Nuclear Elements of Degree 6 in the Free Alternative Algebra

I. R. Hentzel and L. A. Peresi

CONTENTS

We construct five new elements of degree 6 in the nucleus of the free alternative algebra. We use the representation theory of the symmetric group to locate the elements. We use the computer algebra system ALBERT and an extension of ALBERT to express the elements in compact form and to show that these new elements are not a consequence of the known degree-5 elements in the nucleus. We prove that these five new elements and four known elements form a basis for the subspace of nuclear elements of degree 6. Our calculations are done using modular arithmetic to save memory and time. The calculations can be done in characteristic zero or any prime greater than 6, and similar results are expected. We generated the nuclear elements using prime 103. We check our answer using five other primes.

1. INTRODUCTION

An *alternative algebra* is a nonassociative algebra over a field satisfying the *alternative laws* $(x, x, y) = 0$ and $(x, y, y) = 0$, where the *associator* (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$. These algebras are called alternative because the associator is an alternating function of its three arguments. That is,

$$(x, y, z) = (y, z, x) = (z, x, y) = -(y, x, z) = -(x, z, y) = -(z, y, x).$$

Let $F[X]$ be the free nonassociative algebra over the field F in generators $X = \{x_1, x_2, \ldots, x_n\}$. Let $\mathrm{Alt}[X]$ denote the ideal of $F[X]$ generated by the elements $(f_1, f_1, f_2), (f_2, f_1, f_1)$, $(f_1, f_2 \in F[X])$.

Definition 1.1. The *free alternative algebra* in generators X is the quotient algebra

$$\mathrm{ALT}[X] = F[X]/\,\mathrm{Alt}[X].$$

The alternative laws of degree 3 imply identities of degree n. These are the elements in $\mathrm{Alt}[X]$ of degree n.

2000 AMS Subject Classification: Primary 17D05; Secondary 17-04, 17-08, 68W30.

Keywords: Free alternative algebras, nucleus, polynomial identities, computational algebra

Definition 1.2. The *nucleus* of a nonassociative algebra $\mathcal{A}$ is the set

$$N(\mathcal{A}) = \{p \in \mathcal{A} \mid (p,x,y) = (x,p,y) = (x,y,p) = 0, \\ \forall x,y \in \mathcal{A}\}.$$

The *center* of a nonassociative algebra $\mathcal{A}$ is the set

$$C(\mathcal{A}) = \{p \in N(\mathcal{A}) \mid [p,x] = 0, \ \forall x \in \mathcal{A}\}.$$

(The *commutator* $[x,y]$ is defined by $[x,y] = xy - yx$.)

In 1953, it was shown in [Kleinfeld 53] that for any x and y in an alternative algebra, the element $[x,y]^4$ is in the nucleus and this element is nonzero in the free alternative algebra on two or more generators. Subsequently, other authors have found elements of larger and smaller degree in the nucleus, as well as elements in the center. See [Hentzel and Peresi 06a, Hentzel and Peresi 06b] and the references therein.

In the Dniester Notebook, I. P. Shestakov proposed the following problem (see [Filippov et al. 06, Problem 2.121]): Describe the center and the associative center—the nucleus in our terminology—of a free alternative algebra as completely characteristic subalgebras. Are they finitely generated? (A subalgebra $\mathcal{S}$ of an algebra $\mathcal{A}$ is *completely characteristic* if $\psi(\mathcal{S}) \subset \mathcal{S}$ for all homomorphisms $\psi : \mathcal{A} \to \mathcal{A}$.)

Our results in [Hentzel and Peresi 06a, Hentzel and Peresi 06b] and in this paper give a partial solution to this problem by giving a finite basis for the central and nuclear elements of low degree.

In the free alternative algebra over Z_{103}, the elements of smallest degree in the center have degree 7, as we proved in [Hentzel and Peresi 06a]. Furthermore, we obtained all degree-7 central elements.

Definition 1.3. Let p be an element of the free nonassociative algebra $F[X]$ over the field F in generators $X = \{x_1, x_2, \ldots, x_n\}$. We say that p is an *element of the nucleus of the free alternative algebra in generators* X if in the free alternative algebra on generators $X \cup \{x_{n+1}, x_{n+2}\}$ one has that $(p, x_{n+1}, x_{n+2}) = 0$.

In [Hentzel and Peresi 06b] we proved that in the free alternative algebra over Z_{103}, the elements of smallest degree in the nucleus have degree 5. Furthermore, we proved that in the free alternative algebra over Z_{103} on generators $\{a,b,c,d,e\}$, all the nuclear elements of degree 5 are consequences of the alternative identities of degree 5 (i.e., the identities of degree 5 implied by the alternative laws of degree 3) and the nuclear element

$$([a,b][a,c])a - (a[a,b])[a,c]. \tag{1–1}$$

In this paper we prove the following result (the *Jordan product* $x \circ y$ is defined by $x \circ y = xy + yx$, the *Jordan multiplication by* x is denoted by V_x and defined by $V_x(y) = x \circ y$, and the *Jordan associator* $\langle x,y,z\rangle$ is defined by $\langle x,y,z\rangle = (x \circ y) \circ z - x \circ (y \circ z)$).

Theorem 1.4.

(i) *The following elements are in the nucleus of the free alternative algebra over Z_{103} on generators $\{a,b,c,d,e,f\}$:*

$$[[a,b][a,b],b]a, \tag{1–2}$$

$$\begin{aligned} &2[[a,b],a] \circ [[b,c],c] - 2[[a,b],b] \circ [[a,c],c] \\ &\quad - 2[[a,c],a] \circ [[b,c],b] + 2[[a,c],b] \circ [[a,c],b] \\ &\quad + 6[[b,c],a] \circ [[b,c],a] - 6[[a,c],b] \circ [[b,c],a] \\ &\quad + 4[[a,b],c] \circ [[b,c],a] + 3[a,b] \circ [[a,c],[b,c]] \\ &\quad - 3[a,c] \circ [[a,b],[b,c]] - 3[b,c] \circ [[a,c],[a,b]], \end{aligned} \tag{1–3}$$

$$\begin{aligned} &\langle a,a,b\rangle\langle c,c,b\rangle + \langle a,a,c\rangle\langle b,b,c\rangle \\ &\quad - \langle a,b,c\rangle\langle a,b,c\rangle + \langle b,b,a\rangle\langle c,c,a\rangle \\ &\quad + \langle b,b,c\rangle\langle a,a,c\rangle - \langle b,c,a\rangle\langle b,c,a\rangle \\ &\quad - \langle c,a,b\rangle\langle c,a,b\rangle + \langle c,c,a\rangle\langle b,b,a\rangle \\ &\quad + \langle c,c,b\rangle\langle a,a,b\rangle - 72(a,b,c)(a,b,c), \end{aligned} \tag{1–4}$$

$$[V(d^2) - V(d) \circ d, a], \tag{1–5}$$

where

$$V = V_aV_bV_c + V_bV_cV_a + V_cV_aV_b - V_bV_aV_c \\ - V_aV_cV_b - V_cV_bV_a$$

and

$$\begin{aligned} &\mathrm{ALT\,SUM}_{\{b,c,d,e\}} \\ &\quad \Big\{3[[[a,b],a],c][d,e] + [[[a,b],c],a][d,e] \\ &\qquad - 2[[[a,b],c],d][a,e] + 2[[[b,c],a],a][d,e] \\ &\qquad - 4[[[b,c],a],d][a,e]\Big\}, \end{aligned} \tag{1–6}$$

where $\mathrm{ALT\,SUM}_{\{b,c,d,e\}}$ *denotes the alternating sum over all permutations of the variables* $\{b,c,d,e\}$.

(ii) *All the nuclear elements of degree* 6 *are consequences of the alternative identities of degree* 6 *(i.e., the identities of degree* 6 *implied by the alternative* *laws of degree* 3*), the lifted nuclear elements of degree* 6 *(i.e., the nuclear elements of degree* 6 *implied by* (1–1)*), and the nuclear elements* (1–2) *through* (1–6)*.*

The nuclear elements (1–2) through (1–6) are obtained using the representation theory of the symmetric group as well as the computer algebra system Albert [Jacobs et al. 96].

The degree-5 nuclear element (1–1) generates elements of degree 6 in the nucleus (see the last paragraph of this section). We show that the nuclear elements (1–2) through (1–6) are independent of one another and not consequences of the degree-5 nuclear element (1–1).

We shall assume that all algebras are over a field F of characteristic zero or of characteristic greater than the degree of the identities in question. This is necessary for two reasons. The first is that we work with the linearized identities, and the assumption on characteristics ensures that the linearized form of the identities is equivalent to the unlinearized form. The second is that we need the group algebra on the symmetric group to be semisimple. For smaller characteristics, the classical idempotents construction involves dividing by zero. Since the process involves taking the direct sum of copies of the group algebra, one for each association type, the same conditions on characteristic are necessary and sufficient for our calculations. We shall continue to use the definitions and notation that are used in [Hentzel and Peresi 06b].

The *Teichmüller identity*

$$\begin{aligned}(xy, z, w) - (x, yz, w) + (x, y, zw)\\ = x(y, z, w) + (x, y, z)w\end{aligned}$$

holds in any nonassociative algebra $\mathcal{A}$. This identity may be validated by expanding out the associators and seeing that all the terms cancel.

The Teichmüller identity shows that if p is in $N(\mathcal{A})$, then

$$\begin{aligned}p(x, y, z) &= (px, y, z),\\ (xp, y, z) &= (x, py, z),\\ (x, yp, z) &= (x, y, pz),\\ (x, y, zp) &= (x, y, z)p.\end{aligned}$$

In an alternative algebra A, all eight of the above terms are equal. In particular:

$$\begin{aligned}&(x, py, z) = (py, z, x) = p(y, z, x) = p(x, y, z),\\ &(x, yp, z) = (x, y, pz) = (pz, x, y) = p(z, x, y) = p(x, y, z).\end{aligned}$$

It follows that $(x, [p, y], z) = 0$. We have shown that in an alternative algebra A, we have $[A, N(A)] \subset N(A)$.

The degree-6 nuclear elements implied by the known degree-5 nuclear element (1–1) are

$$([a, bd][a, c])a - (a[a, bd])[a, c], \tag{1–7}$$

$$([a, b][a, cd])a - (a[a, b])[a, cd], \tag{1–8}$$

$$\begin{aligned}&([de, b][a, c])a + ([a, b][de, c])a + ([a, b][a, c])(de)\\ &\quad - ((de)[a, b])[a, c] - (a[de, b])[a, c] - (a[a, b])[de, c],\end{aligned} \tag{1–9}$$

$$[([a, b][a, c])a - (a[a, b])[a, c], d]. \tag{1–10}$$

Element (1–7) is obtained by replacing b with bd in (1–1). Element (1–8) is obtained by replacing c with cd in (1–1). Element (1–9) is obtained by first linearizing (1–1) on a to obtain

$$\begin{aligned}&([d, b][a, c])a + ([a, b][d, c])a + ([a, b][a, c])d - (d[a, b])[a, c]\\ &\quad - (a[d, b])[a, c] - (a[a, b])[d, c]\end{aligned}$$

and then replacing d by de. Element (1–10) is obtained by bracketing (1–1) with d.

2. EXTENSIONS TO THE COMPUTER ALGEBRA SYSTEM ALBERT

Albert is a computer algebra system based on the algorithm described in [Hentzel and Jacobs 91]. One gives the program the defining identities of a variety of nonassociative algebras. One also gives the program a polynomial that is to be tested. Using the command `generators`, one supplies the program with the problem type. This refers to the number and degree of variables in this polynomial.

The program builds a finite-dimensional algebra, which is a homomorphic image of the free nonassociative algebra specified by the defining identities, and evaluates the polynomial in this finite-dimensional algebra. This homomorphic image is the free nonassociative algebra in the variety with the given defining identities such that all products of degree larger than the degree specified for that variable are zero.

When the polynomial evaluates to zero, the polynomial is a consequence of the given identities, and the polynomial is an identity in this variety.

When the polynomial does not evaluate to zero, the polynomial is not zero in the free algebra and is not an identity in this variety.

Albert works in positive characteristic. One can specify any characteristic between 2 and 251. Albert does not work in characteristic zero. Using Albert alone, one cannot be sure whether a result is dependent on the particular characteristic chosen or whether it is true in general.

ALBERT has the two commands `save basis` and `save multiplication table`, which dump the basis and the multiplication table of the finite-dimensional homomorphic image of the free algebra created by ALBERT into two files. We use this multiplication table to extend the capabilities of ALBERT to create and simplify nuclear elements.

The program ALBERT is designed to answer specific questions: Given an element p, ALBERT can test whether p is in the nucleus of the free alternative algebra. ALBERT creates a finite-dimensional algebra (which is a homomorphic image of the free alternative algebra with two additional variables x and y) and then uses the multiplication table of this algebra to compute (p, x, y). If (p, x, y) is zero, then p is in the nucleus of the free alternative algebra (see an example in [Hentzel and Peresi 06b, Section 6.1] and another example in Section 4.7 below).

In this section we describe a procedure to create a nuclear element p. The procedure uses the basis and the multiplication table created by ALBERT. The defining identities are the two alternative laws

$$(x, x, y) = 0 \quad \text{and} \quad (x, y, y) = 0.$$

Assume that we are looking for nuclear elements of degree 5. We use ALBERT to build the finite-dimensional homomorphic image of the free alternative algebra using the command `generators` a, b, c, d, e, x, y. We use the `save basis` and `save multiplication table` commands to store the basis and multiplication table in two files.

In the basis file, we find all degree-5 basis elements that contain a, b, c, d, e in some association and permutation. These basis elements appear consecutively as $h_n, \ldots, h_{n+k}$. We use the multiplication table to expand the associator (h_i, x, y) (for $i = n, \ldots, n+k$) into a linear combination of basis elements. The coefficients of this linear combination create the ith row (R_i) of a matrix.

A dependence relation $\sum_{i=n}^{n+k} c_i R_i$ on the rows of this matrix corresponds to a nuclear element $\sum_{i=n}^{n+k} c_i b_i$. This element is generated by working modulo some particular positive characteristic.

Using this procedure, we recovered the nuclear element previously known for degree 5. Unfortunately, when we apply this procedure to find the degree-6 nuclear elements, the problem is too large for ALBERT. We can, however, find the nuclear elements that can be written as homogeneous polynomials in three a's and three b's. The algebra on "generators" a, a, a, b, b, b, x, y is small enough to be created by ALBERT. See Section 4.3.

3. REPRESENTATION TECHNIQUE

Since the problem is too large for ALBERT, we use the representation technique as described in [Hentzel and Peresi 06b]. For each partition of n, we obtain the representation of the group algebra FS_n given by this partition. If d is the degree of this representation, then we represent a multilinear polynomial of degree n by a formal sum of $d \times d$ matrices. The advantage of the representation technique is that it can process higher degrees than techniques that rely on processing the identities as polynomials. The disadvantage is that any nuclear element found is displayed in its encoded representation form. Decoding a multilinear element of degree n produces an expression that may involve $n!\ \mathrm{cat}[n]$ terms. In this paper,

$$\mathrm{cat}[n] = \frac{1}{n}\binom{2n-2}{n-1}.$$

It is called the *Catalan number*. Fortunately, we can use ALBERT to simplify such an expression.

If we have an element p of degree 6 that we wish to prove is in the nucleus, we calculate the type identities of degree 8 implied by the alternative laws. Type identities are the identities formed by lifting the alternative laws in such a way that the set of identities is not excessively redundant (see [Hentzel and Peresi 06b, Section 3]). If in each representation the row space of (p, x, y) is contained in the row space of the type identities, then p is in the nucleus.

The task of creating an element in the nucleus of degree 6 is more complicated. We wish to find elements $f(x_1, x_2, \ldots, x_6)$ such that

$$(f(x_1, x_2, \ldots, x_6), x_7, x_8) = 0$$

in $\mathrm{ALT}[x_1, x_2, \ldots, x_6, x_7, x_8]$. If we work with the representations of degree 8, we are not able to distinguish those permutations that interchange $x_1, x_2, \ldots, x_6$ and leave x_7 and x_8 fixed with those that interchange $x_1, x_2, \ldots, x_7, x_8$. The technique we use keeps x_7 and x_8 special. We attack the problem using the representations of S_6 for permutations of the elements $x_1, x_2, \ldots, x_6$, and we encode the movements of x_7 and x_8 by increasing the number of types. This is not an ideal solution because it requires more memory. As mentioned in [Hentzel and Peresi 06b, Section 5], we can assume that x_7 and x_8 are skew-symmetric.

$T_1 \quad \cdots \quad T_{12012}$	$(T'_1, x_7, x_8) \quad \cdots \quad (T'_{42}, x_7, x_8)$
Augmented type identities of degree 8	Zero matrix
Expansion of associators	$\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$

TABLE 1. Nuclear elements.

Given a multilinear polynomial $f(x_1, x_2, \ldots, x_7, x_8)$, we first sort the terms by association type. Since cat$[8] = 429$, we have 429 association types. For each of these association types, we collect the terms by the positions of the skew-symmetric elements x_7 and x_8. There are $\binom{8}{2} = 28$ distinct positions for x_7 and x_8. There are then $28 \cdot 429 = 12012$ types with which we have to work. The multilinear polynomial is then encoded as an element of the direct sum of 12012 copies of the group algebra FS_6 in the form $(g_1, g_2, \ldots, g_{12012})$.

We create the type identities of degree 8 implied by the alternative laws. Since S_6 permutes only $x_1, x_2, \ldots, x_6$, we have to include the permutations that move x_7 and x_8. We make 28 copies of the type identities. In each copy we interchange x_7 and x_8 with a different pair chosen from $x_1, x_2, \ldots, x_8$.

These augmented type identities of degree 8 are now encoded by the permutation of $x_1, x_2, x_3, x_4, x_5, x_6$ and a number that specifies both the original association type and the positions of x_7 and x_8.

In degree 6 there are 42 association types: $T'_1, \ldots, T'_{42}$. We expand the associator (w_k, x_7, x_8), where w_k is $x_1x_2x_3x_4x_5x_6$ associated in type T'_k. Each expansion has a single I on the right-hand side and exactly two terms on the left-hand side. The two terms are $-(w_kx_7)x_8$ and $w_k(x_7x_8)$. These expansions are appended to the augmented type identities. See Table 1.

Since the group algebra FS_n of the symmetric group S_n is semisimple, it is isomorphic to a direct sum of complete matrix algebras. We use the integral representations obtained by the algorithm given in [Clifton 81].

For each partition of 6, we now obtain a matrix from the group algebra expressions in Table 1 by replacing each element of FS_6 by its representation matrix.

When we reduce this matrix to row canonical form, the nonzero rows that have a leading one in the right-hand portion of the matrix represent the identities of the form (w, x_7, x_8), where w is a multilinear element of degree 6 in $x_1, x_2, \ldots, x_6$.

Among the nonzero rows that have a leading one in the right-hand portion of the matrix occur all expressions of the form (w, x_7, x_8), where w is itself zero. That is, w is a consequence of the alternative identities in degree 6. We can eliminate these trivial identities by comparing the row canonical form of just the alternative identities for degree 6 with the right-hand portion of the matrix. Any additional nonzero row that has a leading one in the right-hand portion of the matrix represents an identity of the form (w, x_7, x_8), where $w \neq 0$. Thus w is a nonzero element of the nucleus.

3.1 Orthogonality Conditions

Suppose that $A_{ij}(t)$ and $B_{hk}(t)$ are inequivalent representations of FS_n. Then

$$\sum_{t \in S_n} A_{ij}(t) B_{hk}(t^{-1}) = 0. \tag{3–1}$$

Furthermore,

$$\sum_{t \in S_n} A_{ij}(t) A_{hk}(t^{-1}) = \frac{n!}{d} \delta_{ik} \delta_{jh}, \tag{3–2}$$

where d is the degree of the representation, and δ_{xy} is the Kronecker delta defined by

$$\delta_{xy} = \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

See [Boerner 63, equations (7.2) and (7.4), p. 77].

The orthogonality conditions (3–1) and (3–2) allow us to create an element in the group algebra FS_n given its representations. In particular, we want the elements

g_{ij}^k whose kth representation is the matrix unit E_{ij} and whose representation is the zero matrix elsewhere.

The representation technique creates an element that is displayed as a single row in a matrix. When we create the polynomial form of this element, we choose to create it when the row is placed across the bottom of the matrix. We get equivalent elements from any fill of the matrix as long as its row space is spanned by the single given row.

To get the matrix unit E_{di} on the bottom row of the $d \times d$ representation $A_{ij}(t)$, we use

$$g_{di}^k = \sum_{t \in S_n} A_{id}(t^{-1})\ t. \tag{3–3}$$

This usually produces a sum with $n!$ terms, which we simplify with ALBERT to an equivalent element modulo the alternative laws that has a compact form.

3.2 Simplification Using ALBERT

ALBERT can be used to simplify the polynomial form of an element. Using the representation technique, we can find nuclear elements that are typically a linear combination of $n!$ cat$[n]$ terms.

An element expressed in this way is certainly a valid solution to the problem, but it is desirable to write the element in a more compact form. We can use ALBERT to simplify the element.

Suppose we wish to know whether the nuclear element can be expressed by polynomials of the form $[[x, y], z] \circ [[u, v], w]$. We add $[[x, y], z] \circ [[u, v], w]$ as an identity in addition to the alternative laws. We test to see whether our element is an identity. If it is an identity, then we know that our element is equivalent modulo the alternative laws to one that can be written as a linear combination of polynomials of the form $[[x, y], z] \circ [[u, v], w]$.

If it is not an identity, then we can add further polynomial forms, and continue testing. When we have finally captured the nuclear element by the alternative laws and the additional polynomial forms, then we turn to the extension of ALBERT. We put the alternative laws as well as the nuclear element into ALBERT as the given identities. We save the basis and the multiplication table. Using the multiplication table, we evaluate the polynomial forms on all possible substitutions and look for dependence relations between these evaluations.

For example, if the nuclear element p is captured by $[[x, y], z] \circ [[u, v], w]$, and the terms of p each involve two a's, two b's, and two c's, then we evaluate

$$\begin{aligned} &[[a, b], a] \circ [[b, c], c], \quad [[a, c], a] \circ [[b, c], b], \\ &[[a, c], b] \circ [[b, c], a], \quad [[b, c], a] \circ [[b, c], a]. \end{aligned}$$

Each evaluation is a linear combination of the basis. The coefficients of this linear combination form a row of a matrix. A linear dependence relation among the rows is either an element that is a consequence of the alternative laws or is a nonzero element of the nucleus. We can test whether it is a consequence of the alternative laws using ALBERT.

Using ALBERT, we can test whether the original element p and this new element are equivalent modulo the alternative laws. We do this by showing that each element is a consequence of the alternative laws and the other element.

4. THE COMPUTATIONS

4.1 Representations of FS_n

The representations of FS_n are given in [Clifton 81]. The algorithm for the matrix representation of a permutation π associated with a particular Young diagram constructs a matrix A_π. The map $\pi \mapsto A_\pi$ is not a representation. However, the map $\pi \mapsto A_I^{-1} A_\pi$ is a representation, where I is the identity permutation.

We arrange the standard tableaus in the "last-number sequence." In a standard tableau, the largest entry must appear at the right end of its row and the bottom of its column. If $i < j$, then those tableaus with largest entry in row i come after those tableaus whose largest entry is in row j. Inductively, those tableaus whose largest entries are both in the same row are themselves sorted by the position of the next-largest element.

Given a Young diagram, if the standard tableaus are $\mathcal{T}_1, \ldots, \mathcal{T}_f$, then the representation has matrices of dimensions $f \times f$. For each permutation π, Clifton's algorithm constructs a matrix $A_\pi = (a_{ij})$ as follows. Apply π to the numbers in tableau $\mathcal{T}_j$. If two numbers occur in the same column of $\mathcal{T}_i$ and the same row of $\pi(\mathcal{T}_j)$, then let $a_{ij} = 0$. If not, then there is a permutation σ that arranges the elements within the columns of $\mathcal{T}_i$ so that in $\sigma(\mathcal{T}_i)$ each element is in the same column that it occupies in $\mathcal{T}_i$ and is in the same row that it occupies in $\pi(\mathcal{T}_j)$. In this case, let $a_{ij} = \mathrm{sgn}(\sigma)$.

Clifton's construction writes the operators on the left: $\pi(X)$. In our work we put the operators on the right:

$(X)\pi$. We use the transposes of the matrices of the representation given by Clifton to adjust for reversing the order of the group action.

Example 4.1. We will calculate the matrix associated with the string *cabed* in the representation 32, which is indexed by the Young diagram

[ ] [ ] [ ]
[ ] [ ]

To encode the string *cabed* as a permutation based on the standard order of *abcde*, we write $1 \to 2$, $2 \to 3$, $3 \to 1$, $4 \to 5$, $5 \to 4$. That is, whatever is in the first position moves to the second position, whatever is in the second position moves to the third position, and so on. This permutation can be written as

$$\pi : \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{matrix}$$

Notice that the bottom row does not match the order of the original string *cabed*.

The standard tableaus in the "last-number sequence" are as follows:

$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$
123	124	134	125	135
45	35	25	34	24

We now apply Clifton's algorithm to calculate the matrix A_π. We obtain

		$\pi(\mathcal{T}_1)$	$\pi(\mathcal{T}_2)$	$\pi(\mathcal{T}_3)$	$\pi(\mathcal{T}_4)$	$\pi(\mathcal{T}_5)$
		231 54	235 14	215 34	234 15	214 35
$\mathcal{T}_1$	123 45	1	0	0	−1	0
$\mathcal{T}_2$	124 35	0	0	0	−1	1
$\mathcal{T}_3$	134 25	0	0	0	−1	0
$\mathcal{T}_4$	125 34	0	−1	1	0	0
$\mathcal{T}_5$	135 24	0	−1	0	0	0

Since

$$A_I = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we obtain

$$A_I^{-1} A_\pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix

$$\left(A_I^{-1} A_\pi\right)^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

is the representation of the string *cabed* for the representation 32.

4.2 Nuclear Elements of Degree 6

There are 11 representations for FS_6. We calculate the row canonical form of the augmented type identities of degree 8 and the expanded associators (see Table 1). We count the number of nonzero rows that have a leading one in the right-hand portion of the matrix. We also count the nonzero rows having leading ones that come from the degree-6 alternative identities. The additional nonzero rows having leading ones are elements that are not zero in the free alternative algebra and must be elements in the nucleus.

Among those elements in the nucleus, we want to distinguish those elements that are already known. That is, they can be obtained from the lifted nuclear elements (1–7) through (1–10). In each representation the ranks have to be related. The rank of the alternative laws is less than or equal to the rank of the alternative laws and the lifted nuclear elements (1–7) through (1–10), and this rank is less than or equal to the rank of the alternative laws and the nuclear elements of degree 6. See Table 2.

There is one new nuclear element in each of the representations given by the partitions 33, 2211, and 21111.

Partition	Alternative laws	Alternative laws and nuclear elements (1–7) through (1–10)	Alternative Laws and nuclear elements
6	41	41	41
51	205	205	205
42	369	372	372
411	406	409	409
33	205	206	207
321	652	658	658
3111	400	403	403
222	202	202	204
2211	360	362	363
21111	194	195	196
111111	36	36	36

TABLE 2. Rank of the matrices.

There are two new nuclear elements in the representation given by the partition 222. The shape of the tableau tells us how the element can be expressed by polynomials with repeated unknowns. In particular, we need as many distinct elements as there are rows in the tableau, and we need as many copies of the ith element as there are boxes in the ith row. We choose to use polynomials with repeated letters rather than multilinear polynomials because it makes for a more compact presentation of the element. A polynomial from partition 33 can be written with unknowns a, a, a, b, b, b. A polynomial from partition 222 can be written with unknowns a, a, b, b, c, c. A polynomial from partition 2211 can be written with unknowns a, a, b, c, d, d. A polynomial from partition 21111 can be written with unknowns a, a, b, c, d, e.

4.3 The Nuclear Elements in Partition 33

We can see that there are two nuclear elements in partition 33. One of them is known because it is a consequence of the lifted nuclear elements (1–7) through (1–10). The other one is new. We could construct this element using (3–3). But fortunately, this problem is small enough that we can do it with Albert (see Section 2).

Using `generators` a, a, a, b, b, b, x, y, we create and save the basis and the multiplication table. Then we look up in the basis table the basis elements that are expressed with exactly three a's and three b's. There are 20 of them. We compute (h_i, x, y) using the multiplication table. The matrix we create has 20 rows. These rows are dependent and there are five dependence relations. These dependence relations generate five elements in the nucleus. These elements have been simplified using Artin's theorem, which says that the subalgebra generated by a and b is associative [Zhevlakov et al. 82, Chapter 2, Theorem 2]. Here are the five elements that are in the nucleus:

$$((([a,bb]b)a)a + ((([b,aa]b)b)a + ((([bb,a]a)a)b + ((([aa,b]a)b)b, \tag{4–1}$$

$$(((b[a,b])b)a)a + ((([bb,a]a)a)b + (((([a,b]a)a)b)b, \tag{4–2}$$

$$((([a,bb]b)a)a + ([bb,aa]b)a + ((a[a,bb])a)b, \tag{4–3}$$

$$(((bb)a)[a,b])a + ((ba)[bb,a])a + (((ba)[a,b])a)b, \tag{4–4}$$

$$[[a,b][a,b],b]a. \tag{4–5}$$

Using Albert, we find that (4–1), (4–2), and (4–3) are consequences of the alternative laws and the lifted nuclear elements (1–7) through (1–10). Therefore none of (4–1), (4–2), and (4–3) are considered new.

Using Albert, we find that neither (4–4) nor (4–5) is a consequence of the alternative laws and the lifted nuclear elements (1–7) through (1–10). We also find that (4–4) and (4–5) are equivalent modulo the alternative laws and the lifted nuclear elements (1–7) through (1–10). Therefore (4–4) or (4–5) is the one new nuclear element in partition 33. Element (4–5) is the element (1–2) in Theorem 1.4.

4.4 The Nuclear Elements in Partition 222

The representation given by partition 222 has degree 5. There are two new nonzero rows that have a leading one in the right-hand side of the matrix. They appear under the following types:

Nuclear Element (a)

$$\begin{array}{c} x(x(x(x(xx)))) \\ 1\;1\;1\;0\;\;-1 \end{array}$$

Nuclear Element (b)

$$\begin{array}{cc} (xx)(x((xx)x)) & x(x((xx)(xx))) \\ 0\,0\,0\,0\,1 & 0\ -1\,0\,0\,0 \\[1ex] x(x(x((xx)x))) & x(x(x(x(xx)))) \\ 0\,0\,0\,0\,1 & 0\,1\,0\ -1\ -2 \end{array}$$

We create the polynomial expression for each of these nuclear elements using (3–3). We then use ALBERT to verify that element (a) can be expressed by polynomials of the form $[[x,y],z] \circ [[u,v],w]$ and $[x,y] \circ [[z,u],[v,w]]$.

There are 90 possible ways to substitute a,a,b,b,c,c for the elements x,y,z,u,v,w. Not all are equivalent to the original element (a). But there is at least one substitution that is. The checking for equivalency is done using ALBERT. The element (a) is equivalent to (1–3).

The element (b) can be expressed by polynomials of the form $\langle x,y,z\rangle \circ \langle u,v,w\rangle$ and $(x,y,z)(u,v,w)$. The element (b) is equivalent to (1–4).

4.5 The Nuclear Element in Partition 2211

There is one nuclear element obtainable from this representation. It is given by the following:

Nuclear Element (c)

$$\begin{array}{c} x(x(x(x(xx)))) \\ 0\,2\,3\,1\ -1\,1\ -5\,1\,4 \end{array}$$

Using (3–3) we construct the polynomial expression for this nuclear element (c). The goal now becomes to write a simpler expression for (c). We consider expressions known to be central elements in Cayley–Dickson algebras (see [Hentzel and Peresi 97]). This led us to verify that element (c) is equivalent to (1–5).

4.6 The Nuclear Elements in Partition 21111

This representation gives the following nuclear element:

Nuclear Element (d)

$$\begin{array}{ccc} (xx)(x(x(xx))) & x((xx)(x(xx))) & x(x((xx)(xx))) \\ 0\,0\,0\,0\,1 & 0\,0\,0\,0\ -1 & 1\,0\,0\,1\,0 \\[1ex] x(x(x((xx)x))) & x(x(x(x(xx)))) & \\ 0\ -2\,0\,0\ -1 & -1\,1\,1\,0\,0 & \end{array}$$

Using (3–3) we obtain the polynomial expression for (d). We then use ALBERT to check that the alternative laws of degree 3 and $[[[x,y],z],u][v,w] = 0$ imply (d). This means that we can capture (d) using a linear combination of the 66 expressions of the form $[[[x,y],z],u][v,w]$, where the arguments x,y,z,u,v,w are replaced by a,a,b,c,d,e in all possible ways.

We evaluate all 66 substitutions of a,a,b,c,d,e into $[[[x,y],z],u][v,w]$ and create the matrix of coefficients with 66 rows. The dependence relations among the rows give elements in the nucleus. Five of the dependence relations are consequences of the alternative laws. The sixth one is (1–6). Therefore element (d) is equivalent to (1–6).

4.7 Further Checking

Using ALBERT we verify that (1–2) through (1–6) are not consequences of the alternative laws. As an example we will present the process for element (1–2). To verify that $[[a,b][a,b],b]a$ is not a consequence of the alternative laws, ALBERT constructs an algebra of dimension 68 (which is a homomorphic image of $\mathrm{ALT}[a,b]$) and verifies that $[[a,b][a,b],b]a$ is not zero in this algebra. Therefore $[[a,b][a,b],b]a$ is not zero in $\mathrm{ALT}[a,b]$.

We verify that $[[a,b][a,b],b]a$ is in the nucleus using ALBERT. We need to show that $([[a,b][a,b],b]a,c,d) = 0$ in $\mathrm{ALT}[a,b,c,d]$. ALBERT constructs an algebra of dimension 4006 (which is a homomorphic image of $\mathrm{ALT}[a,b,c,d]$) and verifies that $([[a,b][a,b],b]a,c,d)$ is zero in this algebra. This implies that $([[a,b][a,b],b]a,c,d)$ is zero in $\mathrm{ALT}[a,b,c,d]$ (see [Hentzel and Jacobs 91, Theorem 1 and its Corollary]). By this process we cannot verify that (1–3) through (1–6) are in the nucleus because the calculations are too large for ALBERT.

We verify that elements (1–2) through (1–6) are independent (modulo the alternative laws and the lifted nuclear elements (1–7) through (1–10)). We give the alternative laws, elements (1–7) through (1–10) and elements (1–3) through (1–6) as the defining identities to ALBERT. We check that (1–2) is not zero. This shows that (1–2) is independent of (1–3) through (1–6). Similarly, we show that each of the elements (1–3) through (1–6) is independent of the other four elements.

Using representation techniques, we verify that elements (1–2) through (1–6) (i) are not a consequence of the alternative laws, (ii) are independent (modulo the alternative laws and the lifted nuclear elements (1–7) through (1–10)), and (iii) are in the nucleus.

To prove (i), we lift the alternative laws to degree 6 and compute the ranks in each representation (see Table 2, second column). Then we separately add each of the elements (1–2) through (1–6) and recompute the ranks. Elements (1–2), (1–5), and (1–6) increase the rank by one in the representation given by partitions 33, 2211, and 21111, respectively. Each of the elements (1–3) and (1–4)

Representation	Partition	Rank
1	8	428
2	71	2996
3	62	8560
4	611	8982
5	53	11984
6	521	27384
7	5111	14962
8	44	5992
9	431	29954
10	422	23959
11	4211	38491
12	41111	14954
13	332	17970
14	3311	23954
15	3221	29933
16	32111	27353
17	311111	8958
18	2222	5981
19	22211	11962
20	221111	8532
21	2111111	2970
22	11111111	418

TABLE 3. Alternative identities: degree 8.

increases the rank by one in the representation given by partition 222.

To prove (ii), we lift the alternative laws to degree 6, add the lifted nuclear elements (1–7) through (1–10) and four of elements (1–2) through (1–6). We compute the ranks in each representation. When we add the remaining element, the rank in only one representation increases. Elements (1–2), (1–5), and (1–6) increase the rank by one in the representation given by partitions 33, 2211, and 21111, respectively. Each of the elements (1–3) and (1–4) increases the rank by one in the representation given by partition 222.

To prove (iii), we lift the alternative laws to degree 8 and compute the ranks in each representation (see Table 3). Now we add (p, x, y), where p is the linearized form of one of the elements (1–2) through (1–6). In all five cases, the ranks remain the same as in Table 3. Therefore $(p, x, y) = 0$.

4.8 Remarks

We use a combination of Albert, the extension to Albert, and representation theory to locate these nuclear elements. The representation theory locates all the nuclear elements of degree 6 in the free alternative algebra over Z_{103}. These nuclear elements are expressed in representation form as a row of a huge matrix.

There is no really good way to represent one of these rows in simplified algebraic form. The shape of the tableau where the nuclear element occurs tells us whether we can express the identity using repeated variables. For element (1–2), this reduces the problem enough so that the creation of the algebraic form can be done just using Albert. For elements (1–3) through (1–6), Albert cannot create the finite homomorphic image of the free alternative algebra, and so the initial algebraic forms are created using the orthogonality conditions. The calculations with the orthogonality conditions are done over the integers.

In the representation form of the element, we first change instances 102 to -1 and 101 to -2, believing that smaller numbers are more likely to be correct for other primes. These initial algebraic forms of the nuclear elements are simplified using Albert modulo 251, and again we change the coefficients to make them small.

At this stage, because of the changes in characteristic, we cannot be sure that our resulting elements are still in the nucleus until we recheck them.

We add identities (p_i, x, y), where p_i $(i = 1, \ldots, 5)$ is the linearized form of elements (1–2) through (1–6), to the degree-8 identities for the free alternative algebra. We calculate the rank of these identities modulo primes 103, 229, 233, 239, 241, 251. In each case, the rank did not increase when the identities are added. This means that (1–2) through (1–6) are nuclear elements modulo the six primes we checked.

We know that we have found all the degree-6 nuclear elements modulo 103. We do not know that we have found all the degree-6 nuclear elements modulo any other characteristic. We do know that our elements are in the nucleus for the primes that we tested.

ACKNOWLEDGMENTS

This paper was written while the author L. A. Peresi was visiting Iowa State University on a grant from FAPESP (Proc. 2006/57531-4). We thank the referee for many suggestions that improved the exposition of this paper.

REFERENCES

[Boerner 63] H. Boerner. *Representations of Groups with Special Consideration for the Needs of Modern Physics*. New York: John Wiley & Sons, 1963.

[Clifton 81] J. Clifton. "A Simplification of the Computation of the Natural Representation of the Symmetric Group S_n." *Proc. Amer. Math. Soc.* 83 (1981), 248–250.

[Filippov et al. 06] V. T. Filippov, V. K. Kharchenko, and I. P. Shestakov. "Dniester Notebook: Unsolved Problems in the Theory of Rings and Modules." Translated by M. R. Bremner and M. V. Kochetov. In *Non-Associative Algebra and Its Applications*, edited by L. Sabinin, L. V. Sbitneva, and I. P. Shestakov, pp. 461–516, Lecture Notes in Pure and Appl. Math. 246. Boca Raton: CRC, 2006.

[Hentzel and Jacobs 91] I. R. Hentzel and D. P. Jacobs. "A Dynamic Programming Method for Building Free Algebras." *Comput. Math. Appl.* 22 (1991), 61–66.

[Hentzel and Peresi 97] I. R. Hentzel and L. A. Peresi. "Identities of Cayley–Dickson algebras." *J. Algebra* 188 (1997), 292–309.

[Hentzel and Peresi 06a] I. R. Hentzel and L. A. Peresi. "Central Elements of Minimal Degree in the Free Alternative Algebra." In *Non-Associative Algebra and Its Applications*, edited by L. Sabinin, L. V. Sbitneva, and I. P. Shestakov, pp. 195–204, *Lecture Notes in Pure and Appl. Math.* 246. Boca Raton: CRC, 2006.

[Hentzel and Peresi 06b] I. R. Hentzel and L. A. Peresi. "The Nucleus of the Free Alternative Algebra." *Experimental Mathematics* 15 (2006), 445–454.

[Jacobs et al. 96] D. P. Jacobs, D. Lee, S. V. Muddana, A. J. Offutt, K. Prabhu, and T. Whiteley. *Version 3.0. Albert's User Guide*. Clemson University, 1996. Available online (www.cs.clemson.edu/~dpj/albertstuff/albert.html).

[Kleinfeld 53] E. Kleinfeld. "Simple Alternative Rings." *Ann. of Math.* 58 (1953), 544–547.

[Zhevlakov et al. 82] K. A. Zhevlakov, A. M. Slin′ko, I. P. Shestakov, and A. I. Shirshov. *Rings That Are Nearly Associative.* New York: Academic Press, 1982.

Irvin R. Hentzel, Department of Mathematics, Room 432 Carver Hall, Iowa State University, Ames, IA 50011 (hentzel@iastate.edu)

Luiz Antonio Peresi, Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, São Paulo-SP, 05508-090, Brazil (peresi@ime.usp.br)

Received September 3, 2007; accepted in revised form February 11, 2008.