Computing Central Values of Twisted *L*-Series: The Case of Composite Levels

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CONTENTS

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We describe a general method to compute weight-$\frac{3}{2}$ modular forms "associated" with a given weight-2 modular form f of level N, and relate its Fourier coefficients to central values of quadratic twists (real and imaginary) of $L(f, s)$. We will focus on examples for levels $N = 27$, $N = 15$, and $N = 75$.

1. INTRODUCTION

Let $f \in S_2(N)$ be a newform of weight 2 and level N. If $f(z) = \sum_{m=1}^{\infty} a(m)q^m$, where $q = e^{2\pi i z}$, and D is a fundamental discriminant, we define the twisted L-series

$$L(f, D, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \left(\frac{D}{m}\right).$$

We will assume that the twisted L-series are primitive (i.e., the corresponding twisted modular forms are newforms). There is no loss of generality in making this assumption: if this were not the case, then f would be a quadratic twist of a newform of smaller level, which we could choose instead.

The question of efficiently computing the family of central values $L(f, D, 1)$, for fundamental discriminants D, has been considered by several authors (see [Gross 87, Böcherer and Schulze-Pillot 90, Pacetti and Tornaría 07a, Pacetti and Tornaría 07b, Mao et al. 07]). By Waldspurger's formula [Waldspurger 81], these values are related to the Fourier coefficients of certain modular forms of weight $\frac{3}{2}$.

Gross [Gross 87] gives a method to construct, for the case of prime level p and provided that $L(f, 1) \neq 0$, a weight-$\frac{3}{2}$ modular form of level $4p$, and gives an explicit version of Waldspurger's formula for the imaginary quadratic twists. In [Böcherer and Schulze-Pillot 90], the authors extend Gross's method to the case of square-free level, but their method works only for a fraction of imaginary quadratic twists (determined by quadratic residue

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conditions). Later, in [Pacetti and Tornaría 07a], the case of level p^2 (p a prime) is considered, and this is used in [Pacetti and Tornaría 07b], provided $p \equiv 3 \pmod 4$, to compute central values for *real* quadratic twists.

In [Mao et al. 07], the nonvanishing condition is removed, and in the case of prime level, two modular forms of weight $\frac{3}{2}$ (one giving the imaginary quadratic twists and another giving the real quadratic twists) are constructed.

The aim of this paper is to show how some of these ideas can be combined to handle the case of composite levels. In the case of odd square-free level N, for instance, this method constructs 2^t modular forms, where t is the number of prime factors of N, whose coefficients give the central values of all the quadratic twists. We will focus on examples for levels $N = 27$, $N = 15$, and $N = 75$, which exhibit our methods for the nonsquare case. For the square case see [Pacetti and Tornaría 07a] and [Pacetti and Tornaría 07b].

2. THE CURVE $27A$

Let f be the modular form of level 27, corresponding to the elliptic curve $X_0(27)$, of minimal equation

$$y^2 + y = x^3 - 7$$

(see [Cremona 08]). The eigenvalue of f for the Atkin–Lehner involution W_{27} is -1, and the sign of the functional equation for $L(f, s)$ is $+1$.

Let $B = (-1, -3)$ be the quaternion algebra ramified at 3 and ∞, and consider the order $R = \langle 1, 3i, \frac{1+3j}{2}, \frac{i+k}{2} \rangle$, a Pizer order of reduced discriminant 27 (see [Pizer 80] for the basic definitions of quaternion algebras, Brandt matrices, and special orders). The class number of left R-ideals for such an order is 2, and representatives for left R-ideals are $\{R, I\}$, where $I = \langle 4, 12i, \frac{7+6i+3j}{2}, \frac{6+13i+k}{2} \rangle$. The eigenvector for the Brandt matrices that corresponds to f is $(1, -1)$, with height 3.

The ternary quadratic forms associated with their right orders are

$$Q_1(x, y, z) = 4x^2 + 27y^2 + 28z^2 - 4xz$$

and

$$Q_2(x, y, z) = 7x^2 + 16y^2 + 31z^2 + 16yz + 2xz + 4xy,$$

respectively.

Note that since the twist of f by the quadratic character of conductor 3 is f itself, we have

$$L(f, -3D, s) = L(f, D, s),$$

D	$c(D)$	$L(f, D, 1)$
-4	1	1.529954
-7	-1	1.156537
-19	-1	0.701991
-31	0	0.000000
-40	-2	1.935256
-43	2	1.866526
-52	1	0.424333
-55	2	1.650392
-67	-1	0.373827
-79	1	0.344267
-88	-2	1.304749
-91	1	0.320766
-103	1	0.301502
-115	-2	1.141352
-127	-2	1.086092
-136	2	1.049540
-139	3	2.335842
-148	1	0.251523
-151	-1	0.249012
-163	-1	0.239670
-184	2	0.902318
-187	-2	0.895051
-199	-3	1.952200

TABLE 1. Coefficients of g and imaginary quadratic twists of $27A$.

for $-3D$ a fundamental discriminant. We will thus assume that $3 \nmid D$.

2.1 Imaginary Quadratic Twists

Let $D < 0$ be a fundamental discriminant. If $\left(\frac{D}{3}\right) = +1$, the sign of the functional equation for $L(f, D, s)$ is -1, so its central value vanishes trivially. Hence we can restrict to the case $\left(\frac{D}{3}\right) = -1$. In this case, we can follow Gross's method, using classical theta series

$$\Theta(Q_i) := \frac{1}{2} \sum_{(x,y,z) \in \mathbb{Z}^3} q^{Q_i(x,y,z)};$$

we obtain a weight-$\frac{3}{2}$ modular form of level $4 \cdot 27$, namely

$$g = \Theta(Q_1) - \Theta(Q_2) = q^4 - q^7 - q^{19} + q^{28} - 2q^{40} + 2q^{43} + \cdots.$$

Table 1 shows the values of the Fourier coefficients $c(D)$ of g and of $L(f, D, 1)$, where $-200 < D < 0$ is a fundamental discriminant such that $\left(\frac{D}{3}\right) = -1$. The Gross type formula

$$L(f, D, 1) = k \, \frac{|c(D)|^2}{\sqrt{|D|}}, \quad D < 0,$$

is satisfied, where $c(D)$ is the $|D|$th Fourier coefficient of g, and

$$k = \frac{1}{3}\cdot\frac{(f,f)}{L(f,1)} = 2L(f,-4,1) \approx 3.0599080741143857498263883\,45.$$

2.2 Real Quadratic Twists

Let $D > 0$ be a fundamental discriminant. In this case, if $\left(\frac{D}{3}\right) = -1$, the sign of the functional equation for $L(f,D,s)$ will be -1, and its central value will vanish trivially. For $\left(\frac{D}{3}\right) = +1$, we will employ a method similar to the one used in [Mao et al. 07] for prime levels. We need to choose an auxiliary prime $l \equiv 3 \pmod 4$ such that $\left(\frac{-l}{3}\right) = -1$ and such that $L(f,-l,1) \neq 0$, for example $l = 7$. Following [Mao et al. 07], we define a generalized theta series

$$\Theta_{-7}(Q_i) := \frac{1}{2}\sum_{(x,y,z)\in\mathbb{Z}^3} \omega_7^{(i)}(x,y,z)\,\omega_3^{(i)}(x,y,z)\,q^{Q_i(x,y,z)/7},$$

where ω_7 and ω_3 are the two kinds of weight function introduced in [Mao et al. 07, Sections 2.2 and 2.3], respectively. The superscript in $\omega_3^{(i)}$ and $\omega_7^{(i)}$ indicates that we are writing the weight functions in the basis corresponding to the quadratic form Q_i.

The weight function of the first kind can be computed as

$$\omega_7^{(1)}(x,y,z) = \begin{cases} 0 & \text{if } 7 \nmid Q_1(x,y,z), \\ \left(\frac{x}{7}\right) & \text{if } 7 \nmid x, \\ \left(\frac{5z}{7}\right) & \text{otherwise;} \end{cases}$$

and

$$\omega_7^{(2)}(x,y,z) = \begin{cases} 0 & \text{if } 7 \nmid Q_2(x,y,z), \\ \left(\frac{3y+5z}{7}\right) & \text{if } 7 \nmid 3y+5z, \\ \left(\frac{6x}{7}\right) & \text{otherwise.} \end{cases}$$

The weight function of the second kind can be computed as

$$\omega_3^{(1)}(x,y,z) = \left(\frac{x+z}{3}\right),$$

and

$$\omega_3^{(2)}(x,y,z) = \left(\frac{2x+y+2z}{3}\right).$$

The generalized theta series will be

$$\Theta_{-7}(Q_1) = -2q^4 + 2q^{13} + 4q^{16} - 4q^{25} + 2q^{28} - 2q^{37} - 4q^{40} + \cdots$$

D	$c_{-7}(D)$	$L(f,D,1)$
1	1	0.588880
13	−3	1.469932
28	−3	1.001590
37	3	0.871301
40	6	3.351961
61	3	0.678585
73	−3	0.620308
76	−3	0.607942
85	0	0.000000
88	−6	2.259892
97	−3	0.538125
109	0	0.000000
124	6	1.903786
133	−3	0.459561
136	6	1.817856
145	6	1.760536
157	6	1.691917
172	0	0.000000
181	−9	3.545457
184	−6	1.562860
193	3	0.381496

TABLE 2. Coefficients of g_{-7} and real quadratic twists of $27A$.

and

$$\Theta_{-7}(Q_2) = q - q^4 - q^{13} + 2q^{16} - 3q^{25} - q^{28} + q^{37} + 2q^{40} + \cdots.$$

Note that $\Theta_{-7}(Q_1) + 2\Theta_{-7}(Q_2) = 2q - 4q^4 + 8q^{16} - 10q^{25} + \cdots$, corresponding to the Eisenstein eigenvector for the Brandt matrices, has nonzero Fourier coefficients only at square indices. Since $\Theta_{-7}(Q_1) + 2\Theta_{-7}(Q_2) \equiv \Theta_{-7}(Q_1) - \Theta_{-7}(Q_2) \pmod 3$, this explains the fact that the coefficients in Table 2, with the exception of $c_{-7}(1)$, are all divisible by 3.

Thus we obtain a modular form of weight $\frac{3}{2}$, namely

$$g_{-7} = \Theta_{-7}(Q_1) - \Theta_{-7}(Q_2) = q + q^4 - 3q^{13} - 2q^{16} + q^{25} - 3q^{28} + 3q^{37} + 6q^{40} + \cdots,$$

and the formula is now

$$L(f,D,1) = k_{-7}\,\frac{|c_{-7}(D)|^2}{\sqrt{|D|}}, \quad D > 0,$$

where $c_{-7}(D)$ is the Dth Fourier coefficient of g_{-7}, and

$$k_{-7} = \frac{1}{3}\cdot\frac{(f,f)}{L(f,-7,1)\sqrt{7}} = L(f,1) \approx 0.58887958342848331910456316\,68.$$

Table 2 shows the values of the Fourier coefficients $c_{-7}(D)$ of g_{-7} and of $L(f,D,1)$, where $0 < D < 200$ is a fundamental discriminant such that $\left(\frac{D}{3}\right) = 1$.

3. THE CURVE $15A$

Let f be the modular form of level 15, corresponding to the elliptic curve $X_0(15)$, of minimal equation

$$y^2 + xy + y = x^3 + x^2 - 10x - 10.$$

The eigenvalues of f for the Atkin–Lehner involutions W_3 and W_5 are $+1$ and -1, and the sign of the functional equation for $L(f, s)$ is $+1$.

The method of Gross, as extended by Böcherer and Schulze-Pillot to the case of square-free levels, requires that the ramification of the quaternion algebra agree with the Atkin–Lehner eigenvalues. In this case, it would be necessary to work with the quaternion algebra ramified at 5 and ∞. To exhibit the generality of our method, we will work with the quaternion algebra ramified at 3 and ∞ instead.

Let $B = (-1, -3)$ be such a quaternion algebra; an Eichler order of level 15 (index 5 in a maximal order) is given by $R = \left\langle 1, i, \frac{1+5j}{2}, \frac{1+i+3j+k}{2} \right\rangle$. The number of classes of left R-ideals is 2, and a set of representatives of the classes is given by $\{R, I\}$, where $I = \left\langle 2, 2i, \frac{3+2i+5j}{2}, \frac{3+i+3j+k}{2} \right\rangle$. The eigenvector for the Brandt matrices corresponding to f is $(1, -1)$, with height 4, and the ternary quadratic forms associated with R and I are

$$Q_1(x, y, z) = Q_2(x, y, z) = 4x^2 + 15y^2 + 16z^2 - 4xz.$$

3.1 Imaginary Quadratic Twists

Let $D < 0$ be a fundamental discriminant. We say that D is of type (s_1, s_2) if $\left(\frac{D}{3}\right) = s_1$ and $\left(\frac{D}{5}\right) = s_2$. We need the sign of the functional equation for $L(f, D, s)$ to be $+1$, so that its central value does not vanish trivially. For this to hold, we need D to be of type $(-, +)$, $(+, -)$, $(+, 0)$, $(0, -)$, or $(0, 0)$ (see [Atkin and Lehner 70]).

Note that the linear combination of classical theta series $\Theta(Q_1) - \Theta(Q_2)$ is trivially zero, since $Q_1 = Q_2$; this reflects the fact that the ramification does not match the Atkin–Lehner eigenvalues. Instead, we set

$$\Theta_1(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_3^{(i)}(x, y, z)\, \omega_5^{(i)}(x, y, z)\, q^{Q_i(x,y,z)},$$

where ω_3 and ω_5 are weight functions of the second kind as in [Mao et al. 07, Section 2.3]. We have $\Theta_1(Q_1) = -\Theta_1(Q_2)$, and hence we obtain a modular form of weight $\frac{3}{2}$ and level $4 \cdot 15^2$, namely

$$g_1 = 2\,\Theta_1(Q_1) = q^4 + q^{16} + 2q^{19} + 2q^{31} + q^{64} + \cdots.$$

D	$c_1(D)$	$L(f, D, 1)$
-4	1	1.596242
-19	2	2.929625
-31	2	2.293549
-79	-2	1.436730
-91	-4	5.354613
-136	-4	4.380053
-139	-2	1.083132
-151	2	1.039203
-184	-4	3.765649
-199	-2	0.905237

TABLE 3. Coefficients of g_1 and imaginary twists of $15A$.

D	$c_{17}(D)$	$L(f, D, 1)$
-3	2	0.921591
-8	-4	1.128714
-15	-2	0.824296
-20	4	1.427722
-23	4	0.665679
-35	-4	1.079257
-47	-4	0.465672
-68	0	0.000000
-83	4	0.350421
-87	4	0.684541
-95	0	0.000000
-107	4	0.308629
-120	-4	1.165730
-123	-4	0.575713
-132	-8	2.222961
-143	-8	1.067876
-152	8	1.035779
-155	8	2.051412
-167	4	0.247042
-168	8	1.970444
-183	0	0.000000
-195	-4	0.91447

TABLE 4. Coefficients of g_{17}, and imaginary twists of $15A$.

The corresponding formula is

$$L(f, D, 1) = k_1 \frac{|c_1(D)|^2}{\sqrt{|D|}}, \quad D < 0 \text{ of type } (-, +),$$

where $c_1(D)$ is the $|D|$th Fourier coefficient of g_1, and

$$k_1 = \frac{1}{4} \cdot \frac{(f, f)}{L(f, 1)} = 2L(f, -4, 1)$$
$$\approx 3.1924844442635670202979381 43;$$

see Table 3.

To obtain the other four types of negative D, we need to choose an auxiliary prime $l \equiv 1 \pmod 4$ such that $\left(\frac{l}{3}\right) = \left(\frac{l}{5}\right) = -1$, and such that $L(f, l, 1) \neq 0$, e.g.,

$l = 17$. We then define the generalized theta series

$$\Theta_{17}(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{17}^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/17},$$

where ω_{17} is the weight function of the first kind defined in [Mao et al. 07, Section 2.2]. Now

$$g_{17} = 2\,\Theta_{17}(Q_1) = 2q^3 - 4q^8 - 2q^{15} + 4q^{20} + 4q^{23} + \cdots$$

is a weight-$\frac{3}{2}$ modular form of level $4 \cdot 15$. As expected by the multiplicity-one theorem of Kohnen [Kohnen 82], this form turns out to be the same as the one constructed by Böcherer and Schulze-Pillot. The formula in this case is

$$L(f, D, 1) = \star\, k_{17}\, \frac{|c_{17}(D)|^2}{\sqrt{|D|}}$$

with $D < 0$ of type $(+,-)$, $(+,0)$, $(0,-)$, or $(0,0)$, and $\star = 1$, 2, 2, or 4 respectively, where $c_{17}(D)$ is the $|D|$th Fourier coefficient of g_{17}, and

$$k_{17} = \frac{1}{4} \cdot \frac{(f,f)}{L(f,17,1)\sqrt{17}} \approx 0.1995302777664729387686211340;$$

see Table 4.

3.2 Real Quadratic Twists

Let $D > 0$ be a fundamental discriminant. In order for the sign of the functional equation of $L(f, D, s)$ to be $+1$, we need D to be of type $(+,+)$, $(0,+)$, $(-,-)$, or $(-,0)$.

For the first two types we need an auxiliary prime $l \equiv 3 \pmod 4$ such that $\left(\frac{-l}{3}\right) = -1$ and $\left(\frac{-l}{5}\right) = +1$, and such that $L(f, -l, 1) \neq 0$, e.g., $l = 19$. Again

$$\Theta_{-19}(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{19}^{(i)}(x,y,z)\, \omega_{5}^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/19},$$

with ω_{19} of the first kind and ω_5 of the second kind. The modular form

$$g_{-19} = 2\,\Theta_{-19}(Q_1) = 2q - 4q^4 + 2q^9 - 8q^{21} + 8q^{24} + \cdots$$

has level $4 \cdot 15 \cdot 5$, and the formula is

$$L(f, D, 1) = \star\, k_{-19}\, \frac{|c_{-19}(D)|^2}{\sqrt{|D|}},$$

$D > 0$ of type $(+,+)$ or $(0,+)$, $\star = 1$ or 2 respectively; $c_{-19}(D)$ is the Dth Fourier coefficient of g_{-19}, and

$$k_{-19} = \frac{1}{4} \cdot \frac{(f,f)}{L(f,-19,1)\sqrt{19}} = \frac{1}{4} L(f,1) \approx 0.08753769014578762644876130241.$$

D	$c_{-19}(D)$	$L(f,D,1)$
1	2	0.350151
21	−8	2.445093
24	8	2.287175
61	16	2.869261
69	−8	1.348902
76	−16	2.570563
109	16	2.146455
124	16	2.012446
129	−8	0.986530
136	0	0.000000
141	−8	0.943616
156	16	3.588416
181	0	0.000000
184	−16	1.652061

TABLE 5. Coefficients of g_{-19}, and real twists of $15A$.

D	$c_{-23}(D)$	$L(f,D,1)$
5	2	1.252737
8	−4	1.980752
17	4	1.358785
53	4	0.769550
65	−4	1.389787
77	−8	2.553816
92	8	2.336367
113	−4	0.527031
137	−4	0.478646
140	8	3.787922
152	0	0.000000
173	−12	3.833492
185	4	0.823795
188	−8	1.634392
197	12	3.592398

TABLE 6. Coefficients of g_{-23}, and real twists of $15A$.

Table 5 shows the values of the coefficients $c_{-19}(D)$ and the central values $L(f, D, 1)$ for $0 < D < 200$ a fundamental discriminant of type $(+,+)$ or $(0,+)$.

For the remaining two types we need an auxiliary prime $l \equiv 3 \pmod 4$ such that $\left(\frac{-l}{3}\right) = +1$ and $\left(\frac{-l}{5}\right) = -1$, and such that $L(f, -l, 1) \neq 0$, e.g., $l = 23$. As before, we define

$$\Theta_{-23}(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{23}^{(i)}(x,y,z)\, \omega_{3}^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/23},$$

with ω_{23} of the first kind and ω_3 of the second kind. The modular form

$$g_{-23} = 2\,\Theta_{-23}(Q_1) = 2q^5 - 4q^8 + 4q^{17} - 4q^{32} + 4q^{53} + \cdots$$

has level $4 \cdot 15 \cdot 3$, and the formula is

$$L(f, D, 1) = \star\, k_{-23}\, \frac{|c_{-23}(D)|^2}{\sqrt{|D|}},$$

$D > 0$ of type $(-,-)$ or $(-,0)$, $\star = 1$ or 2 respectively; $c_{-23}(D)$ is the Dth Fourier coefficient of g_{-23} and

$$k_{-23} = \frac{1}{4} \cdot \frac{(f,f)}{L(f,-23,1)\sqrt{23}} \approx 0.35015076058315050579504520 92\,.$$

Table 6 shows the values of the coefficients $c_{-19}(D)$ and the central values $L(f,D,1)$ for $0 < D < 200$ a fundamental discriminant of type $(-,-)$ or $(-,0)$.

4. THE CURVE 75A

Let f be the modular form of level 75 corresponding to the elliptic curve of minimal equation

$$y^2 + y = x^3 - x^2 - 8x - 7.$$

The eigenvalue of f for the Atkin–Lehner involution W_3 is $+1$, and for W_{25} it is -1; and the sign of the functional equation for $L(f,s)$ is $+1$.

Let $B = (-1,-3)$ be the quaternion algebra ramified at 3 and ∞, and consider the order $R = \langle 1, i, \frac{1+5j}{2}, \frac{i+5k}{2} \rangle$, an Eichler order of level 75 (index 25 in a maximal order). The class number of left R-ideals is 6, and the eigenvector for the Brandt matrices that corresponds to f is $(1,-1,1,-1,0,0)$, with height 6.

The ternary quadratic forms associated with the right orders of the chosen ideal class representatives are

$$\begin{aligned} Q_1(x,y,z) &= Q_2(x,y,z) = 4x^2 + 75y^2 + 76z^2 - 4xz, \\ Q_3(x,y,z) &= Q_4(x,y,z) \\ &= 16x^2 + 19y^2 + 79z^2 + 4xy + 16xz + 2yz, \\ Q_5(x,y,z) &= Q_6(x,y,z) \\ &= 24x^2 + 31y^2 + 39z^2 + 24xy + 12xz + 6yz, \end{aligned}$$

respectively.

We will assume that $5 \nmid D$. Indeed, the twist of f by the quadratic character of conductor 5 is another modular form f' of level 75; thus we have

$$L(f,5D,1) = L(f',D,1),$$

for $5D$ a fundamental discriminant. By applying the same procedure to the modular form f' we can compute the central values for these twists. So, we actually need eight different modular forms of weight $\frac{3}{2}$ to compute all the twisted central values.

4.1 Imaginary Quadratic Twists

Let $D < 0$ be a fundamental discriminant. If the sign of the functional equation for $L(f,D,s)$ is $+1$, the type of D has to be either $(-,+)$ or $(-,-)$.

For the first case, we look at the generalized theta series

$$\Theta_1(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_3^{(i)}(x,y,z)\, \omega_5^{(i)}(x,y,z)\, q^{Q_i(x,y,z)};$$

we obtain the modular form

$$\begin{aligned} g_1 &= 2\Theta_1(Q_1) - 2\Theta_1(Q_3) \\ &= q^4 - 2q^{16} - q^{19} - q^{31} - 2q^{64} + 3q^{76} + 4q^{79} - q^{91} + \cdots. \end{aligned}$$

The formula

$$L(f,D,1) = k_1 \frac{|c_1(D)|^2}{\sqrt{|D|}}, \quad D < 0 \text{ of type } (-,+),$$

is satisfied (see Table 7), where $c_1(D)$ is the $|D|$th Fourier coefficient of g_1 and

$$k_1 = \frac{1}{6} \cdot \frac{(f,f)}{L(f,1)} = 2L(f,-4,1) \approx 4.66953274871871932795120676 1.$$

In the second case, we need to choose an auxiliary prime $l \equiv 1 \pmod 4$ such that $\left(\frac{l}{3}\right) = +1$, $\left(\frac{l}{5}\right) = -1$, and $L(f,l,1) \neq 0$, for example $l = 13$, and define

$$\begin{aligned} &\Theta_{13}(Q_i) \\ &\quad := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{13}^{(i)}(x,y,z) \omega_3^{(i)}(x,y,z) \omega_5^{(i)}(x,y,z) \\ &\qquad\qquad \times q^{Q_i(x,y,z)/13}. \end{aligned}$$

We obtain the modular form

$$\begin{aligned} g_{13} &= 2\Theta_{13}(Q_1) - 2\Theta_{13}(Q_3) \\ &= 3q^7 + 3q^{28} + 3q^{43} + 3q^{52} - 3q^{67} - 6q^{88} + \cdots, \end{aligned}$$

and the formula

$$L(f,D,1) = k_{13} \frac{|c_{13}(D)|^2}{\sqrt{|D|}}, \quad D < 0 \text{ of type } (-,-),$$

is satisfied (see Table 8), where $c_{13}(D)$ is the $|D|$th Fourier coefficient of g_{13} and

$$k_{13} = \frac{1}{6} \cdot \frac{(f,f)}{L(f,13,1)\sqrt{13}} \approx 1.55651091623957310931706892 0.$$

D	$c_1(D)$	$L(f,D,1)$
-4	1	2.334766
-19	-1	1.071264
-31	-1	0.838673
-79	4	8.405816
-91	-1	0.489500
-136	2	1.601637
-139	-2	1.584258
-151	5	9.500030
-184	2	1.376970
-199	-5	8.275360

TABLE 7. Coefficients of g_1, and imaginary twists of $75A$.

D	$c_{13}(D)$	$L(f,D,1)$
-7	3	5.294752
-43	3	2.136291
-52	3	1.942643
-67	-3	1.711423
-88	-6	5.973286
-103	-6	5.521233
-127	-6	4.972248
-148	0	0.000000
-163	3	1.097238
-187	0	0.000000

TABLE 8. Coefficients of g_{13}, and imaginary twists of $75A$.

4.2 Real Quadratic Twists

Let $D > 0$ be a fundamental discriminant. The only possibilities for which the sign of the functional equation for $L(f,D,s)$ is $+1$ are the discriminants D of types $(+,+)$, $(0,+)$, $(+,-)$, and $(0,-)$.

For the first two cases we can use the generalized theta series

$$\Theta_{-19}(Q_i) := \frac{1}{2} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{19}^{(i)}(x,y,z)\, \omega_5^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/19}.$$

Thus we obtain a modular form of weight $\frac{3}{2}$, namely

$$g_{-19} = q + q^4 + q^9 - q^{21} - 2q^{24} - q^{36} - 4q^{49} - q^{61} + \cdots,$$

and the formula is

$$L(f,D,1) = \star\, k_{-19}\, \frac{|c_{-19}(D)|^2}{\sqrt{|D|}},$$

$D > 0$ of type $(+,+)$ or $(0,+)$, $\star = 1$ or 2 respectively, $c_{-19}(D)$ the Dth Fourier coefficient of g_{-19}, and

$$k_{-19} = \frac{1}{6} \cdot \frac{(f,f)}{L(f,-19,1)\sqrt{19}} = L(f,1) \approx 1.4025399402162211198444940\,86;$$

D	$c_{-19}(D)$	$L(f,D,1)$
1	1	1.402540
21	-1	0.612119
24	-2	2.290338
61	-1	0.179577
69	2	1.350768
76	1	0.160882
109	-1	0.134339
124	5	3.148795
129	5	6.174338
136	-6	4.329605
141	2	0.944921
156	-1	0.224586
181	3	0.938250
184	-2	0.413586

TABLE 9. Coefficients of g_{-19}, and real twists of $75A$.

D	$c_{-7}(D)$	$L(f,D,1)$
12	3	2.429270
13	3	1.166984
28	-3	0.795165
33	-6	5.859621
37	0	0.000000
57	-3	1.114626
73	6	1.969859
88	-6	1.794135
93	-3	0.872620
97	9	3.844972
133	-3	0.364847
157	3	0.335805
168	6	2.596999
172	3	0.320828
177	-6	2.530113
193	9	2.725840

TABLE 10. Coefficients of g_{-7}, and real twists of $75A$.

see Table 9.

In the other two cases we can use the generalized theta series

$$\Theta_{-7}(Q_i) := \frac{1}{2} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_7^{(i)}(x,y,z)\, \omega_5^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/7}.$$

We obtain a modular form of weight $\frac{3}{2}$,

$$g_{-7} = 3q^{12} + 3q^{13} - 3q^{28} - 6q^{33} + 6q^{48} - 9q^{52} - 3q^{57} + 6q^{73} + \cdots,$$

satisfying the formula

$$L(f,D,1) = \star\, k_{-7}\, \frac{|c_{-7}(D)|^2}{\sqrt{|D|}},$$

$D > 0$ of type $(+,-)$ or $(0,-)$, $\star = 1$ or 2 respectively, $c_{-7}(D)$ the Dth Fourier coefficient of g_{-7}, and

$$k_{-7} = \frac{1}{6} \cdot \frac{(f,f)}{L(f,-7,1)\sqrt{7}} \approx 0.46751331340540703994816469 50;$$

see Table 10.

5. COMPUTATION

Using the methods of the previous section, we computed the coefficients up to 10^8 for all the theta series of weight $\frac{3}{2}$ corresponding to elliptic curves $27A$ and $15A$. The computations of the theta series for the elliptic curve $75A$ are currently underway, and will be published online at http://www.ma.utexas.edu/cnt/. All the computations were done on a cluster of 2.2-GHz AMD Opteron processors funded by an NSF SCREMS grant and run by the Department of Mathematics of the University of Texas at Austin.[1]

The computation for $27A$ is quite fast. Indeed, the form g_1 is a combination of classical theta series, and was computed in about 4 cpu hours using the standard `qfrep` function of PARI/GP.[2] The form g_7, on the other hand, requires the use of weight functions, and computing it took about 40 cpu hours using a custom `qfrepmod` function written in C for this purpose, together with a collection of GP scripts to compute weight functions. The strategy is to use the fast `qfrepmod` function to compute theta series with congruences where the weight functions are constant, and combine them in a GP script.

The computation for $15A$ is much longer. Indeed, the higher conductor of the weight functions requires too many congruence theta series except in the case $l = 1$. We actually divide the computation of the coefficients of the g_i by the congruence class of its index modulo 60. In particular, we avoid the need to reserve memory for coefficients that are trivially 0 (namely, only half the indices are actual discriminants, and of those, half correspond to quadratic twists with minus sign in the functional equation). Moreover, each computation requires only a fraction of the space to keep all the coefficients in main memory while counting vectors. It also lends itself to a trivial way to parallelize the computation in 30 independent processes.

The computation used 30 cores in the above-mentioned cluster, with a wall time of 26.5 days (this was the time for the two longest-running processes, corresponding to discriminants congruent to 2 and 8 modulo 60). The accumulated running times were as follows:

g_1	0.30 days
g_{17}	110.19 days
g_{-19}	106.88 days
g_{-23}	131.80 days
total	349.17 days

We believe that the running times for all but g_1 are affected by the number of congruences, the combination of which is done by a GP script. Thus, we expect that the times for the last three computations can be improved considerably with a careful rewriting in C of this code.

Note also that the modular form we are calling here g_{17} can also be computed as a difference of two classical theta series by working with the quaternion algebra ramified at 3 and ∞, and this will be much quicker in all cases. Thus, the totality of imaginary quadratic twists could be quickly computed.

6. RANDOM MATRIX THEORY

The purpose of this section is to check some of the various conjectures of [Conrey et al. 02] and [Conrey et al. 06]. We start by stating the conjectures; we checked each of them numerically with the computation discussed in the previous section.

An important comment should be made: in [Conrey et al. 02] and [Conrey et al. 06], the conjectures are stated and checked, in case of nonprime level, only for a fraction of all quadratic twists, namely those that can be computed without weight functions using the methods of [Gross 87] and [Böcherer and Schulze-Pillot 90]. In the case of the real quadratic twists they have been checked using the methods in [Pacetti and Tornaría 07a, Pacetti and Tornaría 07b]. In both cases this has been based on a massive computation of *classical* theta series that was done in [Conrey et al. 06], using ternary quadratic forms data that was computed by the second author with aid from the first author [Tornaría 04], first published in December 2003, and in its final form in January 2004.

In this paper we have shown how to compute, in a few examples, enough weight-$\frac{3}{2}$ modular forms so that one can compute the central values for *all* the quadratic twists. Hence we state the conjectures for all the quadratic twists for which the sign of the functional equation is $+$, and give numerical evidence for the conjectures for all such twists. The task remains of doing a massive

[1] See http://www.ma.utexas.edu/cluster/.

[2] See http://pari.math.u-bordeaux.fr/.

computation like the one done in [Tornaría 04] and [Conrey et al. 06] to check these conjectures for a very large set of quadratic twists in all the possible sign and residue class combinations, for a large number of different elliptic curves.

In order to state the conjectures, fix an elliptic curve E defined over $\mathbb{Q}$. We let $S(X)$ be the set of fundamental discriminants, of absolute value up to X, such that the corresponding quadratic twist of E has positive sign in the functional equation.

We will refine the conjectures of [Conrey et al. 02], sorting the discriminants by congruence classes in addition to sign: for M a positive integer and a an integer, we let

$$S(X;a,M) := \{d \in S(X) : d \equiv a \pmod{M},\ ad > 0\}.$$

Among those, we consider the subset $S_p(X;a,M)$ of prime discriminants, i.e.,

$$S_p(X;a,M) = \{d \in S(X;a,M) :\ d \text{ is prime}\}.$$

We are interested in the subsets

$$S^0(X;a,M) = \{d \in S(X;a,M) : L(E,d,1) = 0\}$$

and

$$S_p^0(X;a,M) = \{d \in S_p(X;a,M) : L(E,d,1) = 0\}$$

of discriminants with twisted central value vanishing (for nontrivial reasons, and to order at least 2, since the sign of the functional equation is $+$).

Conjecture 6.1. *There are constants $c_E^p(a,M) \geq 0$ such that*

$$\frac{\#S_p^0(X;a,M)}{\#S_p(X;a,M)} \sim c_E^p(a,M) \cdot X^{-1/4} (\log X)^{3/8}.$$

We remark that the constant $c_E^p(a,M)$ could be 0, as noted by [Delaunay 07]. In contrast, we believe that the constants $c_E(a,M)$ in the next conjecture should always be positive.

Conjecture 6.2. *There are constants $c_E(a,M) \geq 0$ such that*

$$\frac{\#S^0(X;a,M)}{\#S(X;a,M)} \sim c_E(a,M) \cdot X^{-1/4} (\log X)^{11/8}.$$

a	$\#S^0(X;a,12)$	$\#S(X;a,12)$	$c_E(X;a,12)$
1	295819	7599045	0.07087151
4	145496	3799561	0.06971437
−7	226182	7599088	0.05418776
−4	110886	3799541	0.05313127

TABLE 11. Numerics for $27A$, all discriminants, with $X = 10^8$.

a	$\#S_p^0(X;a,12)$	$\#S_p(X;a,12)$	$c_E^p(X;a,12)$
1	23700	1440021	0.55193748
−7	18233	1440496	0.42447923

TABLE 12. Numerics for $27A$, prime discriminants, with $X = 10^8$.

In Table 11 we give the experimental numerics for

$$c_E(X;a,M) := \frac{\#S^0(X;a,M)}{\#S(X;a,M)} \cdot X^{1/4} (\log X)^{-11/8}$$

for the elliptic curve $27A$, with $M = 12$ and $X = 10^8$. Only the values of a that lead to discriminants in $S(X)$ are displayed. In Table 12 we show the corresponding numerics for prime discriminants, where

$$c_E^p(X;a,M) := \frac{\#S^0(X;a,M)}{\#S(X;a,M)} \cdot X^{1/4} (\log X)^{-3/8}.$$

In Tables 13 and 14 we investigate the dependence on a of the constants $c_E(a,M)$, for the elliptic curve $15A$ and $M = 60$. An interesting phenomenon can be observed in these tables: the constants $c_E(a,M)$ seem to depend only on the square class of a mod M.

Conjecture 6.3. *Let a and b be integers in the same square class modulo M, i.e., $ab > 0$ and there is an integer x*

a	$\#S^0(X;a,60)$	$\#S(X;a,60)$	$c_E(X;a,60)$
1	103871	1583103	0.11945101
49	103201	1583109	0.11868006
4	56689	791596	0.13037667
16	57272	791596	0.13171749
9	53190	1055442	0.09174878
21	53325	1055430	0.09198269
24	45765	527715	0.15788421
36	46085	527707	0.15899059
17	62882	1583163	0.07231117
53	63276	1583149	0.07276489
8	56117	791553	0.12906816
32	55560	791565	0.12778513
5	70561	1266445	0.10143389
20	46229	633300	0.13289532

TABLE 13. Numerics for real quadratic twists of $15A$, with $X = 10^8$.

a	$\#S^0(X;a,60)$	$\#S(X;a,60)$	$c_E(X;a,60)$
-19	75626	1583138	0.08696751
-31	75333	1583128	0.08663111
-4	62536	791570	0.14382867
-16	62999	791558	0.14489573
-23	67381	1583166	0.07748465
-47	67794	1583158	0.07795997
-8	61142	791545	0.14062700
-32	60724	791565	0.13966207
-3	64191	1055419	0.11072710
-27	64178	1055408	0.11070583
-12	41844	527727	0.14435391
-48	41589	527728	0.14347394
-35	72803	1266486	0.10465345
-20	53586	633266	0.15405289
-15	50383	844328	0.10863694
-60	42661	422192	0.18396098

TABLE 14. Numerics for imaginary quadratic twists of $15A$, with $X = 10^8$.

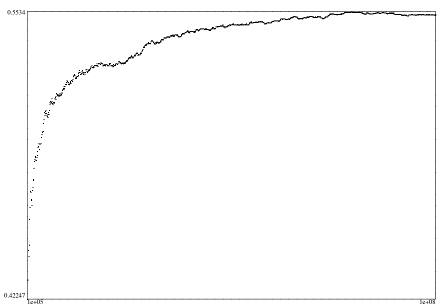

FIGURE 1. Value of $c_E^p(X;+1,1)$ for $27A$.

relatively prime to M such that $a \equiv bx^2 \pmod M$. Then $c_E(a,M) = c_E(b,M)$.

The case of Conjectures 6.1 and 6.2 stated in [Conrey et al. 02] corresponds to the case $M = 1$, with $a = \pm 1$, and moreover restricted to partial subsets of discriminants. In Figures 1 and 2 we show the numerics for the elliptic curve $27A$ in the case of prime discriminants (Conjecture 6.1), and in Figures 3 and 4 we show the numerics in the case of all discriminants (Conjecture 6.2).

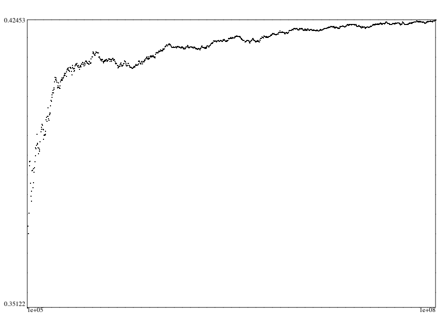

FIGURE 2. Value of $c_E^p(X;-1,1)$ for $27A$.

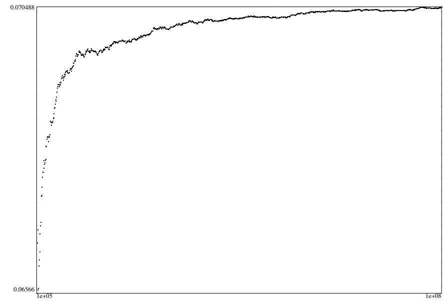

FIGURE 3. Value of $c_E(X;+1,1)$ for $27A$.

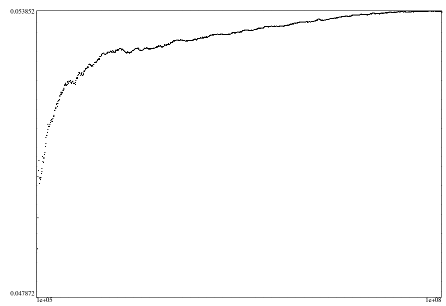

FIGURE 4. Value of $c_E(X;-1,1)$ for $27A$.

For the elliptic curve $15A$ we have the corresponding Figure 5 for the case of prime discriminants, Figures 6 and 7 for the case of all discriminants. By [Delaunay 07], we know that the constant $c_E^p(-1,1)$ is 0; thus we show only positive prime discriminants for this curve. On the other hand, the graphs of $c_E(X;\pm 1,1)$ for this curve seem to be too smooth, as if they had logarithmic growth, for example. We do not have an explanation for this.

We recall another conjecture from [Conrey et al. 02]: let q be a prime and consider the ratios

$$R_q^{\pm}(X) = \frac{\#\left\{d \in S^0(X;\pm 1,1) : \left(\frac{d}{q}\right) = +1\right\}}{\#\left\{d \in S^0(X;\pm 1,1) : \left(\frac{d}{q}\right) = -1\right\}}.$$

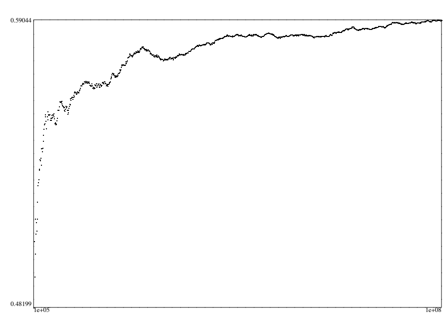

FIGURE 5. Value of $c_E^p(X;+1,1)$ for $15A$.

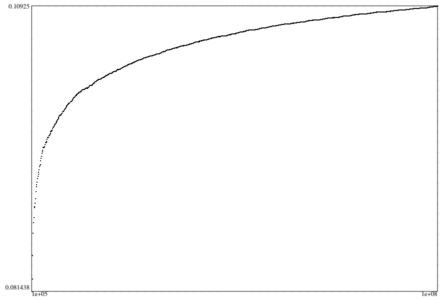

FIGURE 6. Value of $c_E(X;+1,1)$ for $15A$.

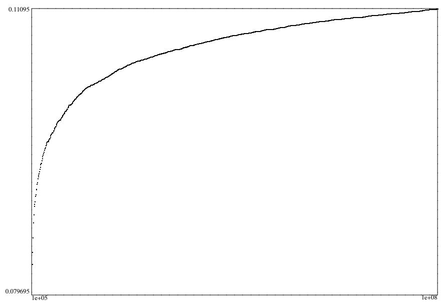

FIGURE 7. Value of $c_E(X;-1,1)$ for $15A$.

Let

$$R_q := \sqrt{\frac{q+1-a_q}{q+1+a_q}},$$

where $a_q = q+1-\#E(\mathbb{F}_q)$.

Conjecture 6.4. *Suppose E has good reduction modulo q. Then*

$$\lim_{X\to\infty} R_q^{\pm}(X) = R_q.$$

As noted in [Conrey et al. 02], the conjectural value R_q of the limit is the square root of the ratio of $\#E(\mathbb{F}_q)$ to $\#E^{\chi}(\mathbb{F}_q)$, where χ is a quadratic character such that $\chi(q) = -1$.

In Figures 8 and 9 we plot, for the elliptic curve $27A$ and for each prime number $q = 2, \ldots, 3571$, the values $R_q^+(10^8) - R_q$ and $R_q^-(10^8) - R_q$, respectively.

In Figures 10 and 11 we do the same for the elliptic curve $15A$. It can be seen on the graphs that these values are close to 0 (the expected limit as X goes to infinity).

In Figures 12 and 13 we plot the distribution of nonzero central values of the twisted L-series of the elliptic curve $27A$ by positive and negative fundamental discriminants, respectively. The same graphs for the elliptic curve $15A$ appear in Figures 14 and 15.

The *central limit conjecture* (see [Conrey et al. 06, Conjecture 3.3]) states that the distribution of nonzero

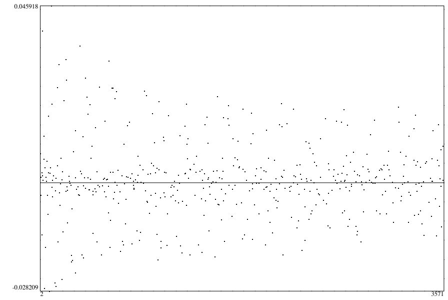

FIGURE 8. The values $R_q^+(10^8) - R_q$ for the elliptic curve $27A$ and $2 \leq q \leq 3571$ prime.

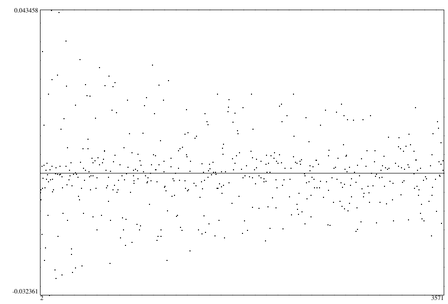

FIGURE 9. The values $R_q^-(10^8) - R_q$ for the elliptic curve $27A$ and $2 \leq q \leq 3571$ prime.

central values of the twisted L-series (scaled in a reasonable way) behaves like a standard Gaussian; concretely, for any pair of real numbers $\alpha < \beta$, the percentage of discriminants $d \in S(X;\pm 1,1)$ with

$$\alpha < \frac{\log(L(E,d,1)) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} < \beta$$

tends to

$$\frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp\left(\frac{-t^2}{2}\right) dt$$

as X tends to infinity. In Figure 16 we plot the value distribution of the twisted L-series of the elliptic curves

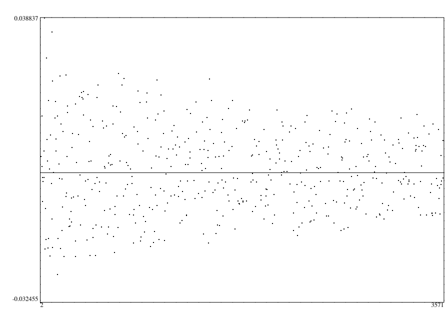

FIGURE 10. The values $R_q^+(10^8) - R_q$ for the elliptic curve $E15A$ and $2 \leq q \leq 3571$ prime.

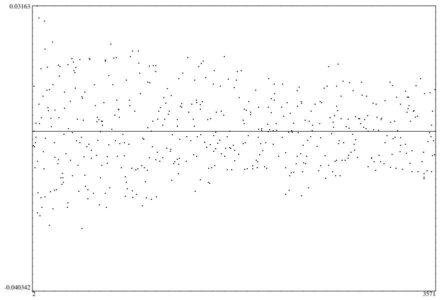

FIGURE 11. The values $R_q^-(10^8) - R_q$ for the elliptic curve $E15A$ and $2 \leq q \leq 3571$ prime.

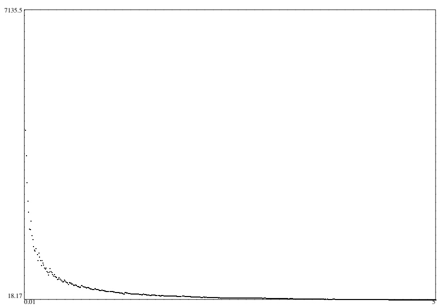

FIGURE 12. Value distribution of $L(27A, d, 1)$ for $0 < d < 10^8$.

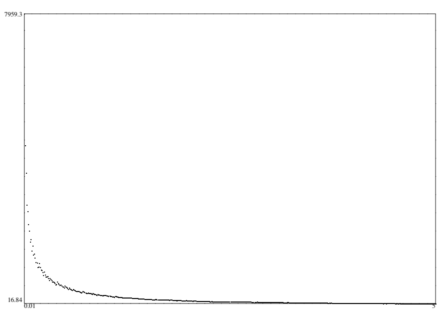

FIGURE 13. Value distribution of $L(27A, d, 1)$ for $0 > d > -10^8$.

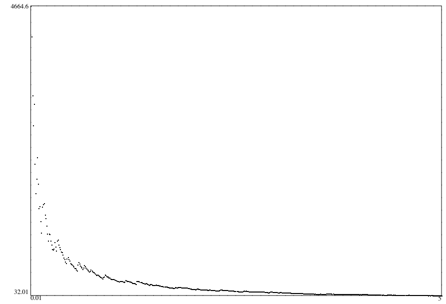

FIGURE 14. Value distribution of $L(15A, d, 1)$ for $0 < d < 10^8$.

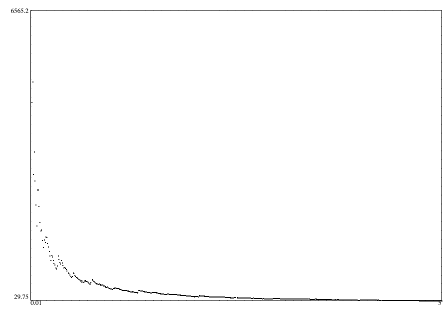

FIGURE 15. Value distribution of $L(15A, d, 1)$ for $0 > d > -10^8$.

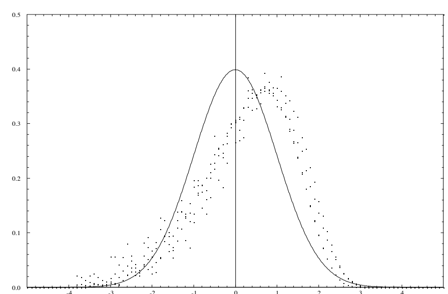

FIGURE 16. Value distribution of $(\log L(E, d, 1) + \frac{1}{2}\log\log|d|)/\sqrt{\log\log|d|}$ for both $27A$ and $15A$, all discriminants, compared to the expected limit, the standard Gaussian.

$27A$ and $15A$ by positive and negative fundamental discriminants compared to the standard Gaussian.

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