The Homotopy Lie Algebra of a Complex Hyperplane Arrangement Is Not Necessarily Finitely Presented

Jan-Erik Roos

CONTENTS

2000 AMS Subject Classification: Primary 16E05, 52C35; Secondary 16S37, 55P62

Keywords: Hyperplane arrangement, homotopy Lie algebra, Yoneda Ext-algebra

We present a theory that produces several examples in which the homotopy Lie algebra of a complex hyperplane arrangement is not finitely presented. We also present examples of hyperplane arrangements in which the enveloping algebra of this Lie algebra has an irrational Hilbert series. This answers two questions of Denham and Suciu.

1. INTRODUCTION

Let $\mathcal{A} = \{H\}$ be a finite set of complex hyperplanes in $\mathbb{C}^n$, i.e., a *complex hyperplane arrangement* in $\mathbb{C}^n$, and let X be the complement of their union in $\mathbb{C}^n$:

$$X = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

The cohomology of X is called the Orlik–Solomon algebra, and the Yoneda Ext-algebra of $H^*(X)$ is a Hopf algebra that is the enveloping algebra of a graded Lie algebra, which is called the homotopy Lie algebra of the arrangement $\mathcal{A}$. In this paper we calculate explicitly this Lie algebra in several cases, and in particular, we show by explicit examples that this Lie algebra is not necessarily finitely presented and not even finitely generated. Furthermore, we present examples of hyperplane arrangements in which the enveloping algebra of this Lie algebra has an irrational Hilbert series. This solves two open problems from [Denham and Suciu 2006, Question 1.7]. We also have some results about how often these two phenomena occur. Some historical remarks are given in Section 8.

2. AN EXPLICIT EXAMPLE

It is useful to begin with an explicit example. Let

$$\mathcal{A} = \{x, y, z, x + y, x + z, y + z\}$$

1058-6458/2008 $ 0.50 per page

be the well-known complex hyperplane arrangement that is the smallest formal arrangement whose Orlik–Solomon algebra is nonquadratic [Shelton and Yuzvinsky 1997, Example 5.1]. We know that the Orlik–Solomon algebra of $\mathcal{A}$ is the quotient of the exterior algebra in six variables $e_1, e_2, e_3, e_4, e_5, e_6$ by the two-sided ideal generated by the four elements

$$(e_2 - e_6)(e_3 - e_6), \quad (e_1 - e_3)(e_3 - e_5),$$
$$(e_1 - e_4)(e_2 - e_4), \quad (e_3 - e_4)(e_4 - e_6)(e_5 - e_6).$$

Thus if we introduce new variables $x_1, x_2, x_3, x_4, x_5, z$ by $x_i = e_i - e_6$ for $1 \leq i \leq 5$ and $z = e_6$, our Orlik–Solomon algebra can be written (we use that $e_1 - e_4 = (e_1 - e_6) - (e_4 - e_6) = x_1 - x_4$, etc.) as a quotient of an exterior algebra:

$$\mathrm{OS}_{\mathcal{A}} = \frac{E(x_1, x_2, x_3, x_4, x_5, z)}{(x_2x_3, (x_1 - x_3)(x_3 - x_5), (x_1 - x_4)(x_2 - x_4), (x_3 - x_4)x_4x_5)},$$

where z does not occur among the relations. Therefore, the Orlik–Solomon algebra decomposes into a tensor product of algebras (all algebras are considered over a field k of characteristic zero):

$$\mathrm{OS}_{\mathcal{A}} = \frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_2x_3, x_1x_3 - x_1x_5 + x_3x_5, x_1x_2 - x_1x_4 + x_2x_4, x_3x_4x_5)} \otimes_k E(z),$$

where $E(z)$ is the exterior algebra in one variable and where we have used that $x_i^2 = 0$.

Thus the Yoneda Ext-algebra[1] of the Orlik–Solomon algebra is the tensor product of the Ext-algebra $\mathrm{Ext}^*_R(k,k)$ of

$$R = \frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_2x_3, x_1x_3 - x_1x_5 + x_3x_5, x_1x_2 - x_1x_4 + x_2x_4, x_3x_4x_5)} \tag{2–1}$$

and the Ext-algebra $\mathrm{Ext}^*_{E(z)}(k,k) = k[Z]$, where the last algebra is the polynomial algebra in one variable Z, dual to z. The last algebra is "innocent," and it therefore follows that the Yoneda Ext-algebra of the Orlik–Solomon algebra is finitely presented if and only if $\mathrm{Ext}^*_R(k,k)$ is also finitely presented, where R is given by (2–1).

But the automorphism of R given by

$$x_1 \mapsto x_1 + x_3, \quad x_2 \mapsto -x_2 + x_3, \quad x_3 \mapsto x_3,$$
$$x_4 \mapsto -x_2 + x_3 + x_4, \quad x_5 \mapsto x_5 + x_3,$$

transforms R into the isomorphic algebra

$$\frac{E(x_1, x_2, x_3, x_4, x_5)}{(x_2x_3, x_1x_5, (x_1 + x_2)x_4, x_3x_4x_5)}, \tag{2–2}$$

[1]For an elementary introduction to the Yoneda Ext-algebra, see [Roos 1982, pp. 112 ff.].

which we will still denote by R. But the algebra (2–2) can now be easily analyzed: it is the "trivial extension" of a Koszul algebra

$$S = \frac{E(x_1, x_2, x_3, x_5)}{(x_2x_3, x_1x_5)} \tag{2–3}$$

by the following cyclic module M over S :

$$M = \frac{S}{(x_1 + x_2, x_3x_5)}.$$

Recall that the trivial extension of any ring Λ by any two-sided Λ-module N is denoted by $\Lambda \propto N$ and consists of the pairs (λ, n) with $\lambda \in \Lambda$ and $n \in N$ with pairwise addition and multiplication

$$(\lambda, n).(\lambda', n') = (\lambda.\lambda', \lambda n' + n\lambda').$$

The Ext-algebra of $R = S \propto M$ can now be analyzed (cf., for example, [Löfwall 1985, Theorem 3]): we have a split extension of Hopf algebras

$$k \to T(s^{-1}\,\mathrm{Ext}^*_S(M,k)) \to \mathrm{Ext}^*_R(k,k) \to \mathrm{Ext}^*_S(k,k) \to k. \tag{2–4}$$

Here S is the Koszul algebra (2–3), and

$$\mathrm{Ext}^*_S(k,k) = k\langle X_1, X_5\rangle \otimes_k k\langle X_2, X_3\rangle$$

is the tensor product of two free algebras in the dual variables X_1, X_5 and X_2, X_3, and therefore it has global dimension 2. Furthermore,

$$s^{-1}\,\mathrm{Ext}^*_S(M,k) = \mathrm{Ext}^{*-1}_S(M,k),$$

and $T(s^{-1}\,\mathrm{Ext}^*_S(M,k))$ is the free algebra on the graded vector space (for the $*$-grading in Ext) $s^{-1}\,\mathrm{Ext}^*_S(M,k)$ and has global dimension 1.

The spectral sequence of extensions of Hopf algebras (2–4) [Roos 1982],

$$\begin{aligned} E^2_{p,q} &= \mathrm{Tor}_p^{\mathrm{Ext}^*_S(k,k)}(k, \mathrm{Tor}_q^{T(s^{-1}\,\mathrm{Ext}^*_S(M,k))}(k,k)) \\ &\Rightarrow \mathrm{Tor}_n^{\mathrm{Ext}^*_R(k,k)}(k,k) \ (= H_n), \end{aligned} \tag{2–5}$$

shows immediately that $\mathrm{Ext}^*_R(k,k)$ has global dimension 3. Furthermore, (2–5) degenerates into a long exact sequence:

$$\begin{aligned} 0 &\to E^2_{2,1} \to H_3 \to E^2_{3,0} \to E^2_{1,1} \to H_2 \to E^2_{2,0} \to E^2_{0,1} \\ &\to H_1 \to E^2_{1,0} \to 0, \end{aligned} \tag{2–6}$$

where the natural maps $H_i \longrightarrow E^2_{i,0}$ are onto. Indeed, the natural ring projection map $S \propto M \longrightarrow S$ is split by the natural ring inclusion $S \longrightarrow S \propto M$, and this leads to a splitting on the Ext-algebra level. Thus we have exact sequences

$$0 \longrightarrow E^2_{i-1,1} \longrightarrow H_i \longrightarrow E^2_{i,0} \longrightarrow 0.$$

For any graded connected algebra A over k, $\operatorname{Tor}^A_1(k,k)$ measures the minimal number of generators of A, $\operatorname{Tor}^A_2(k,k)$ measures the minimal number of relations between these generators, $\operatorname{Tor}^A_3(k,k)$ measures the minimal number of relations between these relations, etc.; cf. [Lemaire 1974, Chapter 1]. Therefore H_1 in (2–6) measures the minimal number of generators of the Ext-algebra $\operatorname{Ext}^*_R(k,k)$, and therefore H_1 is finite-dimensional if and only if the Ext-algebra $\operatorname{Ext}^*_R(k,k)$ is finitely generated. Similarly, H_2 measures the minimal number of relations in a minimal presentation of $\operatorname{Ext}^*_R(k,k)$, and H_3 measures the minimal number of relations between these relations. Since the $E^2_{i,0}$ are all finite-dimensional we are led to the study of

$$E^2_{i,1} = \operatorname{Tor}_i^{\operatorname{Ext}^*_S(k,k)}(k, s^{-1}\operatorname{Ext}^*_S(M,k)), \tag{2–7}$$

for $i \geq 1$, where the left $\operatorname{Ext}^*_S(k,k)$-module structure of $s^{-1}\operatorname{Ext}^*_S(M,k)$ is given by the Yoneda product (cf. [Löfwall 1985, Theorem 3]). Note that underlying our spectral sequence is the Hochschild–Serre spectral sequence and that we are in the skew-commutative setting, whereas [Löfwall 1985] is in the commutative case, but similar (easier) proofs work here in our case.

Thus to show that $\operatorname{Ext}^*_R(k,k)$ is not finitely generated, we have to show that H_1 is infinite-dimensional, i.e., that $E^2_{0,1}$ is finite-dimensional, i.e., that $s^{-1}\operatorname{Ext}^*_S(M,k)$ needs an infinite number of generators as an $\operatorname{Ext}^*_S(k,k)$-module (cf. (2–7)), i.e., we have to study the S-resolutions of $M = S/(x_1+x_2, x_3x_5)$. We also need the extra grading on R and S, so that we should indeed write $R = S \propto s^{-1}M$. Now we denote the S-ideal (x_1+x_2, x_3x_5) by I, so that $M = S/I$.

First we observe that if we apply the functor $\operatorname{Ext}^*_S(.,k)$ to the exact sequence of graded left S-modules

$$0 \longrightarrow I \longrightarrow S \longrightarrow S/I \longrightarrow 0, \tag{2–8}$$

we obtain the isomorphisms

$$\operatorname{Ext}^{*-1,t}_S(I,k) \xrightarrow{\sim} \operatorname{Ext}^{*,t}_S(S/I,k), \tag{2–9}$$

for $* \geq 1$, of left $\operatorname{Ext}^*_S(k,k)$-modules, where we also have inserted the inner grading t that comes from the fact that (2–8) is an exact sequence of graded modules. Note that S is a Koszul algebra, so that only the $\operatorname{Ext}^{i,i}_S(k,k)$ are different from zero, and we still denote them by $\operatorname{Ext}^i_S(k,k)$.

Next we note that the two ideals $I_1 = (x_1+x_2)$ and $I_2 = (x_3x_5)$ in S have zero intersection. Therefore, $I = I_1 \oplus I_2$, and the $\operatorname{Ext}^*_S(k,k)$-module to the left in (2–9) decomposes into a direct sum of $\operatorname{Ext}^*_S(k,k)$-modules:

$$\operatorname{Ext}^{*-1,t}_S((x_1+x_2),k) \oplus \operatorname{Ext}^{*-1,t}_S((x_3x_5),k). \tag{2–10}$$

But x_3x_5 is in the socle of S, and therefore we have as graded S-modules that $(x_3x_5) \xrightarrow{\sim} s^{-2}k$, so that the right summand of (2–10) is isomorphic to $\operatorname{Ext}^{*-1,t}_S(s^{-2}k,k)$, i.e., to $\operatorname{Ext}^{*-1,t-2}_S(k,k)$.

It remains to analyze the left summand of (2–10). But it is easy to see that $\operatorname{Ann}_S((x_1+x_2)) = I$, so that the graded sequence of S-modules

$$0 \longrightarrow s^{-1}I \longrightarrow s^{-1}S \xrightarrow{(x_1+x_2)} S, \tag{2–11}$$

where we multiply on the right by x_1+x_2, is exact. Therefore we have a short exact sequence

$$0 \longrightarrow s^{-1}I \longrightarrow s^{-1}S \longrightarrow (x_1+x_2) \longrightarrow 0 \tag{2–12}$$

of graded left S-modules leading to the isomorphism of left $\operatorname{Ext}^*_S(k,k)$-modules

$$\operatorname{Ext}^{*-1,t}_S(s^{-1}I,k) \xrightarrow{\sim} \operatorname{Ext}^{*,t}_S((x_1+x_2),k) \tag{2–13}$$

for $* \geq 1$. Using (2–9) once more, we obtain that

$$\begin{aligned}\operatorname{Ext}^{*-1,t}_S(s^{-1}I,k) &= \operatorname{Ext}^{*-1,t-1}_S(I,k)\\ &\xrightarrow{\sim} \operatorname{Ext}^{*,t-1}_S(S/I,k),\end{aligned} \tag{2–14}$$

leading to the final isomorphism of left $\operatorname{Ext}_S(k,k)$-modules (combining (2–9), (2–10), (2–13), (2–14))

$$\operatorname{Ext}^{*,t}_S(S/I,k) \xrightarrow{\sim} \operatorname{Ext}^{*-1,t-1}_S(S/I,k) \oplus \operatorname{Ext}^{*-1,t-2}_S(k,k) \tag{2–15}$$

for $* \geq 1$, where the summand $\operatorname{Ext}^{*-1,t-2}_S(k,k)$ is nonzero only if $* = t-1$.

This proves everything, since we see, using (2–15), that $\operatorname{Ext}^{*,t}_S(S/I,k)$ needs a new $\operatorname{Ext}^*_S(k,k)$-generator for $* = t-1$ for each $t = 2,3,4,\dots$.

In particular, if we introduce for any graded module N over a graded k-algebra G the double series

$$P^N_G(x,y) = \sum_{i\geq 0, j\geq 0} \left|\operatorname{Ext}^{i,j}_G(N,k)\right| x^i y^j \tag{2–16}$$

(where as always, for a k-vector space V we denote by $|V|$ its dimension) and if we denote $P^k_G(x,y)$ by $P_G(x,y)$, we then deduce from (2–15) and the fact that $P_S(x,y) = 1/(1-2xy)^2$ that

$$P^{S/I}_S(x,y) = \frac{1}{1-xy} + \frac{xy^2}{(1-xy)(1-2xy)^2},$$

so that

$$P_{S \propto s^{-1} S/I}(x,y) = \frac{P_S(x,y)}{1 - xyP_S^{S/I}(x,y)},$$

leading to the following theorem.

Theorem 2.1. *The Orlik–Solomon algebra of the complex hyperplane arrangement* $\mathcal{A} = \{x, y, z, x+y, x+z, y+z\}$ *is the tensor product of the exterior algebra in one variable with an algebra R whose Yoneda Ext-algebra* $\mathrm{Ext}_R^*(k,k)$ *has a bigraded generating series:*

$$\begin{aligned} P_R(x,y) &= \frac{P_S(x,y)}{1 - xyP_S^M(x,y)} \\ &= \frac{1-xy}{1 - 6xy + 12x^2y^2 - x^2y^3 - 8x^3y^3}, \end{aligned} \tag{2–17}$$

where S and M are defined above. Furthermore, the Ext-algebra $\mathrm{Ext}_R^*(k,k)$ *has global dimension* 3*, and it has five generators in degree* 1 *and needs one new generator in each degree* ≥ 2*. In particular, the homotopy Lie algebra of $\mathcal{A}$ is not finitely generated.*

We will return to this result in the next section.

3. THE HOLONOMY AND HOMOTOPY LIE ALGEBRA OF AN ARRANGEMENT

The analysis of the $\mathcal{A}$ arrangement in Section 2 was intended to give the "simplest possible proof" that the Ext-algebra $\mathrm{Ext}_R^*(k,k)$ is not finitely generated. However, in order to be able to analyze more cases, we need a more general theory. We will here briefly describe the basics of such a theory and apply it as an alternative to our first case and then treat another case of arrangements in which we can prove that the homotopy Lie algebra is also nonfinitely presented.

Note that the graded algebra R of the previous section has Hilbert series $1 + 5z + 7z^2$. Let us now start with any algebra R that is a quotient of an exterior algebra $E(x_1, x_2, \ldots, x_n)$ by a homogeneous ideal J generated by elements of degree ≥ 2. Thus $R = E(x_1, \ldots, x_n)/J$. Let m be the ideal of R generated by $(x_1, \ldots, x_n)$, and consider the exact sequence

$$0 \longrightarrow m/m^2 \longrightarrow R/m^2 \longrightarrow R/m \longrightarrow 0 \tag{3–1}$$

of left R-modules. Now apply the functor $\mathrm{Ext}_R^*(-,k)$ to the exact sequence (3–1). We get a long exact sequence that can be written as an exact sequence of left $\mathrm{Ext}_R^*(k,k)$-modules (we use the Yoneda product):

$$\begin{aligned} 0 &\to s^{-1}\overline{S}_m \to s^{-1}\overline{\mathrm{Ext}}_R^*(R/m^2,k) \\ &\to \mathrm{Ext}_R^*(k,k) \otimes \mathrm{Ext}_R^1(k,k) \to \mathrm{Ext}_R^*(k,k) \to S_m \to 0, \end{aligned} \tag{3–2}$$

where S_m is defined as the image of the natural map

$$\mathrm{Ext}_R^*(k,k) \longrightarrow \mathrm{Ext}_R^*(R/m^2,k), \tag{3–3}$$

and where, for example, $\overline{S}_m$ means that we take the elements of S_m of degrees greater than 0 and where s^{-1} is the "suspension" as before. The Ext-algebra $B = \mathrm{Ext}_R^*(k,k)$ is bigraded, and we recall that its bigraded Hilbert series is denoted by

$$P_R(x,y) = \sum_{i,j \geq 0} \left| \mathrm{Ext}_R^{i,j}(k,k) \right| x^i y^j = B(x,y),$$

where as always, for a k-vector space we denote by $|V|$ its dimension. The subalgebra A of $\mathrm{Ext}_R^*(k,k)$ generated by $\mathrm{Ext}_R^1(k,k)$ is also bigraded, but it is situated on the diagonal, so that the corresponding bigraded Hilbert series is given by

$$A(x,y) = A(xy,1) \stackrel{\text{def}}{=} A(xy).$$

Now take the alternating sum of the two-variable Hilbert series of (3–2). We obtain

$$\begin{aligned} &S_m(x,y) - B(x,y) + xyB(x,y) \left| m/m^2 \right| \\ &\quad - x\overline{P}_R^{R/m^2}(x,y) + x(S_m(x,y) - 1) = 0, \end{aligned} \tag{3–4}$$

where

$$\overline{P}_R^{R/m^2}(x,y) = \sum_{i>0, j \geq i} \left| \mathrm{Ext}_R^{i,j}(R/m^2,k) \right| x^i y^j. \tag{3–5}$$

We now make three fundamental observations:

1. A is a sub Hopf algebra of B, and therefore according to [Milnor and Moore 1965], B is free over A. Thus $S_m = B \otimes_A k$ has bigraded Hilbert series

$$S_m(x,y) = B(x,y)/A(xy). \tag{3–6}$$

2. If $m^3 = 0$, we have an isomorphism of left $\mathrm{Ext}_R^*(k,k)$-modules

$$\overline{\mathrm{Ext}}_R^*(R/m^2,k) \simeq \mathrm{Ext}_R^*(k,k) \otimes \mathrm{Ext}_R^1(R/m^2,k), \tag{3–7}$$

so that

$$\overline{P}_R^{R/m^2}(x,y) = B(x,y)xy^2 \left| m^2/m^3 \right|. \tag{3–8}$$

This follows from [Roos 1979, formula (16)]. Therefore the equality (3–4) can be written

$$\frac{B(x,y)}{A(xy)} = B(x,y) - xyB(x,y)\left|m/m^2\right| + x^2y^2B(x,y)\left|m^2/m^3\right| - x\left(\frac{B(x,y)}{A(xy)} - 1\right), \tag{3–9}$$

which is another way of writing the following (divide by $xB(x,y)$ and use the notation $R(z) = 1 + |m/m^2|z + |m^2/m^3|z^2$ for the Hilbert series of R):

$$1/B(x,y) = (1+1/x)/A(xy) - R(-xy)/x, \tag{3–10}$$

which is a formula due to Löfwall [Löfwall 1986].

3. In the case $m^3 = 0$, the three middle terms of (3–2) are free $\mathrm{Ext}^*_R(k,k)$-modules, so that $s^{-1}\overline{S}_m$ is a third syzygy of a minimal graded $\mathrm{Ext}^*_R(k,k)$-resolution of S_m. We therefore obtain the isomorphism

$$\mathrm{Tor}^B_{i,*}(k,S_m) \simeq \mathrm{Tor}^B_{i-3,*}(k,s^{-1}\overline{S}_m) = \mathrm{Tor}^B_{i-3,*-1}(k,\overline{S}_m) \tag{3–11}$$

for $i \geq 3$. Now apply $\mathrm{Tor}^B_i(k, \,)$ to the exact sequence

$$0 \longrightarrow \overline{S}_m \longrightarrow S_m \longrightarrow k \longrightarrow 0. \tag{3–12}$$

We get the long exact sequence

$$\cdots \to \mathrm{Tor}^B_{n+1}(k,k) \to \mathrm{Tor}^B_n(k,\overline{S}_m) \to \mathrm{Tor}^B_n(k,S_m) \xrightarrow{\varphi_n} \mathrm{Tor}^B_n(k,k) \to \mathrm{Tor}^B_{n-1}(k,\overline{S}_m)\cdots. \tag{3–13}$$

Furthermore, since B is A-flat (which follows from observation 1 above),

$$\mathrm{Tor}^B_n(k,S_m) = \mathrm{Tor}^B_n(k, B\otimes_A k) = \mathrm{Tor}^A_n(k,k),$$

and $\varphi_n : \mathrm{Tor}^A_n(k,k) \to \mathrm{Tor}^B_n(k,k)$ is induced by the natural inclusion $A \to B$, which is split by a ring map in the other direction: divide $B = \mathrm{Ext}^{*,*}_R(k,k)$ by the two-sided ideal generated by $\oplus_{j>i>0}\mathrm{Ext}^{i,j}_R(k,k)$. Thus the maps φ_n in (3–13) are monomorphisms, and (3–13) splits into short exact sequences, using (3–11):

$$0 \longrightarrow \mathrm{Tor}^A_{i,j}(k,k) \longrightarrow \mathrm{Tor}^B_{i,j}(k,k) \longrightarrow \mathrm{Tor}^A_{i+2,j+1}(k,k) \longrightarrow 0. \tag{3–14}$$

Now we can summarize:

Theorem 3.1. *Let R be a quotient of an exterior algebra (finite number of generators in degree 1) by a homogeneous ideal generated by elements of degree ≥ 2. Let m be the augmentation ideal of R. Assume that $m^3 = 0$. Let $B = \mathrm{Ext}^*_R(k,k)$ be the Yoneda Ext-algebra and let A be the subalgebra of B, generated by $\mathrm{Ext}^1_R(k,k)$. Then the exact sequences (3–14) hold. In particular,*

(a) *B is finitely generated if and only if the graded vector space $\mathrm{Tor}^A_{3,*}(k,k)$ has finite dimension.*

(b) *B is finitely presented if and only if the graded vector spaces $\mathrm{Tor}^A_{3,*}(k,k)$ and $\mathrm{Tor}^A_{4,*}(k,k)$ have finite dimension.*

(c) *B is finitely presented and has a finite number of relations between the minimal relations if and only if the graded vector spaces $\mathrm{Tor}^A_{3,*}(k,k)$, $\mathrm{Tor}^A_{4,*}(k,k)$, and $\mathrm{Tor}^A_{5,*}(k,k)$ have finite dimension.*

Note that A is the enveloping algebra of a graded Lie algebra (the holonomy Lie algebra) whose ranks are equal to the ranks of the lower central series (LCS) of the fundamental group of the hyperplane complement (cf. Section 7 below). Note also that in general, the Hilbert series $A(x)$ of A when $m^3 = 0$ is obtained from (3–10): replace x by x/y in that formula and put $y = 0$. This gives

$$1/P_R(x/y,y)|_{y=0} = 1/A(x).$$

We can get an alternative proof of the assertion about the generators of the Ext-algebra in Theorem 2.1 above, using only the formula (2–17) there. The preceding recipe gives in that case that $A(x) = (1-x)/(1-2x)^3$. Now recall that for any graded algebra A we have the following formula for the relation between its Hilbert series $A(z)$ and the Hilbert series $\mathrm{Tor}^A_{i,*}(k,k)(z)$ of the graded Tor (see, for example, [Lemaire 1974, Appendix A2]):

$$\frac{1}{A(x)} = \sum_{i\geq 0}(-1)^i\,\mathrm{Tor}^A_{i,*}(k,k)(x). \tag{3–15}$$

Since A in the case of Theorem 2.1 has global dimension 3 and five generators in degree 1, and seven relations in degree 2, (3–15) gives that

$$\mathrm{Tor}^A_{3,*}(k,k)(x) = x^3/(1-x),$$

so that we see once more that $\mathrm{Tor}^A_{3,i}(k,k)$ is one-dimensional for all $i \geq 3$, and therefore Theorem 3.1(b) gives once more a proof of Theorem 2.1.

Remark 3.2. If $m^3 \neq 0$, but more generally

$$\overline{\mathrm{Ext}}^*_R(R/m^i,k) \longrightarrow \overline{\mathrm{Ext}}^*_R(R/m^{i+1},k), \tag{3–16}$$

is zero for $i \geq 2$, then we have the same conclusion as in Theorem 3.1, but the proof is slightly different, since now $N = \overline{\mathrm{Ext}}^*_R(R/m^2, k)$ is not free as a $B = \mathrm{Ext}^*_R(k,k)$-module, but it has a finite homological dimension, and the corresponding $\mathrm{Tor}^B_i(k, N) \simeq m^{i+2}/m^{i+3}$ are finite-dimensional.

The condition (3–16) is sometimes, but not always, satisfied if $m^4 = 0$, but in the last case, one can prove that the validity of the formula (3–10) is *equivalent* to the assertion that the map (3–16) is zero for $i \geq 2$ (of course, only the case $i = 2$ is important in this case). This will be used below when we study graphic arrangements. Furthermore, the important formula (3–10), which we will now write as

$$1/P_R(x,y) = (1 + 1/x)/R^!(xy) - R(-xy)/x, \quad (3\text{–}17)$$

is still valid under (3–16), but here the Hilbert series $R(z)$ might be a polynomial of degree greater than 2. Note that we have written $R^!$ (instead of A); it is the Koszul dual of R.

Remark 3.3. The formula (3–17) in Remark 3.2 is a special case ($n = 3$) of a whole family of formulas $(3\text{–}18)_n$, $n \geq 3$, valid under certain conditions:

$$\frac{1}{P_R(x,y)} = \frac{(1-(-x)^{2-n})}{R^!(xy)} + R(-xy)(-x)^{2-n}. \quad (3\text{–}18)$$

Indeed, the validity of $(3\text{–}18)_n$ is a consequence of the fact that the so-called Koszul complex $R^! \otimes_k \mathrm{Hom}_k(R,k)$ has only nonzero homology groups in degree 0 and in degree $n-1$. If $m^3 = 0$, this is true for $n = 3$, but if $m^4 = 0$, this is true only under extra conditions. For more details about this, see [Löfwall 1994, Theorem B.4], [Roos 1994], and [Roos 1996]. We will say here that R satisfies L_n if $(3\text{–}18)_n$ holds.

For Orlik–Solomon algebras with $m^4 = 0$, we still have the formula (3–17), since the algebra is the tensor product of an algebra with $m^3 = 0$ and an "innocent" algebra $E[z]$. In Section 6, in which we study the case of graphic arrangements, we will see that for any $n \geq 3$, there are examples in which the condition L_n holds (namely, the Orlik–Solomon algebra of the graphic arrangement corresponding to an $(n+1)$-gon for $n \geq 3$), but also that there are also examples in which none of these conditions is satisfied (however, these cases can sometimes be handled with the method of [Roos 1996]).

4. SOME OTHER HYPERPLANE ARRANGEMENTS

Some of the cases from [Suciu 2000] can be treated in the same way as in Section 3. Here we briefly describe the results for the so-called X_2-arrangement, which is defined by the polynomial

$$xyz(x+y)(x-z)(y-z)(x+y-2z).$$

Now the Orlik–Solomon algebra can be written as a quotient of the exterior algebra in seven variables $E(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ by the ideal generated by five elements:

$$(e_5 - e_7)(e_6 - e_7), \ (e_3 - e_7)(e_4 - e_7), \ (e_2 - e_6)(e_3 - e_6),$$
$$(e_1 - e_5)(e_3 - e_5), \ (e_1 - e_2)(e_2 - e_4).$$

Now isolate e_3, i.e., introduce variables

$$a = e_1 - e_3, \ b = e_2 - e_3, \ c = e_4 - e_3, \ d = e_5 - e_3,$$
$$e = e_6 - e_3, \ f = e_7 - e_3.$$

Now the relations in the Orlik–Solomon algebra do not contain e_3, and this algebra is now a tensor algebra of the quotient

$$R = \frac{E(a,b,c,d,e,f)}{(ab - ac + bc, ad, be, cf, de - df + ef)}$$

with the exterior algebra in one variable $z = e_3$. Therefore, we are led to the analysis of the Yoneda Ext-algebra of the quotient R above, whose Hilbert series is $R(t) = 1 + 6t + 10t^2$. Furthermore, let

$$S = \frac{E(a,b,c,d,e,f)}{(ab - ac + bc, ad, be, cf)}$$

and consider the ring map

$$S \longrightarrow S/(de - df + ef) = R. \quad (4\text{–}1)$$

It is not difficult to show that (4–1) is a so-called Golod map (cf. [Levin 1985] and the literature cited there. In the present case, we are studying the skew-commutative version of Golod, which has been treated with relevant references in [Sköldberg 1999]). One finds that

$$S^!(t) = (1-t)^2/(1-2t)^4,$$

that

$$R^!(t) = (1-t)^4/(1-2t)^5,$$

and more precisely, that $R^!$ has global dimension 5 and that

$$\sum_{i \geq 0} \left| \mathrm{Tor}^{R^!}_{3,i}(k,k) \right| z^i = 5z^4 + \frac{6z^5}{(1-z)}, \quad (4\text{–}2)$$

$$\sum_{i \geq 0} \left| \mathrm{Tor}^{R^!}_{4,i}(k,k) \right| z^i = 2z^6 + \frac{(6-z)z^7}{(1-z)^2}, \quad (4\text{–}3)$$

and

$$\sum_{i\geq 0}\left|\mathrm{Tor}_{5,i}^{R^!}(k,k)\right| z^i = \frac{z^{10}}{(1-z)^4}. \tag{4–4}$$

Thus using the theory from Section 3, we see that the homotopy Lie algebra of the arrangement X_2 is "extremely nonfinitely presented": it needs an infinite number of generators (4–2), and furthermore, the minimal number of relations between a minimal system of generators is infinite (4–3), and the minimal number of relations between the relations is infinite (4–4). However, among the graphic arrangements (see Section 6 for more details) there are more arrangements with a finitely presented Ext-algebra than with an infinitely presented one.

We finish this section with one unsolved case: Recall that the *non-Fano arrangement* is the hyperplane arrangement defined by

$$xyz(x-y)(x-z)(y-z)(x+y-z).$$

In this case, the corresponding R (we have eliminated one variable as above) has Hilbert series $(1+3t)^2$, but the corresponding $R^!(t)$ is rather complicated. Nevertheless, we have managed to calculate the LCS ranks two steps higher than in [Suciu 2000], using the program BERGMAN [Backelin et al. 07]; with the notation of [Suciu 2000] we have $\phi_8 = 3148$ and $\phi_9 = 9857$, but for the last result we needed 64 bits PSL on an AMD opteron machine with 12 GB of internal memory.

5. ARRANGEMENTS WITH IRRATIONAL HILBERT SERIES

In [Denham and Suciu 2006] it is also asked whether the enveloping algebra of the homotopy Lie algebra of an arrangement can have an irrational Hilbert series.

We will here describe one case we have found in which this is conjecturally true and a second case in which this has been proved to be true. This development is rather recent: we found the second case only recently, and the first (more complicated) case is the well-known Mac Lane arrangement, whose amazing homological properties we also discovered recently. The proof in the second case (the first case is probably treated in a similar but more complicated way) is based on ideas of the present paper, but involves many more new ideas and will be presented in another paper in preparation [Roos 2008]. Let us just indicate some details

First case (the Mac Lane arrangement): Recall the Mac Lane arrangement, defined by the annihilation of the polynomial

$$Q = xyz(y-x)(z-x)(z+\omega y)(z+\omega^2 x+\omega y)(z-x-\omega^2 y)$$

in $\mathbb{C}^3$, where $\omega = e^{2\pi i/3}$. It is not difficult to see that with the notation of our Section 2, the Orlik–Solomon algebra of the Mac Lane arrangement is $R \otimes E[z]$, where R is the quotient of the exterior algebra $E(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in seven variables with the ideal generated by the eight quadratic elements

$$\begin{gathered}x_1x_2 - x_1x_4 + x_2x_4,\;\; x_1x_3 - x_1x_5 + x_3x_5,\\ x_1x_6 - x_1x_7 + x_6x_7,\;\; x_2x_3 - x_2x_6 + x_3x_6,\\ x_4x_5 - x_4x_7 + x_5x_7,\;\; x_2x_5, x_4x_6,\;\; x_3x_7.\end{gathered}$$

This ring R has Hilbert series $R(z) = 1 + 7\,z + 13\,z^2$, and therefore the formula (3–17) above can be applied, and the only thing needed to be proved is that the Koszul dual $R^!$ of R has an irrational Hilbert series. But this Koszul dual $R^! = U(g)$ is the quotient of the free associative algebra $k\,\langle X_1, X_2, \ldots, X_7\rangle$ in the seven dual variables by the two-sided ideal generated by the thirteen dual relations among the Lie commutators $[X_i, X_j] = X_iX_j - X_jX_i$ for $i \neq j$:

$$\begin{gathered}[X_1,X_2]+[X_1,X_4],\quad [X_1,X_4]+[X_2,X_4],\\ [X_1,X_3]+[X_1,X_5],\quad [X_1,X_5]+[X_3,X_5],\\ [X_1,X_6]+[X_1,X_7],\quad [X_1,X_7]+[X_6,X_7],\\ [X_2,X_3]+[X_2,X_6],\quad [X_2,X_6]+[X_3,X_6],\\ [X_4,X_5]+[X_4,X_7],\quad [X_4,X_7]+[X_5,X_7],\\ [X_2,X_7],\quad [X_3,X_4],\quad [X_5,X_6],\end{gathered}$$

so that the Lie algebra g (it is called the holonomy Lie algebra) in $R^! = U(g)$ is the quotient of the free Lie algebra in seven variables by the ideal generated by the thirteen Lie commutators above. Now we have the formula

$$\frac{1}{R^!(z)} = \frac{1}{U(g)(z)} = \prod_{n=1}^{\infty}(1-t^n)^{\phi_n},$$

where the ϕ_n are the *lower central series* (LCS) ranks. But these ranks can be calculated by a program by Clas Löfwall [Löfwall 2007], which is called `liedim.m` and runs under Mathematica. It gives (in characteristic zero) the ranks

7, 8, 21, 42, 87, 105, 172, 264, 476, 816, 1516, 2704, 5068, 9312, 17484, ...,

but for the higher ranks you need the C version of the program unless your computer has a large amount of internal memory. Thus the Hilbert series $R^!$ can be calculated in degrees ≤ 15, and in these degrees it is described by the part of following rather amazing formula of degree less than or equal to 15:

$$\frac{1}{R^!(t)} = \frac{(1-2t)^8}{(1-t)^9}(1-t^3)^5(1-t^4)^{18}(1-t^5)^{39}(1-t^6)^{33} \times \prod_{n=4}^{\infty}(1-t^{2n-1})^{28}(1-t^{2n})^{24}.$$

We indicate a possible proof below, but we wish to emphasize again that the preceding formula is for the moment known to hold only (by `liedim`) up to t^{15}.

Second case (this is indeed a quite different case, but it can be seen as a "simplification-degeneration" of the Mac Lane arrangement): In the Mac Lane arrangement above, ω, a primitive third root of unity, satisfies $\omega^2 = -\omega - 1$, so if you replace ω^2 by $-\omega-1$ in the definition of the Mac Lane arrangement above and then (this is brutal!) put $\omega = 1$, you obtain a new arrangement mlease in $\mathbb{C}^3$ defined by the following polynomial with integer coefficients:

$$Q_{\text{eas}} = xyz(y-x)(z-x)(z+y)(z-2x+y)(z-x+2y). \tag{5–1}$$

The amazing thing now is that the corresponding hyperplane arrangement has an almost identical Orlik–Solomon algebra $R_{\text{eas}} \otimes E[z]$, but its homological properties are dramatically different: we have indeed that R_{eas} has all the relations of R *with the exception of the relation x_3x_7, which is replaced by $x_3x_6x_7$*, so that $R = R_{\text{eas}}/(x_3x_7)$, and the Hilbert series is given by $R_{\text{eas}}(t) = 1 + 7\,t + 14\,t^2$, which is close to $R(t) = 1 + 7\,t + 13\,t^2$. But now we have the following theorem.

Theorem 5.1. *The Koszul dual $R^!_{\text{eas}}$ of the hyperplane arrangement* mlease *has the following Hilbert series:*

$$\frac{1}{R^!_{\text{eas}}(t)} = \frac{(1-2\,t)^7}{(1-t)^7}\prod_{n=3}^{\infty}(1-t^n). \tag{5–2}$$

Corollary 5.2. *The Ext-algebra of the algebra R_{eas} corresponding to the hyperplane arrangement* mlease *has a transcendental Hilbert series.*

Proof of Corollary 5.2: The condition L_3 is satisfied, since $m^3 = 0$. □

Sketch of proof of theorem 5.1: We have

$$R^!_{\text{eas}} = R^!/([X_3, X_7]).$$

Furthermore, $R^!_{\text{eas}}$ is the enveloping algebra of a Lie algebra g_{eas}. Now divide out g_{eas} by the Lie element of degree 3: $[X_6,[X_7,X_5]]$. We get an exact sequence of Lie algebras:

$$0 \longrightarrow \ker \longrightarrow g_{\text{eas}} \longrightarrow g_{\text{eas}}/([X_6,[X_7,X_5]]) \longrightarrow 0, \tag{5–3}$$

where ker is defined by (5–3).

We now claim that the Hilbert series of the enveloping algebra of

$$\text{quot} = g_{\text{eas}}/([X_6,[X_7,X_5]])$$

is

$$(1-z)^7/(1-2\,z)^7.$$

Indeed, the underlying Lie algebra has a basis of seven elements $X_1, X_2, \ldots, X_7$ in degree 1. From now on, to simplify we denote the elements $[X_i,[X_j,[X_k,\ldots]]]$ by $ijk\ldots$. With this notation we have the following basis of seven elements in degree 2:

$$42,\ 52,\ 53,\ 63,\ 64,\ 75,\ 76.$$

These seven elements commute pairwise, and we have fourteen elements of degree 3. One can prove that the Lie algebra decomposes in the sense of [Papadima and Suciu 2006], i.e., that the Lie algebra quot in degrees ≥ 2 is a direct sum of seven parts of degree ≥ 2 of free Lie algebras in two variables. One cannot use [Papadima and Suciu 2006] directly, but if S is R_{eas} without the relation $x_4x_5 - x_4x_7 + x_5x_7$, then S comes from the so-called X_2 arrangement, which decomposes [Papadima and Suciu 2006] and has $S^!(z) = (1-z)^3/(1-2z)^5$, and the map $S \to R_{\text{eas}}$ can be analyzed as in Section 4 above.

We now continue analyzing the kernel ker in (5–3). Clearly $675 = [X_6,[X_7,X_5]]$ is in this kernel, and so are also the degree-4 elements $i675 = [X_i,[X_6,[X_7,X_5]]]$ for $i = 1,\ldots,7$. But we get possibly nonzero elements (with different signs) for only $i = 5$ and $i = 6$ (the last one can be written 6775), and in the next degree we similarly get only one element 67775, and so on. Therefore ker is $\leq$ 1-dimensional in each degree ≥ 3. Next we have the formulas

$$\begin{aligned}-[X_5,42] &= [X_6,42] = -[X_4,52] = [X_6,52] = [X_2,53]\\ &= -[X_6,53] = [X_1,63] = -[X_5,63] = -[X_2,64]\\ &= [X_5,64] = -[X_1,75] = [X_6,75] = -[X_4,76]\\ &= [X_5,76] = 675\end{aligned} \tag{5–4}$$

in g_{eas}. The other commutators lie in quot. We now wish to prove that the dimension of ker is exactly 1 in each degree ≥ 3. For this purpose, if we define a graded vector space of dimension 1 in each degree ≥ 3 by

$$V_* = ke_3 \oplus ke_4 \ldots,$$

where the X_i operate as zero, with the exception of $X_5.e_n = e_{n+1}$ and $X_6.e_n = -e_{n+1}$, we get a quot-module V, and by calculating $H^2(\text{quot}, V)$, we find that there is a 2-cocycle $\gamma : \text{quot} \times \text{quot} \longrightarrow V$ on quot with values in V, which in lower degrees starts as in (5–4). Thus we can define a Lie algebra g_γ defined by this cocycle, which sits in the middle of an extension of Lie algebras in which the kernel V is abelian:

$$0 \longrightarrow V \longrightarrow g_\gamma \longrightarrow \text{quot} \longrightarrow 0. \tag{5–5}$$

We now wish to show that g_γ in (5–5) is isomorphic to g_{eas} in (5–3): we use the Hochschild–Serre spectral sequence of the extension (5–5) to show that $\text{Tor}_{2,*}^{U(g_\gamma)}(k,k)$ is concentrated in degree 2.

Now g_{eas} and g_γ are isomorphic (they have "the same" generators and relations). Thus ker is one-dimensional in each degree ≥ 3. This gives the formula (5–2), and Theorem 5.1 follows. □

Remark 5.3. The previous reasoning with explicit cocycles is analogous to, but a little more complicated than, Löfwall's and my version of the Anick solution of the Serre–Kaplansky irrationality problem (cf. [Löfwall and Roos 1980], [Roos 1981, pages 454–456], [Anick 1982]) as well as Lemaire's Bourbaki talk about these questions [Lemaire 1980].

Now what about the Mac Lane arrangement? The results are similar to those of the easier case just described. In fact, the Lie algebra g_{eas} just studied has no center, and the kernel Lie algebra ker of (5–3) is abelian. In order to carry out similar reasoning for the Mac Lane (ML) arrangement, one needs to divide the Lie algebra g_{ML} by the following five cubic Lie algebra elements:

$$\begin{aligned} &[X_5,[X_7,X_3]], \quad [X_6,[X_7,X_3]], \quad [X_7,[X_6,X_3]],\\ &[X_7,[X_6,X_4]], \quad [X_6,[X_7,X_5]], \end{aligned} \tag{5–6}$$

leading again to a quotient Lie algebra quot_{ML}, whose enveloping algebra has Hilbert series $(1-z)^9/(1-2\,z)^8$.

We still get an exact sequence of Lie algebras

$$0 \longrightarrow \ker_{\text{ML}} \longrightarrow g_{\text{ML}} \longrightarrow \text{quot}_{\text{ML}} \longrightarrow 0. \tag{5–7}$$

But now $\ker_{\text{ML}}$ is generated by the five elements (5–6) and is still situated in degrees ≥ 3, where its dimensions are

$$5,\ 18,\ 39,\ 33,\ 28,\ 24,\ 28,\ 24,\ 28,\ 24,\ 28,\ 24,\ 28,\ldots \tag{5–8}$$

and there is still a 2-cocycle describing the extension (5–7).

Now we can use the Löfwall program `liedim` [Löfwall 2007]. Indeed, Clas Löfwall has kindly constructed at our request an extra command `centre[n]`, which gives generators for the center in degree n of a graded Lie algebra given by generators and relations in `liedim`. In our case, one finds (in degrees ≤ 15) that now g_{ML} contains central elements, all situated in $\ker_{\text{ML}}$, and that $[\ker_{\text{ML}}, \ker_{\text{ML}}]$ is contained in the center, which is 1-dimensional in degree 5, 9-dimensional in degree 6, and 4-dimensional in degrees $2n+1$ for $n \geq 3$, and zero-dimensional in even degrees ≥ 8. If we divide out by the center, the Lie algebra $\ker_{\text{ML}}/\text{center}$ is still situated in degrees ≥ 3, but its dimensions there are now

$$5,\ 18,\ 38,\ 24,\ 24,\ 24,\ 24,\ 24,\ 24,\ 24,\ 24,\ 24,\ 24,\ \ldots. \tag{5–9}$$

Furthermore, $\ker_{\text{ML}}/\text{center}$ is abelian, and quot_{ML} operates on it in a similar but more complicated way than for g_{eas} above. But so far, all this has been proved only in degrees ≤ 15.

Although the g_{eas} irrational case is much easier than the g_{ML} case, it still uses a hyperplane arrangement with eight hyperplanes, leading to an algebra $R^!_{\text{eas}}$ in seven variables. One might still wonder whether it would be possible to simplify further, i.e., to obtain a hyperplane arrangement with seven or six hyperplanes and having an irrational series.

However, in [Roos 2000] we have in particular described all homological possibilities for the quotient of an exterior algebra in ≤ 5 variables by an ideal generated by ≤ 3 quadratic forms (the ring-theoretic classification was obtained in [Eisenbud and Koh 1994]). It is only in five variables that we can obtain nonfinitely generated Ext-algebras (only one case, just studied above in Section 2) or Ext-algebras with an irrational Hilbert series (three cases). These three cases are as follows (the numbering of cases is from [Roos 2000]) (in all these three cases the Hilbert series is given by $R(t) = 1 + 5t + 7t^2$):

Case 12: We have

$$R_{12} = \frac{E(x,y,z,u,v)}{(xy,\, xz+yu+zv,\, uv)}$$

with

$$\frac{1}{R^!_{12}(t)} = (1-2t)^2 \prod_{n=1}^{\infty} (1-t^n).$$

Case 20: We have

$$R_{20} = \frac{E(x,y,z,u,v)}{(yz+xu,\, yu+xv,\, zu+yv)}$$

with

$$\frac{1}{R^!_{20}(t)} = \prod_{n=1}^{\infty}(1-t^{2n-1})^5(1-t^{2n})^3.$$

Case 15: We have

$$R_{15} = \frac{E(x,y,z,u,v)}{(yz+xu, xv, zu+yv)}$$

with

$$\frac{1}{R^!_{15}(t)} = (1-2t)\prod_{n=1}^{\infty}(1-t^{2n-1})^3(1-t^{2n})^2.$$

But we cannot see how any of these algebras could arise from some hyperplane arrangement. If we study quotients of $E(x,y,z,u,v)$ with four quadratic forms, there are still three other quotients (this time with Hilbert series $R(t) = 1+5t+6t^2$) that might have irrational $R^!(t)$.

Case 21: We have

$$R_{21} = \frac{E(x,y,z,u,v)}{(yz+xu, yu+xv, zu+yv, uv)}$$

with

$$\begin{aligned}R^!_{21}(t) = {}& 1+5t+19t^2+65t^3+211t^4+667t^5+2081t^6\\ &+6449t^7+19919t^8+61425t^9+189273t^{10}\\ &+583008t^{11}+1795509t^{12}+5529263t^{13}\\ &+17026752t^{14}+52431180t^{15}+161452384t^{16}\\ &+497162060t^{17}+1530914456t^{18}\\ &+4714152439t^{19}+14516309322t^{20}\\ &+44700127353t^{21}+137645268696t^{22}\\ &+423851580822t^{23}+\cdots.\end{aligned}$$

Case 22: We have

$$R_{22} = \frac{E(x,y,z,u,v)}{(yz+xu, yu+xv, zu+yv, zv)}$$

with

$$\begin{aligned}R^!_{22}(t) = {}& 1+5t+19t^2+65t^3+211t^4+666t^5+2071t^6\\ &+6387t^7+19609t^8+60054t^9+183672t^{10}\\ &+561340t^{11}+1714894t^{12}+5237883t^{13}\\ &+15996477t^{14}+\cdots.\end{aligned}$$

Case 33: We have

$$R_{33} = \frac{E(x,y,z,u,v)}{(yz+xu, xv, zu+yv, uv)}$$

with

$$\begin{aligned}R^!_{33}(t) = {}& 1+5t+19t^2+65t^3+212t^4+675t^5+2125t^6\\ &+6653t^7+(21\text{ terms})+483131948638003t^{29}\\ &+1505474194810058t^{30}+\cdots,\end{aligned}$$

but the following formula gives in this last case an indication about theta functions:

$$\begin{aligned}&\frac{1}{(1-t)^2R^!_{33}(t)}\\ &\quad= 1-3t-t^2+t^3+2t^4+3t^5+t^6\\ &\qquad+t^7-t^8-t^9-2t^{10}-t^{11}-3t^{12}-t^{13}-t^{14}\\ &\qquad-t^{15}+t^{17}+t^{18}+2t^{19}+t^{20}+t^{21}+3t^{22}+t^{23}\\ &\qquad+t^{25}+t^{26}-t^{29}-t^{30}-\cdots.\end{aligned}$$

Theta functions are mentioned here because the monomials $2t^4$, $-2t^{10}$, $2t^{19}$, $-2t^{31}$, ... are obtained as $(-1)^{n+1}t^{\frac{3n(n+1)}{2}+1}$ for $n=1,2,\ldots$. Similarly for other terms, leading to the predictions that the coefficient for t^{31} should be -2 and that more precisely, the series should continue as $-2t^{31}-t^{32}-t^{33}-t^{34}-3t^{35}+\cdots$.

But using the program BERGMAN [Backelin et al. 07], we have for the moment been able to calculate the preceding series only in degrees ≤ 30, and no precise theory is in sight. But it is not known whether these last three cases come from some hyperplane arrangements. In higher embedding dimensions $(6,7,\ldots)$, there are of course more irrational series, and as we have indicated, two of them in embedding dimension 7 come from complex hyperplane arrangements.

Remark 5.4. The case $R^!_{20}$ (which comes from Jürgen Wisliceny and whose series was determined up to degree 67 by Czaba Schneider [Schneider 1997, Theorem 6.1]) was completely determined in the super-Lie algebra case in [Löfwall and Roos 1997] (where we had periodicity 4). Here its treatment is easier (periodicity 2). Note that we have described above what happens only in characteristic 0. In Case 20, we have different $R^!_{20}(z)$ in all characteristics, and the same remark seems to be applicable to the cases 21, 22, 33.

6. IRRATIONAL OR NONFINITELY PRESENTED CASES FOR OTHER ARRANGEMENTS ?

In the sections above we have found two classes of unexpected complex hyperplane arrangements. An interesting question is to determine how rare those hyperplane arrangements are. The simplest arrangements are the so-called graphic arrangements: we have a simple graph Γ given with n vertices and t edges. The corresponding hyperplane arrangement $\mathcal{A}_\Gamma$ in $\mathbb{C}^n$ is defined by

$$\mathcal{A}_\Gamma = \{x_i - x_j\},$$

where $i < j$ and (i, j) is an edge of Γ.

Such a graph leads as in Section 2 to an Orlik–Solomon algebra that can be written in the form $R_\Gamma \otimes E(z)$, where R_Γ is a quotient of the exterior algebra in $t-1$ variables by homogeneous forms, and $E(z)$ is the exterior algebra in one variable z. It is therefore sufficient to analyze R_Γ.

It was recently proved [Lima-Filho and Schenck 2006] that the Hilbert series of all $R_\Gamma^!$ are rational of a special form, and therefore it follows that the Hilbert series of the Ext-algebra of the Orlik–Solomon algebra of $\mathcal{A}_\Gamma$ is always rational, at least for those cases in which the Hilbert series of R_Γ has the cube of its maximal ideal equal to 0 (and maybe in all cases; see remarks below).

Indeed, formula (3–17) above can be applied, and it gives an explicit rational formula. But nonfinitely presented Ext-algebras can indeed occur for some graphs. In the book about graphs [Harary 1969], there is at the end an explicit list of simple graphs with up to six vertices. We have gone through that list completely, and we can report part of the results as Theorem 6.1 below (note that it is sufficient to analyze connected graphs, since the Orlik–Solomon algebra decomposes as a tensor product of the algebras corresponding to the connected components of the graph). It is known that the number of simple connected graphs with n vertices increases rapidly with n, according to the following table:

# vertices	2	3	4	5	6	7	8
# graphs	1	2	6	21	112	853	11117

We also use the numbering of the simple connected graphs from the home page of of Brendan McKay.[2]

One finds that graph 4 (in the numbering above for graphs having four vertices) defined by

$$(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)$$

is the only graphical arrangement for a graph with four vertices (the graph of a square) in which the Orlik–Solomon algebra is not a Koszul algebra. However, the corresponding Orlik–Solomon algebra satisfies the condition L_3 (cf. the discussion after Remark 3.3), since $m^4 = 0$. Furthermore, in this case the Ext-algebra is finitely presented, and $\mathrm{Ext}^*_R(k, k)$ has a rational Hilbert series.

[2] http://cs.anu.edu.au/~bdm/data/graphs.html.

For graphs of orders 5 and 6 we have the following theorem (we continue using the numbering of graphs of orders 5 and 6 given by McKay).

Theorem 6.1.

(a) *Among the* 21 *connected graphs with five vertices,* 15 *give rise to Orlik–Solomon algebras (OS-algebras) that are Koszul. Among the six remaining non-Koszul algebras, only one (corresponding to graph 19, which is the graph of a pyramid with a square basis) gives rise to a hyperplane arrangement in which the Ext-algebra of the OS-algebra is not finitely presented; graphs 5, 7, 15, 17, 19 give OS-algebras that satisfy L_3; and graph 14 (the graph of a pentagon) has an OS-algebra that satisfies L_4, and since all $R(z)$ and $R^!(z)$ are rational, the Hilbert series of the* 21 *Ext-algebras are rational.*

(b) *Among the* 112 *connected graphs with six vertices,* 34 *give rise to Orlik–Solomon algebras that are Koszul. Among the* 78 *remaining non-Koszul algebras, only seven (corresponding to graphs 71, 74, 100, 102, 107, 108, 109) have nonfinitely presented Ext-algebras of their OS-algebras, and one (corresponding to graph 98) has a finitely presented Ext-algebra, which, however, has an infinite number of relations between the relations. The condition L_5 is satisfied in one case (the graph of a hexagon; more generally, L_{n-1} is satisfied for the graph of an n-gon). The condition L_4 is satisfied for the graphs 48, 95, 98, and the condition L_3 is satisfied for the graphs 11, 13, 25, 33, 36, 39, 42, 44, 46, 51, 53, 57, 61, 63, 66, 68, 72, 73, 81, 87, 92, 99, 100, 102, 106, 107, 108, 109.*

There are five graphs for which no condition L_n has been verified: 38, 71, 74, 96, 97. But also for these graphs the Hilbert series of the Ext-algebra can be analyzed and proved to be rational, so all these 112 *graphs give rational series.*

Sketch of proof of part of the theorem: (a) The case of graph 19 with five vertices gives rise to the arrangement defined by the polynomial

$$(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5) \\ \times (x_3 - x_5)(x_4 - x_5),$$

corresponding to a pyramid, where the vertex x_5 is at the top of the pyramid, and x_1, x_3, x_2, x_4 are at the base. We have eight factors, and the OS-algebra is in eight variables:

$$\begin{aligned}E(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)/&\\ \big((e_1 - e_7)(e_3 - e_7), (e_2 - e_8)(e_3 - e_8),&\\ (e_4 - e_7)(e_6 - e_7), (e_5 - e_8)(e_6 - e_8),&\\ (e_1 - e_2)(e_2 - e_5)(e_4 - e_5)\big).&\end{aligned} \tag{6–1}$$

Let us now introduce new variables $x_i = e_i - e_5$ for $i \neq 5$ and $z = x_5$. Our algebra (6–1) becomes, as earlier, the tensor product of the exterior algebra $E(z)$ and the algebra in seven variables (x_5 is missing!):

$$\begin{aligned}R = E(x_1, x_2, x_3, x_4, x_6, x_7, x_8)/&\\ \big((x_1x_3 - x_1x_7 + x_3x_7, x_2x_3 - x_2x_8 + x_3x_8,&\\ x_4x_6 - x_4x_7 + x_6x_7, x_6x_8, x_1x_2x_4)\big).&\end{aligned}$$

It is now easy to see that the annihilator of x_6 in R is generated by x_6 and x_8 and similarly that the annihilator of x_8 is generated by x_6 and x_8. Furthermore, the intersection of the two ideals x_6 and x_8 is 0.

Thus the ideal $a = (x_6, x_8)$ is a direct sum, and $S = R/a$ has a linear resolution over R; more precisely, we have $P_R^S(x, y) = 1/(1 - 2xy)$. Now apply a result from [Bøgvad 1995], which says that if $R \to S = R/a$ is an algebra map such that R/a has a linear R-resolution, then the map $R \to S$ is a large map in the sense of [Levin 1980]. In the proof of [Bøgvad 1995, Lemma 2.3b], there is a slight misprint on line 9 of the proof, which should read, "$\ldots \eta : R \to S$ is 1-linear, i.e., that $\mathrm{Tor}^R_{i,j}(S, k) = 0$ if $i \neq j \ldots$."

This has the consequence that the double series $P_R(x, y)$ is equal to $P_R^S(x, y)P_S(x, y)$ (i.e., the change of rings spectral sequence degenerates; cf. [Levin 1980, Theorem 1.1]). In our case, $P_R^S(x, y) = 1/(1 - 2xy)$, and S now becomes the quotient of an exterior algebra in five variables:

$$S = \frac{E(x_1, x_2, x_3, x_4, x_7)}{(x_1x_3 - x_1x_7 + x_3x_7, x_4x_7, x_2x_3, x_1x_2x_4)}.$$

But this algebra is essentially the algebra (2–2). Indeed, let us first make the substitutions $x_1 \mapsto x_1 + x_3$, $x_7 \mapsto x_7 + x_3$ and then interchange x_2 and x_3 and also x_1 and x_7. We get the same algebra as in (2–2); the only difference is that now the last variable is called x_7 (and not x_5 as in Section 2). It follows that the double series of S is given by

$$\frac{1}{P_S(x, y)} = \frac{1 - 6xy + 12x^2y^2 - x^2y^3 - 8x^3y^3}{1 - xy},$$

so that

$$\frac{1}{P_R(x, y)} = \frac{(1 - 2xy)(1 - 6xy + 12x^2y^2 - x^2y^3 - 8x^3y^3)}{1 - xy}. \tag{6–2}$$

From (6–2) we can now read off that $R(z) = 1 + 7z + 17z^2 + 14z^3 = (1 + 2z)(1 + 5z + 7z^2)$ and that $R^!(z) = (1 - z)/(1 - 2z)^4$, so that the formula L_3 holds. One then proves that $\mathrm{gldim}(R^!) = 4$ and that $\mathrm{Tor}^{R^!}_{4,i}(k, k)$ is 1-dimensional for $i \geq 4$ and $\mathrm{Tor}^{R^!}_{3,i}(k, k) = 0$ for $i \neq 3$, so that the algebra in Section 2, which there needs an infinite number of generators, now returns here as a subtle part of a graphic arrangement whose Ext-algebra is slightly better in that it is finitely generated but not finitely presented.

Note in particular that $\mathrm{Ext}^*_R(k, k)$ is not the tensor product of $\mathrm{Ext}^*_S(k, k)$ and the free algebra on two variables of degree 1. The other non-Koszul cases in Theorem 6.1(a) are simpler and treated in a similar way.

(b) For graphs with six vertices, similar procedures are used, and the most complicated case is case 109, in which the arrangement is defined by the polynomial

$$\begin{aligned}&(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)(x_1 - x_6)(x_2 - x_3)(x_2 - x_4)\\ &\quad \times (x_2 - x_5)(x_2 - x_6)(x_3 - x_5)(x_3 - x_6)(x_4 - x_5)\\ &\quad \times (x_4 - x_6),\end{aligned}$$

where we still have a nonfinitely presented Ext-algebra.

But the most interesting pairs of examples (from my point of view) are 107 and 74, which have the same Hilbert series both for the Orlik–Solomon algebra and for the quadratic dual $R^!$, and the first algebra satisfies L_3 and the second does not. But they both have Ext-algebras that are not finitely presented. In particular, the Tors (or Exts) of the Orlik–Solomon algebras differ, the first difference occurring for $\mathrm{Tor}^{\mathcal{OS}}_{4,6}(k, k)$, which has dimension 9 for case 107 and dimension 10 for case 74.

A similar phenomenon occurs for cases 87 and 71 where $\mathrm{Tor}^{\mathcal{OS}}_{4,6}(k, k)$ has dimension 16 for case 87 (but here the Ext-algebra is finitely presented), and the case 71 where $\mathrm{Tor}^{\mathcal{OS}}_{4,6}(k, k)$ has dimension 17 and the Ext-algebra is not finitely presented in that case. The differences in the $\mathrm{Tor}^{\mathcal{OS}}(k, k)$ mentioned above and the $R(z)$ can be found by the program Macaulay 2 [Grayson and Stillman 2008]. Furthermore, the $R^!(z)$ can be found by [Lima-Filho and Schenck 2006]. It remains to study the Koszul complex $R^! \otimes_k \mathrm{Hom}_k(R, k)$, mentioned earlier, which is not done here. □

Remark 6.2. We have also studied many of the 853 cases corresponding to graphs with seven vertices. Everything in sight leads to rational Hilbert series for the Ext-algebras.

Remark 6.3. When we lectured about this at Stockholm University, Jörgen Backelin made some interesting observations:

1. For graphs with five vertices, the only case of non-finitely presented Ext-algebras comes from the case in which you remove two disjoint edges from the complete graph on five vertices (case 19).
2. If you remove three disjoint edges in the complete graph on six vertices, you get case 109, which is the most complicated one for graphs of order 6.
3. This leads to a heuristic surmise that if you remove $[n/2]$ disjoint edges from the complete graph on n vertices ($n \geq 7$), then you should get a very interesting situation.

Remark 6.4. Here is another example, [Lima-Filho and Schenck 2006, Example 1.3], where G is the "one-skeleton of the Egyptian pyramid and the one-skeleton of a tetrahedron sharing a single triangle." We have $1/R^!(t) = (1-2t)^4(1-3t)$ and $R(t) = 1+11t+48t^2+103t^3+107t^4+42t^5$, leading to the formula

$$\frac{1}{P_R(x,y)} = 1 - 11x^2y^2 + 48x^3y^3 - x^2y^3 - 104x^3y^3 + 5x^3y^4 + 112x^4y^4 - 6x^4y^5 - 48\,x^5y^5,$$

i.e., this is another one of the cases in which (3–17) is true but $m^3 \neq 0$.

Furthermore, the global dimension of $R^!$ is 5, and the Ext-algebra is finitely generated but not finitely presented.

Remark 6.5. The preceding results show that the behavior of the graphic arrangements in the non-Koszul case are rather unpredictable (but the irrational case is rare and it is quite probable that it does not occur for graphic arrangements).

Remark 6.6. One of the referees has asked me to recall that for graphic arrangements, the Orlik–Solomon algebra is Koszul $\Leftrightarrow$ the graph is cordal $\Leftrightarrow$ the arrangement is supersolvable $\Leftrightarrow$ the Orlik–Solomon algebra has a quadratic Gröbner basis. For more details about this, see [Schenck and Suciu 2002, Part 6.3] and the literature cited there.

7. QUESTIONS OF MILNOR, GRIGORCHUK, ZELMANOV, DE LA HARPE, AND IRRATIONALITY

In Section 5, we presented two hyperplane arrangements: the Mac Lane arrangement ML and an easier variant mlease, in (5–1), both of which have irrational Hilbert series for the corresponding $R^!$. In Section 5 we also presented the three possibilities for irrational Hilbert series for $R^!$ when R is an arbitrary quotient of an exterior algebra in five variables with an ideal generated by three quadratic forms (the only possibilities): cases R_{12}, R_{20}, and R_{15}. It is seems difficult to achieve similar examples for hyperplane arrangements using five variables.

But mlease can be considered as a higher variant of R_{12}, and similarly, ML can be considered as a higher variant of R_{15}. Indeed, in this last case there are central elements in odd degrees ≥ 3, so that we get the exponents $2, 2, 2, \dots$ in the infinite product formula for case $R^!_{15}(t)$ in Section 5 when we have divided out the center. But so far, we have not found any hyperplane arrangement corresponding (or similar) to the irrational case R_{20} in Section 5.

If such a hyperplane arrangement existed, it would in particular lead to results about growth of groups and groups of finite width. Let me be more precise: First recall that if $\mathcal{A}$ is *any* finite complex hyperplane arrangement in $\mathbb{C}^n$ and if $G = G(\mathcal{A})$ is the fundamental group of the complement of the union of the corresponding hyperplanes in $\mathbb{C}^n$, then G is finitely presented; indeed, the complement has the homotopy type of a finite CW-complex [Orlik 1989, Proposition 5.1]. Let

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$$

be the descending lower central series of G defined inductively by $G_1 = G$ and $G_k = [G_{k-1}, G_1]$ (for $k \geq 2$). We have a structure of a graded Lie ring (which can have torsion),

$$\operatorname{gr}(G) = \bigoplus_{i\geq 1} \frac{G_i}{G_{i+1}},$$

where the graded Lie structure is defined as follows: Let $\bar{x}$ and $\bar{y}$ be elements in G_i/G_{i+1} and G_j/G_{j+1} respectively, and let them be represented by x and y in G_i and G_j. Then $xyx^{-1}y^{-1}$ lies in G_{i+j}, and its image in G_{i+j}/G_{i+j+1} is denoted by $[\bar{x}, \bar{y}]$. It was proved in [Kohno 1983] that we have an isomorphism of graded Lie algebras

$$\operatorname{gr}(G) \otimes_Z \mathbf{Q} \simeq \bigoplus \eta^i, \tag{7–1}$$

where η is the Lie algebra of primitive elements in the subalgebra generated by $\operatorname{Ext}^1_{\mathrm{OS}(\mathcal{A})}(Q,Q)$ of the Yoneda Ext-algebra of the Orlik–Solomon algebra of $\mathcal{A}$.

Now G_{ML} and G_{mlease} are finitely presented groups. Therefore, if we could find a hyperplane arrangement (probably in high embedding dimension) corresponding to the case R_{20} or similar, we would have at the same time found a *finitely presented group* G such that the Lie algebra $\mathrm{gr}(G) \otimes_Z \mathbf{Q}$ is infinite and of finite width (i.e., the dimensions of the η^i in (7–1) are bounded (for further terminology and results we refer to the surveys [de la Harpe 2000], [Bartholdi and Grigorchuk 2000], [Grigorchuk and Pak 2006] and the literature cited there)).

If so, one would probably be close to *finitely presented* groups having intermediate growth. Note that it is not expected that such groups exist (cf. [Grigorchuk and Pak 2006, Conjecture 11.3], where it is stated two lines earlier that the existence of such groups is a major open problem in the field, and [de la Harpe 2000, research problem VI.63]). But our two groups corresponding to the Mac Lane arrangement ML and its easier variant mlease give at least *finitely presented groups* with irrational growth series (Hilbert series) of $U(\mathrm{gr}(G) \otimes_Z \mathbf{Q})$.

8. FINAL REMARKS

It is interesting to note that about 32 years ago, in writing his thesis, Jean-Michel Lemaire [Lemaire 1974] was inspired by the Stallings group-theoretic example [Stallings 1963] (now used again in [Dimca et al. 2006]) to construct a finite simply connected CW-complex X such that the homology algebra of the loop space $H_*(\Omega X, \mathbf{Q})$ is not finitely presented (not even finitely generated). In [Roos 1979], we used a general recipe that in particular could be used to translate Lemaire's results to local commutative ring theory to obtain a local ring (R, m) such that the Yoneda Ext-algebra $\mathrm{Ext}^*_R(k, k)$ was not finitely generated, thereby solving in the negative a problem by Gerson Levin [Levin 1974].

The example in Section 2 is just a skew-commutative variant of my example in [Roos 1979], but with a quick direct proof, which, it is to be hoped, should satisfy mathematicians working with arrangements of hyperplanes. The theory of Section 3 above, combined with more difficult variants of the later developments in the 1980s about a question of Serre–Kaplansky [Lemaire 1980], are here shown to be useful for solving the second problem of Denham–Suciu [Denham and Suciu 2006].

REFERENCES

[Anick 1982] David J. Anick. "A Counterexample to a Conjecture of Serre." *Ann. of Math. (2)* 115:1 (1982), 1–33, and comment, "A Counterexample to a Conjecture of Serre." *Ann. of Math. (2)* 116:3 (1982), 661.

[Backelin et al. 07] Jörgen Backelin et al. "BERGMAN, a Programme for Non-commutative Gröbner Basis Calculations." Available online (http://servus.math.su.se/bergman/).

[Bartholdi and Grigorchuk 2000] Laurent Bartholdi and Rostislav Grigorchuk. "Lie Methods in Growth of Groups and Groups of Finite Width." In *London Math. Soc. Lecture Notes* 275, pp. 1–27. Cambridge: Cambridge Univ. Press, 2000.

[Bøgvad 1995] Rikard Bøgvad. "Some Homogeneous Coordinate Rings That Are Koszul Algebras." arXiv:math.AG/9501011 v2, 1995.

[Denham and Suciu 2006] Graham Denham and Alexander I. Suciu. "On the Homotopy Lie Algebra of an Arrangement." *Michigan Mathematical Journal* 54:2 (2006), 319–340.

[Dimca et al. 2006] Alexandru Dimca, Stefan Papadima, and Alexander I. Suciu. "Non-finiteness Properties of Fundamental Groups of Smooth Projective Varieties." arXiv:math.AG/0609456, 2006.

[Eisenbud and Koh 1994] David Eisenbud and Jee Koh. "Nets of Alternating Matrices and the Linear Syzygy Conjectures." *Adv. Math.* 106:1 (1994), 1–35.

[Grayson and Stillman 2008] Daniel R. Grayson and Michael E. Stillman. "Macaulay 2: A Software System for Research in Algebraic Geometry." Available online (http://www.math.uiuc.edu/Macaulay2/), 2008.

[Grigorchuk and Pak 2006] Rostislav Grigorchuk and Igor Pak. "Groups of Intermediate Growth: An Introduction for Beginners." arXiv:math.GR/0607384, 2006.

[Harary 1969] Frank Harary. *Graph Theory*. Reading: Addison-Wesley, 1969.

[de la Harpe 2000] Pierre de la Harpe. *Topics in Geometric Group Theory*. Chicago: University of Chicago Press, 2000.

[Kohno 1983] T. Kohno. "On the Holonomy Lie Algebra and the Nilpotent Completion of the Fundamental Group of the Complement of Hypersurfaces." *Nagoya Math. J.* 92 (1983), 21–37.

[Lemaire 1974] Jean-Michel Lemaire. *Algèbres connexes et homologie des espaces de lacets*, Lecture Notes in Mathematics, 422. New York: Springer-Verlag, 1974.

[Lemaire 1980] Jean-Michel Lemaire. "Anneaux locaux et espaces de lacets à séries de Poincaré irrationnelles (d'après Anick, Roos, etc.)." In *Bourbaki Seminar, Vol. 1980/81*, pp. 149–156, Lecture Notes in Math. 901. New York: Springer, 1981.

[Levin 1974] Gerson Levin. "Two Conjectures in the Homology of Local Rings." *J. Algebra* 30 (1974), 56–74.

[Levin 1980] Gerson Levin. "Large Homomorphisms of Local Rings. *Math. Scand.* 46 (1980), 209–215.

[Levin 1985] Gerson Levin. "Modules and Golod Homomorphisms." *J. Pure Appl. Algebra* 38:2–3 (1985), 299–304.

[Lima-Filho and Schenck 2006] Paulo Lima-Filho and Hal Schenck. "Holonomy Lie Algebras and the LCS-Formula for Subarrangements of A_n." Preprint available online (http://www.math.tamu.edu/~schenck/glcs.pdf), 2006.

[Löfwall 1985] Clas Löfwall. "On the Homotopy Lie Algebra of a Local Ring." *J. Pure Appl. Algebra* 38:2–3 (1985), 305–312.

[Löfwall 1986] Clas Löfwall. "On the Subalgebra Generated by the One-Dimensional Elements in the Yoneda Ext-Algebra." In *Algebra, Algebraic Topology and Their Interactions (Stockholm, 1983)* pp. 291–338, Lecture Notes in Math. 1183. Berlin: Springer, 1986.

[Löfwall 1994] Clas Löfwall. "Appendix B." In [Roos 1994].

[Löfwall 2007] Clas Löfwall. "`liedim.m`: A Mathematica Programme to Calculate (among Other Things) the Ranks of a Finitely Presented Graded Lie Algebra." Available online (http://www.math.su.se/~clas/liedim/), 2001–2007.

[Löfwall and Roos 1980] Clas Löfwall and Jan-Erik Roos. "Cohomologie des algèbres de Lie graduées et séries de Poincaré–Betti non rationnelles." *C. R. Acad. Sci. Paris Sér. A-B* 290:16 (1980), A733–A736.

[Löfwall and Roos 1997] Clas Löfwall. and Jan-Erik Roos. "A Non-nilpotent 1-2-Presented Graded Hopf Algebra Whose Hilbert Series Converges in the Unit Circle." *Adv. Math.* 130:2 (1997), 161–200.

[Milnor and Moore 1965] John W. Milnor and John C. Moore. "On the Structure of Hopf Algebras." *Ann. of Math. (2)* 81 (1965), 211–264.

[Orlik 1989] Peter Orlik. *Introduction to Arrangements*, CBMS Regional Conference Series in Mathematics, 72. Providence: American Mathematical Society, 1989.

[Papadima and Suciu 2006] Stefan Papadima and Alexander I. Suciu. "When Does the Associated Graded Lie Algebra of an Arrangement Group Decompose?" *Comment. Math. Helv.* 81 (2006), 859–875.

[Roos 1979] Jan-Erik Roos. "Relations between Poincaré–Betti Series of Loop Spaces and of Local Rings." In *Séminaire d'Algèbre Paul Dubreil 31ème année (Paris, 1977–1978)*, pp. 285–322, Lecture Notes in Math. 740. Berlin: Springer, 1979.

[Roos 1981] Jan-Erik Roos. "Homology of Loop Spaces and of Local Rings." In *18th Scandinavian Congress of Mathematicians (Aarhus, 1980)*, pp. 441–468, Progr. Math. 11. Boston: Birkhäuser, 1981.

[Roos 1982] Jan-Erik Roos. "On the Use of Graded Lie Algebras in the Theory of Local Rings." In *Commutative Algebra: Durham 1981 (Durham, 1981)*, pp. 204–230, London Math. Soc. Lecture Note Ser. 72. Cambridge: Cambridge Univ. Press, 1982.

[Roos 1994] Jan-Erik Roos. "A Computer-Aided Study of the Graded Lie Algebra of a Local Commutative Noetherian Ring." *J. Pure Appl. Algebra* 91:1–3 (1994), 255–315.

[Roos 1996] Roos, Jan-Erik, "On Computer-Assisted Research in Homological Algebra." *Mathematics and Computers in Simulation* 42 (1996), 475–490.

[Roos 2000] Jan-Erik Roos. "Homological Properties of Quotients of Exterior Algebras." In preparation. Abstract available at *Abstracts Amer. Math. Soc.* 21 (2000), 50–51.

[Roos 2008] Jan-Erik Roos. "Irrationality of Hyperplane Arrangements and Growth of Groups." In preparation, 2008.

[Schenck and Suciu 2002] Hal Schenck and Alexander Suciu. "Lower Central Series and Free Resolutions of Hyperplane Arrangements." *Trans. Amer. Math. Soc.* 354 (2002), 3409–3433.

[Schneider 1997] Czaba Schneider. "Computing Nilpotent Quotients in Finitely Presented Lie Rings." *Discrete Mathematics and Theoretical Computer Science* 1:1 (1997), 1–16.

[Shelton and Yuzvinsky 1997] Brad Shelton and Sergey Yuzvinsky. "Koszul Algebras from Graphs and Hyperplane Arrangements." *J. London Math. Soc. (2)* 56:3 (1997), 477–490.

[Sköldberg 1999] Emil Sköldberg. "Monomial Golod Quotients of Exterior Algebras. *J. Algebra* 218 (1999) 183–189.

[Stallings 1963] John Stallings. "A Finitely Presented Group Whose 3-Dimensional Integral Homology Is Not Finitely Generated." *Amer. J. Math.* 85 (1963) 541–543.

[Suciu 2000] Alexander I. Suciu. "Fundamental Groups of Line Arrangements: Enumerative Aspects." In *Advances in Algebraic Geometry Motivated by Physics (Lowell, MA, 2000)*, pp. 43–79, Contemp. Math. 276. Providence: Amer. Math. Soc., 2001.

Jan-Erik Roos, Department of Mathematics, Stockholm University, SE–106 91 Stockholm, Sweden (jeroos@math.su.se)

Received May 27, 2007; accepted in revised form September 26, 2007.