Decomposition and Enumeration of Triangulated Surfaces

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CONTENTS

We describe some theoretical results on triangulations of surfaces and we develop a theory on roots, decompositions, and genus surfaces. We apply this theory to describe an algorithm to list all triangulations of closed surfaces with at most a fixed number of vertices. We specialize the theory to the case that the number of vertices is at most 11, and we obtain theoretical restrictions on genus surfaces, allowing us to obtain a list of all triangulations of closed surfaces with at most 11 vertices.

1. INTRODUCTION

The enumeration of triangulations of surfaces (i.e., simplicial complexes whose underlying topological space is a surface) began with [Brückner 1897] at the end of the nineteenth century. This study was continued through the twentieth century by many authors. For example, a complete classification of triangulations of closed surfaces with at most eight vertices was obtained in [Datta 1999] and [Datta and Nilakantan 2002], while the list of such triangulations with at most ten vertices was obtained in [Lutz 2008]. The numbers of triangulations, depending on genus and number of vertices, are collected in [Lutz 2007] and [Sulanke 2007].

We point out that all these studies, as well as this paper, deal with genuine piecewise linear triangulations of surfaces, and not with mere gluings of triangles (for which different techniques must be used).

We will describe here an algorithm to list the triangulations of closed surfaces with at most a fixed number of vertices. This algorithm is based on some theoretical results that are interesting in themselves. By specializing this theory to the case in which the number of vertices is at most 11, we are able to improve the algorithm for this particular case. We have hence written the computer program `trialistgs11` [Amendola 2007], which gives a complete enumeration of all triangulations of closed surfaces with at most 11 vertices. Table 1 gives the detailed

2000 AMS Subject Classification: Primary 57Q15

Keywords: Surface, triangulation, decomposition, listing algorithm

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V	S	T	R	N	V	S	T	R	N
4	S^2	1	1		10	S^2	233	12	221
						T^2	2109	887	1222
5	S^2	1		1		S_2^+	865	865	
						S_3^+	20	20	
6	S^2	2	1	1		$\mathbb{RP}^2$	1210	185	1025
	$\mathbb{RP}^2$	1	1			K^2	4462	1971	2491
						S_3^-	11784	9385	2399
7	S^2	5	1	4		S_4^-	13657	13067	590
	T^2	1	1			S_5^-	7050	7044	6
	$\mathbb{RP}^2$	3	2	1		S_6^-	1022	1022	
						S_7^-	14	14	
8	S^2	14	2	12					
	T^2	7	6	1	11	S^2	1249	34	1215
	$\mathbb{RP}^2$	16	8	8		T^2	37867	9732	28135
	K^2	6	6			S_2^+	113506	93684	19822
						S_3^+	65878	65546	332
9	S^2	50	5	45		S_4^+	821	821	
	T^2	112	75	37		$\mathbb{RP}^2$	11719	1050	10669
	$\mathbb{RP}^2$	134	36	98		K^2	86968	23541	63427
	K^2	187	133	54		S_3^-	530278	298323	231955
	S_3^-	133	133			S_4^-	1628504	1314000	314504
	S_4^-	37	37			S_5^-	3355250	3175312	179938
	S_5^-	2	2			S_6^-	3623421	3596214	27207
						S_7^-	1834160	1833946	214
						S_8^-	295291	295291	
						S_9^-	5982	5982	

TABLE 1. Number of triangulations (T), roots (R), and nonroots (N), with at most 11 vertices, depending on the number of vertices V and the closed surface S triangulated.

numbers of such triangulations. This result was obtained independently in [Lutz and Sulanke 2006].

The aim of this paper is to describe the theory of what we call roots, decompositions, and genus surfaces, and to describe the algorithm based on this theory. The implementation `trialistgs11` of the algorithm is not designed to be as fast as possible: more precisely, our program is slower than the program given in [Lutz and Sulanke 2006].

A triangulation of a closed surface is a *root* if either it has no 3-valent vertex or it is the boundary of the tetrahedron. We will see that each triangulation of a closed surface can be transformed into a unique root by repeatedly contracting edges containing a 3-valent vertex. By uniqueness, roots divide the class of all triangulations of closed surfaces into disjoint subclasses, depending on their root. One can think of roots as irreducible triangulations when only edge-contractions deleting edges containing a 3-valent vertex are allowed. However, there are some differences; for instance, we gain uniqueness (in fact, a triangulation may have more than one irreducible triangulation), but we lose finiteness (indeed, we have infinitely many roots for each surface).

It is worth noting that the number of roots is by far smaller than the number of triangulations, at least as the number of vertices increases (see Table 1). Moreover, we note also that for the sphere S^2, the number of roots is very small; hence roots seem to work better for the sphere S^2 than for other surfaces.

Roughly speaking, a *decomposition* of a closed triangulated surface is obtained by dividing it into a number of disjoint triangulated disks and one triangulated surface (called a *genus surface*) in such a way that at least one disk contains in its interior a maximal-valence vertex of the triangulation. Such a decomposition is called *minimal* if the number of triangles in the genus surface is the smallest possible. We will see that minimal decompositions satisfy many properties proved theoretically. Roughly speaking, the algorithm consists in listing the pieces of such minimal decompositions (using the properties to simplify the search) and then gluing the pieces found.

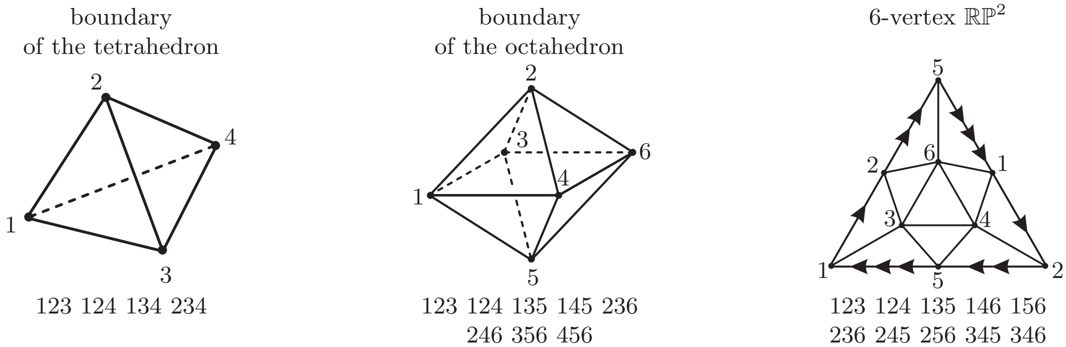

FIGURE 1. Examples of triangulations/roots.

1.1 Definitions and Notation

From now on, S will denote a connected compact surface.

Triangulated Surfaces. A *triangulation* $\mathcal{T}$ *of a (connected compact) surface* S is a simplicial complex whose underlying topological space is the surface S. The vertices of the triangulation $\mathcal{T}$ are usually denoted by numbers, say $1, 2, \ldots, n$; the choice of a (different) number for each vertex is called *labeling*. Obviously, a change in labeling (*relabeling*) modifies neither the triangulation nor the surface.

In dealing with triangulations, there is the problem of deciding whether the underlying topological space of a triangulation belongs to a particular class (in our case, the class of surfaces). This is in general a difficult matter. For instance, there is no algorithm to decide whether the underlying topological space of a given d-dimensional simplicial complex is a d-sphere if $d \geq 5$. In our case, in order to decide whether the underlying topological space of a triangulation is a surface, we can check the property "the link of each vertex is a circle or an interval." The case of the interval is forbidden in the closed case.

Since we deal only with triangulations of surfaces, in order to define a triangulation, it is enough to list the triangles. Hence, for instance, the boundary of the tetrahedron can be encoded by "123 124 134 234"; see Figure 1, on the left. Moreover, the order of the triangles in such lists can be changed arbitrarily; hence it is not restrictive always to choose the lexicographically smallest one.

It is well known [Radó 1925] that each closed surface can be *triangulated*, i.e., it is the underlying topological space of a simplicial complex. This and the following paragraph allow us to forget about the abstract surface and to use the term *triangulated surface*.

Euler Characteristic. For an arbitrary closed (orientable or nonorientable) surface S, the Euler characteristic $\chi(S)$ of S is the alternating sum of the number of vertices $V(\mathcal{T})$, the number of edges $E(\mathcal{T})$, and the number of triangles $T(\mathcal{T})$, i.e., $\chi(S) = V(\mathcal{T}) - E(\mathcal{T}) + T(\mathcal{T})$, of any triangulation $\mathcal{T}$ of S. This definition makes sense because it turns out that it does not depend on the triangulation $\mathcal{T}$ but only on the topological type of the surface S.

Since each triangle contains three edges and each edge is contained in two triangles, we have $2E(\mathcal{T}) = 3T(\mathcal{T})$. Thus, the number of vertices $V(\mathcal{T})$ and the Euler characteristic $\chi(\mathcal{T})$ determine $E(\mathcal{T})$ and $T(\mathcal{T})$, by the formulas $E(\mathcal{T}) = 3V(\mathcal{T}) - 3\chi(\mathcal{T})$ and $T(\mathcal{T}) = 2V(\mathcal{T}) - 2\chi(\mathcal{T})$.

A closed orientable surface S_g^+ of genus g has Euler characteristic $\chi(S_g^+) = 2 - 2g$, whereas a closed nonorientable surface S_g^- of genus g has Euler characteristic $\chi(S_g^-) = 2 - g$. For instance, S_0^+ is the sphere S^2, S_1^+ is the torus T^2, S_1^- is the projective plane $\mathbb{RP}^2$, S_2^- is the Klein bottle K^2. The topological type of a closed surface is completely determined if its Euler characteristic is known (or, equivalently, its genus) and it is known whether it is orientable; hence the notation $S_*^\pm$ above makes sense.

The smallest possible number of vertices $V(\mathcal{T})$ for a triangulation $\mathcal{T}$ of a closed surface S is determined by Heawood's bound [Heawood 1890]

$$V(\mathcal{T}) \geq \left\lceil \frac{1}{2}\left(7 + \sqrt{49 - 24\chi(S)}\right) \right\rceil .$$

It was proved in [Ringel 1955] that this bound is tight for the nonorientable case, and then it was proved in [Jungerman and Ringel 1980] for the orientable case, except for S_2^+, the Klein bottle K^2, and S_3^-, for each of which an extra vertex has to be added.

Notation. Let now $\mathcal{T}$ be a triangulation of a (not necessarily closed) surface S. We will denote by $\partial\mathcal{T}$ the triangulation of the boundary of S induced by $\mathcal{T}$, and by $\mathrm{int}(\mathcal{T})$ the triangulation of the interior of S induced by $\mathcal{T}$. If S is closed, we have $\partial\mathcal{T} = \varnothing$ and $\mathrm{int}(\mathcal{T}) = \mathcal{T}$. For each simplex $\sigma \in \mathcal{T}$ we will denote by $\mathrm{st}(\sigma)$ the open star of σ (i.e., the subtriangulation of $\mathcal{T}$ made up of the simplices containing σ), by $\mathrm{clst}(v)$ the closed star of v (i.e., the closure of $\mathrm{st}(\sigma)$), and by $\mathrm{link}(v)$ the link of v (i.e., the subtriangulation of $\mathrm{clst}(\sigma)$ made up of the simplices disjoint from σ). For each vertex $v \in \mathcal{T}$ we will, moreover, denote by $\mathrm{val}(v)$ the valence of v (i.e., the number of triangles of $\mathcal{T}$ containing v) and by $\deg(v)$ the degree of v (i.e., the number of vertices of $\mathcal{T}$ adjacent to v).

When a subtriangulation $\mathcal{U}$ of $\mathcal{T}$ is considered, we will denote by $\mathrm{val}_{\mathcal{U}}(v)$ the valence of v in $\mathcal{U}$ and by $\deg_{\mathcal{U}}(v)$ the degree of v in $\mathcal{U}$. Note that $\deg(v) = \mathrm{val}(v)$ if $v \in \mathrm{int}(\mathcal{T})$, while $\deg(v) = \mathrm{val}(v) + 1$ if $v \in \partial\mathcal{T}$.

We will denote by $\mathrm{mv}(\mathcal{T})$ the maximal valence of the vertices of $\mathcal{T}$ and by $\mathrm{md}(\mathcal{T})$ the maximal degree of the vertices of $\mathcal{T}$. Note that if $\mathcal{T}$ is closed, $\mathrm{md}(\mathcal{T}) = \mathrm{mv}(\mathcal{T})$. With a slight abuse of notation we will freely move back and forth between closed and open triangles.

Remark 1.1. The boundary of the tetrahedron is the unique closed triangulated surface with maximal vertex-valence 3. The boundary of the octahedron shown in Figure 1, center, is the unique closed triangulated surface with maximal vertex-valence 4 and without 3-valent vertices.

2. ROOTS

We will describe in this section the notion of root of a closed triangulated surface. Triangulated surfaces can be modified by applying a move called a T-*move*: it consists in replacing an open triangle of the triangulation with the open star of a new 3-valent vertex (i.e., one new vertex, three new edges, and three new triangles), as shown in Figure 2. Note that a T-move can be applied for each triangle of a triangulated surface. Conversely, an *inverse* T-move can be applied only if the triangulated surface $\mathcal{T}$ has a 3-valent vertex and the link of this vertex does not already bound a triangle in $\mathcal{T}$ (for otherwise, the new triangle added by the inverse T-move would already be in $\mathcal{T}$); moreover, if the two conditions above are satisfied, an inverse T-move can be applied (the result being indeed a triangulated surface). It is worth noting that the boundary of the tetrahedron is the only triangulated surface having a 3-valent vertex whose link bounds a triangle.

Definition 2.1. A *root* of a closed triangulated surface $\mathcal{T}$ is a triangulation $\mathcal{R}$ obtained from $\mathcal{T}$ by a sequence of inverse T-moves such that no inverse T-move can be applied to it.

Example 2.2. The boundary of the tetrahedron, the boundary of the octahedron, and the unique $\mathbb{RP}^2$ with six vertices, shown in Figure 1, are roots.

Remark 2.3. The boundary of the tetrahedron is the only root with a 3-valent vertex. In fact, as mentioned above, when a closed triangulated surface has a 3-valent vertex, an inverse T-move can be applied unless the added triangle is already in the triangulation, which is then the boundary of the tetrahedron.

Remark 2.4. Remarks 1.1 and 2.3 obviously imply that the boundary of the tetrahedron and the boundary of the octahedron are the only roots with maximal valence at most 4.

Since T-moves are particular edge-contractions, each irreducible closed triangulated surface is a root. But there are finitely many irreducible triangulations of each closed surface [Barnette and Edelson 1988], while each closed surface S has infinitely many roots.

In fact, consider the boundary of the octahedron if $S = S^2$, or an irreducible triangulation of S otherwise; such triangulations are roots. By repeatedly applying edge-expansions creating 4-valent vertices (as shown in Figure 3), we get infinitely many different roots of S (they are roots because no 3-valent vertex appears).

It is worth noting that we have used the boundary of the octahedron (which is not an irreducible triangulation) because we need a root without 3-valent vertices, so that when we apply edge expansions, creating 4-valent ver-

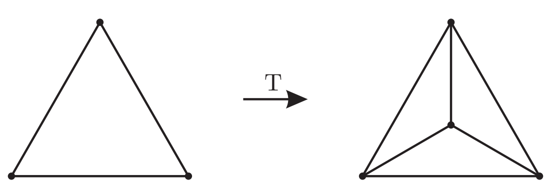

FIGURE 2. T-move.

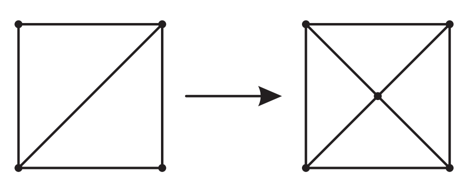

FIGURE 3. How to create a new 4-valent vertex via an edge-expansion.

tices, we get no 3-valent vertex, while the sphere has only one irreducible triangulation [Steinitz and Rademacher 1934], the boundary of the tetrahedron.

Theorem 2.5. *Each closed triangulated surface has exactly one root.*

Proof: Let $\mathcal{T}$ be a closed triangulated surface. In order to prove the existence of a root for $\mathcal{T}$, it is enough to repeatedly apply inverse T-moves until it is possible finally to obtain a root of $\mathcal{T}$ (note that each inverse T-move decreases by one the number of vertices; hence this procedure terminates).

The proof of uniqueness is slightly longer. We prove it by induction on the length of the longest sequence of inverse T-moves needed to get a root from $\mathcal{T}$ (obviously, there is a longest one). If such a sequence has length 0, there is nothing to prove. In fact, $\mathcal{T}$ is already a root, and there can be no other root, because inverse T-moves cannot be applied to it.

Suppose now that if a closed triangulated surface $\mathcal{T}$ has a root $\mathcal{R}$ obtained from $\mathcal{T}$ via a sequence having length n, and n is the maximal length of such sequences, then $\mathcal{R}$ is the unique root of $\mathcal{T}$; and let us prove that if a closed triangulated surface $\mathcal{T}$ has a root $\mathcal{R}$ obtained from $\mathcal{T}$ with a sequence having length $n+1$, and $n+1$ is the maximal length of such sequences, then $\mathcal{R}$ is the unique root of $\mathcal{T}$. In order to do this, consider the sequence

$$\mathcal{T} \xrightarrow{m_1} \mathcal{T}_1 \xrightarrow{m_2} \cdots \xrightarrow{m_{n-1}} \mathcal{T}_{n-1} \xrightarrow{m_n} \mathcal{T}_n \xrightarrow{m_{n+1}} \mathcal{R}$$

of inverse T-moves relating $\mathcal{T}$ to $\mathcal{R}$, and suppose by way of contradiction that another sequence

$$\mathcal{T} \xrightarrow{m_1'} \mathcal{T}_1' \xrightarrow{m_2'} \cdots \xrightarrow{m_{n'-1}'} \mathcal{T}_{n'-1} \xrightarrow{m_{n'}} \mathcal{R}'$$

of inverse T-moves relating $\mathcal{T}$ to another root $\mathcal{R}'$ exists (obviously, $n' \leq n+1$). Now consider the two triangulations $\mathcal{T}_1$ and $\mathcal{T}_1'$, and note that the longest sequence of inverse T-moves from each of them to the respective root has length at most n (because otherwise, we could find a sequence of inverse T-moves from $\mathcal{T}$ to a root with length greater than $n+1$). Hence, we can apply the inductive hypothesis, and we have that $\mathcal{R}$ and $\mathcal{R}'$ are the only roots of $\mathcal{T}_1$ and $\mathcal{T}_1'$, respectively.

In order to prove that $\mathcal{R} = \mathcal{R}'$, the idea is to change the sequences used to obtain $\mathcal{R}$ and $\mathcal{R}'$. Let us call v and v' the (3-valent) vertices removed by the inverse T-moves m_1 and m_1', respectively. If $v = v'$, then $\mathcal{T} = \mathcal{T}'$, and hence $\mathcal{R} = \mathcal{R}'$. Therefore, we suppose $v \neq v'$. Note that

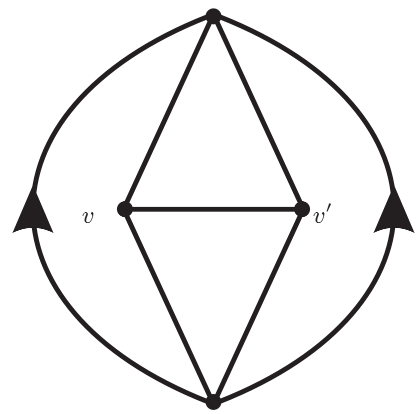

FIGURE 4. If a closed triangulated surface contains two 3-valent adjacent vertices, it is the boundary of the tetrahedron.

v and v' are not adjacent, because otherwise, $\mathcal{T}$ would be the boundary of the tetrahedron (see Figure 4 and note that two vertices can be the endpoints of at most one edge), and this is not the case (for no inverse T-move can be applied to the boundary of the tetrahedron).

Hence, we have $\mathrm{st}(v) \cap \mathrm{st}(v') = \varnothing$, and the inverse T-move removing v' (respectively v) can be applied to $\mathcal{T}_1$ (respectively $\mathcal{T}_1'$); let us continue calling this move m_1' (respectively m_1). In both cases we get the same triangulation, say $\mathcal{T}_2''$. Now let us consider a sequence

$$\mathcal{T}_2'' \xrightarrow{m_3''} \mathcal{T}_3'' \xrightarrow{m_4''} \cdots \xrightarrow{m_{n''-1}''} \mathcal{T}_{n''-1} \xrightarrow{m_{n''}''} \mathcal{R}''$$

of inverse T-moves relating $\mathcal{T}_2''$ to a root $\mathcal{R}''$, and let us use the sequences

$$\begin{tikzcd}
\mathcal{T}_1 \arrow[dr, "m_1'"] \\
& \mathcal{T}_2'' \arrow[r, "m_3''"] & \mathcal{T}_3'' \arrow[r, "m_4'"] & \cdots \arrow[r, "m_{n''-1}''"] & \mathcal{T}_{n''-1} \arrow[r, "m_{n''}''"] & \mathcal{R}'' \\
\mathcal{T}_1' \arrow[ur, "m_1"']
\end{tikzcd}$$

to obtain the roots $\mathcal{R}$ and $\mathcal{R}'$, which are equal to $\mathcal{R}''$ by uniqueness. Hence, we have proved that $\mathcal{R} = \mathcal{R}'$, and we are done. □

Remark 2.6. This theorem implies that the class of closed triangulated surfaces has a partition into (disjoint) subclasses depending on their root. This fact implies that each invariant of a root, and in particular the root itself, is actually an invariant of all the closed triangulated surfaces having that root.

Remark 2.7. For the sake of completeness we will also prove that irreducible triangulations are in general not unique. More precisely, there are triangulations to which

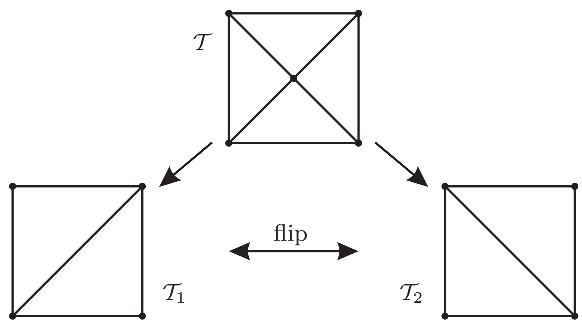

FIGURE 5. Nonuniqueness of irreducible triangulations. In the lower part is shown a flip, while the two moves above are edge-contractions leading to two different irreducible triangulations from $\mathcal{T}$.

we can apply two edge-contractions leading to two different irreducible triangulations. Take, for instance, two different irreducible triangulations (say $\mathcal{T}_1$ and $\mathcal{T}_2$) related by a *flip* (i.e., a move modifying a square made up of two adjacent triangles by changing the diagonal; see Figure 5, bottom). The proof of the existence of such a pair can be found in [Sulanke 2006]. Now, consider the triangulation $\mathcal{T}$ obtained by dividing the square of the flip by using both the diagonals, see Figure 5, above. We can apply two edge-contractions to $\mathcal{T}$ leading to $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, as shown in Figure 5.

We conclude this section by noting that roots could be used, for instance, to try to prove the following conjecture [Duke 1970, Hougardy et al. 2006].

Conjecture 2.8. *Every triangulation of a closed orientable surface with genus at most* 4 *is realizable in* $\mathbb{R}^3$ *by straight edges, flat triangles, and without self-intersections.*

This conjecture is true for spheres (as proved in [Steinitz 1922] and [Steinitz and Rademacher 1934]), while its natural extension to closed orientable surfaces of greater genus is not true [Bokowski and Guedes de Oliveira 2000, Schewe 2008]. Obviously, it makes sense only for orientable closed surfaces, because nonorientable closed surfaces are not embeddable in $\mathbb{R}^3$. Note that if a root is realizable, every triangulation having that root is also realizable; hence in order to prove the conjecture, it would suffice to prove it only for roots.

3. GENUS SURFACES

We will describe in this section the notions of decomposition and genus surface of a closed triangulated surface.

Definition 3.1. A *decomposition* of a closed triangulated surface $\mathcal{T}$ is a triple $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$, with $n \geq 0$, such that:

- $\mathcal{G}, \mathcal{D}, \mathcal{D}_1, \ldots, \mathcal{D}_n$ are subtriangulations of $\mathcal{T}$,
- $\mathcal{D}, \mathcal{D}_1, \ldots, \mathcal{D}_n$ are triangulated disks,
- $\mathrm{int}(\mathcal{D})$ contains a maximal-valence vertex of $\mathcal{T}$,
- $\mathcal{G} \cup \mathcal{D} \cup \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_n = \mathcal{T}$,
- the intersection of each pair of these subtriangulations is either a (triangulated) circle or empty.

The surface $\mathcal{G}$ is called the *genus surface* (of the decomposition), and $\mathcal{D}$ is called the *main disk* (of the decomposition).

First of all, we note that decompositions of closed triangulated surfaces exist. In fact, if $\mathcal{T}$ is a closed triangulated surface, then for each maximal-valence vertex $v \in \mathcal{T}$, we have that $(\mathcal{T} \setminus \mathrm{st}(v), \mathrm{clst}(v), \varnothing)$ is a decomposition of $\mathcal{T}$. We note also that the decompositions of a triangulation $\mathcal{T}$ hold some invariants of $\mathcal{T}$. For instance, the maximal valence of $\mathcal{T}$ is the maximal valence of internal vertices of the main disk only, and the genus of $\mathcal{T}$ can be computed from the genus surface only. As was the case for triangulations, two decompositions are considered equivalent if they are obtained from each other by a relabeling.

Remark 3.2. Genus surfaces are connected triangulated surfaces. In fact, they are obtained from closed triangulated surfaces by removing open triangulated disks whose closures are disjoint.

Definition 3.3. A decomposition $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ of a closed triangulated surface $\mathcal{T}$ is *minimal* if the number of triangles in $\mathcal{G}$ is minimal among all the decompositions of $\mathcal{T}$. In such a case, the genus surface (of the decomposition) is also said to be *minimal*.

By finiteness, minimal decompositions obviously exist.

Example 3.4. The minimal decompositions of a triangulated sphere $\mathcal{T}$ are exactly those of type $(\{T\}, \mathcal{T} \setminus \{T\}, \varnothing)$ such that T is a triangle of $\mathcal{T}$ and a vertex not in T has maximal valence (among those in $\mathcal{T}$). Hence in particular, the unique planar (e.g., disk, annulus) minimal genus surface is the triangle.

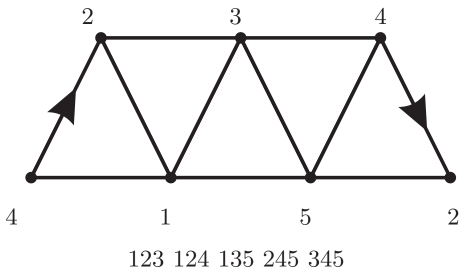

FIGURE 6. The triangulation of the Möbius strip with five vertices.

Example 3.5. The unique decomposition of the (unique) surface $\mathbb{RP}^2$ with six vertices is $(\mathcal{T} \setminus \mathrm{st}(v), \mathrm{clst}(v), \varnothing)$, where v is any vertex of $\mathcal{T}$. The genus surface $\mathcal{T} \setminus \mathrm{st}(v)$ is the Möbius strip with five vertices, shown in Figure 6.

The following obvious remarks will be useful in accelerating the search for genus surfaces of minimal decompositions.

Remark 3.6. The inequality $V(\mathcal{G}) \leq V(\mathcal{T}) - 1$ holds.

Remark 3.7. If

$$(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\}) \quad \text{and} \quad (\mathcal{G}', \mathcal{D}', \{\mathcal{D}'_1, \ldots, \mathcal{D}'_{n'}\})$$

are two decompositions of $\mathcal{T}$ such that $\mathcal{G} \subsetneq \mathcal{G}'$, then $(\mathcal{G}', \mathcal{D}', \{\mathcal{D}'_1, \ldots, \mathcal{D}'_{n'}\})$ is not minimal.

The following easy result will be also useful.

Proposition 3.8. *The following inequalities hold:*

$$T(\mathcal{D}) \geq \mathrm{mv}(\mathcal{T}),$$
$$V(\mathcal{D}) \geq \mathrm{mv}(\mathcal{T}) + 1.$$

Proof: Note that the main disk $\mathcal{D}$ contains a vertex $v \in \mathrm{int}(\mathcal{D})$ with valence $\mathrm{mv}(\mathcal{T})$. Hence $\mathcal{D}$ contains its closed star $\mathrm{clst}(v)$ and then at least $\mathrm{mv}(\mathcal{T})$ triangles and $\mathrm{mv}(\mathcal{T}) + 1$ vertices. $\square$

3.1 Constructing the Main Disk

It is easy to prove (so we leave it to the reader) that each main disk $\mathcal{D}$ can be constructed from the closed star $\mathrm{clst}(v)$ of a maximal-valence vertex $v \in \mathrm{int}(\mathcal{D})$ by repeatedly gluing triangles along the boundary. Such triangles can be glued in two ways, more precisely, along either one or two edges of the boundary. In the first case (*type* I), the number of vertices in the boundary increases by one, the number of vertices in the interior remains fixed, and one 1-valent vertex in the boundary appears. In the second case (*type* II), the number of vertices in the boundary decreases by one, the number of vertices in the interior increases by one, and the valence of each vertex remaining in the boundary does not decrease. See Figure 7.

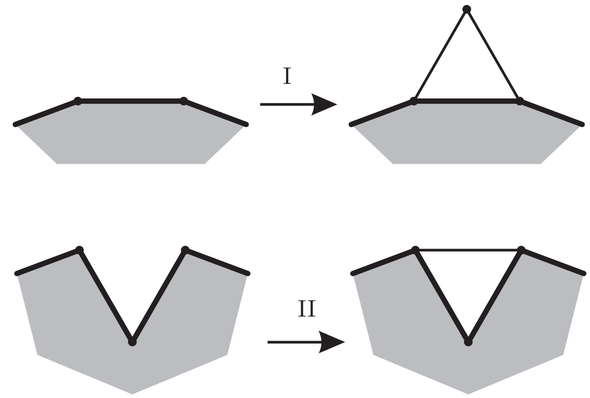

FIGURE 7. Gluing triangles of type I (above) and type II (below).

Remark 3.9. Let us denote by n_{I} (respectively n_{II}) the number of triangles of type I (respectively type II) glued to obtain $\mathcal{D}$. The following properties hold:

1. We have $V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} - n_{\mathrm{II}}$.
2. We have $V(\mathrm{int}(\mathcal{D})) = 1 + n_{\mathrm{II}}$.
3. If $\mathcal{T}$ is a root, then the first triangle glued to $\mathrm{clst}(v)$ must be of type I (because otherwise, a 3-valent vertex in $\mathrm{int}(\mathcal{D})$ would appear), and hence $n_{\mathrm{II}} > 0$ implies $n_{\mathrm{I}} > 0$. More precisely, we have ruled out the boundary of the tetrahedron (having 3-valent vertices), but it has only one decomposition, and we have $n_{\mathrm{I}} = n_{\mathrm{II}} = 0$ for it, however.

Note also that properties 1 and 2 above imply that n_{I} and n_{II} are defined unambiguously, regardless of the maximal-valence vertex v in $\mathrm{int}(\mathcal{D})$ and the order of the gluings.

This simple remark allows us to find other restrictions on genus surfaces.

Proposition 3.10. *Suppose $\mathcal{T}$ is a nonsphere root. Let us call* (A) *the condition "$V(\mathcal{G}) = V(\mathcal{T}) - 1$" and* (B) *the condition "the maximal degree of a vertex in $\mathcal{G}$ is achieved in $\partial\mathcal{G} \cap \mathcal{D}$." Then under the hypotheses made above, the number of vertices in the boundary of the main disk is bounded from below as described by the following table:*

(A)	(B)	$V(\partial\mathcal{D}) \geq$
true	true	$\mathrm{md}(\mathcal{G})+1$
true	false	$\mathrm{md}(\mathcal{G})$
false	true	$\mathrm{md}(\mathcal{G})+3+V(\mathcal{G})-V(\mathcal{T})$
false	false	$\mathrm{md}(\mathcal{G})+2+V(\mathcal{G})-V(\mathcal{T})$

Proof: As we did above, let us denote by n_{I} (respectively n_{II}) the number of triangles of type I (respectively of type II) glued to obtain $\mathcal{D}$.

Suppose that condition (A) holds. In such a case, we have $V(\mathrm{int}(\mathcal{D})) = 1$; hence by applying Remark 3.9(2) and then Remark 3.9(1), we get $V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} \geq \mathrm{mv}(\mathcal{T})$. If condition (B) does not hold, we are done, because $\mathrm{mv}(\mathcal{T}) \geq \mathrm{md}(\mathcal{G})$. Suppose, then, that condition (B) holds. Let us denote by w a vertex in $\partial\mathcal{G} \cap \mathcal{D}$ with maximal degree in $\mathcal{G}$. The valence of w in $\mathcal{D}$ is either 1 or greater than 1. In the first case, we have $n_{\mathrm{I}} > 0$ and hence $V(\partial\mathcal{D}) \geq \mathrm{mv}(\mathcal{T}) + 1 \geq \mathrm{md}(\mathcal{G}) + 1$. In the second case, we have $\mathrm{mv}(\mathcal{T}) \geq \mathrm{val}(w) > \mathrm{md}(\mathcal{G})$ and hence the result.

Suppose now that condition (A) does not hold. In this case, by Remark 3.6, we have $V(\mathcal{G}) < V(\mathcal{T}) - 1$, or equivalently, $0 \geq 2 + V(\mathcal{G}) - V(\mathcal{T})$. By Remark 3.9, we get that the number of vertices in $\partial\mathcal{D}$ is $\mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} - n_{\mathrm{II}}$ and that $n_{\mathrm{II}} = V(\mathrm{int}(\mathcal{D})) - 1 \leq V(\mathcal{T}) - V(\mathcal{G}) - 1$. Now we have two cases to be analyzed, since either $n_{\mathrm{I}} = 0$ or $n_{\mathrm{I}} > 0$.

We analyze the case $n_{\mathrm{I}} = 0$ first. By Remark 3.9(3), we have $n_{\mathrm{II}} = 0$, and hence $\mathcal{D}$ is the closed star of a maximal-valence vertex of $\mathcal{T}$. If now condition (B) does not hold, we have $V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) \geq \mathrm{md}(\mathcal{G}) \geq \mathrm{md}(\mathcal{G}) + 2 + V(\mathcal{G}) - V(\mathcal{T})$. Conversely, if condition (B) does hold, let us denote by w a vertex in $\partial\mathcal{G} \cap \mathcal{D}$ with maximal degree in $\mathcal{G}$; the degree of w in $\mathcal{T}$ is $\mathrm{md}(\mathcal{G})+1$, and hence $V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) \geq \mathrm{val}(w) = \mathrm{md}(\mathcal{G}) + 1 \geq \mathrm{md}(\mathcal{G}) + 3 + V(\mathcal{G}) - V(\mathcal{T})$.

We are left to prove the assertion in the case $n_{\mathrm{I}} \geq 1$. If condition (B) does not hold, we have $V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} - n_{\mathrm{II}} \geq \mathrm{md}(\mathcal{G}) + 2 + V(\mathcal{G}) - V(\mathcal{T})$. Finally, suppose that condition (B) holds. Let us denote by w a vertex in $\partial\mathcal{G}\cap\mathcal{D}$ with maximal degree in $\mathcal{G}$. If $n_{\mathrm{II}} = 0$, we have $V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} \geq \mathrm{mv}(\mathcal{T}) + 1 \geq \mathrm{md}(\mathcal{G}) + 3 + V(\mathcal{G}) - V(\mathcal{T})$. Conversely, suppose $n_{\mathrm{II}} > 0$. We have two cases to be analyzed, in both of which we get our result. Indeed,

- if the valence of w in $\mathcal{D}$ is 1, there are at least two triangles of type I (for otherwise, there would exist a 3-valent vertex in $\mathcal{T}$), and hence $V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} - n_{\mathrm{II}} \geq \mathrm{md}(\mathcal{G}) + 3 + V(\mathcal{G}) - V(\mathcal{T})$;
- if the valence of w in $\mathcal{D}$ is greater than 1, we have $\mathrm{mv}(\mathcal{T}) > \mathrm{md}(\mathcal{G})$, and hence $V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} - n_{\mathrm{II}} > \mathrm{md}(\mathcal{G}) + 2 + V(\mathcal{G}) - V(\mathcal{T})$.

□

3.2 Restrictions on Minimal Genus Surfaces

Minimal genus surfaces satisfy many restrictions: we now describe some of them. Throughout this section, $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ will always denote a decomposition of a closed triangulated surface $\mathcal{T}$.

Proposition 3.11. *If $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal, the link of each edge of $\partial\mathcal{G}$ is made up of a vertex also contained in $\partial\mathcal{G}$.*

Proof: First of all, we note that the link of an edge contained in the boundary of a surface contains exactly one vertex. Now suppose by way of contradiction that $e \subset \partial\mathcal{G}$ is an edge such that $\mathrm{link}(e) = \{v\}$, where $v \in \mathrm{int}(\mathcal{G})$. If we remove the triangle containing e and v from $\mathcal{G}$ and add it to the disk (either $\mathcal{D}$ or $\mathcal{D}_i$, for some $i \in \{1, \ldots, n\}$) to which it is adjacent (in $\mathcal{T}$), we get a decomposition of $\mathcal{T}$ whose genus surface has one triangle fewer than $\mathcal{G}$, which is a contradiction to the hypothesis that $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal. □

Proposition 3.12. *If $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal, each triangle in $\mathcal{G}$ intersects the boundary of $\mathcal{G}$.*

Proof: Suppose by way of contradiction that a triangle (say T) of $\mathcal{G}$ does not intersect the boundary of $\mathcal{G}$. Then $(\mathcal{G} \setminus \{T\}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n, \{T\}\})$ is a decomposition of $\mathcal{T}$ whose genus surface $\mathcal{G} \setminus \{T\}$ has one triangle fewer than $\mathcal{G}$, contradicting the hypothesis that $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal. □

Proposition 3.13. *If $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal, then $\mathcal{G}$ contains no pair of triangles adjacent to each other along an edge and adjacent along edges to different boundary components of $\mathcal{G}$ (see Figure 8, on the left).*

Proof: Suppose by way of contradiction that such a pair exists. If we remove these two triangles from $\mathcal{G}$ and glue (by adding the two triangles) the two disks (distinct by hypothesis) adjacent to $\mathcal{G}$ along the two boundary components adjacent to the two triangles, we get a decomposition of $\mathcal{T}$ whose genus surface $\mathcal{G}'$ has two triangles fewer than $\mathcal{G}$ (see Figure 8, on the right) This contradicts the hypothesis that $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal. □

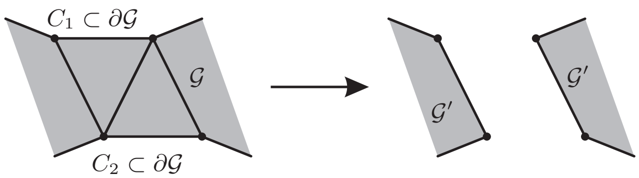

FIGURE 8. Forbidden configuration if C_1 and C_2 are different boundary components of $\mathcal{G}$ (on the left), because of the existence of a genus surface $\mathcal{G}'$ with fewer triangles than $\mathcal{G}$ (on the right).

Remark 3.14. In the proof of the proposition above we need the boundary components (adjacent to the pair of triangles) to be distinct; for otherwise, the operation of *removing and gluing* would lead to an annulus in the complement of the genus surface. Note that the two boundary components may be the same; see, for instance, Example 3.5.

Proposition 3.15. *Suppose $\mathcal{T}$ is a nonsphere root and $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal. Then for each vertex v of $\mathcal{G}$, the following inequalities hold:*

$$\begin{aligned} 3 \leq \deg_{\mathcal{G}}(v) \leq V(\mathcal{T}) - 2 \quad &\text{if } v \in \partial\mathcal{G}, \\ 4 \leq \deg_{\mathcal{G}}(v) \leq V(\mathcal{T}) - 2 \quad &\text{if } v \in \operatorname{int}(\mathcal{G}); \end{aligned}$$

or equivalently,

$$\begin{aligned} 2 \leq \operatorname{val}_{\mathcal{G}}(v) \leq V(\mathcal{T}) - 3 \quad &\text{if } v \in \partial\mathcal{G}, \\ 4 \leq \operatorname{val}_{\mathcal{G}}(v) \leq V(\mathcal{T}) - 2 \quad &\text{if } v \in \operatorname{int}(\mathcal{G}). \end{aligned}$$

Moreover, there exists at least one vertex w in $\partial\mathcal{G}$ with $\deg_{\mathcal{G}}(w) \geq 4$ *(or equivalently,* $\operatorname{val}_{\mathcal{G}}(w) \geq 3$*).*

Proof: Since

$$\operatorname{val}_{\mathcal{G}}(v) = \begin{cases} \deg_{\mathcal{G}}(v) - 1 & \text{if } v \in \partial\mathcal{G}, \\ \deg_{\mathcal{G}}(v) & \text{if } v \in \operatorname{int}(\mathcal{G}), \end{cases}$$

it is enough to prove the following inequalities:

(a) $\deg_{\mathcal{G}}(v) \leq V(\mathcal{T}) - 2$ for each $v \in \mathcal{G}$,

(b) $4 \leq \deg_{\mathcal{G}}(v)$ if $v \in \operatorname{int}(\mathcal{G})$,

(c) $2 \leq \operatorname{val}_{\mathcal{G}}(v)$ if $v \in \partial\mathcal{G}$,

(d) $\operatorname{val}_{\mathcal{G}}(w) \geq 3$ for at least one vertex $w \in \partial\mathcal{G}$.

Inequality (a) is obvious because $\deg_{\mathcal{G}}(v)+1 \leq V(\mathcal{G}) \leq V(\mathcal{T}) - 1$ by Remark 3.6. Inequality (b) is also obvious by Remark 2.3, because $\mathcal{T}$ is a nonsphere root.

We will now prove inequality (c). Suppose by way of contradiction that $\operatorname{val}_{\mathcal{G}}(v) = 1$ for a vertex $v \in \partial\mathcal{G}$. Hence there is exactly one triangle $T \subset \mathcal{G}$ such that $v \in T$. The two edges of T incident to v belong to $\partial\mathcal{G}$; therefore, the third edge of T does not belong to $\partial\mathcal{G}$ because $\mathcal{G}$ is not a disk, and we can remove the triangle T from $\mathcal{G}$, adding it to the disk (either $\mathcal{D}$ or $\mathcal{D}_i$, for some $i \in \{1, \ldots, n\}$) to which it is adjacent (in $\mathcal{T}$). We have thereby obtained a decomposition of $\mathcal{T}$ whose genus surface has one triangle fewer than $\mathcal{G}$, a contradiction to the hypothesis that $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal.

Let us finally prove inequality (d). Suppose by way of contradiction that $\operatorname{val}_{\mathcal{G}}(v) < 3$ for each vertex $v \in \partial\mathcal{G}$. By inequality (c), we have $\operatorname{val}_{\mathcal{G}}(v) = 2$ for each vertex $v \in \partial\mathcal{G}$. Let us consider one of these vertices (say w). It is contained in two triangles (say T_1 and T_2), and it is adjacent to three vertices (denote by v_0 the one contained in $T_1 \cap T_2$, and by v_i the one contained in T_i only, for $i = 1, 2$). By Proposition 3.11, we have $v_0 \in \partial\mathcal{G}$ and hence $\operatorname{val}_{\mathcal{G}}(v_0) = 2$. Therefore, v_0 is contained in T_1 and T_2 only. This implies that $\mathcal{G} = T_1 \cup T_2$ and hence that $\mathcal{G}$ is a disk, contradicting the hypothesis that $\mathcal{T}$ is not a sphere. □

4. LISTING CLOSED TRIANGULATED SURFACES

In this section, we will apply the theory of roots and decompositions to find an algorithm for listing all triangulations of closed surfaces with at most n vertices. Then we will specialize it (by specializing the theory described above) to the case $n = 11$. In fact, we will see that minimal decompositions of closed triangulated surfaces with at most 11 vertices satisfy some stronger theoretical restrictions. Closed triangulated surfaces with at most 12 vertices has been independently listed in [Lutz and Sulanke 2006] using a very subtle lexicographic enumeration approach. The computer program that the authors have written to implement their algorithm is very fast. Perhaps an algorithm mixing their technique and ours would be even faster.

4.1 The Listing Algorithm

First of all, we recall that we have seen in Section 2 that each closed triangulated surface (say $\mathcal{T}$) has exactly one root (Theorem 2.5). Hence, in order to list closed triangulated surfaces, it is enough to list first roots and then nonroots (deducing the latter from the former). Moreover, we have seen in Section 3 that each closed triangulated surface has a (minimal) decomposition, say $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$. Hence, we can start listing triangulated disks and genus surfaces, and then we can glue them to get all closed triangulated surfaces. When the triangulated surface is a root and the decomposition is minimal, that decomposition satisfies some theoretical restrictions that simplify the search. However, there is a drawback that slows down the search: such a decomposition is not unique.

Classical Listing Technique. The basic technique we use to list triangulations is the classical one. We start from the closed star of a maximal-valence vertex v and we repeatedly glue triangles. Each time, we choose an edge in the boundary and we glue a triangle along that edge. In order to do this, we need only to choose the third vertex of the triangle: this vertex can be either a vertex of the current boundary or a new one. For each choice we create a new triangulation, and we repeat the procedure for it. If at some time we violate some property that must be satisfied (e.g., the first vertex is maximally valent, the link of each vertex is always contained in a circle), we go back and attempt to glue other triangles. This technique has been described in detail in [Lutz 2008] and [Lutz and Sulanke 2006].

Relabeling. In order to avoid duplicates, each time we find a triangulated surface $\mathcal{T}$, we check whether $\mathcal{T}$ has already been found, changing the labeling to get the mixed-lexicographically smallest one, where a list of triangles is *mixed-lexicographically smaller* than another if the first vertex has greater valence or, when the first vertices have the same valence, the list is smaller in the lexicographic order.

In a mixed-lexicographically minimal triangulation, the list of triangles starts with the star of vertex 1:

$$123 \quad 124 \quad 135 \quad 146 \quad 157 \quad \dots \quad 1(d-1)(d+1),$$

where $d = \deg(1)$. There are two possibilities:

1. If vertex 1 is in the interior of $\mathcal{T}$, then its link must be a circle, and hence the next triangle is $1d(d+1)$.

2. If vertex 1 is in the boundary of $\mathcal{T}$, then its link is an interval, and hence no more triangles containing vertex 1 appear in the list.

Then the list continues with the remaining triangles not containing vertex 1.

Note that when we list a class of triangulations following the mixed-lexicographic order, the subclass of triangulations with the same maximal valence (say m) is sorted lexicographically, and every triangulation in such a subclass begins with the same m triangles. But note that the list of all triangulations does not follow the lexicographic order.

In order to find the mixed-lexicographically smallest labeling, we carry out the following steps:

1. We list all maximal-valence vertices of $\mathcal{T}$.

2. For each such vertex (say v) we list the vertices in $\operatorname{link}(v)$.

3. For each pair (v, w), where $w \in \operatorname{link}(v)$, we relabel v as 1 and w as 2.

4. We relabel the two vertices in the link of the edge vw to be 3 and 4 (we have two choices).

5. We extend the new labeling in a lexicographically smallest way (sometimes we have two choices, as in the previous step).

6. We search among all such pairs (v, w) for the mixed-lexicographically smallest labeling.

The Listing Algorithm. The algorithm is made up of five steps. Let n be the maximal number of vertices of the closed triangulated surfaces we are searching for.

1. Triangulated disks: The list of triangulated disks can be achieved by applying the classical technique described above. Obviously, we want to list roots; hence we discard triangulated disks with 3-valent vertices in their interior. Listing of triangulated disks with at most n vertices (with n small) is fast and well known; hence we do not describe this step. We have essentially used the same technique of Step 3 below.

2. Triangulated spheres: Minimal decompositions of triangulated spheres are of type $(\{T\}, \mathcal{T} \setminus \{T\}, \varnothing)$, where T is a triangle of $\mathcal{T}$, and a vertex not in T has maximal valence (among those in $\mathcal{T}$); see Example 3.4. Obviously, we have $V(\partial(\mathcal{T} \setminus \{T\})) = 3$; hence in order to list triangulated spheres, we pick out the triangulated disks $\mathcal{D}$ such that $V(\partial\mathcal{D}) = 3$, and we glue the missing triangle to each of them. Moreover, each main disk has a maximal-valence vertex in its interior, so we discard all triangulated spheres with maximal valence greater than this number. Finally, we note that we need to relabel each triangulated sphere found to check that it has not been already found.

3. "Minimal" genus surfaces: In order to list "minimal" genus surfaces, we follow the same classical technique, but we have some restrictions that minimal genus surfaces of roots must fulfill (see, for instance, the results of Sections 2 and 3). Hence we can discard those not fulfilling these restrictions. Note that we will not know whether all genus surfaces found are actually minimal: we know only that they fulfill some restrictions necessary to be minimal and that all minimal genus surfaces are found. Moreover, the number of genus surfaces we will find may be greater than that of closed triangulated surfaces/roots, but this search has the advantage of dealing with a lower number of vertices (at most $n-1$) and triangles necessary to construct genus surfaces instead of closed triangulated surfaces (see Remark 3.6 and Proposition 3.8). Finally, we note that as above, we need to relabel each genus surface found to check that it has not been already found.

4. Gluings: In order to get roots from genus surfaces, we glue the triangulated disks found to each genus surface found (along its boundary components). One such disk is a main disk; hence it contains a maximal-valence vertex in its interior. Note that we must check all possible gluings between the genus surface and the triangulated disk(s): the number of possible gluings of each disk is twice the number of its boundary vertices. Note that we need to check that the result of the gluing is a triangulated surface, which essentially means that we must check that each pair of adjacent vertices of the genus surface is not adjacent in the triangulated disks. Moreover, each main disk has a maximal-valence vertex in its interior, so we discard the triangulations with maximal valence greater than this number. Note also that since we are searching for roots, we discard the triangulations with a 3-valent vertex. Finally, we note that as above, we need to relabel each root found to check that it has not already been found.

5. Nonroots: We know that nonroots can be divided depending on their root (see Remark 2.6). Hence, we start from each root (with at most $n-1$ vertices) and we list the nonroots having that root. The search is quite simple: in order to get all nonroots from a root, it is enough to repeatedly apply T-moves. Obviously, we need to relabel each nonroot found to check that it has not already been found, but the search can be restricted to those having the same root.

4.2 Listing Closed Triangulated Surfaces with at Most 11 Vertices

If the maximal number of vertices of the triangulations we are searching for is at most 11, we can make the listing algorithm faster, because in this case, minimal decompositions of roots fulfill restrictions stronger than in the general case. Throughout this section, $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ will always denote a decomposition of a closed triangulated surface $\mathcal{T}$. Recall that by Remark 3.6, if $V(\mathcal{T}) \leq 11$ then $V(\mathcal{G}) \leq 10$.

We have already noted that we have no useful a priori restriction on the number of disks in $\mathcal{T} \setminus \mathcal{G}$. If instead, we have $V(\mathcal{T}) \leq 11$, then we have the following result.

Proposition 4.1. *If $V(\mathcal{T}) \leq 11$, the number of boundary components of $\mathcal{G}$ is 1 or 2.*

Proof: By definition, $\mathcal{G}$ has nonempty boundary; hence it is enough to prove that $\partial\mathcal{G}$ has at most two components. The main ingredient of the proof is that each component must contain at least three vertices because it is a triangulated circle.

Let we first suppose that $\mathrm{mv}(\mathcal{T}) \leq 4$. By Remark 2.4, the root of $\mathcal{T}$ is the boundary of the tetrahedron or the boundary of the octahedron. In order to get $\mathcal{T}$ from them, we must apply T-moves. It is very easy to prove that only three possibilities for $\mathcal{T}$ arise: the two roots and one nonroot with five vertices. By Remark 3.6, we have $V(\mathcal{G}) \leq V(\mathcal{T}) - 1 \leq 5$, and hence $\mathcal{G}$ has at most one boundary component.

Suppose now that $\mathrm{mv}(\mathcal{T}) > 4$. By Proposition 3.8, we have $V(\mathcal{D}) \geq \mathrm{mv}(\mathcal{T}) + 1 \geq 6$ and then $V(\partial\mathcal{G} \setminus \mathcal{D}) \leq V(\mathcal{T}) - V(\mathcal{D}) \leq 5$. Hence there cannot be more than two components in $\partial\mathcal{G}$, for otherwise, $\partial\mathcal{G} \setminus \mathcal{D}$ would have at least two components containing at least six vertices. □

The following results will prove that if $\mathcal{T}$ is a root with at most 11 vertices and $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1, \ldots, \mathcal{D}_n\})$ is minimal, then only three cases for $\mathcal{T} \setminus (\mathcal{G} \cup \mathcal{D})$ can arise.

Proposition 4.2. *If $V(\mathcal{T}) \leq 11$, each minimal decomposition of $\mathcal{T}$ is of type $(\mathcal{G}, \mathcal{D}, \varnothing)$ or of type $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1\})$, where $V(\partial\mathcal{D}_1)$ is 3 or 4.*

Proof: By Example 3.4, all minimal decompositions of a triangulated sphere are of type $(\{T\}, \mathcal{T} \setminus \{T\}, \varnothing)$, where T is a triangle; hence the proposition is obvious for triangulated spheres. Therefore, suppose $\mathcal{T}$ is not a triangulated sphere (recall that $\mathrm{mv}(\mathcal{T}) \geq 5$ by Remark 2.4). By Proposition 4.1, we have that $\partial\mathcal{G}$ has either one or two boundary components. In the first case, there is nothing to prove. In the second case, the decomposition is $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1\})$. We suppose by way of contradiction that $V(\partial\mathcal{D}_1) > 4$. By Proposition 3.8, we have $V(\mathcal{D}) \geq \mathrm{mv}(\mathcal{T}) + 1 \geq 6$ and $V(\mathcal{D}_1) \leq V(\mathcal{T}) - V(\mathcal{D}) \leq 5$; hence $V(\mathcal{D}_1) = 5$. We also have $6 \leq \mathrm{mv}(\mathcal{T}) + 1 \leq V(\mathcal{D}) \leq V(\mathcal{T}) - V(\partial\mathcal{D}_1) \leq 6$, and hence $V(\mathcal{D}) = 6$, $\mathrm{mv}(\mathcal{T}) = 5$,

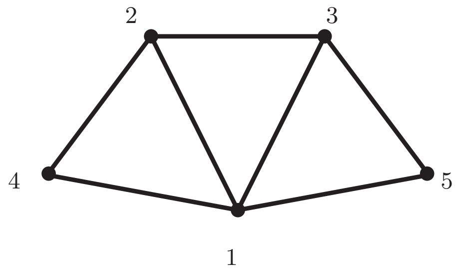

FIGURE 9. The unique triangulated surface $\mathcal{D}_1$ with $V(\partial\mathcal{D}_1) = 5$ and $V(\mathrm{int}(\mathcal{D}_1)) = 0$.

and $\mathcal{D} = \mathrm{clst}(v)$, where v is the maximal-valence vertex of $\mathcal{D}$. Moreover, $V(\mathrm{int}(\mathcal{D}_1)) \le V(\mathcal{T}) - V(\mathcal{D}) - V(\partial\mathcal{D}_1) \le 0$; hence $V(\mathrm{int}(\mathcal{D}_1)) = 0$, and $\mathcal{D}_1$ is the triangulation shown in Figure 9.

Let us denote by w the unique 3-valent vertex of $\mathcal{D}_1$. Since $\mathrm{mv}(\mathcal{T}) = 5$, we have that the valence of w in $\mathcal{G}$ is 2 (it cannot be 1, by Proposition 3.15). Let us denote by T_1 and T_2 the two triangles in $\mathcal{G}$ incident to w, and let w' denote the other vertex in $T_1 \cap T_2$. See Figure 10.

By Proposition 3.11, we have $w' \in \partial\mathcal{G}$; moreover, since w is adjacent in $\mathcal{D}_1$ to all the other boundary vertices of $\mathcal{D}_1$, then w' does not belong to $\mathcal{D}_1$ (for otherwise, $\mathcal{T}$ would not be a triangulation of a surface); hence $w' \in \mathcal{D}$. The edge of T_1 not incident to w is not contained in $\partial\mathcal{D}$ (for otherwise, $\mathcal{D}$ and $\mathcal{D}_1$ would have nonempty intersection); the same holds for T_2, and hence the valence of w' in $\mathcal{G}$ is at least 4. Finally, the valence of w' in $\mathcal{D}$ is 2, because $\mathcal{D} = \mathrm{clst}(v)$. Hence $\mathrm{val}(w') \ge 6$; see Figure 10. This is a contradiction to $\mathrm{mv}(\mathcal{T}) = 5$. $\square$

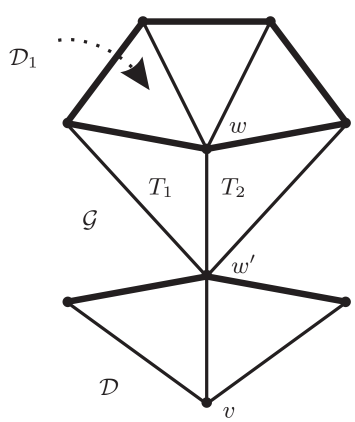

FIGURE 10. If $V(\partial\mathcal{D}_1) > 4$, there is a 6-valent vertex in ∂D.

Proposition 4.3. *If $\mathcal{T}$ is a root and $V(\mathcal{T}) \le 11$, then each minimal decomposition of $\mathcal{T}$ is of type $(\mathcal{G}, \mathcal{D}, \varnothing)$ or of type $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1\})$, where $V(\mathrm{int}(\mathcal{D}_1)) = 0$.*

Proof: By Example 3.4, all minimal decompositions of a triangulated sphere are of type $(\{T\}, \mathcal{T} \setminus \{T\}, \varnothing)$, where T is a triangle; hence the proposition is obvious for triangulated spheres. Therefore, suppose $\mathcal{T}$ is not a triangulated sphere (recall that $\mathrm{mv}(\mathcal{T}) \ge 5$ by Remark 2.4). By Proposition 4.2, we have three possibilities for the decomposition of $\mathcal{T}$. It can be of type $(\mathcal{G}, \mathcal{D}, \varnothing)$, of type $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1\})$, where $V(\partial\mathcal{D}_1) = 4$, or of type $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1\})$, where $V(\partial\mathcal{D}_1) = 3$. For the first type, there is nothing to prove.

Let us analyze the case that the decomposition is of type $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1\})$ and $V(\partial\mathcal{D}_1) = 4$. Suppose by way of contradiction that $V(\mathrm{int}(\mathcal{D}_1)) > 0$. Since $V(\partial\mathcal{D}) \ge 3$, we have $V(\mathrm{int}(\mathcal{D})) \le V(\mathcal{T}) - V(\partial\mathcal{D}_1) - V(\mathrm{int}(\mathcal{D}_1)) - V(\partial\mathcal{D}) \le 3$. We will now use the notation (and results) of Remark 3.9. We have $n_{\mathrm{II}} \le 2$, and then

$$V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} - n_{\mathrm{II}} \ge \begin{cases} \mathrm{mv}(\mathcal{T}) \ge 5 & \text{if } n_{\mathrm{I}} = 0, \\ \mathrm{mv}(\mathcal{T}) + 1 - n_{\mathrm{II}} \ge 4 & \text{if } n_{\mathrm{I}} > 0. \end{cases}$$

Now we can repeat this technique, obtaining $V(\mathrm{int}(\mathcal{D})) \le V(\mathcal{T}) - V(\partial\mathcal{D}_1) - V(\mathrm{int}(\mathcal{D}_1)) - V(\partial\mathcal{D}) \le 2$, $n_{\mathrm{II}} \le 1$, and then

$$V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} - n_{\mathrm{II}} \ge \begin{cases} \mathrm{mv}(\mathcal{T}) \ge 5 & \text{if } n_{\mathrm{I}} = 0, \\ \mathrm{mv}(\mathcal{T}) + 1 - n_{\mathrm{II}} \ge 5 & \text{if } n_{\mathrm{I}} > 0. \end{cases}$$

Hence, we get $V(\partial\mathcal{D}) \ge 5$, $V(\mathrm{int}(\mathcal{D})) = 1$, and then $\mathcal{D} = \mathrm{clst}(v)$, where v is the maximal-valence vertex of $\mathcal{D}$. Moreover, we have $V(\mathcal{D}) \le V(\mathcal{T}) - V(\partial\mathcal{D}_1) - V(\mathrm{int}(\mathcal{D}_1)) \le 6$ and hence $\mathrm{val}(v) = \mathrm{mv}(\mathcal{T}) = 5$.

We also have $\mathcal{D}_1 = \mathrm{clst}(w)$, where w is the vertex in $\mathrm{int}(\mathcal{D}_1)$, because $V(\mathrm{int}(\mathcal{D}_1)) \le V(\mathcal{T}) - V(\mathcal{D}) - V(\partial\mathcal{D}_1) \le 1$, $\mathrm{val}(w) \ge 4$, and $V(\partial\mathcal{D}_1) = 4$. Now note that $V(\mathrm{int}(\mathcal{G})) = 0$ and that each vertex in $\partial\mathcal{G}$ has valence 2 or 3 (for the valence of these vertices in $\mathcal{D}$ or $\mathcal{D}_1$ is 2, the maximal valence in $\mathcal{T}$ is 5, and the valence in $\mathcal{G}$ is at least 2 by Proposition 3.15). With the same technique used at the end of the proof of Proposition 4.2, we can prove that no vertex in $\partial\mathcal{G}$ has valence 2 (the only difference is in the proof that w and w' do not belong to the same disk $\mathcal{D}$ or $\mathcal{D}_1$, where the property $2 \le \mathrm{val}_{\mathcal{G}}(v)\ \forall v \in \partial\mathcal{G}$ from Proposition 3.15 should be used). Hence, all vertices in $\partial\mathcal{G}$ have valence 3. This and the fact that $V(\mathrm{int}(\mathcal{G})) = 0$

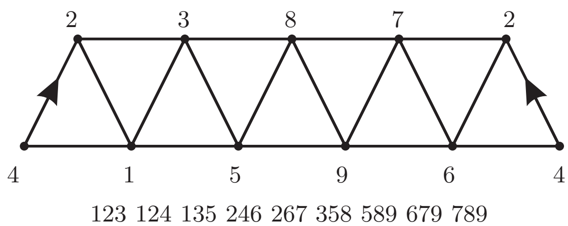

FIGURE 11. The unique genus surface $\mathcal{G}$ such that $V(\mathrm{int}(\mathcal{G})) = 0$, $\mathrm{val}_{\mathcal{G}}(w) = 3$ for each vertex $w \in \partial\mathcal{G}$, and with two boundary components with 4 and 5 vertices respectively.

easily imply that $\mathcal{G}$ is the annulus shown in Figure 11, a contradiction to Example 3.4.

Let us analyze now the case that the decomposition is of type $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1\})$ and $V(\partial\mathcal{D}_1) = 3$. Suppose by way of contradiction that $V(\mathrm{int}(\mathcal{D}_1)) > 0$. Let us denote by w a vertex in $\mathrm{int}(\mathcal{D}_1)$. Obviously, we have $\mathrm{val}(w) \geq 4$. By applying the same technique used to construct main disks (see Remark 3.9, also for notation), we easily get $n_{\mathrm{I}} > 0$ and $n_{\mathrm{II}} = \mathrm{val}(w) + n_{\mathrm{I}} - V(\partial\mathcal{D}_1) \geq 2$. Hence $V(\mathrm{int}(\mathcal{D}_1)) = 1 + n_{\mathrm{II}} \geq 3$, $V(\mathcal{D}_1) \geq 6$, and $V(\mathrm{int}(\mathcal{D})) \leq V(\mathcal{T}) - V(\mathcal{D}_1) - V(\partial\mathcal{D}) \leq 2$.

We now apply Remark 3.9 to $\mathcal{D}$, getting

$$V(\partial\mathcal{D}) = \mathrm{mv}(\mathcal{T}) + n_{\mathrm{I}} - n_{\mathrm{II}} \geq \begin{cases} \mathrm{mv}(\mathcal{T}) \geq 5 & \text{if } n_{\mathrm{I}} = 0, \\ \mathrm{mv}(\mathcal{T}) + 1 - (V(\mathrm{int}(\mathcal{D})) - 1) \geq 5 & \text{if } n_{\mathrm{I}} > 0. \end{cases}$$

Hence we have $V(\mathcal{D}) \geq 6$, and then $V(\mathcal{T}) \geq V(\mathcal{D}) + V(\mathcal{D}_1) = 12$, a contradiction to the hypothesis $V(\mathcal{T}) \leq 11$. □

The two propositions above obviously yield the following result.

Corollary 4.4. *If $\mathcal{T}$ is a root and $V(\mathcal{T}) \leq 11$, then each minimal decomposition of $\mathcal{T}$ is of type $(\mathcal{G}, \mathcal{D}, \varnothing)$ or of type $(\mathcal{G}, \mathcal{D}, \{\mathcal{D}_1\})$, where $\mathcal{D}_1$ is made up of one or two triangles.*

Computational Results. The computer program `trialistgs11` implementing the algorithm described in Section 4.1, specialized for the 11-vertex case with the results of Section 3.2 and of this section, can be found in [Amendola 2007]. Such results have simplified the search: for instance,

- by Proposition 4.2, we search only for genus surfaces with either one or two boundary components, and in the latter case, one of the components must contain at most four vertices;
- by Corollary 4.4, there are only two cases for the triangulated disk $\mathcal{D}_1$ (when $\partial\mathcal{G}$ has two components);
- by Proposition 3.10, we can discard the genus surfaces that fail to satisfy some properties.

V	S	Number	V	S	Number
3	S^2	1	9	T^2	230
				S_2^+	1261
5	$\mathbb{RP}^2$	1		S_3^+	59
				$\mathbb{RP}^2$	28
6	T^2	1		K^2	597
	$\mathbb{RP}^2$	2		S_3^-	6919
				S_4^-	18166
7	T^2	5		S_5^-	18199
	$\mathbb{RP}^2$	6		S_6^-	4994
	K^2	10		S_7^-	78
8	T^2	46	10	T^2	1513
	$\mathbb{RP}^2$	11		S_2^+	50878
	K^2	108		S_3^+	99177
	S_3^-	284		S_4^+	3892
	S_4^-	134		$\mathbb{RP}^2$	356
	S_5^-	3		K^2	3864
				S_3^-	82588
				S_4^-	713714
				S_5^-	3006044
				S_6^-	5672821
				S_7^-	4999850
				S_8^-	1453490
				S_9^-	53484

TABLE 2. Number of genus surfaces with at most 10 vertices, used to list closed triangulated surfaces with at most 11 vertices, depending on the number of vertices V and the closed surface S obtained by gluing disks to their boundary.

The numbers of genus surfaces found with at most 10 vertices are listed in Table 2, while the numbers of roots and nonroots with at most 11 vertices are listed in Table 1.

It is worth noting that we are searching for closed triangulated surfaces with at most 11 vertices; hence we get a list of the genus surfaces needed to construct those triangulated surfaces. If we had searched for closed triangulated surfaces with fewer vertices, we would have obtained a shorter list of genus surfaces.

The computer program carries out the search for a fixed homeomorphism type (i.e., genus and orientability) of the surface each time. The longest case is that of S_7^-, which took twelve days on a 2.33-GHz notebook processor to obtain the list.

ACKNOWLEDGMENTS

I am grateful to Professor Jürgen Bokowski and Simon King for useful discussions during the beautiful period I spent at the Department of Mathematics in Darmstadt. I would also like to thank the Galileo Galilei Doctoral School of Pisa and the DAAD (Deutscher Akademischer Austausch Dienst) for giving me the opportunity to reside in Darmstadt, and Professor Alexander Martin for his assistance. I would also like to thank the referee for his or her useful comments and corrections.

This paper is dedicated to Paolo.

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Received May 7, 2007; accepted September 26, 2007.