_y-cosets gG_y inside the double-coset $G_x\backslash G/G_y$ that satisfy the following condition:

$$g^{-1}Yg\in\mathfrak{g}_{y,-r}\wedge\rho_{y,-r}(g^{-1}Yg)\in V.$$

Let $\tilde{A}$ be the set of diagonal matrices of the form $\mathrm{diag}(\varpi^\lambda,\varpi^{-\lambda})$, where λ is an arbitrary integer. Then G has the decomposition $G=G_x\tilde{A}G_y$ (note the difference with Cartan decomposition, where λ is nonnegative). Using the G_y-invariance of the function $\kappa_{y,-r}$, we obtain, for an arbitrary test function f,

$$\int_G\int_{\mathfrak{g}}f(Y)\kappa_{y,-r}(g^{-1}Yg)\,dY\,dg$$
$$=\int_{G/G_y}\int_{G_y}\int_{\mathfrak{g}}f(Y)\kappa_{y,-r}(y^{-1}h^{-1}Yhy)\,dY\,dy\,dh$$
$$=\mathrm{m}(G_y)\sum_a\sum_h\int_{\mathfrak{g}}f(Y)\kappa_{y,-r}(h^{-1}Yh)\,dY, \tag{4–12}$$

where the outside summation is over $a\in G_x\backslash G/G_y$, and the inside summation is over $h\in G_xaG_y/G_y$. Note that the summation index in the outside sum in fact runs over $\mathbb{Z}$. As with the case of the character, the sum in fact contains only finitely many nonzero terms, since at the moment we are considering only the elliptic elements Y.

The rest of the argument follows the pattern of the proof of Proposition 3.5, taking equation (4–12) as the starting point. It also proceeds case by case. Here we carry out the proof for the test functions $\tilde{f}_{z,n,\pm}$ with $z=s_1$ and $z=t_2$. The other cases are very similar; the results of these calculations are recorded in Tables 10 and 11.

As in Section 3.1, we can continue the chain of equalities (4–12) by writing the integral inside the sum as a sum over $\mathrm{GL}(1,\mathbb{F}_q)$-orbits V:

$$\mathrm{m}(G_y)\sum_a\sum_h\int_{\mathfrak{g}}\tilde{f}_{z,n,\pm}(Y)\kappa_{y,-r}(h^{-1}Yh)\,dY$$
$$=\sum_a\sum_V\kappa|_V\int_{{}^G\mathfrak{h}_{z,n,\pm}}N_{V,\lambda}(Y)dY$$
$$=\sum_V\kappa|_V\sum_{\lambda=-\infty}^{\infty}\int_{{}^G\mathfrak{h}_{z,n,\pm}}N_{V,\lambda}(Y)dY,$$

where the summations over a and h are as in equation (4–12).

Note that since there are, in fact, only finitely many nonzero terms, the permutation of the two sums is valid.

Now it remains to "calculate" the numbers $N_{V,\lambda}(Y)$, i.e., to express them in terms of motivic volumes of some definable sets. This is done by brute force, in a manner similar to the calculation of the character. Our calculation will make it transparent that these numbers are constant on each of the sets ${}^G\mathfrak{h}_{z,n,\pm}$.

We will need the formula for the number of G_y-cosets inside each double coset $G_xa_\lambda G_y$ ($\lambda\in\mathbb{Z}$): the cardinality $\#G_xa_\lambda G_y/G_y$ equals $q^{2\lambda-1}$ if $\lambda>0$, and $q^{2|\lambda|}$ if $\lambda\le 0$, as can be shown using the affine Bruhat decomposition for G (see [Bruhat and Tits 96], for example). Recall that with the notation of Section 2.7.2, we can write $[G_xa_\lambda G_y/G_y]=\mathbb{L}^{2\lambda-1}$ when λ is a positive integer, $[G_xa_\lambda G_y/G_y]=\mathbb{L}^{-2\lambda}$ when $\lambda\le 0$.

Let ha_λ be a representative of a coset $G_xa_\lambda G_y/G_y$; write $h=\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in G_x$.

A statement completely analogous to Lemma 3.2 relates the numbers $N_{V,\lambda}$ to the motivic volumes of the sets $\{h\mid\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda)=(x,0)\}$ and $\{h\mid\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda)=(0,x)\}$, with x square or nonsquare, respectively (of course, we also need to show that these four sets are definable). Then to obtain the virtual motives $\mathcal{N}^{1,\pm}_{z,n,\nu}$ and $\mathcal{N}^{2,\pm}_{z,n,\nu}$, we need to sum these motivic volumes over all values of λ.

The following is the list of possibilities for the element $\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda)$ of $\mathbb{A}^2(\mathbb{F}_q)$ in the two cases $z=t_2$ and $z=s_1$:

Case $z=t_2$: The conditions on a, b, c, d, and λ for $a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda$ to be in $G_{y,-r}$ are

$$\begin{aligned}&-2\lambda+\mathrm{ord}(d^2u\varpi^n-b^2\varepsilon u\varpi^{n+1})\ge -1,\\&2\lambda+\mathrm{ord}(-c^2u\varpi^n+a^2\varepsilon u\varpi^{n+1})\ge 0.\end{aligned} \tag{4–13}$$

z	$\mathcal{N}^{1,\pm}_{z,n,\nu}$	$\mathcal{N}^{2,\pm}_{z,n,\nu}$
s_1	$\mathcal{N}^{1,+}_{z,n,\nu} = \mathcal{N}^{1,-}_{z,n,\nu} = \frac{1}{2}\mathbb{L}^n$ n odd $\mathcal{N}^{1,+}_{z,n,\nu} = \mathcal{N}^{1,-}_{z,n,\nu} = 0$ n even	$\mathcal{N}^{2,+}_{z,n,\nu} = \mathcal{N}^{2,-} = 0$ n odd $\mathcal{N}^{2,+}_{z,n,\nu} = \mathcal{N}^{2,-}_{z,n,\nu} = \frac{1}{2}\mathbb{L}^n$ n even
s_2	$\mathcal{N}^{1,+}_{z,n,\nu} = \mathcal{N}^{1,-} = 0$ n odd $\mathcal{N}^{1,+}_{z,n,\nu} = \mathcal{N}^{1,-}_{z,n,\nu} = \frac{1}{2}\mathbb{L}^n$ n even	$\mathcal{N}^{2,+}_{z,n,\nu} = \mathcal{N}^{2,-}_{z,n,\nu} = \frac{1}{2}\mathbb{L}^n$ n odd $\mathcal{N}^{2,+}_{z,n,\nu} = \mathcal{N}^{2,-}_{z,n,\nu} = 0$ n even

TABLE 10. The virtual motives for the Fourier transform of orbital integrals at elements $Y_{z,n}(u)$, for $z \in \{s_1, s_2\}$. Here $\nu = \mathrm{sgn}(u)$.

z	$\mathcal{N}^{1,\pm}_{z,n,\nu}$	$\mathcal{N}^{2,\pm}_{z,n,\nu}$
t_0	$\mathcal{N}^{1,+}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}^{1,-}_{z,n,\nu} = 0$, if $\zeta^{2n+2} = \nu$ $\mathcal{N}^{1,+}_{z,n,\nu} = 0$ and $\mathcal{N}^{1,-}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise	$\mathcal{N}^{2,+}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}^{2,-}_{z,n,\nu} = 0$ if $\zeta^{2n+2} = \nu$ $\mathcal{N}^{2,+}_{z,n,\nu} = 0$ and $\mathcal{N}^{2,-}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise
t_1	$\mathcal{N}^{1,+}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}^{1,-}_{z,n,\nu} = 0$, if $\zeta^{2n+2} = -\nu$ $\mathcal{N}^{1,+}_{z,n,\nu} = 0$ and $\mathcal{N}^{1,-}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise	$\mathcal{N}^{2,+}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}^{2,-}_{z,n,\nu} = 0$, if $\zeta^{2n+2} = -\nu$ $\mathcal{N}^{2,+}_{z,n,\nu} = 0$ and $\mathcal{N}^{2,-}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise
t_2	$\mathcal{N}^{1,+}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}^{1,-}_{z,n,\nu} = 0$, if $\zeta^{2n+2} = (-1)^{n+1}\nu$ $\mathcal{N}^{1,+}_{z,n,\nu} = 0$ and $\mathcal{N}^{1,-}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise	$\mathcal{N}^{2,+}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}^{2,-}_{z,n,\nu} = 0$, if $\zeta^{2n+2} = (-1)^{n}\nu$ $\mathcal{N}^{2,+}_{z,n,\nu} = 0$ and $\mathcal{N}^{2,-}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise
t_3	$\mathcal{N}^{1,+}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}^{1,-}_{z,n,\nu} = 0$, if $\zeta^{2n+2} = (-1)^{n}\nu$ $\mathcal{N}^{1,+}_{z,n,\nu} = 0$ and $\mathcal{N}^{1,-}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise	$\mathcal{N}^{2,+}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ and $\mathcal{N}^{2,-}_{z,n,\nu} = 0$, if $\zeta^{2n+2} = (-1)^{n+1}\nu$ $\mathcal{N}^{2,+}_{z,n,\nu} = 0$ and $\mathcal{N}^{2,-}_{z,n,\nu} = \frac{\mathbb{L}^{n+1}}{\mathbb{L}+1}$ otherwise

TABLE 11. The virtual motives for the Fourier transform of orbital integrals at elements $Y_{z,n}(u)$, for $z \in \{t_0, t_1, t_2, t_3\}$. (Recall that $\zeta^2 = \mathrm{sgn}(-1)$ and $\nu = \mathrm{sgn}(u)$.)

Applying the reduction map $\rho_{y,-r}$ to $a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda$, we see that there are a few possibilities:

1. n is even: Recall the notation $\nu = \mathrm{sgn}(u)$.

   (a) If $-n/2 < \lambda \le n/2$, then we have $\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,0)$.

   (b) Suppose $\lambda = n/2+1$. If $\mathrm{ord}(d) > 0$, then

   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (x,0)$$

   with $\mathrm{sgn}(x) = \mathrm{sgn}(-\varepsilon u) = -\mathrm{sgn}(-u) = -\zeta^2\mathrm{sgn}(u)$; on the other hand, if $\mathrm{ord}(d) = 0$, we have $a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda \notin G_{y,-r}$.

   (c) Suppose $\lambda = -n/2$. If $\mathrm{ord}(c) > 0$, then

   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,0);$$

   on the other hand, if $\mathrm{ord}(c) = 0$, then

   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,y),$$

   with $\mathrm{sgn}(y) = \mathrm{sgn}(-u) = \zeta^2\nu$.

   Case (b) allows us to calculate the virtual motives responsible for the two orbits of the form $(x,0)$: the virtual Chow motive $\mathcal{N}^{1,+}_{t_2,n,\nu}$ corresponding to the orbit of $(1,0)$ equals 0 if $\zeta^2\nu = 1$, and equals $\frac{\mu(G_y)}{\mu(G_x)}[G_x a_{\frac{n}{2}+1} G_y/G_y] = \frac{1}{\mathbb{L}+1}\mathbb{L}^{n+1}$ if $\zeta^2\nu = -1$. For the orbit of $(\epsilon,0)$, the answer is the reverse: $\mathcal{N}^{1,-}_{t_2,n,\nu}$ equals 0 if $\zeta^2\nu = -1$, and equals $\frac{1}{\mathbb{L}+1}\mathbb{L}^{n+1}$ if $\zeta^2\nu = 1$.

   From case (c), we get the motives corresponding to the other two orbits: $\mathcal{N}^{2,+}_{t_2,n,\nu} = \frac{\mu(G_x)-\mu(G_y)}{\mu(G_x)}[G_x a_{-\frac{n}{2}} G_y/G_y] = \frac{\mathbb{L}}{\mathbb{L}+1}\mathbb{L}^n$ if $\zeta^2\nu = 1$, and 0 if $\zeta^2\nu = -1$, and for $\mathcal{N}^{2,-}_{t_2,n,\nu}$ these answers are reversed.

2. n is odd:

   (a) If $-(n+1)/2 < \lambda < (n+1)/2$, then $\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,0)$.

   (b) If $\lambda = (n+1)/2$, there are two possibilities: if $\mathrm{ord}(d) = 0$, then we have

   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (x,0)$$

   with $\mathrm{sgn}(x) = \mathrm{sgn}(u)$; on the other hand, if $\mathrm{ord}(d) > 0$, then

   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,0).$$

   (c) Suppose $\lambda = -(n+1)/2$. If $\mathrm{ord}(c) = 0$, we have $a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda \notin G_{y,-r}$; if $\mathrm{ord}(c) > 0$, then

   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,y),$$

   with $\mathrm{sgn}(y) = -\mathrm{sgn}(u)$.

   As before, from the calculations in case (b), we see that the virtual Chow motive $\mathcal{N}^{1,+}_{t_2,n,\nu}$ that corresponds to the orbit of $(1,0)$ equals 0 if $\nu = -1$, and equals $\frac{\mu(G_x)-\mu(G_y)}{\mu(G_x)}[G_x a_{\frac{n+1}{2}} G_y/G_y] = \frac{\mathbb{L}}{\mathbb{L}+1}\mathbb{L}^{n+1-1} = \frac{1}{\mathbb{L}+1}\mathbb{L}^n$ if $\nu = 1$. For the orbit of $(\epsilon,0)$, the answer is the reverse. Case (c) yields $\mathcal{N}^{2,+}_{t_2,n,\nu} = \frac{\mu(G_y)}{\mu(G_x)}[G_x a_{-\frac{n+1}{2}} G_y/G_y] = \frac{1}{\mathbb{L}+1}\mathbb{L}^{n+1}$ if $\nu = -1$, and 0 if $\nu = 1$.

Case $z = s_1$: In this case, the conditions (4–13) are replaced with

$$\begin{aligned} -2\lambda + \mathrm{ord}(d^2\varpi^n - b^2\varepsilon u\varpi^n) &\ge -1,\\ 2\lambda + \mathrm{ord}(-c^2 u\varpi^n + a^2\varepsilon u\varpi^n) &\ge 0. \end{aligned} \tag{4–14}$$

1. Suppose n is even. Then we obtain the following:

   (a) If $-n/2 < \lambda \le n/2$, then

   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,0).$$

   (b) If $\lambda = -n/2$, then

   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,y)$$

   with $\mathrm{sgn}(y) = \mathrm{sgn}(u)\mathrm{sgn}(\mathrm{ac}(-c^2 + a^2\varepsilon))$.

   We see that $\mathcal{N}^{1,\pm}_{s_1,n,\nu}$ are both zero when n is even, for any value of ν. In order to find $\mathcal{N}^{2,\pm}_{s_1,n,\nu}$, we need to calculate the ratio of the volume of the subset of G_x defined by the formula $\nexists\eta \ne 0, \mathrm{ac}(-c^2 + a^2\varepsilon) = \eta^2$ to the total volume of G_x. Note that $-c^2 + a^2\varepsilon = -(c^2 - a^2\varepsilon)$. (This formula should be understood as an abbreviation. We should first consider the formula with an extra free variable δ: $\nexists\eta \ne 0, \mathrm{ac}(-c^2 + a^2\delta) = \eta^2$, do the motivic calculation, then plug in our value of ε, and the calculation is very similar to the one in Lemma 3.3, carried out in [Gordon 09].) Note that the expression $c^2 - a^2\varepsilon$ is a square of a nonzero element for exactly half of the elements of G_x. Then this ratio is $\frac{1}{2}$. Therefore, in both cases $\nu = 1$ and $\nu = -1$, the answer is the same:

   $$\mathcal{N}^{2,\pm}_{s_1,n,\nu} = \frac{1}{2}[G_x a_{n/2} G_y/G_y] = \frac{1}{2}\mathbb{L}^{2|-n/2|} = \frac{1}{2}\mathbb{L}^n.$$

2. Suppose n is odd. In this case, we have the following:
   (a) If $-(n+1)/2 < \lambda < (n+1)/2$, then
   $$\rho_{y,-1/2}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (0,0).$$
   (b) If $\lambda = -(n+1)/2$, then the element $a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda$ is not in G_x for any h.
   (c) If $\lambda = (n+1)/2$, then
   $$\rho_{y,-r}(a_\lambda^{-1}h^{-1}Y_{z,n}(u)ha_\lambda) = (x,0),$$
   with $\mathrm{sgn}(x) = \mathrm{sgn}(u)\mathrm{sgn}(\mathrm{ac}(d^2 - \varepsilon b^2))$.

   Similarly to the previous case, we get $\mathcal{N}^{2,\pm}_{s_1,n,\nu} = 0$; $\mathcal{N}^{1,\pm}_{s_1,n,\nu} = \frac{1}{2}\mathbb{L}^n$ (again using Lemma 3.3 to show that the volume of the subset of G_x defined by the formula $\nexists\eta \neq 0, \mathrm{ac}(d^2 - b^2\varepsilon) = \eta^2$ equals half of the volume of G_x.

The calculation in the case $z = s_2$ follows the pattern of the case $z = s_1$, except that there are the same additional complications as we saw in the case of the character. Not surprisingly, the answer is still the same as in the case of s_1, except that the roles of the cases n even and n odd are switched. The results of similar calculations for the remaining ramified cases are summarized in Tables 10 and 11.

The virtual Chow motives $\mathcal{N}^0_{z,n,\nu}$ can be computed following arguments as above. However, since they do not appear in any of our further calculations (all the functions κ in which we are interested vanish at the origin), we omit this calculation. $\square$

Corollary 4.6. *Let $\mathfrak{h}_z$ be ramified elliptic (so $z \in \{t_0, t_1, t_2, t_3\}$). For each $n \in \mathbb{N}$, the Fourier transform $\hat{\mu}_{X_z}$ of the orbital integral at X_z is constant on the set ${}^G\mathfrak{h}_{z',n,\nu}$ for $\nu = \pm$, $z' \in \{s_1, s_2, t_0, t_1, t_2, t_3\}$ and a positive (respectively nonnegative if $z' \in \{t_0, t_1, t_2, t_3\}$) integer n, and we have*

$$\hat{\mu}_{X_z}(\tilde{f}_{z',n,\nu}) = \hat{\varphi}_{X_z}|_{V_0}\mathrm{TrFrob}\mathcal{N}^0_{z',n,\nu} + \sum_{\alpha=\pm}\hat{\varphi}_{X_z}|_{V^{1,\nu}}\mathrm{TrFrob}\mathcal{N}^{1,\alpha}_{z',n,\nu} + \sum_{\alpha=\pm}\hat{\varphi}_{X_z}|_{V^{2,\nu}}\mathrm{TrFrob}\mathcal{N}^{2,\alpha}_{z',n,\nu}.$$

Proof: From Section 4.2, we see that our elements X_z correspond to the point $y = (01)$ in the building, and their depth is $r = \frac{1}{2}$. First, recall that by Proposition 4.1, we have

$$\mu_{X_z}(\tilde{f}_{z',n,\pm}) = \int_G \int_{\mathfrak{g}} \tilde{f}_{z',n,\pm}(gYg^{-1})(\hat{\varphi}_{\bar{X}_z})_{y,-r}(Y).$$

Note that the support of the functions $\tilde{f}_{z,n,\pm}$ for all z and all $n > 0$ is contained in $\mathfrak{h}_{y,-r}$, so Proposition 4.1 is applicable. Now it remains to plug in the function $\kappa = \hat{\varphi}_{\bar{X}_z}$, which is now a $\mathrm{GL}(1, \mathbb{F}_q)$-invariant function on $\mathbb{A}^2(\mathbb{F}_q)$, in Proposition 4.5 to complete the proof. $\square$

5. MOTIVIC PROOF OF THE CHARACTER FORMULA

Now we are ready to prove Theorem 2.5, and to calculate the coefficients that appear in Table 4 in the process. Recall that semisimple character expansion is an equality of two distributions on the topologically nilpotent regular set in the Lie algebra. These distributions are represented by locally integrable functions, which are constant on the sets ${}^G\mathfrak{h}_{z,n,\pm}$; see Proposition 3.5, Corollary 4.4, and Corollary 4.6. We prove the semisimple character expansion by checking the equality on each of these sets.

Let π be a representation as in Section 2.1. We start with the case that π is obtained from a Deligne–Lusztig representation, where the proof is straightforward and requires no consideration of separate cases. If π is of the type $\pi(x, \theta)$ with $x = (0)$ or $x = (1)$, then by Proposition 3.5, we have, for each $z' \in \{s_0, s_1, s_2, t_0, t_1, t_2, t_3\}$,

$$\frac{1}{m({}^G\Gamma_{z',n,\nu})}\Theta_{\pi(x,\theta)}(f_{z',n,\nu}) = -\sum_U Q_T|_U \mathrm{TrFrob}M^{x,U}_{z',n,\nu}.$$

On the other hand, for $z = s_1$ or s_2, by Corollary 4.4,

$$\hat{\mu}_{X_z}(\tilde{f}_{z',n,\pm}) = \sum_U \varphi_{\bar{X}_z}\mathrm{TrFrob}M^{x,U}_{z',n,\pm},$$

where $x = (0)$ if $z = s_1$, and $x = (1)$ if $z = s_2$. As shown in Sections 4.2.1 and 4.2.2,

$$Q_T = (1-q)\hat{\varphi}_{\bar{X}_{s_1}} = (1-q)\hat{\varphi}_{\bar{X}_{s_2}}.$$

It follows immediately that

$$\frac{1}{m({}^G\mathfrak{h}_{z',n,\nu})}\Theta_\pi(\tilde{f}_{z',n,\nu} \circ \mathrm{cay}) = \begin{cases}(1-q)\hat{\mu}_{X_{s_1}}(\tilde{f}_{z',n,\nu}) & \text{if } \pi = \pi(0,\theta),\\ (1-q)\hat{\mu}_{X_{s_2}}(\tilde{f}_{z',n,\nu}) & \text{if } \pi = \pi(1,\theta).\end{cases} \tag{5–1}$$

This proves the theorem in the case that π comes from a Deligne–Lusztig representation.

Let us now turn to the non-Deligne–Lusztig case. Let $\pi = \pi(x, +)$, where $x = (0)$ or $x = (1)$ (the other two cases can be obtained by changing the sign in front of

Q_G everywhere below). Then

$$\frac{1}{m({}^G\mathfrak{h}_{z',n,\nu})}\Theta_\pi(\tilde{f}_{z',n,\nu}\circ \mathrm{cay}) = -\frac{1}{2}\sum_U Q_T(U)\mathrm{TrFrob}M^{x,U}_{z',n,\nu} - \frac{1}{2}\sum_U Q_G(U)\mathrm{TrFrob}M^{x,U}_{z',n,\nu}. \quad (5\text{–}2)$$

As we have seen in equation (5–1), the first term (i.e., the part of the character that comes from inflating Q_T) is a multiple of $\hat{\mu}_{X_{s_1}}(\tilde{f}_{z',n,\nu})$ in the case $\pi = \pi(0,+)$, and of $\hat{\mu}_{X_{s_2}}(\tilde{f}_{z',n,\nu})$ in the case $\pi = \pi(1,+)$. The coefficient is $\frac{q-1}{2}$, since on the left, Q_T appears with the coefficient $-\frac{1}{2}$.

It remains to express the second term, $-\frac{1}{2}\sum_U Q_G(U)\mathrm{TrFrob}M^{x,U}_{z',n,\nu}$, as a linear combination of the Fourier transforms of orbital integrals. In order to do that, recall the functions $\varphi_{(0)}$ and $\varphi_{(1)}$ defined in Section 4.3. By Corollary 4.6, if z' is elliptic, we have

$$\sum_{\substack{l=1,2\\ \alpha=\pm}} \hat{\varphi}_{(0)}|_{V^{l,\alpha}}\mathrm{TrFrob}\mathcal{N}^{l,\alpha}_{z',n,\nu} = \hat{\mu}_{X_{t_0}}(f_{z',n,\nu}) - \hat{\mu}_{X_{t_1}}(f_{z',n,\nu}) + \hat{\mu}_{X_{t_2}}(f_{z',n,\nu}) - \hat{\mu}_{X_{t_3}}(f_{z',n,\nu}) \quad (5\text{–}3)$$

and

$$\sum_{\substack{l=1,2\\ \alpha=\pm}} \hat{\varphi}_{(1)}|_{V^{l,\alpha}}\mathrm{TrFrob}\mathcal{N}^{l,\alpha}_{z',n,\nu} = \hat{\mu}_{X_{t_0}}(f_{z',n,\nu}) - \hat{\mu}_{X_{t_1}}(f_{z',n,\nu}) - \hat{\mu}_{X_{t_2}}(f_{z',n,\nu}) + \hat{\mu}_{X_{t_3}}(f_{z',n,\nu}). \quad (5\text{–}4)$$

Also, if $z' = s_0$, then, since the both functions $\hat{\varphi}_{(0)}$ and $\hat{\varphi}_{(1)}$ vanish at the origin and take opposite values on the orbits $V^{1,+}, V^{1,-}$ and $V^{2,+}, V^{2,-}$, it is easy to see in a way that mimics the proof of Proposition 3.5 in the case $z = s_0$ that the right-hand sides of equations (5–3) and (5–4) vanish on $\mathfrak{h}_{s_0,n}$ for $n > 0$.

We claim that for any z and any positive (respectively nonnegative) n,

$$-\frac{1}{2}\sum_U Q_G(U)\mathrm{TrFrob}M^{x,U}_{z',n,\nu} = c\sum_{\substack{l=1,2\\ \alpha=\pm}} \hat{\varphi}_x|_{V^{l,\alpha}}\mathrm{TrFrob}\mathcal{N}^{l,\alpha}_{z,n,\nu}, \quad (5\text{–}5)$$

with some constant c that we will calculate below (we will see that $c = -\frac{q^2-1}{2^3 q}$). Note that the left-hand side depends on the choice of the vertex x. On the right, it is the constant c and the function φ_x that depend on x. This equation could be called the motivic version of our character formula. It is the core of the proof; here we are comparing the inflation of two functions that live on different reductive quotients. Note that once we prove this claim, Theorem 2.5 will follow immediately. On the other hand, the proof of the claim is automatic: all we need to do is plug in the values of the functions φ_x from Section 4.3, the values of Q_G from Section 2.3, and the motivic coefficients from Tables 6 and 7 if $x = (0)$ and Tables 8 and 9 if $x = (1)$ on the left, and from Tables 10 and 11 on the right.

The equality (5–5) has to be checked on each of the sets $\mathfrak{h}_{z',n,\nu}$ (recall that $\nu = \pm$).

Observe that the function $\hat{\varphi}_{(0)}$ vanishes at $V^{1,\pm}$, and the function $\hat{\varphi}_{(1)}$ vanishes at $V^{2,\pm}$, so that the right-hand side in any case has only two nonzero terms. On the left, since Q_G vanishes at the identity, there are also only two nonzero terms.

For $z' = s_0$ the equality we want to prove is trivial, since both sides vanish, as discussed above. For $z' = s_1$ and $z = s_2$, the equality also turns out to be trivial. On the right, the two nonzero opposite values of the function $\hat{\varphi}_{(0)}$ or $\hat{\varphi}_{(1)}$ appear with the same coefficient, so the right-hand side equals zero. On the left, the function Q_G takes opposite values on U_1 and U_ε, and they also appear on the left-hand side with equal coefficients, so the left-hand side of equation (5–5) is also zero. This, however, gives no information about the constant c.

For $z' \in \{t_0, t_1, t_2, t_3\}$, we see from Tables 10 and 11 that only one of the nonzero values of $\hat{\varphi}_x$ appears with a nonzero coefficient, and we see from Tables 6, 7, 8, and 9 that also only one nonzero term appears on the left. The left-hand side of equation (5–5) equals $-\frac{1}{2}\sqrt{q}\zeta^3 q^n \mathrm{SIGN}_1$, where $\mathrm{SIGN}_1 = 1$ if the coefficient $M^{x,U_1}_{z',n,\nu}$ is nonzero, and $\mathrm{SIGN}_1 = -1$ if $M^{x,U_\varepsilon}_{z',n,\nu}$ is nonzero.

The right-hand side equals

$$c\frac{2^2}{q-1}\sqrt{q}\zeta^3\frac{q^{n+1}}{q+1}\mathrm{SIGN}_2 \quad \text{if } x = (0),$$
$$c\frac{2^2}{q-1}\sqrt{q}\zeta\frac{q^{n+1}}{q+1}\mathrm{SIGN}_2 \quad \text{if } x = (1),$$

where SIGN_2 is the sign that depends on which one of the virtual Chow motives $\mathcal{N}^{1,2,\pm}_{z',n,\nu}$ is nonzero. Let $c = -\frac{q^2-1}{2^3 q}\zeta^2$, which is the constant that appears in Table 4. Then, to finish the proof of the theorem it remains to show that on every set $\mathfrak{h}_{z',n,\nu}$ with $z' \in \{t_0,\dots,t_3\}$, we have the identity $\mathrm{SIGN}_1 = \zeta^2\mathrm{SIGN}_2$ in the case $x = (1)$, and $\text{SIGN}_1 = \text{SIGN}_2$ in the case $x = (0)$. Here is the comparison of the two SIGNs in the case $x = (0)$:

$$z = t_0\colon \text{SIGN}_1 = \nu\zeta^{2n},\ \text{SIGN}_2 = \zeta^{2(n+1)}.$$
$$z = t_1\colon \text{SIGN}_1 = -\nu\zeta^{2n},\ \text{SIGN}_2 = -\nu\zeta^{2(n+1)}.$$
$$z = t_2\colon \text{SIGN}_1 = \nu\zeta^{2n}(-1)^n,\ \text{SIGN}_2 = \nu\zeta^{2(n+1)}(-1)^n.$$
$$z = t_3\colon \text{SIGN}_1 = \nu\zeta^{2n}(-1)^{n+1},\ \text{SIGN}_2 = \nu\zeta^{2(n+1)}(-1)^{n+1}.$$

We see that in all cases, $\text{SIGN}_1 = \zeta^2\text{SIGN}_2$, which completes the proof in the case $x = (0)$.

The case $x = (1)$ is identical, except that we need to use Tables 8 and 9 instead of Tables 6 and 7 to calculate SIGN_1, and the first column of Tables 10 and 11 instead of the second column to calculate SIGN_2.

6. FINAL COMMENTS

6.1 Theorem 2.5 and Our Choices

Let us begin by reviewing all the choices made in this paper before the proof of Theorem 2.5.

First, in the preamble to Section 2, we began with an odd prime p, a p-adic field $\mathbb{K}$, a prime ℓ different from p (e.g.,, $\ell = 2$), and an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

Next, in Section 2.2 we fixed an additive character $\bar{\psi} : \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell$ and a square root $\sqrt{q}$ of q in $\bar{\mathbb{Q}}_\ell$. These choices (via Gauss sums) determined a fourth root of unity $\zeta \in \bar{\mathbb{Q}}_\ell$ such that $\zeta^2 = \text{sgn}(-1)$ (see Remark 2.1). This, in turn, determined how we labeled the two representations σ_+ and σ_- in the Lusztig series for (T, θ_0) and therefore our definition of Q_G (see Section 2.3), and therefore our definition of $\pi(0,+)$, $\pi(0,-)$, $\pi(1,+)$, and $\pi(1,-)$ (see Remark 2.2).

Independently, we fixed cocycles $\{s_1, s_2, t_0, t_1, t_2, t_3\}$ in $Z^1(\mathbb{K}, N)$ such that their cohomology classes lay in the kernel of the map $H^1(\mathbb{K}, N) \to H^1(\mathbb{K}, G)$ induced by inclusion $N \to G$. This choice determined a uniformizer ϖ for $\mathbb{K}$ and a nonsquare unit ε in $\mathcal{O}_\mathbb{K}$ (see Remark 2.4). We remind the reader that if $\text{sgn}(-1) = -1$, then the cohomology class for t_0 equals the cohomology class for t_1, and the cohomology class for t_2 equals the cohomology class for t_3.

Finally, for each $z \in \{s_1, s_2, t_0, t_1, t_2, t_3\}$ we chose an element $X_z \in \mathfrak{g}$ with minimal nonnegative depth in its Cartan subalgebra; these are listed in Table 3. This last step amounted to the choice of a unit v in $\mathcal{O}_\mathbb{K}$ (see Section 2.8).

Also, the Fourier transform of the orbital integral μ_{X_z} is taken with respect to an additive character $\psi : \mathbb{K} \to \bar{\mathbb{Q}}_\ell$ with conductor $\mathcal{O}_\mathbb{K}$ such that the induced additive character of $\mathbb{F}_q$ is $\bar{\psi}$. Also, we must use compatible Killing forms, and the correct measures everywhere, as we did.

With all these choices that we made, the values of the coefficients $c_z(\pi)$ in our semisimple character expansion are presented in Table 4. From that table we make three observations.

1. If π is induced from a Deligne–Lusztig representation, then for each cocycle z, $c_z(\pi)$ is a rational function of q with integer coefficients.

2. If the cocycle z is unramified (by which we mean that the Cartan T_z is unramified) then $c_z(\pi)$ is a rational function in q with integer coefficients for every depth-zero supercuspidal irreducible representation π.

3. If π is a depth-zero supercuspidal irreducible representation but π is not induced from a Deligne–Lusztig representation and if the cocycle z is ramified (by which we mean that the Cartan T_z is ramified), then $c_z(\pi)$ is a rational number for every p, but it is *not* a rational function of q. Instead, in this case, $c_z(\pi)$ is a rational function of q multiplied by $\text{sgn}_q(-1)$. Observe that $\text{sgn}_q(-1)$ cannot be expressed as a rational function in q.

It is reasonable to ask whether these properties would continue to hold had we made different choices above. The answer is affirmative. To see why, we say a few words about how the semisimple character expansion would change if the definition of the X_z's were modified. Since we have gathered complete information about the Fourier transforms of all regular elliptic orbital integrals $\hat{\mu}_X$ (when X has minimal nonnegative depth in its Cartan subalgebra) evaluated at topologically nilpotent elements Y, we can explore the dependence of the coefficients in the semisimple character expansion of the orbits we chose. (Of course, each X_z can be replaced by any element in the same adjoint orbit as X_z, since the distributions $\hat{\mu}_{X_z}$ and Θ_π are invariant under the group action.)

To begin, consider the local constancy of $X \mapsto \hat{\mu}_X(f)$ for a fixed Schwartz function f supported by topologically nilpotent elements. Using equations (4–6) and (3–1) we see that

$$\int_G \int_\mathfrak{g} f(\text{Ad}(g)Y)\,(Q_T)_{(0),0}(Y)\,dY\,dg = (1-q)\hat{\mu}_{X_{s_1}(v)}(f). \tag{6–1}$$

Moreover, since Q_T (see Section 2.3) does not depend on $v \in \mathcal{O}_\mathbb{K}^*$, it follows that $\hat{\mu}_{X_{s_1}(v)}(f)$ is independent of v;

thus, for any f as above,

$$\int_{\mathcal{O}_{\mathbb{K}}^*} \hat{\mu}_{X_{s_1(v)}}(f)\,dv = (q-1)\hat{\mu}_{X_{s_1}}(f), \tag{6–2}$$

where the Haar measure on $\mathcal{O}_{\mathbb{K}}^*$ is chosen such that $\mathcal{O}_{\mathbb{K}}^*$ has measure $q-1$.

From our point of view, it would have been more natural to rewrite equation (6–1) (and therefore the first line in Table 4) in the form

$$\Theta_{\pi(0,\theta)}(f) = \int_{\mathcal{O}_{\mathbb{K}}^*} \hat{\mu}_{X_{s_1(v)}}(f)\,dv, \tag{6–3}$$

from which we see that our choice for v was completely unimportant here. (See Sections 2.1 and 2.3 for the definition of $\pi(0,\theta)$.) Similarly, we can rewrite the last line of Table 4 in the form

$$\Theta_{\pi(1,\theta)}(f) = \int_{\mathcal{O}_{\mathbb{K}}^*} \hat{\mu}_{X_{s_2(v)}}(f)\,dv. \tag{6–4}$$

These expressions depend on our choice for the cocycles s_1 and s_2 only. Therefore, once the cocycle s_1 is chosen, the element X_{s_1} may be replaced by *any* element in the Cartan subalgebra uniquely determined by the cocycle s_1 as long as that element has minimal nonnegative depth in that Cartan subalgebra; likewise for the cocycle s_2.

Regarding the totally ramified Cartan subgroups, things are a bit more subtle. From Table 4 we see that if f is a Schwartz function supported by topologically nilpotent elements, then

$$\int_G \int_{\mathfrak{g}} f(\mathrm{Ad}(g)Y)\,(Q_G)_{(01),\frac{1}{2}}(Y)\,dY\,dg = \frac{q^2-1}{2^2 q}\left(\hat{\mu}_{X_{t_0}}(f) - \hat{\mu}_{X_{t_1}}(f) + \hat{\mu}_{X_{t_2}}(f) - \hat{\mu}_{X_{t_3}}(f)\right).$$

Invoking the local constancy of $X \mapsto \hat{\mu}_X(f)$, we may rewrite the equation above in the form

$$\begin{aligned}\frac{2q}{q+1}\int_G \int_{\mathfrak{g}} f(\mathrm{Ad}(g)Y)\,(Q_G)_{(0),0}(Y)\,dY\,dg &= \int_{\{\mathrm{sgn}(v)=+1\}} \hat{\mu}_{X_{t_0(v)}}(f)\,dv \\ &- \int_{\{\mathrm{sgn}(v)=+1\}} \hat{\mu}_{X_{t_1(v)}}(f)\,dv \\ &+ \int_{\{\mathrm{sgn}(v)=+1\}} \hat{\mu}_{X_{t_2(v)}}(f)\,dv \\ &- \int_{\{\mathrm{sgn}(v)=+1\}} \hat{\mu}_{X_{t_3(v)}}(f)\,dv.\end{aligned}$$

Similar observations apply to representations induced from $G_{(1)}$. In this way we see that only $\mathrm{sgn}(v)$ was important in our definition of the ramified orbits appearing in Theorem 2.5. More precisely, once the cocycle t_0 is chosen, the element X_{t_0} may be replaced by $X_{t_0}(v)$ as long as $\mathrm{sgn}(v)=1$; likewise for the cocycles t_1, t_2, and t_3. However, if $\mathrm{sgn}(v)=-1$ and $z \in \{t_0,t_1,t_2,t_3\}$, then $\hat{\mu}_{X_z}$ is *not* equal to $\hat{\mu}_{X_{t_1(v)}}(Y)$, even when restricted to functions supported by topologically nilpotent elements of $\mathfrak{g}$.

Having dealt with v, we return to our choice of nonsquare unit ε. From the local constancy of the Fourier transform of the orbital integrals we have the following immediate consequence. For any Schwartz function f supported by topologically nilpotent elements of $\mathfrak{g}$,

$$\int_{\{\mathrm{sgn}(\delta)=-1\}} \hat{\mu}_{\left[\begin{smallmatrix}0 & 1\\ \delta\varpi & 0\end{smallmatrix}\right]}(f)\,d\delta = \frac{q-1}{2}\hat{\mu}_{X_{t_2}}(f). \tag{6–5}$$

Similar observations hold for all our orbits. Motivated by the motivic integration view of things, we might have replaced each of our orbital integrals with an integral over v and δ (ε is a particular value of δ). In fact, that is exactly what we did when we used motivic integration with parameters in Section 3.3.2 and allowed the parameter to vary over the set of all nonsquares in the residue field. The result is easily related back to orbital integrals appearing in representation theory because of relations of the following form:

$$\begin{aligned}\int_{\{\mathrm{sgn}(\delta)=-1\}}\int_{\{\mathrm{sgn}(v)=+1\}} \hat{\mu}_{\left[\begin{smallmatrix}0 & v\\ \delta\varpi v & 0\end{smallmatrix}\right]}(f)\,dv\,d\delta \\ = \frac{(q-1)^2}{2^2}\hat{\mu}_{X_{t_2}}(f).\end{aligned} \tag{6–6}$$

Similar observations hold for all our orbital integrals. Consequently, replacing any of our orbital integrals with these "smeared" orbital integrals would have the effect of changing the coefficients in the semisimple character expansion in very simple ways.

In summary, we have shown in this section that the three observations made above concerning the rationality of $c_z(\pi)$ are completely independent of all our choices.

Also, since we have been switching freely between polynomials in q and virtual Chow motives, we notice that the coefficients $c_z(\pi)$ can always be interpreted as elements of the ring Mot, as we do in Table 12. Indeed, if we replace the polynomials in q with the corresponding elements of the ring Mot as we have been doing so far, we can see that the denominators of $c_z(\pi)$ are invertible in the ring Mot. Moreover, a motivic expression for ζ^2 is

$c_z(\pi)$	$z=s_1$	$z=s_2$	$z=t_0$	$z=t_1$	$z=t_2$	$z=t_3$
$\pi=\pi(0,\theta)$	$\mathbb{L}-1$	0	0	0	0	0
$\pi=\pi(1,\theta)$	0	$\mathbb{L}-1$	0	0	0	0
$\pi=\pi(0,+)$	$\frac{\mathbb{L}-1}{2}$	0	$+\frac{\mathbb{M}}{2^3\mathbb{L}}$	$-\frac{\mathbb{M}}{2^3\mathbb{L}}$	$+\frac{\mathbb{M}}{2^3\mathbb{L}}$	$-\frac{\mathbb{M}}{2^3\mathbb{L}}$
$\pi=\pi(0,-)$	$\frac{\mathbb{L}-1}{2}$	0	$-\frac{\mathbb{M}}{2^3\mathbb{L}}$	$+\frac{\mathbb{M}}{2^3\mathbb{L}}$	$-\frac{\mathbb{M}}{2^3\mathbb{L}}$	$+\frac{\mathbb{M}}{2^3\mathbb{L}}$
$\pi=\pi(1,+)$	0	$\frac{\mathbb{L}-1}{2}$	$+\frac{\mathbb{M}}{2^3\mathbb{L}}$	$-\frac{\mathbb{M}}{2^3\mathbb{L}}$	$-\frac{\mathbb{M}}{2^3\mathbb{L}}$	$+\frac{\mathbb{M}}{2^3\mathbb{L}}$
$\pi=\pi(1,-)$	0	$\frac{\mathbb{L}-1}{2}$	$-\frac{\mathbb{M}}{2^3\mathbb{L}}$	$+\frac{\mathbb{M}}{2^3\mathbb{L}}$	$+\frac{\mathbb{M}}{2^3\mathbb{L}}$	$-\frac{\mathbb{M}}{2^3\mathbb{L}}$

TABLE 12. Motives for the coefficients $c_z(\pi)$ appearing in Theorem 2.5. Here we write $\mathbb{M}$ for $(\mathbb{L}^2-1)(1-\mathbb{S})$ in order to save space.

found by considering the 0-dimensional variety defined by the equation $x^2=-1$ (this variety appears for the same purpose in [Hales 05b]): if $\mathbb{S}$ denotes the class of $x^2=-1$ in the ring Mot, then $\zeta^2=-\mathrm{TrFrob}(1-\mathbb{S})$. Note, moreover, that this motive appears exactly when we study depth-zero supercuspidal representations π that are not induced from Deligne–Lusztig representations and when we consider cocycles z for which the corresponding Cartan T_z is ramified. We will have more to say about this phenomenon in Section 6.2.

6.2 Theorem 2.5 and Endoscopy

This paper has focused on the motivic nature of the values of characters of depth-zero supercuspidal representations of p-adic SL(2), on the motivic nature of the Fourier transform of some associated orbital integrals, and on the relations between the associated motives in the Chow ring. In particular, the techniques used in the paper make it clear that once these motives are determined, the character formula of Theorem 2.5 admits a proof that could easily be automated. While it is seems promising to illustrate a strategy showing that certain results from local harmonic analysis can be proved algorithmically, this method of proof does not explain some of the striking patterns in Table 4. In this section we explain these patterns.

First, one must note that although Table 4 is a 6×6 matrix, the rank of this matrix is 4. It is easy to understand why the rank of the matrix is at most 5: it follows immediately from the nilpotent characters expansion that characters of depth-zero supercuspidal representations of $\mathrm{SL}(2,\mathbb{K})$ span a space of dimension at most 5, since that is the number of nilpotent orbits in $\mathfrak{sl}(2,\mathbb{K})$ when the residual characteristic of $\mathbb{K}$ is odd.

But why is the rank of Table 4 exactly 4? Going back to Section 2.3, observe that characters of cuspidal representations of $\mathrm{SL}(2,\mathbb{F}_q)$ are linearly dependent when restricted to unipotent elements. In fact, the set $\{\mathrm{trace}\,\sigma_\theta, \mathrm{trace}\,\sigma_+, \mathrm{trace}\,\sigma_-\}$ admits exactly one linear relation on unipotent elements: if $g\in\mathfrak{sl}(2,\mathbb{F}_q)$ is unipotent, then

$$\mathrm{trace}\,\sigma_+(g)+\mathrm{trace}\,\sigma_-(g)=\mathrm{trace}\,\sigma_\theta(g). \tag{6–7}$$

(This relation is best understood through Lusztig's work, but this would take us too far afield.) Two linear relations involving characters of depth-zero supercuspidal representations on topologically nilpotent elements of $\mathfrak{sl}(2,\mathbb{K})$ follow directly from this observation: if f is supported by topologically nilpotent elements, then

$$\Theta_{\pi(0,+)}(\mathrm{cay}^* f)+\Theta_{\pi(0,-)}(\mathrm{cay}^* f)=\Theta_{\pi(0,\theta)}(\mathrm{cay}^* f) \tag{6–8}$$

and

$$\Theta_{\pi(1,+)}(\mathrm{cay}^* f)+\Theta_{\pi(1,-)}(\mathrm{cay}^* f)=\Theta_{\pi(1,\theta)}(\mathrm{cay}^* f). \tag{6–9}$$

Together with the fact that the Fourier transform of our orbital integrals are linearly independent on functions supported by topologically nilpotent elements (which can be seen using the techniques of [Cunningham and Hales 04, Section 1]), this explains why the rank of Table 4 is exactly 4.

But a deeper understanding of Table 4 begins with the following observation: if f is supported by topologically nilpotent elements, then

$$\begin{aligned}\Theta_{\pi(0,\theta)}(\mathrm{cay}^*(f)) + \Theta_{\pi(1,\theta)}(\mathrm{cay}^* f) \\ = (q-1)\left(\hat{\mu}_{X_{s_1}}(f) + \hat{\mu}_{X_{s_2}}(f)\right);\end{aligned} \tag{6–10}$$

moreover, the right-hand side is the Fourier transform of the stable distribution

$$\mu^{\mathrm{st}}_{X_{s_1}} := \mu_{X_{s_1}} + \mu_{X_{s_2}}, \tag{6–11}$$

and so it follows from the work of Waldspurger that $\hat{\mu}^{\mathrm{st}}_{X_{s_1}}$ is a stable distribution. Thus, the sum of characters on the left-hand side is a stable distribution on the set of topologically nilpotent elements. In fact,

$$\{\pi(0,\theta), \pi(1,\theta)\}$$

is an L-packet (see [Labesse and Langlands 79, Section 12]). Likewise, from Table 4 we see that if f is supported by topologically nilpotent elements, then

$$\begin{aligned}\Theta_{\pi(0,+)}(\mathrm{cay}^* f) + \Theta_{\pi(1,+)}(\mathrm{cay}^* f) + \Theta_{\pi(1,+)}(\mathrm{cay}^* f) \\ + \Theta_{\pi(1,-)}(\mathrm{cay}^* f) = (q-1)\hat{\mu}^{\mathrm{st}}_{X_{s_1}}(f),\end{aligned} \tag{6–12}$$

which is the same stable distribution appearing above. In fact,

$$\{\pi(0,+), \pi(0,-), \pi(1,+), \pi(1,-)\}$$

is an L-packet (see [Labesse and Langlands 79, Section 12]). Thus, if π is a depth-zero supercuspidal irreducible representation of $\mathrm{SL}(2,\mathbb{K})$, then the L-packet containing π has cardinality two if π is induced from a Deligne–Lusztig representation; otherwise, the L-packet containing π has cardinality four.

Now we are ready to understand the pattern seen in Table 4 through the theory of endoscopy. There are exactly five endoscopic groups H for G; four elliptic endoscopic groups, and one nonelliptic endoscopic group. The elliptic endoscopic groups are SL(2) itself, and three copies of U(1), one for each quadratic extension of $\mathbb{K}$. In Table 13 we write $\mathrm{U}_\varepsilon(1)$ for the special unitary group splitting over $\mathbb{K}(\sqrt{\varepsilon})$, $\mathrm{U}_\varpi(1)$ for the special unitary group splitting over $\mathbb{K}(\sqrt{\varpi})$, and $\mathrm{U}_{\varepsilon\varpi}(1)$ for the special unitary group splitting over $\mathbb{K}(\sqrt{\varepsilon\varpi})$. The nonelliptic endoscopic group for G is GL(1), which plays the role of a Levi subgroup of SL(2); had we considered nonsupercuspidal depth-zero representations in this paper, it would have played an important part.

In Table 13 we pick one good elliptic element Y_H from the Lie algebra $\mathfrak{h}$ of each elliptic endoscopic group for G.

H	SL(2)	$\mathrm{U}_\varepsilon(1)$	$\mathrm{U}_\varpi(1)$	$\mathrm{U}_{\varepsilon\varpi}(1)$
Y_H	$\left[\begin{smallmatrix}0 & 1\\ \varepsilon & 0\end{smallmatrix}\right]$	$\sqrt{\varepsilon}$	$\sqrt{\varpi}$	$\sqrt{\varepsilon\varpi}$
X_H	X_{s_1}	X_{s_1}	X_{t_0}	X_{t_2}
$\mu^{G,H}_{Y_H}$	$\mu^{\mathrm{st}}_{X_{s_1}}$	$\mu^{\mathrm{sgn}}_{X_{s_1}}$	$\mu^{\mathrm{sgn}}_{X_{t_0}}$	$\mu^{\mathrm{sgn}}_{X_{t_2}}$

TABLE 13. Elliptic endoscopic groups H for SL(2) over $\mathbb{K}$; one SL(2)-regular element Y_H from each $\mathfrak{h} = \mathrm{Lie}H$; an image $X_H \in \mathfrak{sl}(2,\mathbb{K})$ under the Langlands–Shelstad map; and the κ-orbital integral determined by Y_H.

Each $Y_H \in \mathfrak{h}$ is $\mathfrak{g}$-regular; for each Y_H we choose an image $X_H = X_z$ in $\mathfrak{g}$ from the list of elements appearing in Theorem 2.5 such that $\Delta_{\mathfrak{g},\mathfrak{h}}(X_H, Y_H) = 1$, where $\Delta_{\mathfrak{g},\mathfrak{h}}$ is the Langlands–Shelstad transfer factor for the pair $(\mathfrak{g},\mathfrak{h})$. Each $Y_H \in \mathfrak{h}$ thus determines a κ-orbital integral on $\mathfrak{g}$ according to the formula

$$\mu^{G,H}_{Y_H} = \sum_{X'} \Delta_{\mathfrak{g},\mathfrak{h}}(X', Y_H)\mu_{X'}, \tag{6–13}$$

where the sum is taken over adjoint orbits in $\mathfrak{g}$. In our cases the results are

$$\begin{aligned}\mu^{G,\mathrm{SL}(2)}_{Y_{\mathrm{SL}(2)}} &= \mu_{X_{s_1}} + \mu_{X_{s_2}} = \mu^{\mathrm{st}}_{X_{s_1}},\\ \mu^{G,\mathrm{U}_\varepsilon(1)}_{Y_{\mathrm{U}_\varepsilon(1)}} &= \mu_{X_{s_1}} - \mu_{X_{s_2}} = \mu^{\mathrm{sgn}}_{X_{s_1}},\\ \mu^{G,\mathrm{U}_\varpi(1)}_{Y_{\mathrm{U}_\varpi(1)}} &= \mu_{X_{t_0}} - \mu_{X_{t_1}} = \mu^{\mathrm{sgn}}_{X_{s_1}},\\ \mu^{G,\mathrm{U}_{\varepsilon\varpi}(1)}_{Y_{\mathrm{U}_{\varepsilon\varpi}(1)}} &= \mu_{X_{t_2}} - \mu_{X_{t_3}} = \mu^{\mathrm{sgn}}_{X_{t_2}},\end{aligned}$$

as recorded in Table 13. It is now clear from Table 4 that when restricted to Schwartz functions supported by topologically nilpotent elements, the Fourier transforms of these four distributions span exactly the same space spanned by the characters of depth-zero supercuspidal representations of G (on the pullback by the Cayley transform of the same space of functions). In fact, the distributions

$$\{\hat{\mu}^{\mathrm{st}}_{X_{s_1}}, \hat{\mu}^{\mathrm{sgn}}_{X_{s_1}}, \hat{\mu}^{\mathrm{sgn}}_{X_{s_1}}, \hat{\mu}^{\mathrm{sgn}}_{X_{t_2}}\}$$

are linearly independent on the set of Schwartz functions supported by topologically nilpotent elements, and they provide a *natural* basis for the characters of depth-zero supercuspidal representations of G (on the pullback by the Cayley transform of the same space of functions).

	$H = SL(2)$	$H = \mathrm{U}_{\varepsilon}(1)$	$H = \mathrm{U}_{\varpi}(1)$	$H = \mathrm{U}_{\varepsilon\varpi}(1)$
$\pi = \pi(0,\theta)$	$+\frac{\mathbb{L}-1}{2}$	$+\frac{\mathbb{L}-1}{2}$	0	0
$\pi = \pi(1,\theta)$	$+\frac{\mathbb{L}-1}{2}$	$-\frac{\mathbb{L}-1}{2}$	0	0
$\pi = \pi(0,+)$	$+\frac{\mathbb{L}-1}{2^2}$	$+\frac{\mathbb{L}-1}{2^2}$	$+\frac{\mathbb{L}^2-1}{2^3\mathbb{L}}(1-\mathbb{S})$	$+\frac{\mathbb{L}^2-1}{2^3\mathbb{L}}(1-\mathbb{S})$
$\pi = \pi(0,-)$	$+\frac{\mathbb{L}-1}{2^2}$	$+\frac{\mathbb{L}-1}{2^2}$	$-\frac{\mathbb{L}^2-1}{2^3\mathbb{L}}(1-\mathbb{S})$	$-\frac{\mathbb{L}^2-1}{2^3\mathbb{L}}(1-\mathbb{S})$
$\pi = \pi(1,+)$	$+\frac{\mathbb{L}-1}{2^2}$	$-\frac{\mathbb{L}-1}{2^2}$	$+\frac{\mathbb{L}^2-1}{2^3\mathbb{L}}(1-\mathbb{S})$	$-\frac{\mathbb{L}^2-1}{2^3\mathbb{L}}(1-\mathbb{S})$
$\pi = \pi(1,-)$	$+\frac{\mathbb{L}-1}{2^2}$	$-\frac{\mathbb{L}-1}{2^2}$	$-\frac{\mathbb{L}^2-1}{2^3\mathbb{L}}(1-\mathbb{S})$	$+\frac{\mathbb{L}^2-1}{2^3\mathbb{L}}(1-\mathbb{S})$

TABLE 14. Motives for the *unique* coefficients $c_H(\pi)$ appearing in Theorem 6.1.

The result is Theorem 6.1, which enjoys one advantage over Theorem 2.5: the coefficients are *unique* with the choices made above. Moreover, regarding these choices, the techniques of Section 6.1 apply here too, as do all other techniques from this paper.

Theorem 6.1. *Let $\mathbb{K}$ be a p-adic field with $p \neq 2$. For each depth-zero supercuspidal representation π of G and for each elliptic endoscopic group H for G there is a good elliptic $Y_H \in \mathrm{Lie}H$ with minimal nonnegative depth in $\mathrm{Lie}H$ and a unique rational number $c_H(\pi)$ such that*

$$\Theta_\pi(\mathrm{cay}^* f) = \sum_H c_H(\pi)\hat{\mu}^{G,H}_{Y_H}(f) \qquad (6\text{–}14)$$

for all Schwartz functions f supported by topologically nilpotent elements in $\mathfrak{g}$. Moreover, the coefficients are motivic, in the sense explained in this paper. Motives for the coefficients $c_H(\pi)$ are given in Table 14.

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REFERENCES

[Adler 98] Jeffrey Adler. "Refined Anisotropic K-Types and Supercuspidal Representations." *Pacific Journal of Mathematics* 185:1 (1998), 1–32.

[Bruhat and Tits 96] François Bruhat and Jacques Tits. "Groupes réductifs sur un corps local." *Inst. Hautes Études Sci. Publ. Math.* 41 (1972), 5–251.

[Bushnell and Henniart 96] C. Bushnell and G. Henniart. "Local Tame Lifting for GL(N) I: Simple Characters." *Inst. Hautes Études Sci. Publ. Math.* 83 (1996), 105–233.

[Cluckers and Loeser 08] R. Cluckers amd F. Loeser. "Constructible Motivic Functions and Motivic Integration." *Invent. Math.* 173:1 (2008), 23–121.

[Cluckers et al. 07] R. Cluckers, T.C. Hales, and F. Loeser. "Transfer Principle for the Fundamental Lemma." Preprint available online (http://arxiv.org/abs/0712.0708), 2007.

[Cunningham and Hales 04] Clifton Cunningham and Thomas Hales. "Good Orbital Integrals." *Represent. Theory* 8 (2004), 414–457.

[Deligne and Lusztig 76] Pierre Deligne and George Lusztig. "Representations of Reductive Groups over Finite Fields." *Annals of Mathematics* 103 (1976), 103–161.

[Denef and Loeser 01] J. Denef and F. Loeser. "Definable Sets, Motives and p-adic Integrals." *J. Amer. Math. Soc.* 14:2 (2001), 429–469.

[Digne and Michel 91] François Digne and Jean Michel. *Representations of Finite Groups of Lie Type*, London Mathematics Society Student Texts 21. Cambridge, UK: Cambridge University Press, 1991.

[Gordon 04] Julia Gordon. "Motivic Nature of Character Values of Depth Zero Representations." *Int. Math. Res. Not.* 34 (2004), 1735–1760.

[Gordon 09] Julia Gordon. "An Overview of Arithmetic Motivic Integration." To appear in *Ottawa Lectures on Representations of Reductive Algebraic Groups*, Fields Institute Monograph Series, 2009.

[Hales 05a] Thomas Hales. "Orbital Integrals Are Motivic." *Proc. Amer. Math. Soc.* 133:5 (2005), 1515–1525.

[Hales 05b] Thomas Hales. “What Is Motivic Measure?” *Bull. Amer. Math. Soc. (N.S.)* 42:2 (2005), 119–135.

[Harish-Chandra 70] Harish-Chandra (with notes by G. van Dijk). *Harmonic Analysis on Reductive p-adic Groups*, Springer Lecture Notes in Mathematics 162. New York: Springer-Verlag, 1970.

[Labesse and Langlands 79] J.-P. Labesse and R. P. Langlands. “L-Indistinguishability for SL(2).” *Canad. J. Math.* 31:4 (1979), 726–785.

[Lusztig 85] George Lusztig. “Character Sheaves, I–V.” *Advances in Mathematics* 56 (1985), 193–297; *Advances in Mathematics* 57 (1985), 226–265; *Advances in Mathematics* 57 (1986), 266–315; *Advances in Mathematics* 59 (1986), 1–63; *Advances in Mathematics* 61 (1986), 103–155.

[Moy and Prasad 94] Allen Moy and Gopal Prasad. “Unrefined Minimal K-Types for p-adic Groups.” *Inventiones Mathematicae* 116 (1994), 393–408.

[Murnaghan 96] Fiona Murnaghan. “Characters of Supercuspidal Representations of Classical Groups.” *Ann. Sci. École Norm. Sup. (4)* 29:1 (1996), 49–105.

[Sally and Shalika 68] P. J. Sally Jr. and J. A. Shalika. “Characters of the Discrete Series of Representations of SL(2) over a Local Field.” *Proc. Nat. Acad. Sci. U.S.A.* 61 (1968), 1231–1237.

[Scholl 94] A. J. Scholl. “Classical Motives.” In *Motives (Seattle, WA, 1991)*, pp. 163–187, Proc. Sympos. Pure Math. 55, Part 1. Providence: Amer. Math. Soc., 1994.

[Vigneras 03] Marie-France Vigneras. “Schur Algebras of Reductive p-adic Groups I.” *Duke Math. J.* 116:1 (2003), 35–75.

[Yu 02] Jiu-Kang Yu. “Smooth Models Associated to Concave Functions in Bruhat–Tits Theory.” Preprint, 2002.

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