Tropical Polytopes and Cellular Resolutions

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CONTENTS

Tropical polytopes are images of polytopes in an affine space over the Puiseux series field under the degree map. This viewpoint gives rise to a family of cellular resolutions of monomial ideals that generalize the hull complex of Bayer and Sturmfels [Bayer and Sturmfels 98], instances of which improve upon the hull resolution in the sense of being smaller. We also suggest a new definition of a face of a tropical polytope, which has nicer properties than previous definitions; we give examples and provide many conjectures and directions for further research in this area.

1. INTRODUCTION

In this paper, continuing the work in [Develin and Sturmfels 04] and [Joswig 05], we investigate tropical polytopes, which are the natural tropical analogue of ordinary polytopes in Euclidean space. We give examples of strange behavior in this genus of objects and ponder their facet descriptions; we also describe how tropical polytopes embody a family of resolutions of monomial ideals, which includes the hull complex of [Bayer and Sturmfels 98].

The *tropical semiring* is given by the real numbers $\mathbb{R}$ together with the operations of tropical addition $\oplus$ and tropical multiplication $\odot$, defined by $a \oplus b = \max(a, b)$ and $a \odot b = a + b$. Let $\mathbb{R}^d$ be a tropical semimodule under tropical addition $\oplus$ (which takes the componentwise maximum of two vectors) and tropical scalar multiplication $\odot$ (which adds a constant to each coordinate). It proves more convenient to mod out by tropical scalar multiplication and work in *tropical projective space* $\mathbb{TP}^{d-1} := \mathbb{R}^d/(1, 1, \ldots, 1)$, where we will typically choose the coordinatization given by $x_1 = 0$.

The *tropical convex hull* of a set of points V in tropical projective space, denoted by $\operatorname{tconv}(V)$, consists of all tropical linear combinations of those points (not just those with coefficients between 0 and 1, since 0 and 1 have no meaning in the tropics). A *tropical polytope* is the tropical convex hull of a finite set of points; it consists of all tropical linear combinations $c_1 \odot v_1 \oplus c_2 \odot v_2 \cdots \oplus c_k \odot v_k$.

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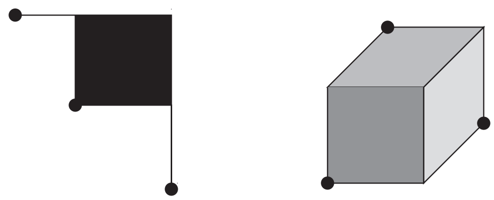

FIGURE 1. Two examples of tropical polytopes in two- and three-dimensional space, respectively. The polytope on the right consists of the union of the three indicated facets of a cube. The vertices are marked.

Tropical polytopes have a natural decomposition as complexes of ordinary polytopes; see [Develin and Sturmfels 04] for more details. Some examples of tropical polytopes are shown in Figure 1. They enjoy many useful properties, such as being contractible and having a tropical Farkas lemma.

This tropical Farkas lemma says that any point not in a tropical polytope can be separated from that polytope by a tropical hyperplane. A *tropical hyperplane* is given by a linear form $\bigoplus c_i \odot x_i$; this form is said to vanish if the maximum encoded by this tropical expression is achieved at least twice, and the corresponding tropical hyperplane is defined to be the locus of vanishing. This is a fan with apex $(-c_1, \ldots, -c_d)$, and is polar to the simplex given by the convex hull of the standard basis vectors e_i (of which there are d living in $\mathbb{TP}^{d-1}$). Thus, each tropical hyperplane divides $\mathbb{TP}^{d-1}$ into d *sectors* indexed by the d coordinates; see Figure 2. A point x lies in the (closed) sector indexed by coordinate i if $x_i - c_i$ is maximized among all $x_j - c_j$. Note that all hyperplanes are translates of each other, meaning that all that is needed to specify a hyperplane is its apex.

One manner in which tropical (discrete, algebraic) geometry arises naturally is as the image of ordinary (discrete, algebraic) geometry over the Puiseux series field with real exponents $K := \mathbb{R}[[t^\alpha]]$ (consisting of locally finite power series with a highest exponent and real coefficients and exponents) under the degree map $\deg : K \to \mathbb{R}$ sending an element to its leading (highest) exponent. (Sometimes it proves more convenient to define K as the Puiseux series field with rational exponents, in which case the tropical objects in $\mathbb{R}^d$ or $\mathbb{TP}^{d-1}$ are the topological closures of the images of the lifted objects in K^d.) In fact, tropical polytopes are images of polytopes in K^d. We will discuss this point of view in Section 2, and as an application, we will present a family of cellular resolutions of monomial ideals in Section 3.

Michael Joswig, in his seminal paper [Joswig 05], used these hyperplanes to propose a face structure of tropical polytopes; in Section 4, we investigate this structure and raise some issues with it, presenting an alternative definition that is both more intuitive and more practical. We will discuss more examples in Section 5 and future directions in Section 6.

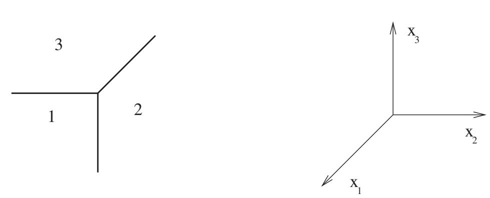

FIGURE 2. A tropical hyperplane in $\mathbb{TP}^2$.

2. POLYTOPES OVER THE PUISEUX SERIES FIELD

Let $K = \mathbb{R}[[t^\alpha]]$ be the Puiseux series field with real exponents, as defined above. It is naturally an ordered field, where $a < b$ if the leading coefficient of $b - a$ is positive; its positive elements K^+ constitute the set of all elements with positive leading coefficient. As such, the usual theory of discrete geometry applies in K^d, and in particular we can define the convex hull of a point set as the set of all affine combinations of the points. Indeed, tropical polytopes are simply the images of objects up above:

Proposition 2.1. *Suppose $P = \operatorname{tconv}(v_1, \ldots, v_k) \subset \mathbb{TP}^{d-1}$, where each v_i has first coordinate zero. Define lifts $\overline{v_i} \in \deg^{-1}(v_i)$ such that all leading coefficients of the $\overline{v_i}$ are positive, and define $\overline{P} = \operatorname{conv}(\{\overline{v_i}\})$. Then $P = \deg(\overline{P})$.*

Proof: First, we show that $P \subset \deg(\operatorname{conv}(\{\overline{v_i}\}))$. Suppose that we have a point $x = \bigoplus c_i \odot v_i \in P$. Since $P \subset \mathbb{TP}^{d-1}$, we can add a constant to each c_i such that the largest c_i, without loss of generality c_1, is equal to 0. It is easy to lift the c_i's to $\overline{c_i} \in K$ with $\deg(\overline{c_i}) = c_i$ such that $\overline{c_i} > 0$ and $\sum \overline{c_i} = 1$: lift every $c_i < 0$ to t^{c_i}, and lift $c_i = 0$ to $(1 - \sum_{c_j < 0} t^{c_j})/|\{j \mid c_j = 0\}|$.

But then we claim that $\deg(\sum \overline{c_i}\overline{v_i}) = x$. This follows immediately: since there is no cancellation of the leading terms, since all leading coefficients of the $\overline{c_i}$'s and $\overline{v_i}$'s are positive, for each coordinate j we have

$$\begin{aligned}\deg\left(\sum_i \overline{c_{ij}}\,\overline{v_{ij}}\right) &= \max(\deg(\overline{c_{ij}}\,\overline{v_{ij}})) \\ &= \max((c_i)_j + (v_i)_j) = x_j,\end{aligned}$$

as desired.

For the reverse direction, using the same logic, it is easy to see that $\deg(\sum \alpha_i \overline{v_i}) = \bigoplus \deg(\alpha_i) \odot v_i$, again due to the lack of cancellation of leading terms, which shows that $\deg(\mathrm{conv}(\{\overline{v_i}\})) \subset P$, completing the proof. $\square$

We call the polytope $\overline{P}$ a *lift* of P. Note that we can take any lifts of the vertices of the polytope, as long as the lifted points have positive leading coefficients; the lift operation always preserves convex hulls. As an aside, the stipulation that each v_i have first coordinate zero is merely for simplicity; we could ignore this and just look at the facial structure of the cone generated by the lifts of the vertices. Giving each v_i first coordinate zero amounts to slicing this cone with the hyperplane $\overline{x_1} = 1$.

In a sense, each of these lifts yields a candidate for the face lattice of the tropical polytope P. The problem is that when the points are not in (tropically) general position, the lifts can have different combinatorial structures.

Let $A = [a_{ij}]$ be a $d \times d$ matrix whose columns are considered as d points in $\mathbb{TP}^{d-1}$. The *tropical determinant* of A is defined by the formula $\oplus_{\sigma \in S_d}(\odot a_{i,\sigma_i})$, where S_d denotes the group of permutations of d elements. We say that A is *tropically nonsingular* if the maximum in its tropical determinant is attained uniquely. In this case, the *tropical sign* of A is defined (as in [Joswig 05]) to be the sign of the permutation that attains the maximum. Otherwise, the tropical sign is defined to be 0. Let $\overline{A}$ be a $d \times d$ matrix with entries in K^+ whose degree is A. If the tropical sign of A is not zero, then the sign of the unique permutation that attains the maximum in the tropical determinant is also the sign of the leading term of the determinant of $\overline{A}$. This observation leads to the following.

Lemma 2.2. *For a tropical polytope P with at least d vertices in $\mathbb{TP}^{d-1}$, the oriented matroid structure of any lift $\overline{P}$ must refine the partial oriented matroid structure of P given by the tropical signs on each subset of the d vertices.*

On the other hand, there may be a point configuration whose oriented matroid refines the partial oriented matroid of vertices of P but cannot be obtained as a lift. The oriented matroid of "the model" that we will see in Example 4.3 is the same as that of a square pyramid with two points at the cone point, but this point configuration cannot be attained as a lift, since distinct points must be lifted to distinct points.

Proposition 2.3. *If a tropical polytope P with at least d vertices in $\mathbb{TP}^{d-1}$ is in general position, then all lifts $\overline{P}$ are simplicial and have the same oriented matroid structure.*

Proof: Let V be a matrix whose columns are vertices of P. The assumption that P is in general position (in the sense of [Block and Yu 06, Proposition 4]) implies that all maximal $(d \times d)$ submatrices of V are tropically nonsingular. By the previous lemma, the tropical signs of these submatrices determine the chirotope of all the lifts $\overline{P}$. Moreover, since these signs are all nonzero, the lifts are simplicial. $\square$

It is still possible for lifts of a nongeneric tropical polytope to be simplicial and have the same face lattice. For an example, see the tropical $(2, 4)$-hypersimplex (Example 5.2).

The geometric objects that form the relevant atoms of a theory of tropical faces are the images of faces of lifts under the degree map, which we call *fatoms* (face atoms). In this paper, we will discuss different ways to combine these to form faces. A crucial step is the following, which provides the link between lifted hyperplanes and tropical hyperplanes.

Proposition 2.4. *Let $H = \{X \in K^d : f(X) := f_1X_1 + \cdots + f_dX_d = 0\} \subset K^d$ be a hyperplane, and define $H^+ := \{X \in K^d : f(X) \geq 0\}$. The image of H under the degree map is a tropical hyperplane with apex $(-\deg(f_1), \ldots, -\deg(f_d))$, the image of $H^+ \cap (K^+)^d$ consists of the union of the closed sectors indexed by $\{i : f_i > 0\}$, and the image of $H \cap (K^+)^d$ is the boundary of this tropical half-space. (If some f_i is equal to 0, then the apex of this tropical hyperplane has ith coordinate equal to infinity.)*

Proof: Let $z = \deg(X)$. If $f(X) = 0$, then the leading term of $f(X)$ has to cancel. The leading exponent of f_iX_i is $\deg(f_i) + z_i$, so the maximum of these d expressions has to be achieved at least twice. This means that z is in the indicated tropical hyperplane. Conversely, if z is in the tropical hyperplane, then $f(t^z)$ has ties in the leading terms, and it is trivial to adjust leading coefficients and fill in subleading terms to find a lift of z that lies on H.

Suppose now that $X \in H^+ \cap (K^+)^d$. If z is not in any closed sector indexed by $f_i > 0$, then each f_i for which the maximum of all $\deg(f_i) + z_i$ is achieved has negative leading coefficient. Since these are the terms that contribute to the leading term of $f(X)$, $f(X)$ must be negative, a contradiction. Conversely, if z is in a closed sector indexed by, without loss of generality, $f_1 > 0$, then $\deg(f_i) + z_i$ is maximized by $i = 1$. Therefore $f_1 X_1$ contributes to the leading term of $f(X)$. Finding a lift with sufficiently large leading coefficient of X_1 then ensures that $f(X) > 0$. This completes the proof of the second assertion.

The third statement can be proven the same way as the first, and by noting that the leading term of $f(X)$ can cancel for $X \in (K^+)^d$ if and only if $\deg(X)$ lies in both the sectors $\{i : f_i > 0\}$ and sectors $\{j : f_j < 0\}$. $\square$

Armed with this, we can prove a crucial step in this discussion, namely that the boundary of the lifted polytope indeed maps to the boundary of the tropical polytope.

Proposition 2.5. *Let $\overline{P}$ be a lift of the tropical polytope $P \subset \mathbb{TP}^{d-1}$. Then the image of the boundary of $\overline{P}$ under the degree map is precisely the boundary of P.*

Proof: Every point in the boundary of $\overline{P}$ lies in some facet. Since the degree map preserves convex hulls, it must map to a point in the convex hull of the vertices of this facet. However, by Proposition 2.4, these vertices all lie in a tropical hyperplane that bounds P (namely the image of the hyperplane defining the facet of the lifted polytope), and since tropical hyperplanes are convex, the image of the point in question must also lie on the boundary of P.

For the converse, suppose we have some point v on the boundary of P. This point is the image of some point in $\overline{P}$, since lifts preserve convex hulls; however, a priori, this point need not lie on the boundary. Consider a tropical hyperplane H with apex v. Not every open sector of this hyperplane contains a point in P, since that would imply that v is in the interior of P. Therefore, we can partition the sectors of this hyperplane into a pair (A, B) such that the A-sectors contain P, with B nonempty. Suppose without loss of generality that $A = \{1, \dots, k\}$ and $B = \{k+1, \dots, d\}$. Define a linear functional f on the lift space via

$$f = c_1 t^{-v_1} x_1 + \cdots + c_k t^{-v_k} x_k - t^{-v_{k+1}} x_{k+1} - \cdots - t^{v_d} x_d,$$

where the c_i's are positive real constants. Since P lies in the union of the A-sectors of H, every vertex w of P has $w_i - v_i$ maximized for some $i \in [k]$. Therefore, the leading term of the expression given by $f(\overline{w})$ contains some positive summand from among the first k terms of f. So, if we make the c_i's large enough, we can ensure that $f(\overline{w}) > 0$ for all vertices $\overline{w}$ of $\overline{P}$. Fix a set of such c_i's.

But we can easily lift the point v to a point $\overline{v}$ with $f(\overline{v} = 0)$, for instance, simply taking $\overline{v}_i = \frac{1}{c_i} t^{v_i}$ for $i \in [k]$ and $\overline{v}_i = \frac{k}{d-k} t^{v_i}$ otherwise. Hence v has a lift that lies outside of $\overline{P}$.

So, we have constructed a lift of v that lies inside of $\overline{P}$ and a lift that lies outside of it. The line segment between these two lifts, each of which has positive leading coefficient, consists entirely of other lifts of v, and must intersect the boundary of $\overline{P}$ somewhere, completing the proof. $\square$

The degree map is well behaved; for instance, it also does not increase the dimension of a face.

Proposition 2.6. *If $\overline{F}$ is a k-face of $\overline{P}$, then the image of $\overline{F}$ under the degree map is at most k-dimensional.*

Proof: Suppose we have a k-face $\overline{F}$ of $\overline{P}$. Triangulate this with its original vertex set, dividing it into a number of k-simplices that are the convex hulls of $k+1$ lifts of vertices of P. The image of each such simplex under the degree map, by Proposition 2.1, is the convex hull of $k+1$ points in $\mathbb{TP}^{d-1}$, and is therefore at most k-dimensional (see, for example, [Develin and Sturmfels 04]). Therefore, the image of $\overline{F}$ under the degree map is the finite union of k-dimensional things, and is therefore k-dimensional. $\square$

The fatoms for which the dimension is preserved under this map are particularly crucial to our theory.

Definition 2.7. A *k-fatom* of a tropical polytope P is a k-dimensional piece of the boundary of P that is the image of a k-dimensional face $\overline{F}$ of some lift $\overline{P}$ of P.

Next, we discuss the dual formulation of a tropical polytope. Tropical polytopes are typically considered as the convex hulls of point sets, and when expressed in this fashion they obtain a natural polyhedral decomposition as detailed in [Develin and Sturmfels 04]. Joswig [Joswig 05] noted that each tropical polytope is the bounded intersection of a finite number of half-spaces, and that the apices of these half-spaces are drawn from an easily computable set (the pseudovertices of the polytope, which are the vertices of the polyhedral decomposition of [Develin and Sturmfels 04]).

Our goal is to come up with a reasonably succinct list of tropical half-spaces whose intersection is P. One natural attempt is to take a lift of the polytope, find the facet-defining half-spaces whose intersection is that lift, and then map the whole setup down to $\mathbb{TP}^{d-1}$. This does not work in general; the problem is that the intersection of the images of the half-spaces under the degree map can be larger than the degree of the intersections. In other words, there could be a point in $\mathbb{TP}^{d-1}$ that does not lift anywhere in $\overline{P}$ but that lifts to a (different) point inside each facet-defining half-space of $\overline{P}$; see Example 4.2 for an example. However, this process does work for the interior of the polytope:

Proposition 2.8. *Let $P \subset \mathbb{TP}^{d-1}$ be a full-dimensional tropical polytope. Take a generic lift $\overline{P}$, and consider the images of its facet-defining half-spaces, which are tropical half-spaces in $\mathbb{TP}^{d-1}$. Then the intersection of the interiors of these tropical half-spaces is precisely the interior of P.*

Proof: First, note that P is contained in each tropical half-space, since $\overline{P}$ is contained in each of its facet-defining half-spaces. If we have a point in the interior of P, there is a ball surrounding it that is entirely in P. It therefore cannot be on the boundary of any tropical half-space.

For the reverse direction, suppose we have a point $x = (x_1, \ldots, x_d)$ in the interior of each tropical half-space. Since the boundary of P is contained in the union of the boundaries of the tropical half-spaces by Proposition 2.5, x cannot lie in the boundary of P. So we need only to show that $x \in P$. First, we claim that any lift of x lies in each lifted half-space. Suppose without loss of generality that the half-space in question consists of sectors $\{1, \ldots, k\}$ of the hyperplane with apex $(0, \ldots, 0)$. Since x is in the interior of the tropical half-space, the maximum of $x_1, \ldots, x_d$ is achieved only in the first k coefficients.

The lifted half-space is defined by $f_1 X_1 + \cdots + f_d X_d \geq 0$, where each f_i has degree zero. Let us evaluate this on any lift $\overline{x}$ of x. Since $\deg(X_i) = x_i$, the leading terms of these d terms have degrees $x_1, \ldots, x_d$, and thus only the first k terms have a chance of contributing to the leading term. Furthermore, we know that $f_i > 0$ for $i \in [k]$, since the half-space is to map to the union of the first k sectors, and $X_i > 0$ by the definition of a lift. Therefore, the leading term of $f_i X_i$ is positive for $i \in [k]$, and so the leading term of $f(\overline{x})$ is positive; therefore, $\overline{x}$ is in this lifted half-space.

Therefore, any lift $\overline{x}$ is in the interior of all facet-defining half-spaces of $\overline{P}$, and is therefore in (the interior of) $\overline{P}$. So $x \in \deg(\overline{P}) = P$, which completes the proof. □

If P is pure, this process often results in an actual half-space description of P.

Proposition 2.9. *Suppose that the intersection of all tropical half-spaces in Proposition 2.8, which is a polyhedral complex, is pure. Then it is equal to P.*

Proof: The intersection contains P, and their interiors are the same. Since the intersection is pure, it is equal to the closure of its interior; P contains this closure, since it is closed, and so the two must be the same. □

Indeed, we conjecture that this is true in general:

Conjecture 2.10. *If P is pure, then the half-spaces from a generic lift $\overline{P}$ of P map to tropical half-spaces whose intersection is P itself. If in addition, P is generic, then any lift works.*

Conjecture 2.11. *A tropical polytope P is pure and full-dimensional if and only if it has a half-space description in which the apices of these half-spaces are in general position with respect to the tropical semiring $(\mathbb{R}, \min, +)$.*

In ordinary polytope theory, getting a half-space description of a polytope is important for many reasons. Perhaps foremost is that it provides a way for checking whether a point is in a polytope; simply evaluate the relevant linear functionals on that point. However, in tropical geometry, it is easy to check whether a point is in a polytope from the vertex description:

Proposition 2.12. [Develin and Sturmfels 04] *Let P be the tropical convex hull of $v_1, \ldots, v_k \in \mathbb{TP}^{d-1}$, and let x be a point in $\mathbb{TP}^{d-1}$. For each v_i, define $c_i = \min_j(x_{ij} - v_{ij})$. Then $x \in P$ if and only if $\sum c_i v_i = x$.*

Because of this proposition, the half-space description we search for is less crucial tropically than it is normally. However, it is still important; for instance, it is easy to intersect two polytopes given by half-space descriptions (take the union of the sets of half-spaces in question). At press time there is no shortcut like the above proposition for intersecting tropical polytopes, so a half-space description would allow us to accomplish this.

Another natural question to ask is the following:

Question 2.13. How does one obtain a vertex description of a tropical polytope from its hyperplane description?

If the tropical polytope is pure, then lifting each tropical hyperplane to a hyperplane up above does the trick; this will create some polytope $\overline{P}$ in K^d, and by Proposition 2.9 its image in $\mathbb{TP}^{d-1}$ will be exactly equal to P, and the vertices of P will be drawn from among the degrees of the vertices of $\overline{P}$ (there may be some redundancy, especially if there is redundancy in the hyperplane description). If the polytope is not pure, it is trickier. Computing the extreme points of a tropically convex object is relatively simple geometrically—they are the points for which a closed sector of the hyperplane with that apex contains only P—but can these be computed directly from the hyperplane description?

3. CELLULAR RESOLUTIONS OF MONOMIAL IDEALS

In [Bayer and Sturmfels 98], the authors defined the hull complex, a complex that yields a resolution of a monomial ideal I. The construction is simple: lift each generator $x^{\mathbf{a}} \in k[x_1, \ldots, x_{d-1}]$ to a vector $t^{\mathbf{a}} \subset \mathbb{R}^{d-1}$, where t is some large number. Form then a polyhedron P_t by adding the positive orthant to the convex hull of these vectors. For t sufficiently large, Bayer and Sturmfels showed that this polyhedron has constant combinatorial type, and that the complex of bounded faces of P_t yields a cellular resolution of the monomial ideal. We give a brief review of the process that leads from a complex to a resolution.

Given any polytopal complex P with the vertices labeled by generators $x^{\mathbf{a}}$, we label each face by the least common multiple of the generators corresponding to its vertices. We then form a chain complex as follows: each face F of the polytopal complex corresponds to a generator lying in homological degree equal to its dimension. This chain complex P_X is to be graded, with the degree of F equal to the label of F (which we denote by x_F); each generator maps to the appropriately homogenized signed sum of the generators corresponding to its facets, i.e.,

$$\partial F = \sum_{F' \in F} \pm \frac{x_F}{x_{F'}} F',$$

where the sum runs over all facets F' of F and the signs are chosen so that $\partial^2 = 0$.

Given any multidegree $\mathbf{b}$, the complex $X_{\leq \mathbf{b}}$ is defined to be the subcomplex of this chain complex obtained by taking the generators with x_F dividing $x^{\mathbf{b}}$. The key result about cellular resolutions is the following:

Proposition 3.1. [Bayer and Sturmfels 98] *The chain complex P_X is a free resolution (called a cellular resolution) of I if and only if the subcomplex $X_{\leq \boldsymbol{b}}$ is acyclic for all $b \in \mathbb{Z}^n$.*

Bayer and Sturmfels go on to prove that the hull complex satisfies this condition. Indeed, their proof works essentially verbatim if we lift each generator $x^{\mathbf{a}}$ to any vector in K^d that maps to $\mathbf{a}$ under the degree map, i.e., a d-vector of elements in the Puiseux series field with specified leading exponents. Note that we can evaluate the facial structure of the resulting polyhedron in K^d, without having to plug in a specific value for t.

Theorem 3.2. *Let P be the tropical polytope given by the convex hull of the points $(0, \boldsymbol{a})$, where $\boldsymbol{a}$ ranges over the exponent vectors of the generators of a monomial ideal I in $k[x_1, \ldots, x_{d-1}]$. Take any lift $\overline{P}$ of P and add the positive orthant in the last $d-1$ coordinates, $\{0\} \times (K^+)^{d-1}$, to $\overline{P}$ to obtain a polyhedron $\overline{P}^+$. Then the complex of bounded faces of $\overline{P}^+$ yields a cellular resolution of I.*

Proof: We have only to check that for each $\mathbf{b} = (b_1, \ldots, b_{d-1})$, the bounded faces of $\overline{P}^+$ with labels dividing $\mathbf{b}$ form an acyclic complex. Let the coordinates of the space K^d that $\overline{P}^+$ lives in be given by $x_0, \ldots, x_{d-1}$, and consider the linear functional on K^d given by $f(x) = t^{-b_1}x_1 + \cdots + t^{-b_{d-1}}$. Then on a given vertex of $\overline{P}^+$, $f(x)$ has positive leading exponent if and only if that generator does not divide $x^{\mathbf{b}}$. Thus the bounded faces in the half-space given by $f(x) \leq t^{1/2}$ are precisely those in the subcomplex $X_{\leq \mathbf{b}}$. Applying a projective transformation sending this hyperplane to infinity yields $X_{\leq \mathbf{b}}$ as the complex of bounded faces of some polytope, which is acyclic. Therefore, each $X_{\leq \mathbf{b}}$ is acyclic, and thus by Proposition 3.1, this complex yields a cellular resolution of I. □

These cellular resolutions can be smaller than the hull resolution. For instance, in a previous paper [Develin 04], the first author showed that no face of the hull complex can have more than $(d-1)!$ vertices; this is not true for these cellular resolutions. For example, if we had seventeen million generators of a three-variable ideal all lying on the same tropical hyperplane, we could find a lift such that these all lay on a single facet.

These lifts include the hull complex as a special case, where each point is simply lifted to $t^{\mathbf{a}}$. Like the hull complex, each of them includes the *Scarf complex* as a subcomplex; this complex is a simplicial complex where a set of vertices forms a face if the corresponding set of generators has a unique least common multiple among all least common multiples of sets of generators.

Proposition 3.3. *Suppose that a set S of the generators of I has a unique least common multiple. Then S forms a (simplicial) bounded face in every $\overline{P}^+$.*

Proof: The proof is by induction. We claim that for every $v \in S$, $S \setminus \{v\}$ has a unique least common multiple. Suppose it did not; let T be another set with the same least common multiple. Then $T \cup \{v\}$ has the same least common multiple as $S \setminus \{v\} \cup v$, so since S has a unique least common multiple, we must have $T \cup \{v\} = S$. The only other set for which this is true is $T = S$, but this contradicts the statement that S has a unique least common multiple.

Therefore, by induction on the dimension, every proper subset of S forms a face. Let the least common multiple of S be $x^{\mathbf{b}}$, and as before, consider the linear functional given by $f(x) = \sum t^{-b_i} x_i$. No generator not in S divides $x^{\mathbf{b}}$ (otherwise, adding it to S gives the same least common multiple), and so the hyperplane $f(x) = t^{1/2}$ separates the vertices in S from the vertices outside of S. Again applying a projective transformation to map this hyperplane to infinity, the induced subcomplex of bounded faces on S must be acyclic. Since it contains every proper subset of S as a face, it must also contain S itself. So S is a bounded face of $\overline{P}^+$ as desired. □

The following result follows from [Miller and Sturmfels 04, Proposition 6.26] and Propositions 2.3 and 3.3.

Proposition 3.4. *If the exponent vectors of the minimal generators of a monomial ideal I are in general position tropically, then I is a generic monomial ideal in the sense of [Miller and Sturmfels 04, Chapter 6].*

Bayer and Sturmfels referred to the hull complex as a canonical free resolution of a monomial ideal. Here, however, we see that the hull complex is just one of a family of resolutions arising from different lifts of the corresponding tropical polytope. For instance, in Example 4.3 (Figure 3), we come up with different free resolutions of the corresponding monomial ideal. Note that we can always obtain a simplicial resolution by taking a generic lift. It would be interesting to answer the following questions:

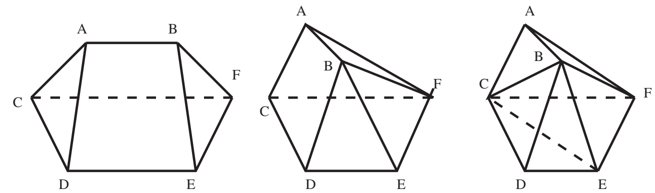

FIGURE 3. Various lifts of the model.

Question 3.5. What does the family of lifts of a given tropical polytope look like? Are there generalized hull resolutions we can get only by lifting to nonconvergent power series? Is there a reasonable algorithm for picking a small resolution from among these hull complexes?

Note that the faces that are bounded upon adding the positive orthant to $\overline{P}$ correspond to faces of the polytope with direction $\{\{2, \ldots, d\}, 1\}$ for some set S, i.e., those whose defining linear functional has all positive coefficients in all but the first coordinate.

4. FACES OF TROPICAL POLYTOPES

In order to find a good notion of *faces* of tropical polytopes, let us enumerate some desirable properties:

- The faces should be extreme sets, in the sense that there should not be two points in the polytope not in a face such that the tropical line segment connecting them intersects the face.
- The vertices of a tropical polytope (its extreme points) should be faces.
- The face lattice should be graded, and should have height equal to the dimension of the tropical polytope.
- The homology of the face lattice should be that of a sphere.
- The intersection of two faces should be a face of both, or at least a union of faces of both.

4.1 Joswig's Facet Definition

While [Develin and Sturmfels 04] gave a canonical decomposition of a tropical polytope as an ordinary polytopal complex dual to a subdivision of a product of two simplices, this decomposition was larger than desired by Michael Joswig. For instance, according to this decomposition, the convex hull of three points in two-space could have as many as six edges. In [Joswig 05], Joswig therefore proposed the following definition of a facet of a tropical polytope, to which we ascribe his initial.

Definition 4.1. A *J-facet* of a tropical polytope P is the convex hull of a maximal set of vertices of P contained in a closed half-space containing P. The *J-face lattice* of a tropical polytope is given by the intersection lattice of the vertex sets of these J-facets, and a *J-face* of a tropical polytope is given by the convex hull of one of these intersections.

Joswig's definition looks at the maximal fatoms, and defines these to be the facets; it then assumes that, as in ordinary geometry, it makes sense to recover the vertex sets of faces by combinatorially intersecting the vertex sets of facets. This definition works well for two-dimensional tropical polytopes; the polytope formed by the convex hull of n points in convex position will have face lattice identical to that of an n-gon. However, in larger dimensions, things go awry. Although the J-faces are themselves tropical polytopes, they do not have any of the properties listed above. Moreover, a J-facet may not be the intersection of the half-space defining it with the polytope itself, and the intersection of the facet-defining half-spaces is not necessarily equal to the polytope itself. In this section, we present examples exhibiting several issues with this definition.

Our first example is in $\mathbb{TP}^2$, where Joswig's theory is plainly correct; it highlights a difference between ordinary polytopes and tropical polytopes.

Example 4.2. Let P be the tropical convex hull of $\{(0,3,0),(0,1,1),(0,2,3)\} \subset \mathbb{TP}^2$.

Discussion. This polytope is a tropical triangle in the plane, whose vertices are the solid circles in Figure 4. It evidently has three facets, and three facet-defining half-spaces, one for each edge. These facet-defining half-spaces have apices given by the dotted circles, which are $(0,1,2)$, $(0,3,3)$, and $(0,3,1)$; the half-space at $(0,1,2)$ is given by sector 2, the one at $(0,3,3)$ by sector 1, and the one at $(0,3,1)$ by sectors 2 and 3.

However, the intersection of these three facet-defining half-spaces is bigger than the polytope; it also contains a ray starting at (0, 3, 0) and emanating downward. As stated in [Joswig 05], every tropical polytope is in fact the intersection of a finite number of half-spaces; to obtain P, we need to add a fourth half-space. One with apex $(0,3,0)$ would work.

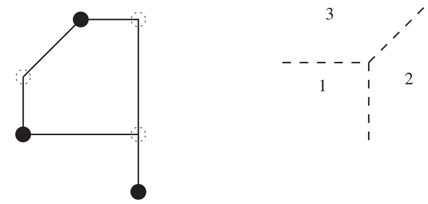

FIGURE 4. A tropical triangle in $\mathbb{TP}^2$.

This problem seems fundamental to the nature of tropical polytopes. The problem here is not one of genericity; moving the point $(0,3,0)$ infinitesimally results in a combinatorially identical tropical polytope. Rather, this is just a way in which tropical polytopes are different from ordinary polytopes.

Note also that the two facets involving $(0,3,0)$ intersect in a line segment, not a point. One way of expressing this is that each J-facet is defined by a tropical hyperplane that splits space into two parts, each a union of sectors of the hyperplane; P is contained in one of these parts, and the J-facet in question is contained in the boundary between the two parts. In this case, the facet connecting $(0,3,0)$ and $(0,2,2)$ contains the extra data that sector 1 contains P, and that this facet is contained in the boundary between sector 1 and the union of sectors 2 and 3. This extra data can also be computed by finding a lift for which the J-facet lifts to a facet and using Proposition 2.4.

Using this extra data, the two facets ought to intersect in the single point $(0,3,0)$; in essence, they are on different sides of the rest of the line segment (see Section 4.3 for more discussion). Joswig gets around this by stating that the intersection of two facets is defined to be the convex hull of the intersection of their vertex sets; as we will see later, though, this definition has major problems. We will give a definition that provides a better solution to this apparent problem in Section 4.2.

Example 4.3. Let P be the tropical convex hull of

$$(A,B,C,D,E,F) = (0201,0210,0125,0134,0143,0152) \subset \mathbb{TP}^3,$$

where $wxyz$ means the point (w,x,y,z).

Discussion. This example will prove to be very illuminating throughout this paper, and so we give it a name: we call it *the model*. The model is a tropical 3-polytope lying in three-space, shown in Figure 5. Its natural polyhedral decomposition [Develin and Sturmfels 04] consists

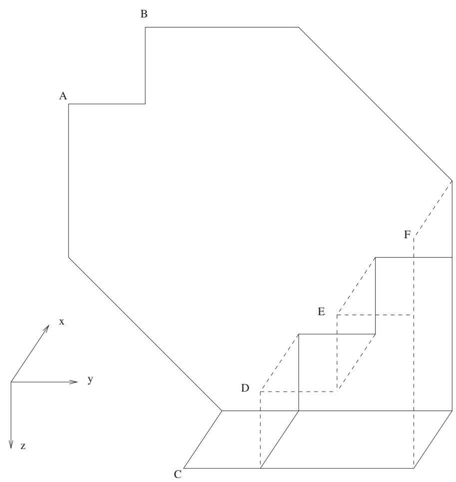

FIGURE 5. The model. The three-dimensional part is the three cubes in the lower right-hand corner of the figure; it also contains a pair of vertical two-dimensional flaps near vertices C and F and an extensive top flap connecting vertices A and B to the other vertices.

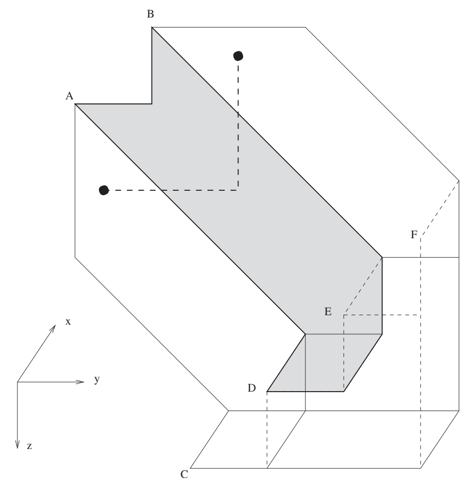

FIGURE 6. The J-facet ABED and two points that demonstrate that it is not an extreme set.

of three three-dimensional cells, all unit cubes, and a number of two-dimensional flaps.

This polytope exhibits a number of the problems mentioned earlier in this section. The vertex sets of the J-facets as defined by Joswig are ABDC (hyperplane with apex 0235), ABED (0244), ABFE (0253), ABFC (0255), and CDEF (0155). Consider the facet with vertices ABED. This comprises the center third of the upper flap, as well as two boundary squares connecting the apex 0244 with the vertices 0134 and 0143. This facet is not the intersection of the polytope with its facet-defining hyperplane, which includes some more of the upper flap; taking two points on opposite sides of ABED yields two points not in the facet for which the tropical line segment containing them intersects the facet. See Figure 6.

The fundamental problem here is that in some lifts, ABED itself is not a facet of the lift. In those lifts, we can take two points not in the lift of ABED for which the tropical line segment connecting them (these could be the lifts of 0242 and 0224, for instance) pierces the convex hull of ABED.

Investigating these facets further, we find that the facets ABDC, ABED, ABFE, and ABFC intersect pathologically. To be precise, their intersections are two-dimensional; the convex hulls of ABDC and ABED intersect in the convex hull of ABD, a two-dimensional object. The intersection of ABED and ABFC is the "top flap" portion of ABED, which is not the convex hull of the intersection of their vertex sets (AB); indeed, this is not a J-face at all according to the definition. This last problem again is unavoidable, and can be explained by the fact that these facets lie in different directions, and thus do not morally intersect two-dimensionally.

But the problem with ABDC and ABED, which are faces lying in the same direction, is not of this flavor. Simply put, two-dimensional faces should not intersect two-dimensionally. Looking at ABDC, its edges are clearly given by AB, BD, DC, and CA; any reasonable person would call it a square. Yet ABED's edges are clearly AB, BE, ED, and DA, meaning that AD is also an edge; this is clearly false, as can be seen by an examination of the square ABDC.

Furthermore, according to Joswig's definition, A and B are not actually vertices of this polytope. If we intersect the facet vertex sets setwise, we never get the singletons A and B, but merely the atom AB. So according to Joswig's definition, this object is a pyramid over a square, with vertices AB (apex), C, D, E, and F. But this misses some aspects of the tropical polytope, since A and B are both vertices in the sense that if we remove either, the convex hull changes. It is merely the weirdness of the tropical structure that ensures that there is no facet-defining hyperplane that contains one of them but not the other.

However, there certainly exist hyperplanes that contain P in a half-space and intersect it only in, for instance, vertex A. In every way other than Joswig's definition, A appears to be a vertex; indeed, it is a vertex of every lift. It should be a vertex of P.

It is worth noting that in ordinary geometry, we can obtain a hyperplane that defines a face $F \cap G$ by finding hyperplanes that define F and G and adding their linear functionals. This operation does not work in tropical geometry, where (a) the directions of F and G may be different, and (b) there is no reasonable way to take linear combinations of hyperplanes.

In this example, the J-face lattice is actually reasonable; it is graded, with height equal to the dimension of the polytope, namely 3. Indeed, as previously mentioned, it is the face lattice of an Egyptian pyramid. However, this is easily breakable by expanding the example.

Example 4.4. Let P be the tropical convex hull of

$$\begin{aligned}&(A, B, C, D, E, F, G, H, I)\\&\quad = (0301, 0310, 0224, 0233, 0242, 0158, 0167, 0176, 0185)\\&\qquad \subset \mathbb{TP}^3.\end{aligned}$$

Discussion. This polytope (Figure 7) is merely a three-tiered version of the two-tiered previous example. The J-facets are ABFI, ABDC, ABED, CDEGF, CDEHG, and CDEIH. By Joswig's definition, taking setwise intersections of these J-facets, we have a chain of faces given by $D \subset CDE \subset CDEG \subset CDEGF$. This chain is too long to live in the boundary of a 3-polytope. The problem geometrically, of course, is that CDEG and CDEGF are both two-dimensional; by adding the third tier, we merely ensured that D would actually be a vertex by Joswig's definition.

One possible solution would be to treat the facets as maximal elements of a cell complex, evaluating their faces by looking at the facets and taking the collection of all of their faces as the cells (for instance, the faces of ABDC are AB, BD, DC, and CA). Under this definition, the J-facets ABDC and ABED would not intersect in the triangle ABD, but rather in the union of the edge AB and the vertex D. This breaks the condition that two faces should intersect in a single face, but restores the grading to the face lattice in a rather crude way. However, even

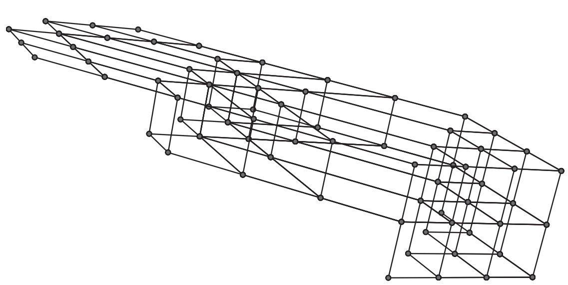

FIGURE 7. A tropical 3-polytope that is a more complicated version of the model.

if we do this, the homology of the face lattice (which was not even definable if the lattice wasn't graded) will not be that of an appropriately dimensioned sphere; for instance, in the model, the homology turns out to be $\mathbb{Z}$ in dimension 1, and 0 in all other dimensions.

4.2 A New Definition of Faces of a Tropical Polytope

As outlined in the previous section, Joswig's definition has several undesirable properties. In this section, we introduce a new definition that deals with these problems.

Definition 4.5. A *k-face* F of P is a minimal subset of the boundary of P such that for any lift of P, F is always the union of k-fatoms emanating from this lift, i.e., the union of images of k-faces of the lift under the degree map.

If P is in general position, then by Proposition 2.3, the face lattice of P is the same as that of any lift. In this case, the face lattice is determined by combinatorially intersecting the vertex sets of facets, and this new definition is the same as Joswig's. In essence, this definition gets rid of many of the problems with Joswig's definition in the nongeneric case via the following method: whenever two J-facets fundamentally intersect improperly, they are declared to be part of the same facet of P. Note also that it is clear that the 0-faces of P are always precisely the vertices.

It is illuminating to reinvestigate the model; recall that this is the convex hull of the points $(0201, 0210, 0125, 0134, 0143, 0152)$. By this definition, the model P has three facets; the upper shell comprising the convex hull of $\{A, B, C, F\}$, the lower face comprising the convex hull of $\{C, D, E, F\}$, and the "underbelly" given by the union of the convex hulls of $\{A, B, C, D\}$, $\{A, B, D, E\}$, and $\{A, B, E, F\}$.

The problematic pairs of faces that intersect improperly, such as $(ABCD, ABDE)$, have all been combined into the underbelly. This underbelly is a facet because in every lift, it is the union of facets of the lifted polytope. For instance, it could be the union of the faces $\{ABC, BCD, BDE, BEF\}$, or $\{ACD, ABDE, BEF\}$, or $\{ACD, ADE, ABEF\}$, or numerous other possibilities depending on the lift. However, in every lift, it is the union of images of facets. Furthermore, it is minimal with this property; for instance, the J-facet ABDE is not the union of facets in some lifts, such as $\{ABC, BCD, BDE, BEF\}$. This is because it overlaps other J-facets in an essential manner.

Our new definition also handles inessential overlaps elegantly. Consider the underbelly and the upper shell. Setwise, these intersect in an awkward two-dimensional region, but this overlap does not cause them to be united into the same face, since in all lifts ABCF is partitioned and the underbelly is partitioned independently. This is because the surface ABCF always partitions the top portion of the lift into facets with different directions.

For any lift $\overline{P}$ of P, the facets of P partition the boundary of $\overline{P}$ in a natural way; every face of $\overline{P}$ has an image under the degree map that is contained in a face of P. This opens the door to an improved definition of the intersection of two faces of P; in particular, it would be nice if the underbelly and the upper shell were in the union of the edges CA, AB, and BF rather than in the awkward two-dimensional region that is their setwise intersection. The following does the trick.

Definition 4.6. Let P be a tropical polytope, and let F and G be faces. Then the intersection of F and G is defined as follows: take any lift $\overline{P}$ of P, and consider the lifts $\overline{F}$ and $\overline{G}$, i.e., the unions of all faces of the same dimension that map into F. The *intersection* of F and G is the image under the degree map of the intersection of these two lifts.

To see this in action, consider the various lifts of the model shown in Figure 3. In each of these lifts, each of the three facets is a union of various faces of the lifts. Their intersections, however, are always the same, each a union of edges. Note that each edge contained in one of these intersections is an edge in each lift.

In practice, this definition seems to yield extremely sensible results. The complete results for the model show three 2-faces, seven 1-faces (AB, AC, CD, DE, EF, CF, BF), and six 0-faces (the vertices); see Figure 8. This, along with many other examples and intuition, leads us to formulate the following conjectures about our new definition, each of which represents an improvement on Joswig's.

Conjecture 4.7.

1. *The k-faces of tropical polytopes are extreme sets.*

2. *The (topological) boundary of a k-face is a union of $(k-1)$-faces. Thus, the faces fit together to form a cell complex.*

3. *The homology of this cell complex is that of a sphere.*

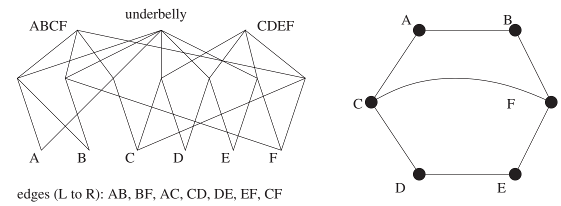

FIGURE 8. The face lattice of the model (L) and an artist's rendition of it as a cell complex (R). The latter consists of a flat hexagon with a couple of squares puckering up, forming a "pita pocket."

4. *The intersection of two faces is well-defined (i.e., does not depend on the lift), and is itself a contractible union of faces. Note that in some cases, a (for instance) 3-face in a lift that maps to a two-dimensional object under the degree map may be part of the lifts of two 3-faces of P. In this case, though, its contribution to the intersection will still be two-dimensional.*

5. *The k-faces of tropical polytopes are always contractible.*

6. *The faces of a tropical polytope do not depend on the provided vertex set. In other words, if we consider P as the convex hull of a different set of points (and form lifts by lifting this different set of points), the k-faces of P will be the same for all k.*

4.3 Directions

Item 3 of Conjecture 4.7 essentially states that the boundary of a tropical polytope can be partitioned into a natural cell complex, which triangulates this spherical boundary. Of course, since a tropical polytope need not be pure, its boundary need not be homeomorphic to a sphere. However, there is some natural sense in which we can modify a tropical polytope to obtain a ball: pumping some air into its interior will "inflate" the lower-dimensional parts of the boundary. The intersection rule given in the previous section will ensure that faces that are morally on opposite sides of these lower-dimensional parts will intersect properly, i.e., as they would upon inflation.

Another natural sense in which the boundary is a sphere is given by lifting the polytope. Any lift $\overline{P}$ of a tropical polytope P is of course an ordinary polytope, homeomorphic to a ball, and thus the boundary $\partial\overline{P}$ is homeomorphic to a sphere. This boundary is subdivided by the boundary complex of $\overline{P}$, and maps to the boundary of P; in this fashion, each lift of P provides a subdivision of the boundary of P. Again, different faces (even

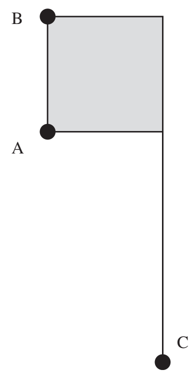

FIGURE 9. A tropical triangle in $\mathbb{TP}^2$, not in general position.

from the same lift) can map to parts of P that overlap setwise, but do not do so morally (upon inflation). Consider, for instance, the model; the 2-fatoms ABCF and ABDE appear to intersect in a two-dimensional region. This should not be the case, and the reason is that ABDE covers the bottom side of this region, while ABCF covers the top side. Meanwhile, ABCD and ABDE do intersect in a two-dimensional region, since both lie on the bottom side of this flap.

In this case, it is easy to define the concept of *direction*. ABCF is cut out by a hyperplane with apex 0255, and the polytope lies in sector 1 of this hyperplane, with the face lying in the boundary between 1 and $\{2, 3, 4\}$. So the 2-face has direction given by $\{1, 234\}$. Similarly, ABDE is cut out by a hyperplane with apex 0244; the polytope lies in the sectors indexed by 2, 3, and 4, and the face lies in the boundary between the union of these sectors and sector 1. So ABDE has direction given by $\{234, 1\}$. ABCD similarly has direction $\{234, 1\}$, so ABDE and ABCD fundamentally overlap, while the overlap between either and ABCF is illusory.

These directions also correspond to equations of hyperplanes $f \cdot x \geq 0$ defining the faces of the lift that make these 2-fatoms, via Proposition 2.4. The first set in the direction is given by those i for which $f_i > 0$ in all lifts, while the second set is given by those i for which $f_i < 0$ in all lifts. Otherwise, i is in neither set. This happens in the following example:

Example 4.8. Let P be the convex hull of $\{A, B, C\} = \{002, 003, 010\} \subset \mathbb{TP}^2$ (see Figure 9).

Discussion. In this example, vertical edge AB has direction given by $\{2, 1\}$, while edge AC has direction $\{23, 1\}$ and edge BC has direction $\{1, 23\}$. Note that as before, AC and BC are on opposite sides of their setwise one-dimensional overlap.

In all of these situations thus far, it is clear what the direction would be. This concept is an intuitive notion as to when k-fatoms intersect improperly, and when they merely appear to intersect improperly. However, there are potential problems. If k is fairly small, different lifts may have different minimal direction sets (R, S); for instance, a fatom in some large-dimensional polytope could conceivably lie in the boundary between sectors 1 and 2, but also in the boundary between sectors 1 and 3, with the polytope lying in 1. It makes no sense for the direction here to be $\{1, \varnothing\}$. Another important question in direction theory is to figure out how to compute the intersections of faces with different directions. For instance, ABCF and ABDE in the model have complementary directions. How do we use this information to determine that they intersect in edge AB?

Indeed, determining the direction of edge AB, and of lower-dimensional faces in general, is a tricky one. Even fixing the lift, there are many different face-defining functionals. For instance, consider the *hull lift* of the model, where we lift the point abcd to (t^a, t^b, t^c, t^d). This lift has as two of its facets ABCF and ABDE, which are defined by the following linear functionals (the notation (65-21) represents the coefficient $t^6 + t^5 - t^2 - t$, and so on):

$$\begin{aligned}
\text{ABDE: } & -(65-21)x_1 + (43-10)x_2 + (2-1)x_3 \\
& + (2-1)x_4 \geq 0, \\
\text{ABCF: } & (74-21)x_1 - (52-10)x_2 - (2-1)x_3 \\
& - (2-1)x_4 \geq 0.
\end{aligned}$$

So far, so good; the directions indicated by the sign vectors of the coefficients agree with the sectors of the corresponding tropical hyperplane in which the polytope is contained, and this is true of any lift for which these are faces. However, we can find a linear functional cutting out the lifted edge AB by taking any positive linear combination of these two functionals; this produces many possible sign vectors. The set that can be obtained from this lift alone is $\{(+ - --), (+ - 00), (+ - ++), (0 - ++), (-0++), (-+++)\}$, and others such as $(+-+-)$ can be obtained from different lifts. What is the true direction of this edge?

Some of these hyperplanes, namely the set $\{(+ - --), (+ - 00), (+ - ++), (- + ++)\}$, map to tropical hyperplanes whose intersection with the model polytope P is not just the edge AB. Indeed, if we perturb the facet-defining functionals of ABDE and ABCF infinitesimally, they will map to the same tropical hyperplanes, but their intersection with the lifted polytope will decrease in dimension. Similarly, the hyperplane $(+-00)$ maps to the degenerate tropical hyperplane with apex at $(02\infty\infty)$; this is just the hyperplane $x-w=2$, whose intersection with P is two-dimensional (although it intersects the vertex set of P in only the vertices A and B).

The sign vectors $(0-++)$ and $(-0++)$ are the ones that best embody the Platonic ideal of a direction for AB. The apex of the tropical hyperplane that defines AB should, morally speaking, be the point (0211). If we place a tropical hyperplane there, it intersects P precisely in the edge AB. Furthermore, the rest of P is contained in (the union of) sectors 3 and 4. Meanwhile, the edge AB lies entirely in the intersection between sector 1 and the union of sectors 3 and 4, but also lies entirely in the intersection between sector 2 and the union of sectors 3 and 4. Therefore, from considering this hyperplane, both $\{34, 1\}$ and $\{34, 2\}$ are reasonable candidates for the direction of AB. By translating this hyperplane in the positive-1 or positive-2 direction, we can easily obtain tropical hyperplanes that suggest either of these as the unique direction of AB.

Question 4.9. Is there always a natural apex for a (tropical) hyperplane defining a k-fatom of a tropical polytope P?

If there were, we could use these to determine (more or less) directions, and use these directions to determine many things, among them which fatoms should be part of the same face (an alternative definition that should be equivalent to the one presented here, made without having to consider all lifts of the polytope), and what the intersection of two faces should be (i.e., their moral intersection, as opposed to their setwise intersection). The study of tropical polytopes via directions deserves more investigation; note that such a thing is possible in tropical geometry because the set of edge directions of tropical polytopes viewed as ordinary polyhedral complexes is restricted to the set of 0/1-vectors (with a similar statement about higher-dimensional faces).

5. EXAMPLES

In this section, we present two more examples of tropical polytopes.

Example 5.1. Let P be the convex hull of

$$\{A, B, C, D, E\} = (0101, 0011, 0002, 0001, 0110) \subset \mathbb{TP}^3,$$

as depicted in Figure 10.

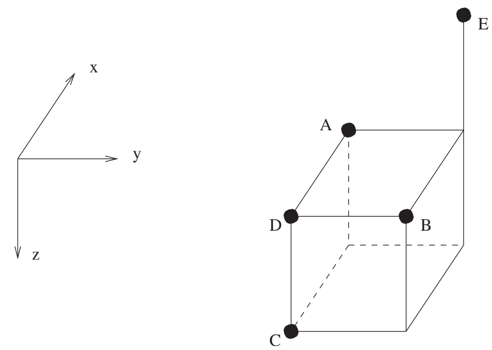

FIGURE 10. A five-vertex tropical polytope in $\mathbb{TP}^3$: cube with pendant edge.

Discussion. This is a cube with a pendant edge. We previously encountered the convex hull of the vertices ABC; this tropical polytope consists of the union of three facets of a cube surrounding a vertex (0112 here). Adding the point D realizes the unit cube as a tropical polytope; its facets are simply the convex hull of ABC, which consists of three facets of the cube, and the convex hulls of ABD, ACD, and BCD, each of which is another facet of the cube. This is a perfectly normal tropical tetrahedron (albeit not one in general position). All lifts of this polytope are simplices.

Adding the pendant vertex, however, produces a variety of possible lifts. There exist lifts for which the lifted point $\overline{E}$ lies in various places with respect to the tetrahedron ABCD; for instance, it could be coplanar with 0, e_1, and e_2 (since these four points lie in a tropical hyperplane), or with e_1, e_2, and e_3. Even if it is in general position, it can be beyond different faces of the tetrahedron. See Figure 11 for some possibilities.

In all, the face lattice of this tropical 3-polytope is depicted in Figure 12. This is a tetrahedron formed by ABCD, with the edge AB subdivided by point E. It has four facets: two triangles ACD and BCD, and two squares ABCE and ABDE. The two squares intersect in the union of the edges AE and BE. Its f-vector is $(5, 7, 4)$. The fact that point E subdivides edge AB is interesting; intuitively, this makes sense, since the point through which point E is connected to the rest of the polytope lies

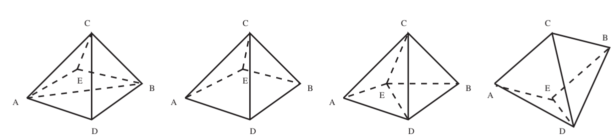

FIGURE 11. Four lifts of the cube with pendant vertex. Two are bipyramids and two are square pyramids.

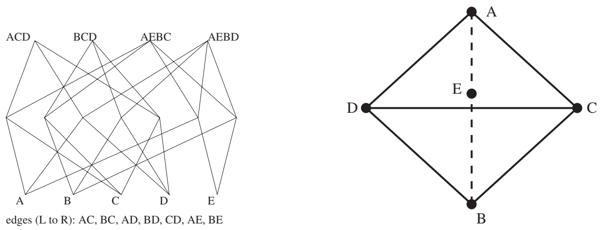

FIGURE 12. The face lattice of a cube with pendant edge (L) and its realization as a cell complex (R), a tetrahedron with one edge subdivided.

on edge AB. If we had placed point E on point AB, the diagram would be the same, except that point E would not appear as a vertex at all.

Example 5.2. Let P be the convex hull of

$$\{A,B,C,D,E,F\} = (0011, 0101, 0110, 1001, 1010, 1100) \subset \mathbb{TP}^3,$$

as depicted in Figure 13.

Discussion. This polytope is an instance of a *tropical hypersimplex*, to be precise, the $(2,4)$-hypersimplex, whose vertex set consists of all 4-tuples with two 1's and two 0's. These (n,d)-hypersimplices always lie in a hyperplane; to be precise, one with apex $\mathbf{0}$. However, their hull lifts are bona fide hypersimplices, and in particular are full-dimensional; their face lattice will accordingly be d-dimensional. Indeed, it turns out that every lift of this object is a $(2,4)$-hypersimplex, which is an octahedron, so its face lattice is simply that of an octahedron. In general, the face lattice of the tropical hypersimplex is identical to that of the corresponding ordinary hypersimplex.

Interestingly, four of the facet-defining tropical hyperplanes are the same (the one with vertex $\mathbf{0}$); this hyperplane cuts out different facets by virtue of taking different sectors to constitute the relevant half-space.

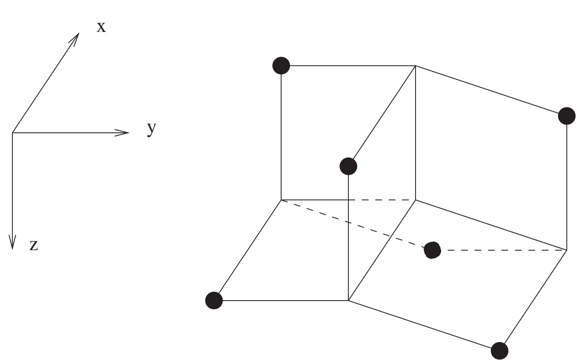

FIGURE 13. A tropical $(2,4)$-hypersimplex, i.e., octahedron.

6. FURTHER QUESTIONS

We have described how to compute the faces of a tropical polytope. In ordinary polytope theory, each face is the intersection of a polytope with the boundary of a half-space including the polytope; is there an analogue of this here? Clearly the bounding objects are not, in general, hyperplanes (i.e., the underbelly of the model), but they should fall into a reasonable set of geometric objects.

The notion of sign of a tropical determinant introduced in [Joswig 05] and also used in Lemma 2.2 and Proposition 2.3 is related to the facial structure defined in this paper. What is the precise relationship? Can we use tropical determinants to aid us in defining directions and thus intrinsically (i.e., without reference to the family of all lifts) defining faces?

A priori, there is only a finite number of (combinatorially different) possible lifts of a tropical polytope. Is there a good way to enumerate these? The connection between tropical chirotopes and oriented matroids of lifted point configurations as seen in Lemma 2.2 and Proposition 2.3 can help with this problem.

The various parts of Conjecture 4.7 all seem to be true based on experimental evidence in three dimensions. Proving these nice properties would certainly be an important step in understanding tropical polytopes.

What do k-faces of tropical polytopes look like? Unlike ordinary polytopes, these need not be isomorphic (as tropical objects) to polytopes in tropical projective k-space. Are they combinatorially isomorphic?

The octahedron (Example 5.2) is an example of a tropical regular polytope, one for which the face lattice is (combinatorially) transitive on the complete flags of faces. Can we classify these in general? Are there any other than simplices, the octahedron in dimension three, and polygons?

In addition to their half-space description, tropical polytopes can be formed by intersecting hyperplanes in another way.

Theorem 6.1. [Develin and Sturmfels 04] *The convex hull of $\{v_1, \dots, v_k\}$ consists of the union of the bounded regions formed by the tropical hyperplane arrangement given by putting a negated hyperplane at each v_i.*

How does this formulation interact with the hyperplane description of P? This provides a natural decomposition of P as a polytopal complex; can we use the polytopes here (the types of [Develin and Sturmfels 04]) to find the faces of P? Joswig [Joswig 05] showed that the 0-faces of this polytopal complex include all the apices of hyperplanes that are needed to cut out P.

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