Matrix-Valued Orthogonal Polynomials Related to SU(N+1), Their Algebras of Differential Operators, and the Corresponding Curves

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CONTENTS

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We give two examples of algebras of differential operators associated with families of matrix-valued orthogonal polynomials arising from representations of $\mathrm{SU}(N+1)$. The first gives a commutative algebra and the second a noncommutative one.

1. INTRODUCTION

The study of certain representations of $\mathrm{U}(N)$, considered in [Grünbaum et al. 02] for $N = 2$ and fully explored in [Pacharoni and Tirao 07b], leads to a second-order differential operator D with matrix-valued coefficients and a family of orthogonal matrix-valued polynomials $\{P_n\}_{n\geq 0}$ that are common eigenfunctions of this ordinary differential operator with a matrix-valued eigenvalue Λ_n:

$$DP_n^* = P_n^* \Lambda_n.$$

Different representations of $\mathrm{U}(N)$ give rise to different differential operators, each one with its corresponding families of matrix-valued orthogonal polynomials. Thus one obtains large sets of examples of the type contemplated in [Durán 97]. A detailed look at two such examples arising from the representation theory of $\mathrm{U}(N)$ is given in [Grünbaum et al. 05]. The theory of matrix-valued orthogonal polynomials, without any consideration of differential equations, goes back to [Kreĭn 71, Kreĭn 49]. The question raised in [Durán 97] is a matrix version of the question raised and settled in the scalar case in [Bochner 29]; see also [Routh 84]. By considering this noncommutative setup one obtains a situation that is much richer than the scalar one. For a sample of applications of matrix-valued orthogonal polynomials, see [Sinap and Van Assche 96].

Starting in [Castro and Grünbaum 06] one considers *for a fixed family of matrix-valued orthogonal polynomials* the algebra of all such differential operators D. Each

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differential operator in the algebra will bring in its own sequence of matrix-valued eigenvalues, which can be denoted by $\Lambda_n(D)$. This algebra of matrix-valued differential operators and the algebra of matrix-valued eigenvalues are isomorphic, as we will show in Section 3.2.

In [Castro and Grünbaum 06] one finds a discussion of this general question in a few instances in which the family of matrix-valued orthogonal polynomials $\{P_n\}_{n\geq 0}$ is not related to any underlying group. These examples were obtained previously in [Durán and Grünbaum 04] and [Grünbaum 03].

In Section 2 we introduce two differential operators obtained by conjugating another pair of "original" differential operators arising from the representation theory in [Grünbaum et al. 05]. Section 3 is devoted to the first one and Section 4 to the second.

In Section 3.1 we give another conjugation of one of the original operators and compute a sequence of orthogonal polynomials that are common eigenfunctions of it. Those polynomials are then related to those obtained using the new tools introduced in [Tirao 03] in terms of the matrix version of Gauss's equation.

In Section 3.2 we return to the conjugation in Section 2 and compute its corresponding family of orthogonal polynomials and consider an algebra of differential operators attached to this sequence. This family of orthogonal polynomials is fully explored in the appendix, Section 5. In Sections 4.1 and 4.2 this process is repeated for the second original operator.

Using the families detailed in the appendix, we present convincing computational evidence that leads us to state the following: In the first example the algebra of operators is commutative, while in the second example this is not so. In each case, we can exhibit the generators of the algebra and a basis for the space of operators of a given order. Finally, in the first example we can write down explicitly a polynomial relation among the two generators of the algebra, while in the second example we give a collection of some of the relations among the five generators of the algebra.

2. FROM THE CASIMIR OPERATOR TO SOME EXAMPLES OF ORDINARY DIFFERENTIAL OPERATORS WITH MATRIX COEFFICIENTS

In this section we give an extremely sketchy indication of the way in which one goes from the Casimir operator, namely a differential operator acting on matrix-valued functions on the group $\mathrm{SU}(N+1)$, to certain ordinary differential operators acting on matrix-valued functions defined in the interval $[0,1]$. There are many details missing in the presentation below, and the reader is advised to consult either [Grünbaum et al. 05] or even better [Pacharoni and Tirao 07b, Grünbaum et al. 02] for a full account.

The main goal here is to give the explicit differential operators that will play a crucial role in the rest of the paper.

In this paper, following the lines of [Grünbaum et al. 05], we use two different skew-Hermitian matrix-valued forms, namely

$$(P,Q)=\int_{\mathbb{R}} P(t)\,W(t)\,Q^*(t)\,dt$$

and

$$\langle P,Q\rangle=(P^*,Q^*)^*.$$

Here $W(t)$ is a positive definite matrix-valued weight.

Two matrix-valued functions are called *orthogonal* if $(P,Q)=\Theta$, the null matrix of the appropriate dimension. On the other hand, and for reasons explained in [Grünbaum et al. 05], a differential operator D is called *symmetric* if

$$\langle DP,Q\rangle=\langle P,DQ\rangle,$$

for all matrix-valued polynomials P and Q.

In the first example, considering representations π of $\mathrm{GL}(N,\mathbb{C})$ that correspond to *one-step* N-tuples of the form

$$\pi=(\underbrace{m+2,\dots,m+2}_{k},\underbrace{m,\dots,m}_{N-k}),\quad 1\leq k\leq N-1,$$

and making the changes $N=\beta+1$ and $m=\alpha$, we have one family of examples that depends on three parameters α,β,k, where the differential operator is given by

$$\begin{aligned}D=t(1-t)\frac{d^2}{dt^2}+\Bigg[&\begin{pmatrix}\alpha+3&0&0\\0&\alpha+2&0\\0&0&\alpha+1\end{pmatrix}\\&-t\begin{pmatrix}\alpha+\beta+4&0&0\\0&\alpha+\beta+3&0\\0&0&\alpha+\beta+2\end{pmatrix}\Bigg]\frac{d}{dt}\\&+\frac{1}{1-t}\begin{pmatrix}-2(\beta-k+1)&2(\beta-k+1)&0\\0&-(\beta-k+2)&\beta-k+2\\0&0&0\end{pmatrix}\\&+\frac{t}{1-t}\begin{pmatrix}0&0&0\\k+1&-(k+1)&0\\0&2k&-2k\end{pmatrix}.\end{aligned}$$

This operator is symmetric with respect to the matrix weight function

$$W(t) = t^\alpha(1-t)^\beta \begin{pmatrix} w_1 t^2 & 0 & 0 \\ 0 & w_2 t & 0 \\ 0 & 0 & w_3 \end{pmatrix}, \quad \alpha,\beta > -1,$$

where

$$w_1 = \prod_{j=0}^{k-2} \frac{(\alpha-j-1)(\alpha-j)}{(k-j-1)(k-j)},$$
$$w_2 = k(\alpha-k)\prod_{j=0}^{k-2} \frac{(\alpha-j-1)(\alpha-j)}{(k-j)(k-j+1)},$$
$$w_3 = \prod_{j=0}^{k-1} \frac{(\alpha-j-1)(\alpha-j)}{(k-j)(k-j+1)}.$$

The example above arises naturally in the context of representation theory with β and k natural numbers. It is, however, possible to consider both D and $W(t)$ under the conditions $\beta > -1$ and $1 \le k \le \beta$.

The name *one-step* for the $(\beta+1)$-tuples or partitions discussed above is very natural if one looks at the corresponding Young diagrams; see [Vilenkin and Klimyk 92]. The same is true in the second type of example, discussed later in this section.

This operator is not yet written in *hypergeometric form* (the coefficients are not matrix polynomials of degree less than or equal to the corresponding order of differentiation). Just as in [Grünbaum et al. 03a], we proceed to do an appropriate conjugation by a certain matrix-valued function, namely

$$\Psi^*(t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & 0 \\ 0 & 0 & (1-t)^2 \end{pmatrix},$$

such that we get

$$\widetilde{W}(t) = \Psi(t)W(t)\Psi^*(t)$$

for the new weight function. The new differential operator becomes $\tilde{D}F = (\Psi^*)^{-1}D(\Psi^*F)$, where

$$\widetilde{D} = \widetilde{A}_2(t)\frac{d^2}{dt^2} + \widetilde{A}_1(t)\frac{d}{dt} + \widetilde{A}_0(t),$$

with $\widetilde{A}_2$, $\widetilde{A}_1$, $\widetilde{A}_0$ given by

$$\widetilde{A}_2(t) = t(1-t),$$
$$\widetilde{A}_1(t) = \begin{pmatrix} \alpha+3 & 0 & 0 \\ -1 & \alpha+2 & 0 \\ 0 & -2 & \alpha+1 \end{pmatrix} - t\begin{pmatrix} \alpha+\beta+4 & 0 & 0 \\ 0 & \alpha+\beta+5 & 0 \\ 0 & 0 & \alpha+\beta+6 \end{pmatrix},$$
$$\widetilde{A}_0(t) = \begin{pmatrix} 0 & 2(\beta-k+1) & 0 \\ 0 & -(\alpha+\beta-k+2) & \beta-k+2 \\ 0 & 0 & -2(\alpha+\beta-k+3) \end{pmatrix},$$

written now in hypergeometric form. This differential operator is symmetric with respect to the new weight function $\widetilde{W}(t)$ and will be the starting point of the discussion in Section 3.2.

As a remark connecting the matrix weight given above with some considerations in [Durán and Grünbaum 04, Pacharoni and Tirao 07a], note that $\widetilde{W}(t)$ admits the factorization

$$\widetilde{W}(t) = \frac{\rho(t)}{\rho(\frac{1}{2})}T(t)\widetilde{W}\left(\frac{1}{2}\right)T^*(t)$$

with $T\left(\frac{1}{2}\right) = I$ and $\rho(t) = t^\alpha(1-t)^\beta$. The matrix $T(t)$ solves the equation

$$T'(t) = \left(\frac{A}{t} + \frac{B}{1-t}\right)T(t)$$

with

$$A = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

To obtain the second example, we consider representations π of $\mathrm{GL}(N,\mathbb{C})$ that correspond to *two-step* N-tuples of the form

$$\pi = (\underbrace{m+2,\dots,m+2}_{k_1},\underbrace{m+1,\dots,m+1}_{k_2-k_1},\underbrace{m,\dots,m}_{N-k_2}),$$

$1 \le k_1 < k_2 \le N-1$, and replacing N and m as above, we have one family of examples that depends on four parameters α, β, k_1, k_2, the differential operator D being specified in Figure 1.

$$D = t(1-t)\frac{d^2}{dt^2} + \left[\begin{pmatrix} \alpha+3 & 0 & 0 & 0 \\ 0 & \alpha+2 & 0 & 0 \\ 0 & 0 & \alpha+2 & 0 \\ 0 & 0 & 0 & \alpha+1 \end{pmatrix} - t\begin{pmatrix} \alpha+\beta+4 & 0 & 0 & 0 \\ 0 & \alpha+\beta+3 & 0 & 0 \\ 0 & 0 & \alpha+\beta+3 & 0 \\ 0 & 0 & 0 & \alpha+\beta+2 \end{pmatrix}\right]\frac{d}{dt}$$
$$+ \frac{1}{1-t}\begin{pmatrix} k_1+k_2-2(\beta+1) & \frac{(k_2-k_1+2)(\beta-k_2+1)}{k_2-k_1+1} & \frac{(k_2-k_1)(\beta-k_1+2)}{k_2-k_1+1} & 0 \\ 0 & -(\beta-k_1+2) & 0 & \beta-k_1+2 \\ 0 & 0 & -(\beta-k_2+1) & \beta-k_2+1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$+ \frac{t}{1-t}\begin{pmatrix} 0 & 0 & 0 & 0 \\ k_2+1 & -(k_2+1) & 0 & 0 \\ k_1 & 0 & -k_1 & 0 \\ 0 & \frac{k_1(k_2-k_1+2)}{k_2-k_1+1} & \frac{(k_2-k_1)(k_2+1)}{k_2-k_1+1} & -(k_1+k_2) \end{pmatrix}.$$

FIGURE 1. The two-step differential operator D (see text for details).

This operator is symmetric with respect to the matrix weight function

$$W(t) = t^\alpha(1-t)^\beta \begin{pmatrix} w_1 t^2 & 0 & 0 & 0 \\ 0 & w_2 t & 0 & 0 \\ 0 & 0 & w_3 t & 0 \\ 0 & 0 & 0 & w_4 \end{pmatrix},$$

with $\alpha, \beta > -1$, where

$$w_1 = \frac{k_2-k_1+1}{k_2}\binom{\beta}{k_2-1}\binom{\beta+1}{k_1-1},$$

$$w_2 = \frac{k_2-k_1+2}{k_2+1}\binom{\beta}{k_2}\binom{\beta+1}{k_1-1},$$

$$w_3 = \frac{k_2-k_1}{k_2}\binom{\beta}{k_2-1}\binom{\beta+1}{k_1},$$

$$w_4 = \frac{k_2-k_1+1}{k_2+1}\binom{\beta}{k_2}\binom{\beta+1}{k_1}.$$

Once again, both D and $W(t)$ can be considered for $\beta > -1$ and $1 \le k_1 < k_2 \le \beta$.

As in the previous example, this operator is not yet written in hypergeometric form. A possible conjugation in this case is given by the matrix-valued function (see [Grünbaum et al. 05])

$$\Psi^*(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & \frac{k_2-k_1+2}{k_2-k_1+1} & \frac{k_2-k_1}{k_2-k_1+1} & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & 0 & 0 \\ 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & (1-t)^2 \end{pmatrix},$$

and we get

$$\widetilde{W}(t) = \Psi(t)W(t)\Psi^*(t)$$

for the new weight function. The new differential operator becomes $\tilde{D}F = (\Psi^*)^{-1}D(\Psi^*F)$, where

$$\widetilde{D} = \widetilde{A}_2(t)\frac{d^2}{dt^2} + \widetilde{A}_1(t)\frac{d}{dt} + \widetilde{A}_0(t), \qquad (2\text{–}1)$$

with $\widetilde{A}_2$, $\widetilde{A}_1$, $\widetilde{A}_0$ given in Figure 2, written now in hypergeometric form. This differential operator is symmetric with respect to the new weight function $\widetilde{W}(t)$ and will be the starting point of the discussion in Section 4.2.

Again, as a remark connecting the matrix weight given above with some considerations in [Durán and Grünbaum 04, Grünbaum et al. 05], note that $\widetilde{W}(t)$ admits the factorization

$$\widetilde{W}(t) = \frac{\rho(t)}{\rho\left(\frac{1}{2}\right)}T(t)\widetilde{W}\left(\frac{1}{2}\right)T^*(t)$$

$$\widetilde{A}_2(t) = t(1-t),$$

$$\widetilde{A}_1(t) = \begin{pmatrix} \alpha+3 & 0 & 0 & 0 \\ -1 & \alpha+2 & 0 & 0 \\ -1 & 0 & \alpha+2 & 0 \\ 0 & -\frac{k_2-k_1+2}{k_2-k_1+1} & -\frac{k_2-k_1}{k_2-k_1+1} & \alpha+1 \end{pmatrix} - t\begin{pmatrix} \alpha+\beta+4 & 0 & 0 & 0 \\ 0 & \alpha+\beta+5 & 0 & 0 \\ 0 & 0 & \alpha+\beta+5 & 0 \\ 0 & 0 & 0 & \alpha+\beta+6 \end{pmatrix},$$

$$\widetilde{A}_0(t) = \begin{pmatrix} 0 & \frac{(k_2-k_1+2)(\beta-k_2+1)}{k_2-k_1+1} & \frac{(k_2-k_1)(\alpha-k_1+2)}{k_2-k_1+1} & 0 \\ 0 & -(\alpha+\beta+2)+k_2 & 0 & \beta-k_1+2 \\ 0 & 0 & -(\alpha+\beta+3)+k_1 & \beta-k_2+1 \\ 0 & 0 & 0 & -2(\alpha+\beta+3)+k_1+k_2 \end{pmatrix},$$

FIGURE 2. $\widetilde{A}_2$, $\widetilde{A}_1$, $\widetilde{A}_0$ from (2–1).

with $T\left(\frac{1}{2}\right) = I$ and $\rho(t) = t^\alpha(1-t)^\beta$. The matrix $T(t)$ solves the equation

$$T'(t) = \left(\frac{A}{t} + \frac{B}{1-t}\right)T(t)$$

with

$$A = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{k_1-k_2-2}{2(k_2-k_1+1)} \\ 0 & 0 & \frac{1}{2} & \frac{k_1-k_2}{2(k_2-k_1+1)} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & 0 & \frac{k_1-k_2-2}{2(k_2-k_1+1)} \\ 0 & 0 & -1 & \frac{k_1-k_2}{2(k_2-k_1+1)} \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

The point of these conjugations is to ensure that the new differential operator will have matrix-valued coefficients that are polynomials in t of degrees not higher than the corresponding order of differentiation. This new form of the differential operator is referred to as a *hypergeometric form.*

Given a differential operator there may be more than one way of conjugating it into a hypergeometric form. In the next section we will introduce another operator, denoted by D, which is obtained from the one we called D at the beginning of this section by means of a conjugation that is different from the one used to produce $\widetilde{D}$ out of D. We will retain the symbol $W(t)$ to denote the new weight matrix obtained as in this section by conjugation of the original $W(t)$. A similar step will be taken in Section 4.1.

3. THE ONE-STEP EXAMPLE

3.1 Generating the Polynomial Eigenfunctions of the Differential Operator

We retain the symbol D for the new differential operator given in *hypergeometric form* for the *one-step* example of the previous section. It is now given by the expression

$$D = t(1-t)\frac{d^2}{dt^2} + (X - tU)\frac{d}{dt} + V,$$

where

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(\alpha+\beta-k+2) & 0 \\ 0 & 0 & -2(\alpha+\beta-k+3) \end{pmatrix},$$

$$U = \begin{pmatrix} \alpha+\beta+4 & -1 & \frac{2}{\alpha+\beta-k+2} \\ 0 & \alpha+\beta+5 & -\frac{2(\alpha+\beta-k+3)}{\alpha+\beta-k+2} \\ 0 & 0 & \alpha+\beta+6 \end{pmatrix},$$

and

$$X = \begin{pmatrix} \frac{(\alpha+1)(\alpha+\beta-k+4)}{\alpha+\beta-k+2} & \frac{(\alpha+1)(\alpha+\beta-k+4)}{(\alpha+\beta-k+2)(\alpha+\beta-k+3)} & 0 \\ \frac{2(\beta-k+1)}{\alpha+\beta-k+2} & C_1(\alpha,\beta,k) & \frac{2(\alpha+2)}{\alpha+\beta-k+4} \\ 0 & \frac{(\beta-k+2)(\alpha+\beta-k+2)}{(\alpha+\beta-k+3)(\alpha+\beta-k+4)} & C_2(\alpha,\beta,k) \end{pmatrix},$$

where

$$C_1(\alpha,\beta,k) = \alpha+2+\frac{2(\alpha+2)}{\alpha+\beta-k+4} - \frac{2(\alpha+1)}{\alpha+\beta-k+2}$$

and

$$C_2(\alpha,\beta,k) = \alpha+3-\frac{2(\alpha+2)}{\alpha+\beta-k-4}$$

The conjugation that was used above to produce D was chosen so that V turns out to be diagonal. The coefficients are not so pleasant as those we will use in the next subsection, but it will be easier to find a relation between some matrix-valued orthogonal polynomials whose adjoints are eigenfunctions of this new D and the *matrix hypergeometric function*, introduced in [Tirao 03].

The operator D above and the new weight $W(t)$ are such that the (unique) sequence of monic matrix-valued polynomials $\{Q_n\}_{n\geq 0}$ with respect to $W(t)$ satisfies

$$DQ_n^* \equiv t(1-t)\frac{d^2}{dt^2}Q_n^* + (X - tU)\frac{d}{dt}Q_n^* + VQ_n^* = Q_n^*\Gamma_n,$$

where $\Gamma_n = -n^2 + n(I-U) + V$.

Now, if $\{P_n\}_{n\geq 0}$ is a sequence given by

$$Q_n(t) = S_n^{-1}P_n(t), \quad \det S_n \neq 0,$$

we have

$$DP_n^*(t) = P_n^*(t)(S_n^*)^{-1}\Gamma_n S_n^*.$$

We now make the genericity assumption that the eigenvalues of Γ_n are different for all n. Then by choosing S_n^* as the matrix whose columns are the eigenfunctions of Γ_n (unique up to scaling), we have that the $P_n^*(t)$ satisfy

$$\begin{aligned} DP_n^* &\equiv t(1-t)\frac{d^2}{dt^2}P_n^* + (X - tU)\frac{d}{dt}P_n^* + VP_n^* \\ &= P_n^*\Lambda_n, \end{aligned} \tag{3–1}$$

with

$$\Lambda_n = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} = (S_n^*)^{-1}\Gamma_n S_n^*,$$

and the values of t_i are given in (3–3). If we put $P_n^*(t) = \sum_{j=0}^{n} A_j^n t^j$, the equations satisfied by A_j^n, $j = n$, $n-1, \ldots, 0$, are

$$\begin{aligned} \Gamma_n A_n^n &= A_n^n \Lambda_n \\ \Gamma_j A_j^n - A_j^n\Lambda_n &= -(j+1)(X+j)A_{j+1}^n, \end{aligned} \tag{3–2}$$

for $j = n-1, \ldots, 0$.

All these equations are of Sylvester type; see [Gantmacher 60]. Under the genericity assumption made earlier, the first equation has a solution A_n^n that is unique up to the choice of three scalars. Each of the equations that follow has a unique solution A_j^n if we assume an extra genericity assumption, namely for each n the spectrum of each Γ_j, $j = 0, \ldots, n-1$, is disjoint from the spectrum of Λ_n. For a careful treatment of the nongeneric case, the reader can consult [Pacharoni and Román 07].

If $\{P_n\}_{n\geq 0}$ is a family of matrix-valued polynomials satisfying (3–1) and D is symmetric, it follows that

$$\langle DP_n^*, P_m^*\rangle = \langle P_n^*\Lambda_n, P_m^*\rangle = (P_m, P_n)\Lambda_n$$

and

$$\langle P_n^*, DP_m^*\rangle = \langle P_n^*, P_m^*\Lambda_m\rangle = \Lambda_m^*(P_m, P_n).$$

Now, under the genericity assumptions made earlier, the spectrum of Λ_n and that of Λ_m^* are disjoint if $n \neq m$. The classical uniqueness result for Sylvester's equations gives that P_n and P_m are orthogonal if $n \neq m$.

It is now our purpose to obtain a relation between the matrix-valued orthogonal polynomials $\{P_n\}_{n\geq 0}$ introduced above and the matrix hypergeometric function. We use the tools in [Tirao 03]. For a warmup, the reader can consult [Grünbaum et al. 03b], where these tools are used in the same fashion as below. The main idea is to replace elements in $M(3,\mathbb{C})$ by vectors in $\mathbb{C}^9$. We will denote this map by vec. It will be important to replace right and left multiplication in $M(3,\mathbb{C})$ by linear maps in $\mathbb{C}^9$.

This allows us to rewrite the differential equation above as the following equivalent differential equation:

$$t(1-t)\frac{d^2}{dt^2}\operatorname{vec}(P_n^*) + (C - t\tilde{U})\frac{d}{dt}\operatorname{vec}(P_n^*) - \tilde{T}\operatorname{vec}(P_n^*) = \Theta,$$

where C and $\widetilde{U}$ are the 9×9 matrices obtained by the rules (see [Horn and Johnson 91])

$$C = X\otimes I, \quad \widetilde{U} = U\otimes I, \quad \text{and} \quad \widetilde{T} = V\otimes I - I\otimes\Lambda_n^*.$$

Then $\widetilde{T}$ is given as follows:

$$\tilde{T} = \operatorname{diag}\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9\},$$

where

$$\begin{aligned} t_1 &= -n^2 - n(\alpha+\beta+3), \\ t_2 &= -n^2 - n(\alpha+\beta+4) - (\alpha+\beta-k+2), \\ t_3 &= -n^2 - n(\alpha+\beta+5) - 2(\alpha+\beta-k+3), \\ t_4 &= -n^2 - n(\alpha+\beta+3) + \alpha+\beta-k+2, \\ t_5 &= -n^2 - n(\alpha+\beta+4), \\ t_6 &= -n^2 - n(\alpha+\beta+5) - (\alpha+\beta-k+4), \\ t_7 &= -n^2 - n(\alpha+\beta+3) + 2(\alpha+\beta-k+3), \\ t_8 &= -n^2 - n(\alpha+\beta+4) + \alpha+\beta-k+4, \\ t_9 &= -n^2 - n(\alpha+\beta+5). \end{aligned} \tag{3–3}$$

Continuing with the strategy in [Tirao 03], we need to find matrices A and B such that

$$\tilde{U} = I + A + B \quad \text{and} \quad \tilde{T} = AB.$$

The fact that $\tilde{T}$ is diagonal will make this easier than it would be otherwise. This was the reason for choosing the conjugation that made V diagonal. The introduction of A and B allows us to rewrite our new equation in a form that is very much like the classical hypergeometric equation of Euler and Gauss; see (3–4).

The factorization in the last equation is not unique, and we look for A in the form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ \Theta & A_{22} & A_{23} \\ \Theta & \Theta & A_{33} \end{pmatrix},$$

where each A_{ij} is a 3×3 diagonal matrix. Once we have found A, the matrix B will be given by the expression

$$B = \tilde{U} - A - I.$$

With this form for B, the factorization above gives a number of conditions on A.

We put

$$A_{ii} = \begin{pmatrix} \phi_i & 0 & 0 \\ 0 & \varphi_i & 0 \\ 0 & 0 & \psi_i \end{pmatrix}, \quad i = 1, 2, 3,$$

with $\phi_i, \varphi_i, \psi_i$ to be determined later.

The elements of the diagonal of A_{12} are of the form

$$\frac{\gamma_1}{-\gamma_1 - \gamma_2 + \alpha + \beta + 4},$$

with $\gamma = \phi, \varphi, \psi$ in each entry.

If we denote by

$$\omega_{13} = \frac{2}{\alpha + \beta - k + 2}$$

and

$$\omega_{23} = -\frac{2(\alpha + \beta - k + 3)}{\alpha + \beta - k + 2}$$

the elements of the last column of U, the elements of the diagonal of A_{23} turn out to be of the form

$$\frac{-\omega_{23}\gamma_2}{-\gamma_2 - \gamma_3 + \alpha + \beta + 5},$$

with $\gamma = \phi, \varphi, \psi$ in each entry.

The elements of the diagonal of A_{13} are a bit more complicated. They are given by

$$-\frac{\omega_{13}\gamma_1}{-\gamma_1 - \gamma_3 + \alpha + \beta + 5} - \frac{\omega_{23}\gamma_1}{(-\gamma_1 - \gamma_3 + \alpha + \beta + 5)(-\gamma_1 - \gamma_2 + \alpha + \beta + 4)} - \frac{\omega_{23}\gamma_1\gamma_2}{(-\gamma_1 - \gamma_3 + \alpha + \beta + 5)(-\gamma_1 - \gamma_2 + \alpha + \beta + 4)(-\gamma_2 - \gamma_3 + \alpha + \beta + 5)},$$

with $\gamma = \phi, \varphi, \psi$ in each entry.

The parameters $\phi_i, \varphi_i, \psi_i$, $i = 1, 2, 3$, are subject to the following conditions, resulting from the factorization above:

$$\begin{cases} \phi_1 = -n \quad \text{or} \quad \phi_1 = n + \alpha + \beta + 3, \\ \varphi_1^2 - (\alpha + \beta + 3)\varphi_1 + t_2 = 0, \\ \psi_1^2 - (\alpha + \beta + 3)\psi_1 + t_3 = 0, \end{cases}$$

$$\begin{cases} \phi_2^2 - (\alpha + \beta + 4)\phi_2 + t_4 = 0, \\ \varphi_2 = -n \quad \text{or} \quad \varphi_2 = n + \alpha + \beta + 4, \\ \psi_2^2 - (\alpha + \beta + 4)\psi_2 + t_6 = 0, \end{cases}$$

$$\begin{cases} \phi_3^2 - (\alpha + \beta + 5)\phi_3 + t_7 = 0, \\ \varphi_3^2 - (\alpha + \beta + 5)\varphi_3 + t_8 = 0, \\ \psi_3 = -n \quad \text{or} \quad \psi_3 = n + \alpha + \beta + 5. \end{cases}$$

We have reached the main point of [Tirao 03], namely, that the vector $\operatorname{vec}(P_n^*)$ satisfies the matrix hypergeometric equation

$$t(1-t)\frac{d^2}{dt^2}\operatorname{vec}(P_n^*) + (C - t(I + A + B))\frac{d}{dt}\operatorname{vec}(P_n^*) - AB\operatorname{vec}(P_n^*) = \Theta. \tag{3–4}$$

Here, by abuse of notation, Θ denotes the null vector.

The eigenvalues of C are $\{\alpha+1, \alpha+2, \alpha+3\}$, each with multiplicity 3, so if $\alpha > -1$, we meet the conditions required to make sense of the following definition:

$$\begin{aligned}(C, A, B)_{i+1} &= (C + iI)^{-1}(A + iI)(B + iI)(C + (i-1)I)^{-1} \\ &\quad \times (A + (i-1)I)(B + (i-1)I) \cdots C^{-1}AB\end{aligned}$$

for all $i \geq 0$ and $(C, A, B)_0 = I$.

Then we define the *matrix hypergeometric function* as

$${}_2F_1(C, A, B; t) = \sum_{i \geq 0} (C, A, B)_i \frac{t^i}{i!}.$$

It is proved in [Tirao 03] that ${}_2F_1(C, A, B; t)$ is analytic on $|t| < 1$, and the analytic solutions at $t = 0$ of (3–4) are given by

$$\operatorname{vec}\big(\Phi(t)\big) = {}_2F_1(C, A, B; t)\operatorname{vec}\big(\Phi(0)\big).$$

The function ${}_2F_1(C, A, B; t)$ is not a polynomial function as in the classical case, but nevertheless we have that

$$\operatorname{vec}\big(P_n^*(t)\big) = {}_2F_1(C, A, B; t)\operatorname{vec}\big(P_n^*(0)\big) \tag{3–5}$$

is a vector-valued polynomial of degree n in t.

In (3–5) we can give explicitly the value of $P_n^*(0)$: Following the strategy explained around (3–2), we have

$P_n^*(0) = A_0^n$, and we obtain

$$P_n^*(0) = \begin{pmatrix} p_{11}(n) & p_{12}(n) & p_{13}(n) \\ 0 & p_{22}(n) & p_{23}(n) \\ 0 & 0 & p_{33}(n) \end{pmatrix},$$

where

$$p_{11}(n) = \frac{(-1)^n(\alpha+\beta-k+n+2)(\alpha+\beta-k+n+3)(\alpha+1)_n}{(\alpha+\beta-k+3)(\alpha+\beta-k+2)(\alpha+\beta+n+3)_n},$$

$$p_{12}(n) = \frac{2(-1)^n n(\beta-k+1)(\alpha+\beta-k+n+3)(\alpha+2)_{n-1}}{(k+n+1)(\alpha+\beta-k+4)(\alpha+\beta-k+2)(\alpha+\beta+n+4)_{n-1}},$$

$$p_{13}(n) = \frac{(-1)^n n(n-1)(\beta-k+1)(\beta-k+2)(\alpha+3)_{n-2}}{(k+n)(k+n+1)(\alpha+\beta-k+3)(\alpha+\beta-k+4)(\alpha+\beta+n+5)_{n-2}},$$

$$p_{22}(n) = \frac{(-1)^{n+1}(\alpha+\beta-k+2)(\alpha+\beta-k+n+4)(\alpha+2)_n}{n(\alpha+\beta-k+4)(\alpha+\beta+n+4)_n},$$

$$p_{23}(n) = \frac{(-1)^{n+1}(\beta-k+2)(\alpha+\beta-k+2)(\alpha+3)_{n-1}}{(k+n)(\alpha+\beta-k+4)(\alpha+\beta+n+5)_{n-1}},$$

$$p_{33}(n) = \frac{(-1)^n(\alpha+\beta-k+2)(\alpha+\beta-k+3)(\alpha+3)_n}{n(n+1)(\alpha+\beta+n+5)_n},$$

and $(a)_n$ denotes the shifted factorial defined by

$$(a)_n = a(a+1)\cdots(a+n-1)$$

for $n > 0,\ (a)_0 = 1,\ (a)_{-1} = (a)_{-2} = 0$, with a any real or complex number.

3.2 The Algebra of Operators

For convenience we use in this section the conjugation of the second-order differential operator D arising from the representation theory introduced in Section 2 instead of that introduced in Section 3.1.

This will give rise to a new family of orthogonal polynomials that are easily related to those obtained in the previous section. This new family has certain computational advantages compared to the previous one, but in principle one could use either of them.

The resulting operator is

$$D_1 = t(1-t)\frac{d^2}{dt^2} + \left[\begin{pmatrix} \alpha+3 & 0 & 0 \\ -1 & \alpha+2 & 0 \\ 0 & -2 & \alpha+1 \end{pmatrix} - t\begin{pmatrix} \alpha+\beta+4 & 0 & 0 \\ 0 & \alpha+\beta+5 & 0 \\ 0 & 0 & \alpha+\beta+6 \end{pmatrix}\right]\frac{d}{dt} + \begin{pmatrix} 0 & 2(\beta-k+1) & 0 \\ 0 & -(\alpha+\beta-k+2) & \beta-k+2 \\ 0 & 0 & -2(\alpha+\beta-k+3) \end{pmatrix}.$$

This operator has a sequence of (nonmonic) matrix-valued orthogonal polynomials $\{P_n\}_{n\geq 0}$ and

$$D_1 P_n^* = P_n^* \Lambda_n(D_1), \quad n = 0, 1, 2, \ldots,$$

where the eigenvalue can be chosen (by an argument like that given in Section 3.1) to be the same as that in Section 3.1,

$$\Lambda_n(D_1) = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix},$$

and the values of $t_i,\ i = 1, 2, 3$, are given in (3–3).

An explicit expression of the coefficients of these $\{P_n\}_{n\geq 0}$, using (3–2), is given in Section 5.1. These expressions feature some denominators whose nonvanishing is equivalent to the genericity assumptions made above.

The main goal of this section is to study the structure of the following set:

$$\mathcal{D} = \{D : DP_n^* = P_n^* \Lambda_n(D),\ n = 0, 1, 2, \ldots\}.$$

Here, D is a differential operator of *arbitrary order*. Using the fact that the leading coefficient of $P_n^*(t)$ is a nonsingular matrix, we can see that $D = \sum_{j=0}^r F_j(t)\partial_t^j$, $F_j(t)$ a matrix polynomial of degree less than or equal to $j,\ j = 0, \ldots, r$, with r the order of D. Clearly, $\mathcal{D}$ is both a complex vector space and an algebra under composition.

We observe that the map between differential operators and the corresponding eigenvalues given by

$$\Lambda_n : \mathcal{D} \longrightarrow M(3, \mathbb{C}), \quad n = 0, 1, 2, \ldots,$$

is a *faithful representation*. The property $\Lambda_n(D_1 D_2) = \Lambda_n(D_1)\Lambda_n(D_2)$ with $D_1, D_2 \in \mathcal{D}$ is easy to prove. If $\Lambda_n(D) = \Theta$ with $D \in \mathcal{D}$ for all n, then $DP_n^* = \Theta$, and we can easily conclude that $D = \Theta$.

For all the calculations in the rest of this section it is very useful to begin with monic matrix-valued orthogonal polynomials and then return to those considered at the beginning. It is important to note that the algebra $\mathcal{D}$ is independent of the choice of the family of matrix-valued orthogonal polynomials. The eigenvalues are changed by an n-dependent conjugation as noted earlier.

We now set out to solve the equations $DP_n^* = P_n^*\Lambda_n(D)$, where $D = \sum_{j=0}^r F_j(t)\partial_t^j$, $F_j(t)$ a matrix polynomial of degree less than or equal to $j,\ j = 0, \ldots, r$, with r the order of D and P_n^* as above.

In the following table, obtained by direct computations, we exhibit the number of *new* linearly independent differential operators that appear as one increases the order of the operators in question:

order	0	1	2	3	4	5	6	7	8	9	10	11	12
dimension	1	0	2	0	3	0	3	0	3	0	3	0	3

There are no odd-order differential operators, two linearly independent second-order differential operators, and three new linearly independent differential operators in each subsequent even order. We denote by $\mathfrak{D}_{2i}$ the space of differential operators of order less than or equal to $2i$ that have our family of orthogonal polynomials as their eigenfunctions.

A possible second-order differential operator linearly independent of D_1 is the following:

$$D_2 = \begin{pmatrix} t(1-t) & 0 & 0 \\ t/2 & t(1-t)/2 & 0 \\ 0 & -t & 0 \end{pmatrix} \frac{d^2}{dt^2}$$

$$+ \left[\begin{pmatrix} \alpha+\beta-k+4 & \beta-k+1 & 0 \\ -(\alpha+\beta-k+4)/2 & (\alpha+4)/2 & (\beta-k+2)/2 \\ 0 & -(\alpha+\beta-k+5) & -(\beta-k+2) \end{pmatrix} \right.$$

$$\left. -t \begin{pmatrix} \alpha+\beta+4 & \beta-k+1 & 0 \\ 0 & (\alpha+\beta+5)/2 & (\beta-k+2)/2 \\ 0 & 0 & \alpha+\beta+6 \end{pmatrix} \right] \frac{d}{dt}$$

$$+ \begin{pmatrix} 0 & -k(\beta-k+1) & 0 \\ 0 & k(\alpha+\beta-k+2)/2 & -k(\beta-k+2)/2 \\ 0 & 0 & k(\alpha+\beta-k+3) \end{pmatrix}.$$

The eigenvalue associated with D_2 is

$$\Lambda_n(D_2) = \begin{pmatrix} -n^2-n(\alpha+\beta+3) & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & k(\alpha+\beta-k+3) \end{pmatrix},$$

where $L = \frac{1}{2}\big(-n^2-n(\alpha+\beta+4)+k(\alpha+\beta-k+2)\big)$. This means in particular that $[D_1, D_2] = D_1D_2 - D_2D_1 = \Theta$.

We come now to the problem of exhibiting, for each value of $i = 2, 3, \ldots,$ a set of three linearly independent differential operators that will have order $2i$ and will span a subspace $\mathfrak{R}_{2i}$ such that

$$\mathfrak{D}_{2i} = \mathfrak{D}_{2i-2} \oplus \mathfrak{R}_{2i}.$$

In the case of $i = 2$ we can choose the set $\mathfrak{R}_4$ consisting of $\{D_1^2, D_2^2, D_1D_2\}$. One can easily check that these operators are linearly independent. Moreover, it is also easy to check, using the isomorphic bijection between differential operators and the corresponding eigenvalues, that the set $\{D_1^i, D_2^i, D_1^{i-1}D_2\}$, $i = 2, 3, \ldots,$ is linearly independent and can be chosen as a basis for $\mathfrak{R}_{2i}$. Since all these operators are obtained from D_1 and D_2, we can conjecture that the algebra $\mathcal{D}$ is abelian and coincides with the subalgebra $\mathcal{A}$ generated by $\{I, D_1, D_2\}$.

We turn our attention now to finding relations among these generators. One relation follows from what we said earlier regarding $\mathfrak{R}_6$. The operator $D_1D_2^2$ cannot be linearly independent of the elements in the basis of $\mathfrak{R}_6$. One is thus inclined to look for $s_1, s_2, \ldots, s_9$ such that

$$s_1I + s_2D_1 + s_3D_2 + s_4D_1^2 + s_5D_2^2 + s_6D_1D_2 + s_7D_1^3 + s_8D_2^3 + s_9D_1^2D_2 = D_1D_2^2. \tag{3–6}$$

The result is apparently rather unilluminating. Namely, we obtain

$$\begin{aligned}
s_1 &= 0,\\
s_2 &= -\frac{1}{3}k(k+1)(\alpha+\beta-k+3)(\alpha+\beta-k+2),\\
s_3 &= \frac{1}{3}k(k+1)(\alpha+\beta-k+3)(\alpha+\beta-k+2),\\
s_4 &= -\frac{1}{3}k(\alpha+\beta-k+3),\\
s_5 &= -\frac{2}{3} - \frac{1}{3}\alpha - k(\beta+1) + k^2 - k\alpha - \frac{1}{3}\beta - \frac{4}{3}k,\\
s_6 &= \frac{2}{3} + \frac{1}{3}\alpha + \frac{4}{3}k(\beta+1) - \frac{4}{3}k^2 + \frac{4}{3}k\alpha + \frac{1}{3}\beta + 2k,\\
s_7 &= 0, \quad s_8 = \frac{2}{3}, \quad s_9 = \frac{1}{3}.
\end{aligned} \tag{3–7}$$

Now something remarkable happens: If we put (3–7) in (3–6), we obtain the following amazing factorization:

$$\big(D_1 - D_2\big)\big(D_2 - k(\alpha+\beta-k+3)\big) \times \big(D_1 - 2D_2 + (1+k)(\alpha+\beta-k+2)\big) = \Theta. \tag{3–8}$$

We are grateful to an anonymous referee for pointing out a way to make this factorization quite transparent. The argument goes as follows: Since the algebra of operators generated by D_1 and D_2 is isomorphic to the quotient of $\mathbb{C}[x,y]$ (the algebra of polynomials in two commuting variables) by the ideal of polynomials $p \in \mathbb{C}[x,y]$ that satisfy $p\,(D_1, D_2) = \Theta$ and the mapping $\Lambda_n : \mathcal{D} \to M(3, \mathbb{C})$ is a faithful representation as indicated earlier, it follows that the condition $p\,(D_1, D_2) = \Theta$ is equivalent to

$$\Lambda_n(p(D_1, D_2)) = p(\Lambda_n(D_1), \Lambda_n(D_2)) = \Theta,$$

for $n = 0, 1, 2, \ldots$.

Since the matrices $\Lambda_n(D_1)$ and $\Lambda_n(D_2)$ are diagonal with eigenvalues $t_1(n)$, $t_2(n)$, $t_3(n)$, and $r_1(n) = t_1(n)$, $r_2(n) = \frac{1}{2}(t_2(n)+(k+1)(\alpha+\beta-k+2))$, $r_3(n) = k(\alpha+\beta-k+3)$, respectively, it is easy to spot three polynomials that should be included as factors of any polynomial in the ideal in question. These polynomials are

$$\begin{aligned}
x - y &\quad \text{from the entry} \quad (1,1),\\
x - 2y + (1+k)(\alpha+\beta-k+2) &\quad \text{from the entry} \quad (2,2),\\
y - k(\alpha+\beta-k+3) &\quad \text{from the entry} \quad (3,3).
\end{aligned}$$

These are the factors that appear in the simplest of the relations that is obtained by multiplying these factors and replacing x, y by D_1, D_2, namely (3–8).

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(\alpha+\beta-k_1+3) & 0 & 0 \\ 0 & 0 & -(\alpha+\beta-k_2+2) & 0 \\ 0 & 0 & 0 & -(2\alpha+2\beta-k_1-k_2+6) \end{pmatrix},$$

$$U = \begin{pmatrix} \alpha+\beta+4 & -1 & -1 & \frac{2\alpha+2\beta-k_1-k_2+6}{(\alpha+\beta-k_2+2)(\alpha+\beta-k_1+3)} \\ 0 & \alpha+\beta+5 & 0 & -\frac{(k_1-k_2)(\alpha+\beta-k_1+4)}{(\alpha+\beta-k_1+3)(k_1-k_2-1)} \\ 0 & 0 & \alpha+\beta+5 & -\frac{(k_1-k_2-2)(\alpha+\beta-k_2+3)}{(k_1-k_2-1)(\alpha+\beta-k_2+2)} \\ 0 & 0 & 0 & \alpha+\beta+6 \end{pmatrix},$$

$$X = \begin{pmatrix} C_{11}(\alpha,\beta,k_1,k_2) & \frac{(\alpha+1)(\alpha+\beta-k_2+3)}{(\alpha+\beta-k_2+2)(\alpha+\beta-k_1+3)} & \frac{(\alpha+1)(\alpha+\beta-k_1+4)}{(\alpha+\beta-k_2+2)(\alpha+\beta-k_1+3)} & 0 \\ \frac{(k_1-k_2)(\beta-k_1+2)}{(\alpha+\beta-k_1+3)(k_1-k_2-1)} & C_{22}(\alpha,\beta,k_1,k_2) & C_{23}(\alpha,\beta,k_1,k_2) & \frac{(k_1-k_2)(\alpha+2)}{(\alpha+\beta-k_2+3)(k_1-k_2-1)} \\ \frac{(k_1-k_2-2)(\beta-k_2+1)}{(k_1-k_2-1)(\alpha+\beta-k_2+2)} & C_{32}(\alpha,\beta,k_1,k_2) & C_{33}(\alpha,\beta,k_1,k_2) & \frac{(k_1-k_2-2)(\alpha+2)}{(\alpha+\beta-k_1+4)(k_1-k_2-1)} \\ 0 & \frac{(\alpha+\beta-k_1+3)(\beta-k_2+1)}{(\alpha+\beta-k_1+4)(\alpha+\beta-k_2+3)} & \frac{(\alpha+\beta-k_2+2)(\beta-k_1+2)}{(\alpha+\beta-k_1+4)(\alpha+\beta-k_2+3)} & C_{44}(\alpha,\beta,k_1,k_2) \end{pmatrix}.$$

FIGURE 3. Matrices defining V, U, and X for (4–1).

One can consider this curve as the analogue of the Burchnall–Chaundy curve, an algebraic-geometric object associated with a commutative algebra of differential operators acting on scalar functions. For a sample of the literature on this beautiful and deep subject, which traces its origin to some early work of J. Burchnall and T. Chaundy, see [Ince 26] and its references. This has become more recently a very active area, and one can consult [Krichever 78, Mumford 77, Van Moerbeke and Mumford 79].

It is worth noting that in the scalar case, as long as the eigenfunctions are orthogonal polynomials and the differential operators are of order two, the algebra is trivial. See [Miranian 05] for a proof.

It is important to note that it is only the product of the three factors in (3–8), and not the product of any smaller subset of them, that vanishes.

4. THE TWO-STEP EXAMPLE

4.1 Generating the Polynomial Eigenfunctions of the Differential Operator

We consider now the example resulting from the *two-step situation* described in the second half of Section 2. This subsection proceeds along the lines of Section 3.1.

As in the one-step case, we retain the symbol D for the new differential operator given in *hypergeometric form* by the expression

$$D = t(1-t)\frac{d^2}{dt^2} + (X - tU)\frac{d}{dt} + V, \qquad (4–1)$$

where V, U, and X are defined in Figure 3, in which

$$C_{11}(\alpha,\beta,k_1,k_2) = \frac{(\alpha+1)(\alpha+\beta-k_1+4)(\alpha+\beta-k_2+3)}{(\alpha+\beta-k_2+2)(\alpha+\beta-k_1+3)},$$

$$C_{23}(\alpha,\beta,k_1,k_2) = \frac{(k_1-k_2)(\beta-k_1+2)}{(k_1-k_2-1)(\alpha+\beta-k_1+3)(\alpha+\beta-k_2+3)},$$

$$C_{32}(\alpha,\beta,k_1,k_2) = \frac{(k_1-k_2-2)(\beta-k_2+1)}{(k_1-k_2-1)(\alpha+\beta-k_1+4)(\alpha+\beta-k_2+2)}.$$

The elements C_{22}, C_{33}, C_{44} are given in Section 5.2.

A sequence of (nonmonic) matrix-valued orthogonal polynomials $\{P_n\}_{n\geq 0}$ is then obtained by solving the following differential equation:

$$t(1-t)\frac{d^2}{dt^2}P_n^* + (X - tU)\frac{d}{dt}P_n^* + VP_n^* = P_n^*\Lambda_n,$$

where

$$\Lambda_n = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix}$$

and t_i, $i = 1, \ldots, 4$, are given below in (4–2). The reasoning behind this choice is exactly the same as in Section 3.1 and it is not repeated here.

Proceeding as in the previous section along the lines in [Tirao 03], we replace the differential operator by another operator acting on functions that take values on $\mathbb{C}^{16}$ and consider left and right multiplication by matrices in $M(4,\mathbb{C})$ as linear maps in $\mathbb{C}^{16}$. Thus we can consider the following equivalent differential equation:

$$t(1-t)\frac{d^2}{dt^2}\operatorname{vec}(P_n^*) + (C - t\tilde{U})\frac{d}{dt}\operatorname{vec}(P_n^*) - \tilde{T}\operatorname{vec}(P_n^*) = \Theta,$$

where C and $\widetilde{U}$ are the 16×16 matrices obtained from X and U respectively, in the same manner as was done earlier. A similar procedure is applied to obtain the matrix $\widetilde{T}$, and we have

$$\tilde{T} = \operatorname{diag}\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}\},$$

where

$$\begin{aligned}
t_1 &= -n^2 - n(\alpha+\beta+3),\\
t_2 &= -n^2 - n(\alpha+\beta+4) - (\alpha+\beta-k_1+3),\\
t_3 &= -n^2 - n(\alpha+\beta+4) - (\alpha+\beta-k_2+2),\\
t_4 &= -n^2 - n(\alpha+\beta+5) - (2\alpha+2\beta-k_1-k_2+6),\\
t_5 &= -n^2 - n(\alpha+\beta+3) + \alpha+\beta-k_1+3,\\
t_6 &= -n^2 - n(\alpha+\beta+4),\\
t_7 &= -n^2 - n(\alpha+\beta+4) + k_2-k_1+1,\\
t_8 &= -n^2 - n(\alpha+\beta+5) - (\alpha+\beta-k_2+3),\\
t_9 &= -n^2 - n(\alpha+\beta+3) + \alpha+\beta-k_2+2,\\
t_{10} &= -n^2 - n(\alpha+\beta+4) + k_1-k_2-1,\\
t_{11} &= -n^2 - n(\alpha+\beta+4),\\
t_{12} &= -n^2 - n(\alpha+\beta+5) - (\alpha+\beta-k_1+4),\\
t_{13} &= -n^2 - n(\alpha+\beta+3) + 2\alpha+2\beta-k_1-k_2+6,\\
t_{14} &= -n^2 - n(\alpha+\beta+4) + \alpha+\beta-k_2+3,\\
t_{15} &= -n^2 - n(\alpha+\beta+4) + \alpha+\beta-k_1+4,\\
t_{16} &= -n^2 - n(\alpha+\beta+5).
\end{aligned} \tag{4–2}$$

Now we find a pair of matrices A, B as in the previous section satisfying the nonlinear matrix equations

$$\tilde{U} = I + A + B \quad \text{and} \quad \tilde{T} = AB.$$

As in the previous section, we pick

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14}\\ \Theta & A_{22} & \Theta & A_{24}\\ \Theta & \Theta & A_{33} & A_{34}\\ \Theta & \Theta & \Theta & A_{44} \end{pmatrix}$$

and

$$B = \tilde{U} - A - I,$$

where each A_{ij} is a 4×4 diagonal matrix. Observe that $A_{23} = \Theta$, a fact that will have important consequences.

We put

$$A_{ii} = \begin{pmatrix} \phi_i & 0 & 0 & 0\\ 0 & \varphi_i & 0 & 0\\ 0 & 0 & \psi_i & 0\\ 0 & 0 & 0 & \xi_i \end{pmatrix},$$

for $i = 1, 2, 3, 4$, with ϕ_i, φ_i, ψ_i, ξ_i to be determined later.

The elements of the diagonal of A_{12} are of the form

$$\frac{\gamma_1}{-\gamma_1 - \gamma_2 + \alpha + \beta + 4},$$

with $\gamma = \phi, \varphi, \psi, \xi$ in each entry.

The elements of the diagonal of A_{13} are of the form

$$\frac{\gamma_1}{-\gamma_1 - \gamma_3 + \alpha + \beta + 4},$$

with $\gamma = \phi, \varphi, \psi, \xi$ in each entry.

If we denote by

$$\begin{aligned}
\omega_{14} &= \frac{2\alpha+2\beta-k_1-k_2+6}{(\alpha+\beta-k_2+2)(\alpha+\beta-k_1+3)},\\
\omega_{24} &= -\frac{(k_1-k_2)(\alpha+\beta-k_1+4)}{(\alpha+\beta-k_1+3)(k_1-k_2-1)},\\
\omega_{34} &= -\frac{(k_1-k_2-2)(\alpha+\beta-k_2+3)}{(k_1-k_2-1)(\alpha+\beta-k_2+2)}
\end{aligned}$$

the elements of the last column of U, the elements of the diagonal of A_{34} are of the form

$$\frac{-\omega_{34}\gamma_3}{-\gamma_3 - \gamma_4 + \alpha + \beta + 5},$$

with $\gamma = \phi, \varphi, \psi, \xi$ in each entry, and the elements of the diagonal of A_{24} are of the form

$$\frac{-\omega_{24}\gamma_2}{-\gamma_2 - \gamma_4 + \alpha + \beta + 5},$$

with $\gamma = \phi, \varphi, \psi, \xi$ in each entry.

The elements of the diagonal of A_{14} are a bit more complicated. They are given by

$$\begin{aligned}
&- \frac{\omega_{14}\gamma_1}{-\gamma_1 - \gamma_4 + \alpha + \beta + 5} - \frac{\omega_{24}\gamma_1}{(-\gamma_1 - \gamma_4 + \alpha + \beta + 5)(-\gamma_1 - \gamma_2 + \alpha + \beta + 4)}\\
&- \frac{\omega_{34}\gamma_1}{(-\gamma_1 - \gamma_4 + \alpha + \beta + 5)(-\gamma_1 - \gamma_3 + \alpha + \beta + 4)}\\
&- \frac{\omega_{24}\gamma_1\gamma_2}{(-\gamma_1 - \gamma_4 + \alpha + \beta + 5)(-\gamma_1 - \gamma_2 + \alpha + \beta + 4)(-\gamma_2 - \gamma_4 + \alpha + \beta + 5)}\\
&- \frac{\omega_{34}\gamma_1\gamma_3}{(-\gamma_1 - \gamma_4 + \alpha + \beta + 5)(-\gamma_1 - \gamma_3 + \alpha + \beta + 4)(-\gamma_3 - \gamma_4 + \alpha + \beta + 5)},
\end{aligned}$$

with $\gamma = \phi, \varphi, \psi, \xi$ in each entry.

$$
\begin{aligned}
p_{11}(n) &= \frac{(-1)^n(\alpha+\beta-k_1+n+3)(\alpha+\beta-k_2+n+2)(\alpha+1)_n}{(\alpha+\beta-k_1+3)(\alpha+\beta-k_2+2)(\alpha+\beta+n+3)_n},\\
p_{12}(n) &= \frac{(-1)^n n(k_2-k_1)(\beta-k_1+2)(\alpha+\beta-k_2+n+2)(\alpha+2)_{n-1}}{(k_1+n)(k_2-k_1+1)(\alpha+\beta-k_1+3)(\alpha+\beta-k_2+3)(\alpha+\beta+n+4)_{n-1}},\\
p_{13}(n) &= \frac{(-1)^n n(k_2-k_1+2)(\beta-k_2+1)(\alpha+\beta-k_1+n+3)(\alpha+2)_{n-1}}{(k_2+n+1)(k_2-k_1+1)(\alpha+\beta-k_1+4)(\alpha+\beta-k_2+2)(\alpha+\beta+n+4)_{n-1}},\\
p_{14}(n) &= \frac{(-1)^n n(n-1(\beta-k_2+1)(\beta-k_1+2)(\alpha+3)_{n-2}}{(k_1+n)(k_2+n+1)(\alpha+\beta-k_1+4)(\alpha+\beta-k_2+3)(\alpha+\beta+n+5)_{n-2}},\\
p_{22}(n) &= \frac{(-1)^{n+1}(\alpha+\beta-k_2+n+4)(\alpha+\beta-k_1+3)(\alpha+2)_n}{n(\alpha+\beta-k_2+3)(\alpha+\beta+n+4)_n},\\
p_{24}(n) &= \frac{(-1)^{n+1}(\beta-k_2+1)(\alpha+\beta-k_1+3)(\alpha+3)_{n-1}}{(k_2+n+1)(\alpha+\beta-k_2+3)(\alpha+\beta+n+5)_{n-1}},\\
p_{33}(n) &= \frac{(-1)^{n+1}(\alpha+\beta-k_1+n+4)(\alpha+\beta-k_2+2)(\alpha+2)_n}{(\alpha+\beta-k_1+4)(\alpha+\beta+n+4)_n},\\
p_{34}(n) &= \frac{(-1)^{n+1}(\beta-k_1+2)(\alpha+\beta-k_2+2)(\alpha+3)_{n-1}}{(k_1+n)(\alpha+\beta-k_1+4)(\alpha+\beta+n+5)_{n-1}},\\
p_{44}(n) &= \frac{(-1)^n(\alpha+\beta-k_1+3)(\alpha+\beta-k_2+2)(\alpha+3)_n}{n(n+1)(\alpha+\beta+n+5)_n}.
\end{aligned}
$$

FIGURE 4. The p_{ij} from the matrix of (4–3).

The parameters $\phi_i, \varphi_i, \psi_i, \xi_i$, $i = 1, 2, 3, 4$, are subject to the following conditions:

$$
\begin{cases}
\phi_1 = -n \quad \text{or} \quad \phi_1 = n+\alpha+\beta+3,\\
\varphi_1^2 - (\alpha+\beta+3)\varphi_1 + t_2 = 0,\\
\psi_1^2 - (\alpha+\beta+3)\psi_1 + t_3 = 0,\\
\xi_1^2 - (\alpha+\beta+3)\xi_1 + t_4 = 0,
\end{cases}
$$

$$
\begin{cases}
\phi_2^2 - (\alpha+\beta+4)\phi_2 + t_5 = 0,\\
\varphi_2 = -n \quad \text{or} \quad \varphi_2 = n+\alpha+\beta+4,\\
\psi_2^2 - (\alpha+\beta+4)\psi_2 + t_7 = 0,\\
\xi_2^2 - (\alpha+\beta+4)\xi_2 + t_8 = 0,
\end{cases}
$$

$$
\begin{cases}
\phi_3^2 - (\alpha+\beta+4)\phi_2 + t_9 = 0,\\
\varphi_3^2 - (\alpha+\beta+4)\varphi_3 + t_{10} = 0,\\
\psi_3 = -n \quad \text{or} \quad \psi_3 = n+\alpha+\beta+4,\\
\xi_3^2 - (\alpha+\beta+4)\xi_3 + t_{12} = 0,
\end{cases}
$$

$$
\begin{cases}
\phi_4^2 - (\alpha+\beta+5)\phi_4 + t_{13} = 0,\\
\varphi_4^2 - (\alpha+\beta+5)\varphi_4 + t_{14} = 0,\\
\psi_4^2 - (\alpha+\beta+5)\psi_4 + t_{15} = 0,\\
\xi_4 = -n \quad \text{or} \quad \xi_4 = n+\alpha+\beta+5.
\end{cases}
$$

Now $\mathrm{vec}(P_n^*)$ satisfies the matrix hypergeometric equation

$$
t(1-t)\frac{d^2}{dt^2}\,\mathrm{vec}(P_n^*) + (C - t(I+A+B))\frac{d}{dt}\,\mathrm{vec}(P_n^*) - AB\,\mathrm{vec}(P_n^*) = \Theta.
$$

The eigenvalues of C are $\{\alpha+1, \alpha+2, \alpha+3\}$ with $\alpha+1$ and $\alpha+3$ with multiplicity 4, and $\alpha+2$ with multiplicity 8. So if $\alpha > -1$, we meet the conditions (see [Tirao 03]) of the definition of the hypergeometric function, as in the one-step case.

The function ${}_2F_1(C, A, B; t)$ is not a polynomial function, as in the classical case, but nevertheless we get, as in [Tirao 03], the following family of vector-valued polynomials of degree n in t:

$$
\mathrm{vec}\big(P_n^*(t)\big) = {}_2F_1(C, A, B; t)\,\mathrm{vec}\big(P_n^*(0)\big).
$$

In this case we can also give explicitly the value of $P_n^*(0)$, as before, using (3–2):

$$
P_n^*(0) = \begin{pmatrix} p_{11}(n) & p_{12}(n) & p_{13}(n) & p_{14}(n)\\ 0 & p_{22}(n) & 0 & p_{24}(n)\\ 0 & 0 & p_{33}(n) & p_{34}(n)\\ 0 & 0 & 0 & p_{44}(n) \end{pmatrix}, \qquad (4\text{–}3)
$$

where the p_{ij} are given in Figure 4.

Note that $p_{23}(n) = 0$. This is related to the vanishing of the $(2, 3)$-block of A.

4.2 The Algebra of Operators

In this case, just as in the one-step example, we use a different conjugation of the second-order differential operator arising from representation theory and introduced at the end of Section 2. This is done only for computational convenience, and all these operators (and the corresponding orthogonal polynomials) are related to each other by simple conjugations.

$$A_2(t) = t(1-t),$$

$$A_1(t) = \begin{pmatrix} \alpha+3 & 0 & 0 & 0 \\ -1 & \alpha+2 & 0 & 0 \\ -1 & 0 & \alpha+2 & 0 \\ 0 & -\frac{k_2-k_1+2}{k_2-k_1+1} & -\frac{k_2-k_1}{k_2-k_1+1} & \alpha+1 \end{pmatrix} - t \begin{pmatrix} \alpha+\beta+4 & 0 & 0 & 0 \\ 0 & \alpha+\beta+5 & 0 & 0 \\ 0 & 0 & \alpha+\beta+5 & \\ 0 & 0 & 0 & \alpha+\beta+6 \end{pmatrix},$$

$$A_0(t) = \begin{pmatrix} 0 & \frac{(k_2-k_1+2)(\beta-k_2+1)}{k_2-k_1+1} & \frac{(k_2-k_1)(\beta-k_1+2)}{k_2-k_1+1} & 0 \\ 0 & -(\alpha+\beta+2)+k_2 & 0 & \beta-k_1+2 \\ 0 & 0 & -(\alpha+\beta+3)+k_1 & \beta-k_2+1 \\ 0 & 0 & 0 & -2(\alpha+\beta+3)+k_1+k_2 \end{pmatrix}.$$

FIGURE 5. A_2, A_1, A_0 from (4–4).

The operator now is

$$D_1 = A_2(t)\frac{d^2}{dt^2} + A_1(t)\frac{d}{dt} + A_0(t), \qquad (4\text{–}4)$$

with A_2, A_1, A_0 given in Figure 5.

The eigenvalue associated with D_1 can be chosen, as before, in the form

$$\Lambda_n(D_1) = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_3 & 0 & 0 \\ 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix},$$

where t_i, $i = 1, 2, 3, 4$, are given in (4–2). Note that t_2 and t_3 above do not appear now in the natural order.

An explicit expression of the coefficients of these $\{P_n\}_{n\geq 0}$, using (3–2), is given in Section 5.3. These expressions feature some denominators whose nonvanishing is equivalent to the genericity assumptions implicitly made in Section 4.1.

We now consider the analogue of the algebra $\mathcal{D}$ from Section 3.2. Recall that these differential operators have matrix-valued polynomial coefficients of the appropriate degrees.

One finds that the commutator of the operator D_1, within the class of operators of order less than or equal to two, has dimension four. A basis is made up of three operators of order two and the identity. One of the linearly independent operators is given by

$$D_2 = B_2(t)\frac{d^2}{dt^2} + B_1(t)\frac{d}{dt} + B_0(t), \qquad (4\text{–}5)$$

with B_2, B_1, B_0 given in Figure 6.

The eigenvalue associated with D_2 is

$$\Lambda_n(D_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_n & 0 \\ 0 & 0 & 0 & \eta_n \end{pmatrix},$$

where to save space we have defined

$$\zeta_n = -\frac{(n+k_1+1)(n+\alpha+\beta-k_1+3)(k_1-k_2-1)}{k_1-k_2},$$
$$\eta_n = -(n+k_1+1)(n+\alpha+\beta-k_1+4).$$

The last second-order linearly independent operator is given by

$$D_3 = C_2(t)\frac{d^2}{dt^2} + C_1(t)\frac{d}{dt} + C_0(t), \qquad (4\text{–}6)$$

with C_2, C_1, C_0 given in Figure 7.

The eigenvalue associated with D_3 is

$$\Lambda_n(D_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa_n & 0 & 0 \\ 0 & 0 & \mu_n & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix},$$

where again to save space we have defined

$$\kappa_n = -\frac{(n+k_2+2)(n+\alpha+\beta-k_2+2)}{k_1-k_2-2},$$
$$\mu_n = (n+k_1+1)(n+\alpha+\beta-k_1+3),$$
$$\lambda_n = n^2 + n(\alpha+\beta+5) + k_1(\alpha+\beta-k_1+2) + 2\alpha + 2\beta - k_2 + 6.$$

We show in the following table, by direct calculations, the growth of the dimensions of the space of differential operators in $\mathcal{D}$ as the order increases:

order	0	1	2	3	4	5	6	7	8	9	10	11	12
dimension	1	0	3	0	6	0	6	0	6	0	6	0	6

There are no odd-order differential operators, three linearly independent second-order differential operators, given by D_1, D_2, and D_3, and six new linearly independent differential operators in each subsequent even order. We denote by $\mathfrak{D}_{2i}$ the space of differential operators of

$$B_2(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k_1-k_2-1}{k_1-k_2}t & 0 & \frac{k_1-k_2-1}{k_1-k_2}t(1-t) & 0 \\ 0 & \frac{k_1-k_2-2}{k_1-k_2-1}t & \frac{1}{k_1-k_2-1}t & t(1-t) \end{pmatrix},$$

$$B_1(t) = \begin{pmatrix} -(\beta-k_1+2) & 0 & -(\beta-k_1+2) & 0 \\ 0 & -\frac{(\beta-k_1+2)(k_1-k_2-2)}{k_1-k_2-1} & -\frac{\beta-k_1+2}{k_1-k_2-1} & -(\beta-k_1+2) \\ \frac{(\alpha+\beta-k_1+3)(k_1-k_2-1)}{k_1-k_2} & -\frac{(\beta-k_2+1)(k_1-k_2-2)}{(k_1-k_2-1)(k_1-k_2)} & b_{33} & -\frac{\beta-k_2+1}{k_1-k_2} \\ 0 & \frac{(k_1-k_2-2)(\alpha+\beta-k_1+4)}{k_1-k_2-1} & \frac{\alpha+\beta-k_1+4}{k_1-k_2-1} & \alpha+\beta-k_1+4 \end{pmatrix}$$
$$- t \begin{pmatrix} 0 & 0 & -(\beta-k_1+2) & 0 \\ 0 & 0 & 0 & -(\beta-k_1+2) \\ 0 & 0 & \frac{(\alpha+\beta+5)(k_1-k_2-1)}{k_1-k_2} & -\frac{\beta-k_2+1}{k_1-k_2} \\ 0 & 0 & 0 & \alpha+\beta+6 \end{pmatrix},$$

$$B_0(t) = \begin{pmatrix} 0 & 0 & (1+k_1)(\beta-k_1+2) & 0 \\ 0 & 0 & 0 & (1+k_1)(\beta-k_1+2) \\ 0 & 0 & -\frac{(1+k_1)(\alpha+\beta-k_1+3)(k_1-k_2-1)}{k_1-k_2} & \frac{(1+k_1)(\beta-k_2+1)}{k_1-k_2} \\ 0 & 0 & 0 & -(1+k_1)(\alpha+\beta-k_1+4) \end{pmatrix},$$

where

$$b_{33} = \alpha+\beta-k_1+3+\frac{\beta-k_1+2}{k_2-k_1+1}-\frac{\alpha+2}{k_1-k_2}.$$

FIGURE 6. B_2, B_1, B_0 from (4–5).

$$C_2(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{k_1-k_2-2}t & \frac{1}{k_1-k_2-2}t(1-t) & 0 & 0 \\ -t & 0 & -t(1-t) & 0 \\ 0 & -t & 0 & -t(1-t) \end{pmatrix},$$

$$C_1(t) = \begin{pmatrix} \beta-k_1+1 & \frac{\beta-k_2+1}{k_1-k_2-1} & \frac{(k_1-k_2)(\beta-k_1+2)}{k_1-k_2+1} & 0 \\ \frac{\alpha+\beta-k_2+2}{k_1-k_2-2} & c_{22} & \frac{(\beta-k_1+2)(k_1-k_2)}{(k_1-k_2-2)(k_1-k_2-1)} & \frac{(\beta-k_1+2)(k_1-k_2-1)}{k_1-k_2-2} \\ -(\alpha+\beta-k_1+3) & \frac{\beta-k_2+1}{k_1-k_2-1} & -c_{33} & 0 \\ 0 & -c_{33}-\frac{1}{k_1-k_2-1} & \frac{k_1-k_2}{k_1-k_2-1} & -(\alpha+\beta-k_1+3) \end{pmatrix}$$
$$- t \begin{pmatrix} 0 & -\frac{\beta-k_2+1}{k_1-k_2-1} & \frac{(k_1-k_2)(\beta-k_1+2)}{k_1-k_2-1} & 0 \\ 0 & \frac{\alpha+\beta+5}{k_1-k_2-2} & 0 & \frac{(\beta-k_1+2)(k_1-k_2-1)}{k_1-k_2-2} \\ 0 & 0 & -(\alpha+\beta+5) & 0 \\ 0 & 0 & 0 & -(\alpha+\beta+6) \end{pmatrix},$$

$$C_0(t) = \begin{pmatrix} 0 & \frac{(2+k_2)(\beta-k_2+1)}{k_1-k_2-1} & -\frac{(1+k_1)(k_1-k_2)(\beta-k_1+2)}{k_1-k_2-1} & 0 \\ 0 & -\frac{(\alpha+\beta-k_2+2)(2+k_2)}{k_1-k_2-2} & 0 & \frac{(\beta-k_1+2)(k_2+2-k_1^2+k_1k_2)}{k_1-k_2-2} \\ 0 & 0 & (1+k_1)(\alpha+\beta-k_1+3) & -(\beta-k_2+1) \\ 0 & 0 & 0 & c_{44} \end{pmatrix},$$

where

$$c_{22} = \beta-k_1+3+\frac{\beta-k_1+2}{k_1-k_2-1}+\frac{\alpha+2}{k_1-k_2-2},$$
$$c_{33} = \alpha+\beta-k_1+4+\frac{\beta-k_1+2}{k_1-k_2-1},$$
$$c_{44} = 2\alpha+2\beta-k_1-k_2+6+k_1(\alpha+\beta-k_1+3).$$

FIGURE 7. C_2, C_1, C_0 from (4–6).

order less than or equal to $2i$ that have our family of orthogonal polynomials as their eigenfunctions.

We come now to the problem of exhibiting, for each value of $i = 2, 3, \dots,$ a set of six linearly independent differential operators that will have order $2i$ and will span a subspace $\mathfrak{R}_{2i}$ such that $\mathfrak{D}_{2i} = \mathfrak{D}_{2i-2} \oplus \mathfrak{R}_{2i}$.

For each $i = 2, 3, \dots,$ $\mathfrak{R}_{2i}$ has a basis consisting of four differential operators that commute with each other and two extra ones that do not commute with any other element in the base. This decomposition of $\mathfrak{R}_{2i}$ is spelled out in the table below.

We denote the two extra operators in $\mathfrak{R}_4$ alluded to above by E and F. The corresponding eigenvalues are given by

$$\Lambda_n(E) = (n+\alpha+\beta-k_1+3)(n+\alpha+\beta-k_2+3) \times (n+k_1)(n+k_1+1)E_{23},$$

and

$$\Lambda_n(F) = (n+\alpha+\beta-k_1+4)(n+\alpha+\beta-k_2+2) \times (n+k_2+1)(n+k_2+2)E_{32},$$

where E_{ij} denotes the 4×4 matrix with (i,j) entry equal to 1 and other entries equal to zero.

In the following table we summarize the decomposition of $\mathfrak{R}_{2i}$ discussed above:

	COMM	NON-COMM
$\mathfrak{R}_0$	I	
$\mathfrak{R}_2$	D_1, D_2, D_3	
$\mathfrak{R}_4$	$D_1^2, D_2^2, D_3^2, D_1D_2$	E, F
$\mathfrak{R}_6$	$D_1^3, D_2^3, D_3^3, D_1^2D_2$	D_1E, D_2F
$\mathfrak{R}_8$	$D_1^4, D_2^4, D_3^4, D_1^3D_2$	D_1^2E, D_2^2F
$\vdots$	$\vdots$	$\vdots$
$\mathfrak{R}_{2i}$	$D_1^i, D_2^i, D_3^i, D_1^{i-1}D_2$	$D_1^{i-2}E, D_2^{i-2}F$

This is, of course, not the only possible choice of a basis. The operators E and F themselves are given by

$$E = G_4(t)\frac{d^4}{dt^4} + G_3(t)\frac{d^3}{dt^3} + G_2(t)\frac{d^2}{dt^2} + G_1(t)\frac{d}{dt} + G_0(t)$$

and

$$F = H_4(t)\frac{d^4}{dt^4} + H_3(t)\frac{d^3}{dt^3} + H_2(t)\frac{d^2}{dt^2} + H_1(t)\frac{d}{dt} + H_0(t),$$

with $G_4(t)$ and $H_4(t)$ given by

$$G_4(t) = \frac{(\beta-k_1+2)(k_1-k_2)}{(\beta-k_2+1)}t^2 \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{k_1-k_2-2}(1-t) & 0 & \frac{1}{k_1-k_2-2}(1-t)^2 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{k_1-k_2-1} & 0 & -\frac{1}{k_1-k_2-2}(1-t) & 0 \end{pmatrix}$$

and

$$H_4(t) = \frac{(\beta-k_2+1)(k_1-k_2-2)}{(\beta-k_1+2)}t^2 \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{k_1-k_2}(1-t) & \frac{1}{k_1-k_2}(1-t)^2 & 0 & 0 \\ \frac{1}{k_2-k_1+1} & \frac{1}{k_2-k_1+1}(1-t) & 0 & 0 \end{pmatrix}.$$

The remaining coefficients are too unpleasant to be displayed here.

The operators D_1, D_2, and D_3 are symmetric with respect to $W(t)$ at the end of Section 2, whereas E and F can be seen not to be symmetric by considering the coefficients $G_4(t)$ and $H_4(t)$. The symmetry of E and F would require

$$WG_4 = G_4^*W \quad \text{and} \quad WH_4 = H_4^*W,$$

which is easily seen to be false.

In contrast to the one-step example, in this case we have found many relations among the generators

$$\{I, D_1, D_2, D_3, E, F\}.$$

We display a few of them below:

$$D_2\big(D_1 + D_3 - k_1(\alpha+\beta-k_1+3)\big) = \Theta,$$
$$\big[(k_1-k_2)D_2 + (k_1-k_2-1)D_3\big] \times \big[-D_1 + (k_1-k_2-1)D_2 + (k_1-k_2-2)D_3 + (1+k_2)(\alpha+\beta-k_2+2)\big] = \Theta,$$

$$D_1E + ED_3 = \big(k_1(\alpha+\beta-k_1+2)+1+k_2\big)E,$$
$$ED_1 - D_1E = (k_1-k_2-1)E,$$
$$FD_1 + D_3F = \big(k_1(\alpha+\beta-k_1+2)+1+k_2\big)F,$$
$$D_1F - FD_1 = (k_1-k_2-1)F,$$
$$D_2E = \Theta,$$
$$FD_2 = \Theta.$$

$$x_{11} = (-1)^l\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-2-\beta-\alpha-2n)(5n^2+2n+2k-2l+2ln^2+2ln-3kl^2+5kl+k^2l^2 - 3k^2l+7nk+2k^2+2l^2-l\beta^2-3l\beta+3l^2\beta+4kl\beta+3k^2n+n\alpha^2+3n\alpha+5n^2\alpha+k^2n^2+7kn^2 + 2kn^3+n^2\alpha^2+2n^3\alpha-2kln^2-2k^2nl+4\alpha kn^2+2\alpha k^2n+2\alpha^2kn+2ln^2\beta-2\alpha knl+2\alpha ln\beta + 2knl\beta+n^4+2nl\beta+8\alpha kn+2\alpha ln+4n^3+2\alpha kl+3\alpha k^2+3\alpha k+\alpha^2k^2-2\alpha k^2l+\alpha^2k+l^2\beta^2 - 2kl^2\beta+2\alpha kl\beta)\Big/(\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n)\Gamma(l-2-\beta-\alpha-2n)(k+n+1)(k+n)),$$

$$x_{12} = -\frac{(-1)^l(n\alpha+\alpha k+l\beta+2l+nk-kl+2k+n^2+2n)\Gamma(-\beta-3-\alpha-2n)\Gamma(l-2-\alpha-n)\Gamma(n+1)}{(k+n)\Gamma(n-l+1)\Gamma(l-3-\beta-\alpha-2n)\Gamma(-\alpha-n-1)\Gamma(l+1)},$$

$$x_{13} = \frac{(-1)^l\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-4-\alpha-2n)}{\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n-2)\Gamma(l-4-\beta-\alpha-2n)},$$

$$x_{21} = \frac{(-1)^l(n\alpha+\alpha k+l\beta+2l-\beta-2+nk-kl+2k+n^2+n)\Gamma(-2-\beta-\alpha-2n)\Gamma(l-2-\alpha-n)\Gamma(n+1)}{(k+n)(k+n+1)\Gamma(n-l+1)\Gamma(l-3-\beta-\alpha-2n)\Gamma(-\alpha-n)\Gamma(l)},$$

$$x_{22} = (-1)^l\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)(-4lnk+4n^2+4n+4k+12l-ln^2+4ln+2kl^2 - 10kl+k^2l+2nk-2k^2-4l^2+2k\beta+2l\beta^2+10l\beta-2l^2\beta+2l\alpha\beta+\alpha k\beta-3kl\beta+n^2\beta-k^2n + n\alpha^2+2n\beta+4n\alpha+2n^2\alpha+3nl\beta+kn\beta+\alpha kn-\alpha ln+n\alpha\beta+n^3-3\alpha kl-\alpha k^2+4\alpha k+4l\alpha+\alpha^2k)\Big/ (2\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l-3-\beta-\alpha-2n)(\beta+1-k)(k+n)),$$

$$x_{23} = -\frac{(-1)^l(\beta+3+\alpha-k+n-l)\Gamma(-\beta-4-\alpha-2n)\Gamma(l-2-\alpha-n)\Gamma(n+1)}{(\beta+1-k)\Gamma(l-4-\beta-\alpha-2n)\Gamma(-\alpha-n-2)\Gamma(n-l+1)\Gamma(l+1)},$$

$$x_{31} = \frac{(-1)^l\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-2-\beta-\alpha-2n)}{(k+n)(k+n+1)\Gamma(l-1)\Gamma(n-l+1)\Gamma(-\alpha-n)\Gamma(l-4-\beta-\alpha-2n)},$$

$$x_{32} = \frac{(-1)^l(\beta+3+\alpha-k+n-l)\Gamma(-\beta-3-\alpha-2n)\Gamma(l-2-\alpha-n)\Gamma(n+1)}{(k+n)(\beta+1-k)\Gamma(l-4-\beta-\alpha-2n)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l)},$$

$$x_{33} = \frac{(-1)^l(\beta+4+\alpha-k+n-l)(\beta+3+\alpha-k+n-l)\Gamma(-\beta-4-\alpha-2n)\Gamma(l-2-\alpha-n)\Gamma(n+1)}{(\beta+1-k)(\beta+2-k)\Gamma(l-4-\beta-\alpha-2n)\Gamma(-\alpha-n-2)\Gamma(n-l+1)\Gamma(l+1)}.$$

FIGURE 8. The x_{ij} from (5–1). Here Γ denotes the Euler gamma function.

$$C_{22} = (18+k_1^2\alpha\beta-k_2^2\alpha^2+21\alpha-k_2^2\alpha\beta+9k_2\alpha\beta-k_1^2k_2\alpha+k_2\alpha^3+k_1k_2^2\alpha+k_2\alpha\beta^2-k_1\alpha\beta^2+2k_2\alpha^2\beta-2k_1\alpha^2\beta - k_1k_2\alpha-11k_1\alpha\beta+k_1^2\alpha^2-27k_1+12\beta+15k_2-5k_2^2+8\alpha^2+8k_1^2+\alpha^3-k_1\alpha^3-k_1k_2-27k_1\alpha+6k_1^2\alpha - 4k_2^2\alpha-9k_1\alpha^2+7k_2\alpha^2-2k_1^2k_2+2k_1k_2^2+17k_2\alpha+2\beta^2+\alpha\beta^2+10\alpha\beta+2\alpha^2\beta-2k_1\beta^2-15k_1\beta+2k_2\beta^2 + 11k_2\beta+2k_1^2\beta-2k_2^2\beta)/((k_2-k_1+1)(\alpha+\beta-k_1+3)(\alpha+\beta-k_2+3)),$$

$$C_{33} = (16+k_1^2\alpha\beta-k_2^2\alpha^2+18\alpha-k_2^2\alpha\beta+9k_2\alpha\beta-k_1^2k_2\alpha+k_2\alpha^3+k_1k_2^2\alpha+k_2\alpha\beta^2-k_1\alpha\beta^2+2k_2\alpha^2\beta-2k_1\alpha^2\beta + 5k_1k_2\alpha-11k_1\alpha\beta+k_1^2\alpha^2-18k_1+14\beta+4k_2-10k_2^2+8\alpha^2+3k_1^2+\alpha^3-k_1\alpha^3+9k_1k_2-21k_1\alpha+3k_1^2\alpha - 7k_2^2\alpha-9k_1\alpha^2+7k_2\alpha^2-2k_1^2k_2+2k_1k_2^2+11k_2\alpha+2\beta^2+\alpha\beta^2+10\alpha\beta+2\alpha^2\beta-2k_1\beta^2-15k_1\beta+2k_2\beta^2 + 11k_2\beta+2k_1^2\beta-2k_2^2\beta)/((k_2-k_1+1)(\alpha+\beta-k_1+4)(\alpha+\beta-k_2+2)),$$

$$C_{44} = (24+23\alpha-k_2\alpha\beta+k_1k_2\alpha-k_1\alpha\beta-7k_1+17\beta-10k_2+8\alpha^2+\alpha^3+3k_1k_2-5k_1\alpha-k_1\alpha^2-k_2\alpha^2-6k_2\alpha + 3\beta^2+\alpha\beta^2+11\alpha\beta+2\alpha^2\beta-3k_1\beta-3k_2\beta)/((\alpha+\beta-k_1+4)(\alpha+\beta-k_2+2)).$$

FIGURE 9. Values of C_{22}, C_{33}, C_{44} of Section 4.1.

$$
\begin{aligned}
x_{11} = {} & (-1)^l \Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-2-\beta-\alpha-2n)(-ln^2k_1 - ln^2k_2 + \alpha k_2 l\beta - k_2 l^2\beta - k_1 l^2\beta + 5n^2 + 2k_1 l\beta \\
& + \alpha k_1 l\beta + 2k_2 l\beta + 2nk_2 + 5nk_1 + 2k_1k_2 - l\beta^2 + 3l^2\beta + 3n^2k_2 + 4n^2k_1 - 3l\beta + 3nk_1k_2 - k_1 l^2 - 2l \\
& + 2n + 3\alpha k_1 + 2k_1 + 2l^2 - 2k_2 l^2 + 4k_2 l + k_1 l + lnk_2 - lnk_1 + 2ln + 2ln^2 + k_1k_2l^2 - 3k_1k_2 l + l^2\beta^2 \\
& + \alpha^2 k_1 + \alpha^2 k_1 k_2 + 3\alpha k_1 k_2 + k_1 n^3 + 2\alpha k_2 l + 2\alpha k_1 k_2 n - \alpha k_1 nl - \alpha k_2 nl + k_1 nl\beta + k_2 n^3 + 5n^2\alpha \\
& + 2nl\beta + 4n^3 + k_1k_2n^2 + k_2 nl\beta + 3\alpha k_2 n + 2\alpha k_2 n^2 + 5\alpha k_1 n + 2\alpha k_1 n^2 + 2\alpha nl + \alpha^2 k_1 n + n\alpha^2 k_2 \\
& + 3n\alpha + n\alpha^2 - 2k_1k_2nl - 2\alpha k_1 k_2 l + 2\alpha nl\beta + 2n^3\alpha + n^4 + 2ln^2\beta + n^2\alpha^2)/ \\
& \qquad (\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n)\Gamma(l-2-\beta-\alpha-2n)(n+k_2+1)(k_1+n)),
\end{aligned}
$$

$$
x_{12} = \frac{(-1)^{l+1}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)(2l+2k_1+\alpha k_1+l\beta+n^2+n\alpha+2n-k_1 l+nk_1)}{\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l-3-\beta-\alpha-2n)(k_1+n)},
$$

$$
x_{13} = \frac{(-1)^{l+1}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)(2k_2+2+\alpha+l+n^2+3n+\alpha k_2+nk_2+n\alpha+l\beta-k_2 l)}{(n+k_2+1)\Gamma(l-3-\beta-\alpha-2n)\Gamma(-\alpha-n-1)\Gamma(n-l+1)\Gamma(l+1)},
$$

$$
x_{14} = \frac{(-1)^l\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-4-\alpha-2n)}{\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n-2)\Gamma(l-4-\beta-\alpha-2n)},
$$

$$
x_{21} = \frac{(-1)^l\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-2-\beta-\alpha-2n)(n\alpha+\alpha k_1+l\beta+2l-\beta-2+nk_1-k_1 l+2k_1+n^2-n)}{(k_1+n)(n+k_2+1)\Gamma(l)\Gamma(n-l+1)\Gamma(-\alpha-n)\Gamma(l-3-\beta-\alpha-2n)},
$$

$$
\begin{aligned}
x_{22} = {} & (-1)^l\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)(ln^2k_1 - ln^2k_2 + k_2^2k_1 l + 2nk_1^2 l - 2k_2^2 l - k_1^2k_2 l + \alpha k_2 l\beta \\
& - k_2 l^2\beta + k_1 l^2\beta + 4n^2 + \alpha k_1\beta + 2k_1k_2\beta - k_2^2 l\beta - k_1 l\beta^2 - 6k_1 l\beta - \alpha k_1 l\beta + \alpha k_1 k_2\beta + k_1^2 l\beta + k_2 l\beta^2 + 3k_2 l\beta \\
& + 2nk_2 + 2k_1k_2 - 4k_1^2 - k_1^2\beta n + 2l\beta^2 + 2\alpha l\beta - 2l^2\beta + 3n^2k_2 - 2nk_2^2 - 3n^2k_1 + 10l\beta + 2k_1\beta - k_1^2 l^2 + 4k_1^2 l \\
& - 2k_2^2k_1 - 4nk_1^2 + 2k_1^2k_2 + 5nk_1k_2 + 4k_1 l^2 + 12l + 4n + 4\alpha k_1 + 4\alpha l + 4k_1 - 4l^2 - 2k_1^2\beta - 2k_2 l^2 + 2k_2 l \\
& - 12k_1 l - 4lnk_1 + 4ln - ln^2 + k_1k_2l^2 - k_1k_2 l - \alpha^2 k_1^2 - k_1^2\beta\alpha + \alpha^2 k_1 - 4\alpha k_1^2 + \alpha^2 k_1 k_2 + 3\alpha k_1 k_2 + \alpha k_1^2 k_2 \\
& - \alpha k_2^2 k_1 - k_1 n^3 - 5\alpha k_1 l + 2\alpha k_2 l + k_1k_2n\beta + 3\alpha k_1 k_2 n + \alpha k_1 nl - \alpha k_2 nl + n^2\beta - k_1 nl\beta - \alpha k_1 n\beta + \alpha k_2 n\beta \\
& + k_2 n^3 + 2n^2\alpha - k_1^2 n^2 - k_2^2 n^2 + 2n\beta - k_2^2 k_1 n + k_1^2 k_2 n + 3nl\beta - k_1 n^2\beta - k_1 n\beta + k_2 n^2\beta + 2k_2 n\beta + n^3 \\
& + 2k_1k_2n^2 + k_2 nl\beta + 3\alpha k_2 n + 2\alpha k_2 n^2 - 2\alpha k_1 n - 2\alpha k_1 n^2 - n\alpha k_2^2 - 2\alpha k_1^2 n - \alpha nl - \alpha^2 k_1 n \\
& + n\alpha^2 k_2 + 4n\alpha + n\alpha^2 + n\alpha\beta - 2k_1k_2nl + 2\alpha k_1^2 l - 2\alpha k_1 k_2 l)/ \\
& \qquad (\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l-3-\beta-\alpha-2n)(\beta+1-k_2)(k_1+n)(k_2-k_1+2)),
\end{aligned}
$$

$$
x_{23} = \frac{(-1)^{l+1}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)}{(n+k_2+1)\Gamma(l)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l-4-\beta-\alpha-2n)},
$$

$$
x_{24} = \frac{(-1)^{l+1}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-4-\alpha-2n)(\beta+3+\alpha-k_2+n-l)}{(\beta+1-k_2)\Gamma(l-4-\beta-\alpha-2n)\Gamma(-\alpha-n-2)\Gamma(n-l+1)\Gamma(l+1)},
$$

$$
x_{31} = \frac{(-1)^l\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-2-\beta-\alpha-2n)(n\alpha+\alpha+\alpha k_2+l\beta+l-\beta+nk_2-k_2 l+2k_2+n^2+2n)}{(k_1+n)(n+k_2+1)\Gamma(l)\Gamma(n-l+1)\Gamma(-\alpha-n)\Gamma(l-3-\beta-\alpha-2n)},
$$

$$
x_{32} = \frac{(-1)^{l+1}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)}{(k_1+n)\Gamma(l)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l-4-\beta-\alpha-2n)},
$$

$$
\begin{aligned}
x_{33} = {} & (-1)^{l+1}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)(6 + ln^2k_1 - 2lnk_2^2 - ln^2k_2 + k_2^2k_1 l - 5k_2^2 l \\
& - k_1^2k_2 l + \alpha k_2 l\beta - k_2 l^2\beta + k_1 l^2\beta + 6n^2 - \alpha k_1\beta + \alpha k_2^2\beta - 2k_1k_2\beta + 2\alpha k_2\beta - k_2^2 l\beta - k_1 l\beta^2 - 5k_1 l\beta \\
& - \alpha k_1 l\beta - \alpha k_1 k_2\beta + k_1^2 l\beta + k_2 l\beta^2 + 4k_2 l\beta + \alpha^2 + 16nk_2 - 14nk_1 - 10k_1k_2 + 2k_1^2 + 6k_2^2 + \alpha\beta \\
& + 7n^2k_2 + 5nk_2^2 - 7n^2k_1 - l\beta - 2k_1\beta + k_1^2 l - 2k_2^2k_1 + 3nk_1^2 + 2k_1^2k_2 + k_2^2 l^2 + 2k_2^2\beta - 9nk_1k_2 + k_1 l^2 \\
& - 3l + 4k_2\beta + 11n + 10\alpha k_2 - 6\alpha k_1 - \alpha l + 12k_2 - 8k_1 + 5\alpha - k_2 l^2 - 2k_1 l - 2lnk_2 + 2lnk_1 - 4ln \\
& - ln^2 - k_1k_2l^2 + 5k_1k_2 l + \alpha^2 k_2^2 - \alpha^2 k_1 + 2\alpha^2 k_2 + \alpha k_1^2 + 5\alpha k_2^2 - \alpha^2 k_1 k_2 - 7\alpha k_1 k_2 + \alpha k_1^2 k_2 + 2\beta \\
& - \alpha k_2^2 k_1 - 2\alpha k_2^2 l - k_1 n^3 - \alpha k_2 l - k_1k_2n\beta - 3\alpha k_1 k_2 n + \alpha k_1 nl - \alpha k_2 nl + n^2\beta - k_1 nl\beta - \alpha k_1 n\beta \\
& + \alpha k_2 n\beta + k_2 n^3 + 2n^2\alpha + k_1^2 n^2 + k_2^2 n^2 + 3n\beta - k_2^2 k_1 n + k_1^2 k_2 n + k_2^2 n\beta - nl\beta - k_1 n^2\beta - 3k_1 n\beta \\
& + k_2 n^2\beta + 4k_2 n\beta + n^3 - 2k_1k_2n^2 + k_2 nl\beta + 9\alpha k_2 n + 2\alpha k_2 n^2 - 8\alpha k_1 n - 2\alpha k_1 n^2 + 2n\alpha k_2^2 + \alpha k_1^2 n \\
& - \alpha nl - \alpha^2 k_1 n + n\alpha^2 k_2 + 7n\alpha + n\alpha^2 + n\alpha\beta + 2k_1k_2nl + 2\alpha k_1 k_2 l)/ \\
& \qquad (\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l-3-\beta-\alpha-2n)(n+k_2+1)(k_1-k_2)(\beta+2-k_1)),
\end{aligned}
$$

FIGURE 10. The x_{ij} values from (5–2), part 1.

$$x_{34} = \frac{(-1)^{l+1}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-4-\alpha-2n)(\beta+4+\alpha-k_1+n-l)}{\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n-2)\Gamma(l-4-\beta-\alpha-2n)(\beta+2-k_1)},$$

$$x_{41} = \frac{(-1)^{l}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-2-\beta-\alpha-2n)}{(k_1+n)(n+k_2+1)\Gamma(l-1)\Gamma(n-l+1)\Gamma(-\alpha-n)\Gamma(l-4-\beta-\alpha-2n)},$$

$$x_{42} = \frac{(-1)^{l}(\beta+3+\alpha-k_2+n-l)\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)}{(k_1+n)(\beta+1-k_2)\Gamma(l)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l-4-\beta-\alpha-2n)},$$

$$x_{43} = \frac{(-1)^{l}(\beta+4+\alpha-k_1+n-l)\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-3-\alpha-2n)}{(n+k_2+1)(\beta+2-k_1)\Gamma(l)\Gamma(n-l+1)\Gamma(-\alpha-n-1)\Gamma(l-4-\beta-\alpha-2n)},$$

$$x_{44} = \frac{(-1)^{l}\Gamma(n+1)\Gamma(l-2-\alpha-n)\Gamma(-\beta-4-\alpha-2n)(\beta+3+\alpha-k_2+n-l)(\beta+4+\alpha-k_1+n-l)}{\Gamma(l+1)\Gamma(n-l+1)\Gamma(-\alpha-n-2)\Gamma(l-4-\beta-\alpha-2n)(\beta+1-k_2)(\beta+2-k_1)}.$$

FIGURE 11. The x_{ij} values from (5–2), part 2.

The expressions of EF and FE are obtained in terms of the elements $\{I, D_2, D_3\}$. For instance, we have

$$\begin{aligned}\frac{(k_1-k_2-1)^3}{(k_1-k_2)^2}FE &= D_2\big[(k_1-k_2)D_2 + (k_1-k_2-1)(\alpha+\beta-2k_1+3)\big]\\ &\quad\times\big[(k_2-k_1)D_2 - (k_1-k_2-1)^2(\alpha+\beta-k_1-k_2+1)\big]\\ &\quad\times\big[D_2+D_3-(\alpha+\beta-k_1-k_2+2)\big].\end{aligned}$$

The set of all relations among the generators (something we have not found) should play the role of the Burchnall–Chaundy curve.

We close our look at this two-step example by stating the conjecture that $\mathcal{D}$ coincides with the subalgebra $\mathcal{A}$ generated by $\{I, D_1, D_2, D_3, E, F\}$.

5. APPENDIX

5.1 Explicit Expression of $\{P_n\}_{n\geq 0}$ for the One-Step Example

The polynomials introduced in Section 3.2 satisfy $D_1(P_n^*) = P_n^*\Lambda_n(D_1)$ and can be expressed, after an appropriate choice of the three parameters, as

$$P_n^*(t) = \sum_{l=0}^{n} A_l^n t^{n-l},$$

where

$$A_l^n = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \qquad (5\text{–}1)$$

with the x_{ij} given in Figure 8.

5.2 Some Constants

In this subsection we give the values of C_{22}, C_{33}, C_{44} of Section 4.1. They appear in Figure 9

5.3 Explicit Expression of $\{P_n\}_{n\geq 0}$ for the Two-Step Example

The polynomials introduced in Section 4.2 satisfy $D_1(P_n^*) = P_n^*\Lambda_n(D_1)$ and can be expressed, after an appropriate choice of the four parameters, as

$$P_n^*(t) = \sum_{l=0}^{n} A_l^n t^{n-l},$$

where

$$A_l^n = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} \qquad (5\text{–}2)$$

with the x_{ij} given in Figures 10 and 11.

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