A Tentative Classification of Bijective Polygonal Piecewise Isometries

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CONTENTS

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We aim to give a classification of Euclidian bijective polygonal piecewise isometries with a finite number of compact polygonal atoms. We rely on a specific type of triangulation process that enables us to describe a notion of combinatorial type similar to its one-dimensional counterpart for interval exchange maps. Moreover, it is possible to handle all the possible piecewise isometries, given two combinatorial types. We show that most of the examples treated in the literature of piecewise isometries can be retrieved by systematic computations. The computations yield new examples with apparently interesting behavior, but they still have to be studied in more detail. We also exhibit a new class of maps, the piecewise similarities, which fit nicely in this framework and whose behavior is shown to be highly nontrivial.

1. INTRODUCTION

Piecewise isometries (PWIs) are simply defined objects consisting of a partition of a domain of $\mathbb{R}^d$ with an isometry attached to each piece. Such simple maps may yield sophisticated dynamics though of zero entropy [Buzzi 01]. Of particular interest is the case in which the map is bijective, i.e., the partition is mapped onto another partition (perhaps up to the boundaries of the atoms). In one dimension, we have the interval translation maps and interval exchange maps (IEMs), which have been intensively studied [Boshernitzan and Kornfeld 95, Rauzy 79], with heavy use of renormalization techniques. With higher-dimensional domains, the situation is much more complicated for several reasons.

First, there is obviously much more freedom in the choice of the partitions than in the one-dimensional case. We will restrict our investigation to the two-dimensional cases with polygonal partitions. Then the isometry group is bigger, and in contrast to the one-dimensional case, we have rotations. A translation of the torus is an easy example of a PWI of higher dimension and has been extensively studied. Still, this example does not involve any rotations.

To our knowledge, the first examples involving both translations and rotations appeared in engineering problems related to overflow in digital filters [Chua and Lin 88]. Starting from these observations, Adler, Kitchens, and Tresser [Adler et al. 01] introduced a one-parameter family of PWIs of the rhombus. An interesting feature of these maps is the coexistence of numerous "periodic islands" and minimal dynamics. For a few values of the parameter they were able to exhibit self-similarity and hence to describe the dynamics.

For all other values, though, they provided more questions than answers. Independently, Goetz and Boshernitzan tackled the case of two half-planes and developed the fruitful idea of self-similarity in other examples (see [Goetz and Sammis 01]). Poggiaspalla and Goetz [Goetz and Poggiaspalla 04] constructed and studied yet another family of examples (towers) and found partial self-similarity. PWIs also appeared naturally in a more arithmetic context after work by Vivaldi and Lowenstein [Lowenstein et al. 97, Lowenstein and Vivaldi 98, Koupstov et al. 03] on discretized rotations.

One of the difficulties in understanding the phenomena of importance is related to the lack of large interesting classes and the relatively poor number of "typical" examples. This is especially true for bijective PWIs. Indeed, even in the polygonal case, it is not easy to find bijective PWIs. Whether, given a set of polygons, it is possible to arrange them in two distinct ways to draw the same figure (as in a tangram) is in itself a nice combinatorial problem.

A first classification of PWIs was proposed by Ashwin and Fu [Ashwin and Fu 02] but there was no attempt at being systematic. We propose a systematic way to describe *all* the PWIs with polygonal domains and partitions. PWIs are naturally embedded in the larger class of piecewise similarities (PWSs). To our knowledge, those maps have not been studied, although they are a possible generalization of the so-called affine interval exchange maps (see, for example, [Camelier and Gutierrez 97]). We provide an algorithmic and geometric description of the set of PWSs.

The point of view we have adopted mimics the roadmap followed for IEMs. We try to distinguish the "combinatorial" aspect and the "real-parameter" aspect. To be more specific, recall that an IEM is easily described by the *permutation* of the intervals and the *lengths* of the intervals.

For any permutation, each set of lengths yields an IEM. This point of view is fruitful as soon as we deal with renormalization. Induction on a well-chosen interval yields a new IEM (i.e., a new permutation and a new set of lengths). The dynamics of this renormalization are interesting from the combinatorial point of view (e.g., Rauzy classes; see [Rauzy 79]) and more generally since it is closely related to generalizations of continued-fraction algorithms.

In two dimensions, the first difficulty is to decide what should play the role of the "combinatorial type" (i.e., of the permutation). It is not a restriction to limit the analysis to triangulations of a triangular domain. But still, it is not obvious how one is to decide what a combinatorial type should be.

To overcome this difficulty, we introduce a specific type of triangulation, namely, triangulations by bisections (so-called nice triangulations). They are "nice" because all bijective PWIs with polygonal domain and partition can be defined using such partitions (see Proposition 3.3), and they prove relatively easy to manipulate. Each such partition is described by a set of combinatorial data, which we call the combinatorial type, and a set of continuous parameters. Notice that it is fairly simple to enumerate all the combinatorial types. It is thus possible to describe by linear equations all the bijective PWSs, and thus recover all the PWIs compatible with a given pair of partition types.

The paper will be structured as follows:

In Section 2, we introduce the notation and definitions used to manipulate partitions and PWI.

In Section 3, we propose a framework using the idea of triangulation by successive bisections. The idea is the following. We start with an initial triangle with unspecified angles. Then we choose one of its three vertices and cut the triangle from this vertex to the opposite side. The construction itself is associated with a continuous (three-

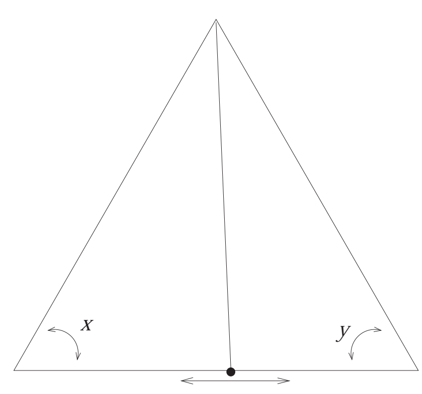

FIGURE 1. By splitting a triangle from a vertex, we are led to a system with three degrees of freedom: x, y, and z in the figure.

dimensional) family of parameters, namely the angles of the two triangles given by the splitting. These parameters are not all independent of one another. We have two degrees of freedom for the triangle itself and one more for the bisector; see Figure 1.

Since the two atoms of the new partition are triangles, the procedure can be iterated in each atom. Finally, a type of partition is described by a sequence (or a tree) of bisections. Roughly speaking, it is a way to organize a triangular partition. We give a formal definition and explain how to make intuitive use of it through lists of triangles. We also stress the fact that this object is not a partition of a triangle; it describes a continuous family of partitions (for n triangles we have $n+1$ parameters). If most partitions are of only one type, it can happen that a partition belongs to several types. An alternative way to specify completely a partition of a given type is to give the angles of all the triangles under consistency constraints. Indeed, to a type with n triangles we can associate an $(n+1)$-dimensional simplex in $\mathbb{R}^{3n}$ (cf. Lemma 3.8).

In Section 4, we consider two types of partition with the same number of triangles and a combinatorial description of how the two partitions should be sent one onto the other, that is, which triangle to which triangle, and for each triangle, which vertex on which vertex. From that, we derive the linear system of equations that the angles must satisfy in order for the map to be a PWS. We investigate the form of the solutions and try to give hints for a rough classification.

We naturally distinguish PWIs among the PWSs, and it is simple to decide from the combinatorial data whether the solution will preserve orientation. Another important feature of a solution is whether it is isolated in the parameter space. In this case, arithmetic properties are natural to consider; we have rational angles. Notice that conversely, if the angles of a partition are not rational, then the PWI is included in a continuous family (this answers a question formulated by A. Goetz). Finally, if the solution is not isolated, we stress the dimension of the simplex.

In Section 5, we illustrate this formalism with explicit computations for partitions with two and three triangles. Even for such low numbers of atoms, the number of solutions is amazingly high, and a huge amount of nontrivial behavior arises. In fact, our method appears to be a valuable source of new examples. We decided (see Section 5.1) to do an exhaustive study of the two-triangle cases. The result is that nothing really surprising arises. Then (Section 5.2) we selected some cases with three triangles, among which appear "old friends" and new examples. The analysis carried out in this section is computationally intensive and requires the use of a computer. We chose the computer algebra software Mathematica, release 5, to help us. A tool box of Mathematica functions was developed in order to handle most of the process. For the sake of conciseness, we will not include the listings of the functions, but the interested reader may find the Mathematica notebook at [Bressaud and Poggiaspalla 06].

Finally, in Section 6, we give a few hints for further investigations.

2. NOTATION AND DEFINITIONS

We denote by $\mathbb{R}^2$ the Euclidean plane. Given three points a, b, and c, the segment $[ab]$ is the convex hull of $\{a,b\}$; the triangle $[abc]$ is the convex hull of $\{a,b,c\}$. We denote by (abc) the interior of $[abc]$. The boundary of the triangle $[abc]$ is the set $\partial[abc] = [abc] \setminus (abc) = [ab] \cup [bc] \cup [ca]$.

Definition 2.1. A polygonal domain (or polygon) P is a compact subset of $\mathbb{R}^2$ whose boundary is a finite union of segments. For simplicity, we will always assume that it is simply connected and that it is the closure of its interior. We will write (P) to denote the interior of P.

Let $\mathcal{P} = (P_1, \ldots, P_n)$ be a finite collection of polygons. We say that it is an *essential partition* of the polygon P if $P = \bigcup_{i=1}^n P_i$ and for all $i \neq j$, $(P_i) \cap (P_j) = \varnothing$. We say that it is a *triangulation* if for all i, P_i is a triangle.

As a consequence, the intersections $P_i \cap P_j \subset \partial P_i \cap \partial P_j$ are finite unions of segments. We denote by $\mathrm{Seg}(\mathcal{P})$ the minimal list of segments such that $\bigcup_{s\in \mathrm{Seg}(\mathcal{P})} s = \bigcup_{i=1}^n \partial P_i$. We denote by $|\mathcal{P}|$ the number of polygons in the collection and by $s(\mathcal{P})$ the number of segments in $\mathrm{Seg}(\mathcal{P})$.

Let $\mathcal{P} = (P_1, \ldots, P_n)$ be an essential partition of a polygon. If they have a segment in common, we can *glue* two elements P_i and P_j to obtain a new partition:

$$\mathcal{P} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{j-1}, P_{j+1}, \ldots, P_n, P_i \cup P_j).$$

If $(Q_1, \ldots, Q_m)$ is an essential partition of P_i, then we can *cut* P_i to obtain a new partition, $(P_1, \ldots, P_{i-1}, Q_1, \ldots, Q_m, P_{i+1}, \ldots, P_n)$. Both operations preserve the property of being an essential partition.

Definition 2.2. A piecewise isometry (respectively, similarity, affine map) of a polygon P is a map f from P to P

such that there is an essential partition $\mathcal{P} = (P_1, \ldots, P_n)$ of P into polygons, called atoms, and a list $(f_1, \ldots, f_n)$ of isometries (similarities, affine maps) such that for all $i = 1, \ldots, n$, the restriction of f to the interior of P_i is f_i, i.e.,

$$f_{|(P_i)} = f_i.$$

Standard definitions of piecewise isometries (see, for example, [Adler et al. 01, Goetz 98, Goetz and Poggiaspalla 04]) usually include the boundaries of the atoms. In the present work, we will not be interested in the behavior of the map on the boundary segments. The dynamics on the images of the singular set have shown very interesting behavior, though, but our aim is, at least as a starting point, to consider concepts that are as global as possible in an attempt to classify the maps themselves rather than to give a detailed study of each of them.

Definition 2.3. We say that a polygonal piecewise isometry (or similarity) is *essentially bijective* if the image of the initial partition is itself an essential partition.

We denote by $\mathcal{S}$ the set of all the essentially bijective polygonal piecewise similarities, and $\mathcal{I}$ the subset of all the essentially bijective polygonal piecewise isometries.

Moreover, if for $f \in \mathcal{S}$ and $i \neq j$, $f_i = f_j$ and $P_i \cap P_j$ contains a segment, we could glue P_i and P_j without changing the map. In the following, we may or may not make the identification, depending on the context.

The aim of this paper is to give an algorithmic way to classify the elements of the set $\mathcal{S}$. In order to do so, we will switch to a formalism whereby all the polygons are triangles, given by a specific triangulation scheme.

3. TRIANGULATION BY BISECTION AND COMBINATORIAL TYPES

3.1 Triangulation by Bisection

To tackle the problem of the classification of polygonal piecewise similarities in a computationally reasonable way, we consider only triangulations, and moreover a specific type of triangulation.

Definition 3.1. A *nice triangulation* $\mathcal{P}$ of a triangle T into n triangles has the following property: There is a sequence of triangulations $(\mathcal{P}_i)_{i=0,\ldots,n}$ such that $\mathcal{P}_0 = (T)$, $\mathcal{P}_n = \mathcal{P}$, and for all $0 \leq i < n$, $\mathcal{P}_i$ is obtained from $\mathcal{P}_{i+1}$ by gluing two elements along a common side.

Notice that the point is that at each step, all the elements of the partition remain triangles. Two triangles that are glued in the sequence will sometimes be called *twins* in the sequel.

Definition 3.2. A sequence of triangulations $(\mathcal{P}_n, \ldots, \mathcal{P}_0)$ in which $\mathcal{P}_n$ is a nice triangulation is called a *gluing path* or a *gluing chain*.

To build a nice triangulation, one can also follow the steps of a gluing chain in reverse order, starting from the original triangle (T), each time cutting one of the triangles into two triangular pieces, i.e., along a bisector. The reversed sequence $(\mathcal{P}_0, \ldots \mathcal{P}_n)$ will be called a *bisection path*. Figure 2 shows some gluing paths if we start from the top partitions, and it shows bisection paths if we start from the bottom partitions. Figure 2 also shows that a nice triangulation may have several gluing paths.

We call $\mathcal{S}_T$ (respectively, $\mathcal{I}_T$) the sets of polygonal piecewise similarities (isometries) such that the initial partitions and their images are nice triangulations in the sense given above.

Any polygonal partition obviously has a refinement that is a nice triangulation. The delicate point in the proof of the following result is that we must find a refinement of the partition that is a nice triangulation *and* whose image is also a nice triangulation.

The following result ensures that we have no loss of generality in considering nice triangulations.

Proposition 3.3. *We have $\mathcal{S} = \mathcal{S}_T$.*

This result holds, provided that we naturally glue together two contiguous atoms if the same map is defined on both.

Proof: Notice first that a polygon can always be included in a triangle and that any partition can be triangulated. We will need the following lemma.

Lemma 3.4. *For every family S of segments in a triangle T there is an essential partition by bisections (i.e., a nice triangulation) $\mathcal{P}$ such that $S \subset \mathrm{Seg}(\mathcal{P})$.*

Proof: We proceed by induction on the number of segments. Given a segment, we can prolong it until we reach the boundary of T. If it lands on a vertex, we are done. Otherwise, we have to draw a segment from the opposite vertex to the landing point of the prolonged segment and cover the latter.

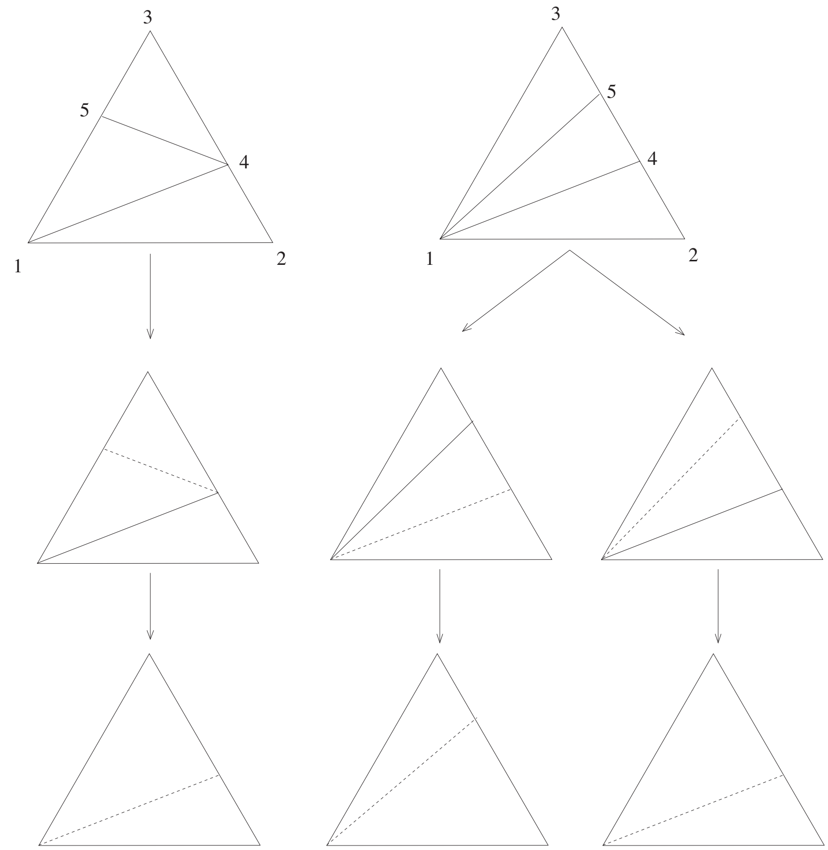

FIGURE 2. Unlike the first triangulation on the left, in the triangulation on the right, which illustrates the triangle list $((1,2,4)(1,4,5)(1,5,3))$, we see that we have two possible gluing paths to recover the initial triangle. Each gluing is depicted in the figures as dashed lines.

Now let us suppose we have $N+1$ segments and that the statement is true for N. Then, let s be one of these $N+1$ segments. Either we can prolong s to reach a vertex of the triangle T or we cannot. If we can, then we are left with two triangles containing at most N segments, and the induction applies.

If the prolonged segment lands on the point x_0 on a side of T, then we split T by joining x_0 to the opposite vertex. We are led to two triangles each containing at most $N+1$ segments. If both triangles contain fewer than $N+1$ segments, we are done.

Only the triangle containing s could have $N+1$ segments. If that is the case, we know that s can be continued to reach a vertex of the triangle, and thus the above argument applies, which completes the proof of the lemma. □

We return to the proof of the proposition. Now we know that any polygonal partition can be refined to become a nice triangulation. Thus, any polygonal piecewise similarity can be viewed as defined on a nice triangulation, by cutting atoms and adding redundancy. But the image of such a partition may not be nice.

Let P be the polygonal partition of a polygonal piecewise similarity f and let Q be its image. Let P' be a refinement of P such that P' is nice; by the lemma, we can always find such a partition. Then the image Q' of P' is a triangular partition refining Q. It may not be nice, but we can find a nice partition Q'' refining Q'.

Then the preimage P'' of the partition Q'' is triangular and refines the nice partition P'. Let P'_i be an atom of P'; it is partitioned into triangles, and this subpartition G_i is in correspondence up to only one similarity f_i with a subpartition of an atom of Q'.

This latter atom was partitioned when Q'' was chosen in such a way that $f_i(G_i)$ is nice. Hence G_i is nice. The argument holds for all possible G_i, and we conclude that P'' is nice, since it is a refinement of a nice partition P' obtained by refining all the triangles in a nice way. The map f extended on P'' remains essentially the same but maps a nice triangulation onto a nice triangulation. □

Remark 3.5. Notice that it is possible to give a bound on the number of triangles needed in the nice triangulation in terms of the number of segments needed to describe the initial partition.

3.2 Combinatorial Types

Given a gluing path now consider not only a specific partition given by this path but *all* the possible partitions that can be constructed with combinatorially equivalent paths. From now on, all the triangles will be oriented counterclockwise, and for the sake of clarity, we name $(1, 2, 3)$ the vertices of the initial triangle. With no loss of generality, we can assume that the vertices 1 and 2 are, respectively, the points $(0, 0)$ and $(1, 0)$ of the real plane, the point 3 remaining free in the upper half-plane.

Let us look at a simple example, the triangle labeled $(1, 2, 3)$, bisected by a segment starting from the vertex 1 and landing on the opposite side, thus creating a fourth vertex, which we call 4. Then, we choose to bisect the triangle $(1, 4, 3)$ with a segment starting at 4 and landing on the side (13). We would like to stress the fact that this description is "combinatorial." We did not mention the continuous information needed actually to describe a partition of a triangle. In other words, there is a continuous family of partitions associated with this description.

In order to describe a particular one we would have to specify the initial triangle (i.e., the position of the point labeled 3, or the angles at the points 1 and 2), the angle between the segments (12) and (14), and finally the angles between the segments (43) and (45).

Note that the list $((1, 2, 4)(1, 4, 5)(5, 4, 3))$ provides all the information needed to trace the gluing path of the triangles (the underlines show the common sides):

$$(1,2,4),(1,\underline{4},\underline{5}),(\underline{5},\underline{4},3) \to (1,2,4),(1,4,3),$$
$$(\underline{1},2,\underline{4}),(\underline{4},3,\underline{1}) \to (1,2,3).$$

The list thus corresponds to a continuous family of partitions. Note also that we can choose a different set of parameters, for instance the nine angles linked by a linear system of equations.

We will now formalize this notion. We want to get rid of the continuous parameters. We will then work with "combinatorial triangles," which are merely lists of vertices. A bisection of a triangle $(1, 2, 3)$ can be described by a list of two triangles. If we call 4 the "new" vertex, we have three cases, depending on whether 4 belongs to (13), (12), or (23). The list will be respectively $((2, 3, 4), (4, 1, 2))$, $((3, 1, 4), (4, 2, 3))$, or $((1, 2, 4), (4, 3, 1))$.

A bisection path of a partition of a triangle $(1, 2, 3)$ corresponds to a growing list of triangles. At each stage, we have the names of the triangles in $\mathcal{P}_i$, where the new vertex created at stage i is called $i + 3$. A sequence of $n+1$ bisections provides a list of n triangles, the names of the vertices ranging from 1 to $n + 2$. Given the final list of triangles, it is easy to recover the path by gluing the two triangles containing the vertex with highest index, and so on, as seen in the example above.

We will say that two bisections paths are *combinatorially* the same if they produce the same sequence of lists $(\mathcal{P}_i)_{i=0,...,n}$, or equivalently, if they produce the same final list.

To a combinatorial bisection path yielding n triangles we associate a map t from $A_n = \{1, 2, \dots, 3n\}$ onto $V_n = \{1, 2, 3, 4, \dots, n + 2\}$ that describes the final list of triangles:

$$\big((t(1), t(2), t(3)), \dots, (t(3n-2), t(3n-1), t(3n))\big).$$

It will be convenient to identify a combinatorial bisection path with such a map. Notice that not all such maps correspond to a bisection path. A map will be called *admissible* if that is the case.

Remark 3.6. The number of bisection paths yielding n triangles is bounded by $3^n(n - 1)!$. Indeed, at each step, the next bisection is determined by the choice of the triangle and of one of its vertices.

We will say that two bisection paths are *equivalent* if they correspond to the same partitions when the continuous parameters vary. More precisely, they are equivalent if for each partition obtained with the first path and a fixed set of parameters, it is possible to choose the parameters of the other one to obtain the same partition.

This equivalence relation, which we will denote by R, takes into account two technical points. Firstly, if during the sequences of bisections we get two nonoverlapping triangles (T') and (T''), we can bisect them separately in any order. Different orders will lead to final lists with the same structure: only the names of the vertices depend on the order in which these bisections are done. Secondly, if a triangle is cut twice (or more) from the same vertex, then the order in which the splitting is done does not matter. Nonetheless, it will change the names of the vertices and may also affect the order in which the triangles are listed.

Definition 3.7. A *combinatorial type of partition* is an equivalence class (for R) of combinatorial bisection paths.

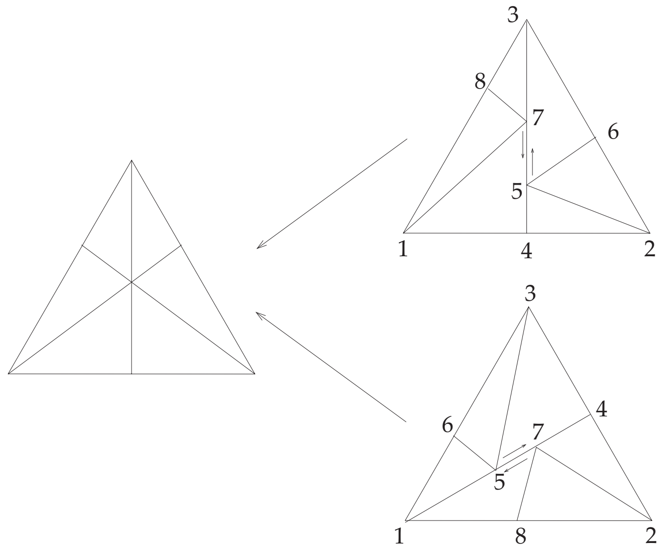

FIGURE 3. The partition on the left can be given by any of the two combinatorial types on the right.

In τ we choose once for all a representative list map denoted by t_τ. We will denote by $|\tau|$ the number of triangles described by the list t_τ.

If t is an admissible map corresponding to a bisection path of type τ, we may write $t \in \tau$, but for conciseness, we may also call the map associated to any representative of a combinatorial type and preferably the selected representative t_τ a "partition of combinatorial type."

Hence, a partition type may have several gluing paths. An easy example is given by the list $((1,2,4)(1,4,5)(1,5,3))$. We can choose to glue the two upper triangles first and then the remaining one, or we can glue the two lower triangles first. Both gluing paths are valid and lead to the same type; see Figure 2.

It is also worth noticing that a partition may be given by more than one combinatorial type, as shown in Figure 3. (Figure 10 provides an exhaustive list of combinatorial types with three triangles.)

In the description of a combinatorial type through a formal list of triangles, the names of the vertices (except 1, 2, and 3) and the order in which the triangles are listed do not matter. It is not very difficult to check that given such a list it is possible to recover a bisection path. At each stage we glue two triangles that have two common vertices, one of them being distinct from 1, 2, and 3 and appearing in no other triangle of the list. In the following, it will be convenient for us to associate to each type τ a particular list of triangles written in the order given by t_τ.

3.3 Alternative Description

The number of combinatorial types with n triangles is bounded from above by the number of combinatorial bisection paths, i.e., $3^n(n-1)!$. To enumerate all the combinatorial types, it is certainly more efficient to use the point of view described below, which will not, however, be needed in the sequel.

Let $\mathcal{T}$ be the set of finite planar rooted trees G with set of vertices V, set of edges E, and root $r \in V$. As a rooted tree, G has no vertex of degree 2, and its root r has degree more than 1. Call $L \subset V$ the subset of leaves (i.e., vertices with degree 1) and set $n = |L|$. We say that G is a labeled tree if it comes with a map that associates a value in $\{1, 2, 3\}$ to the root and a value in $\{-1, +1\}$ to each vertex in $V \setminus (L \cup \{r\})$.

We claim that there is a bijection between the combinatorial types with n triangles and the labeled trees with n leaves. This representation is simply a way to avoid the redundancy described by the equivalence relation introduced above.

However, we do not want to enter into the details here. Let us just say that, roughly speaking, the root represents the triangle $(1, 2, 3)$; the label of the root tells us from which vertex of the triangle the first bisection is done. Each edge starting from the root goes to a vertex that represents a triangle. There can be more than two edges when the triangle is cut into more than two triangles from the same vertex.

Then starting from the triangle associated to a vertex, descending edges starting from this vertex describe how the triangle is cut. Notice that at each vertex except the root we can decide to do the next bisection from only two of the three vertices, since it is not allowed to use the same vertex again. For the purposes of this paper, it is not necessary to develop this formalism any further.

To summarize, we have three different objects that can be considered as abstraction layers of an intuitive concept:

- We consider a nice triangulation of a triangle called $(1, 2, 3)$ up to similarities. However, we decided, for the sake of clarity, to assume that 1 and 2 are fixed and that 3 is in the upper half-plane.
- Then we consider a combinatorial type of the bisection path generating the nice triangulation. It is represented by a list map, which formalizes an equivalence up to a set of continuous parameters (for instance, the angles).
- Finally, since a given partition can be described by different bisection paths, we have another equivalence relation among the bisection paths, and a combinatorial type is represented by one of the above lists (or alternatively by a labeled tree).

3.4 Partitions in a Combinatorial Type

Since by the definition of a type, for a given combinatorial type τ, the set of partitions following this type does not depend on the bisection path chosen, it can be parameterized by the angles of the $n = |\tau|$ triangles. There are n triangles and hence $3n$ angles, under linear constraints. Given a partition, we can consider the angle vector $A \in]0, \pi[^{3|\tau|}$. Its coordinates are ordered according to the map $t \in \tau$, since the order of the list provides an order on the vertices and hence on the angles. The choice of the order is in itself unimportant, but it must be made once and for all before any further computations.

Given a list map $t \in \tau$, we call $A(\tau)$ the subset of $]0, \pi[^{3|\tau|}$ of the angles attained by all the partitions following t. We have the following lemma:

Lemma 3.8. *$A(\tau)$ is a convex subset of $]0, \pi[^{3|\tau|}$ of dimension $|\tau| + 1$.*

Proof: This proof is easily done by induction. We put $n = |\tau|$. If we have only one triangle, we need two parameters to describe it. Now suppose we have n triangles and by the induction hypothesis $n + 1$ parameters. To add one more triangle, we have to split one of the existing ones. Thus, we let all the $n + 1$ parameters be fixed and choose one triangle to bisect. When we cut a triangle, we have only one degree of freedom, which is the position of the landing point of the bisector. We then have $n + 1$ triangles and $n + 2$ parameters, which completes the induction. □

We now write all the equations that the angles have to satisfy. First, we have the following consistency condition for each of the n triangles: For all $j = 0, \dots, n - 1$, if we call α_{3j+k} the angle at the vertex k in the triangle j, we have

$$\sum_{k=1}^{3} \alpha_{3j+k} = \pi, \quad j \in \{0, \dots, n-1\}. \tag{3–1}$$

All these equations are clearly independent, since each deals with a separate set of angles. Moreover, each created vertex v lies on a side, and so the sum of the angles around it must be π:

$$\sum_{i:t(i)=v} \alpha_i = \pi, \quad v \in \{4, \dots, n+2\}. \tag{3–2}$$

The equations of this set are independent as well. Each involves a separate set of two angles. If two angles are in an equation, then they cannot be in another one, since a different equation deals with a different point.

We express these conditions using matrices. Condition (3–1) is expressed by the $(n + 1) \times 3n$ matrix $C(n)$:

$$C_{i,j}(n) = \begin{cases} 1 & \text{if } j = 3i-2, 3i-1, 3i, \\ 0 & \text{otherwise.} \end{cases} \tag{3–3}$$

The $(n-1) \times 3n$ matrix $V(\tau)$ will express condition (3–2):

$$V_{i,j}(\tau) = \begin{cases} 1 & \text{if } t(i) = j+3, \;\; j \in \{1, \dots, n-1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3–4}$$

It is easy to see that no row of the matrix (3–3), which always has three contiguous 1's, can be expressed in terms of a combination of rows of the matrix (3–4). Indeed, each row of the latter contains two 1's, and the matrix never has two 1's in the same column (that is, each angle is used only once). We thus have $2n - 1$ independent equations. By the lemma, this is enough to describe the system.

We can write the constraints in the compact form

$$A > 0, \quad CA = \pi\mathbf{1}, \quad \text{and} \quad VA = \pi\mathbf{1}.$$

In the following, we will need to ensure that the exterior triangle $(1, 2, 3)$ remains unchanged by the PWS. If we call its angles (α, β, γ), then certainly

$$\alpha = \sum_{i:t(i)=1} \alpha_i, \quad \beta = \sum_{i:t(i)=2} \alpha_i, \quad \gamma = \sum_{i:t(i)=3} \alpha_i. \quad (3\text{–}5)$$

We introduce an additional matrix E giving two of these angles. Indeed, the consistency of the triangle $(1,2,3)$ being already encoded in the matrix C, the third angle of $(1,2,3)$ does not give any information. The matrix E is of dimension $2 \times 3n$ and is defined by

$$E_{i,j}(\tau) = \begin{cases} 1 & \text{if } t(i) = j, \quad j \in \{1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = EA.$$

Finally, we remark that taking any other t' in the class τ would result only in permutations of the rows of the matrices.

A pair (τ, A), where τ is a combinatorial type and $A \in A(\tau)$, is all we need in order to determine a nice triangulation of the triangle $(1,2,3)$. The vertex 3 is determined by the angles EA. We will use the following notation.

Definition 3.9. We will denote by (τ, A) the partition of the triangle $(1,2,3)$, the vertices 1 and 2 being fixed and its angles determined by EA. The partition is constructed by the bisection process described by τ.

4. TYPES OF MAPS

We want to enumerate the bijective piecewise similarities with a given number of triangles. We suppose the partitions to be nice in the sense of Definition 3.1, and we associate to each of them a combinatorial type. A piecewise similarity maps each vertex of each triangle of the first partition to a vertex of a corresponding triangle of the target partition.

From the point of view of combinatorial types, the map is a permutation on the vertices. Not all possible permutations are allowed, though the triangles themselves can be permuted as well as the vertices inside a triangle. But a triple of vertices forming a triangle must still correspond to a triangle after the permutation. We will say that a permutation $\Sigma \in S_{3n}$ is *admissible* if there are $(\sigma, s_1, \ldots, s_n) \in S_n \times S_3^n$ such that Σ can be written in the form

$$\Sigma(3i + k) = 3(\sigma(i) - 1) + s_i(k), \quad 1 \leq i \leq n, \ 1 \leq k \leq 3.$$

We denote by $\overline{S}_n \subset S_{3n}$ the subset of admissible permutations on $3n$ elements. If $\Sigma \in \overline{S}_n$, we also denote by Σ its $3n \times 3n$ permutation matrix. Notice that $|\overline{S}_n| = n!6^n$.

Given two combinatorial types τ and τ' with $|\tau| = |\tau'|$, two angle vectors A and A', and an admissible permutation Σ, we consider the piecewise *affine* map f_Σ mapping the triangles of (τ, A) onto the triangles of (τ', A') in the order prescribed by Σ. Precisely, for $t \in \tau$ and $t' \in \tau'$ we have for all $1 \leq i \leq n$ that the image of the triangle $(t(3i)t(3i+1)t(3i+2))$ is the triangle $(t'(\Sigma(3i))t'(\Sigma(3i+1))t'(\Sigma(3i+2))$.

Remark 4.1. Notice that the identity permutation in S_{3n} may yield a nontrivial map if the types are distinct.

An affine map is a similarity if and only if it preserves the angles of a nondegenerate triangle. Hence the map f_Σ is a PWS if and only if $A' = \Sigma A$. According to Section 3, the triangles partitioned by (τ, A) and (τ', A') are the same if and only if $E(\tau)A = E(\tau')A'$. It is then natural to make the following definition.

Definition 4.2. For every integer n, all combinatorial types τ, τ' with $|\tau| = |\tau'| = n$, and every admissible permutation $\Sigma \in \overline{S}_n$, we denote by $\mathcal{A}(\tau, \tau', \Sigma)$ the set of solutions in $A \in]0, \pi[^{3n}$ of the equations

$$\begin{aligned} E(\tau)A &= E(\tau')\Sigma A, \\ C(n)A &= C(n)\Sigma A = \pi, \\ V(\tau)A &= V(\tau')\Sigma A = \pi, \end{aligned} \quad (4\text{–}1)$$

where $E(\tau), V(\tau), C(n)$ are the matrices defined above.

We see that given τ, τ', and Σ, the angle vectors $A \in \mathcal{A}(\tau, \tau', \Sigma)$ are such that A and ΣA describe two partitions with similar triangles. It is then easy to find the piecewise similarity corresponding to this transformation.

Conversely, we can always consider any given piecewise similarity f to be defined on nice triangulations; cf. Proposition 3.3. Both partitions, possibly up to a similarity, can be described by two combinatorial types τ and τ' and two angle vectors. As discussed above, the map corresponds to some permutations Σ of the angles. Then certainly f is included in the set of PWSs given by the solutions $\mathcal{A}(\tau, \tau', \Sigma)$. We shall also use the same notation $\mathcal{A}(\tau, \tau', \Sigma)$ to denote the set of the corresponding maps. We summarize this remark in the following proposition.

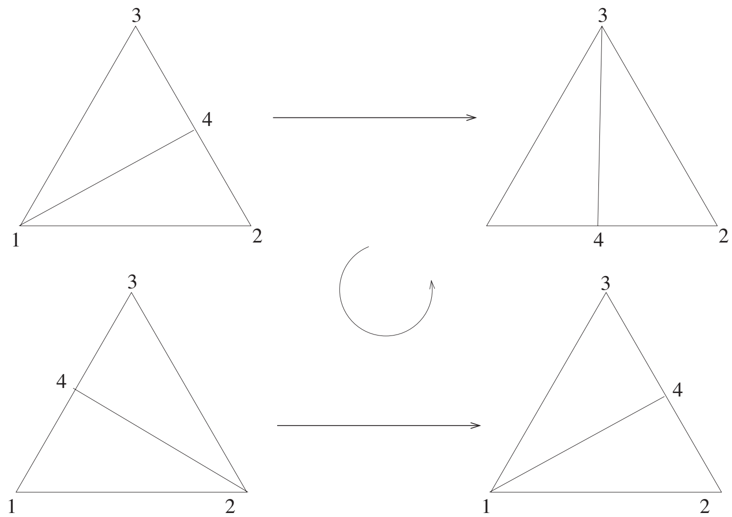

FIGURE 4. The top plot shows the type of pair under consideration. The bottom plot shows the same type but up to a counterclockwise rotation of $\pi/3$ applied to both triangles. We recognize the swapped pair of Section 5.1.1.

Proposition 4.3. *We have*

$$\mathcal{S} = \bigcup_{\tau,\tau',\Sigma} \mathcal{A}(\tau,\tau',\Sigma),$$

where τ and τ' are two combinatorial types with the same number of triangles and $\Sigma \in \overline{S}_{|\tau|}$.

It can happen that two different pairs of types lead to the same map (possibly up to a similarity). For instance, the top and bottom pairs of combinatorial types pictured in Figure 4 are "equivalent" and will give the same maps up to a similarity.

We will say that two pairs (τ_1, τ_2) and (τ_1', τ_2') are *equivalent* if there exists a similarity S such that for all partitions (P_1, P_2) and (P_1', P_2') following these types we have

$$(P_1, P_2) = (SP_1', SP_2').$$

These partitions are nondegenerate, i.e., all their atoms have nonempty interiors. This equivalence can be expressed combinatorially in terms of permutations of the vertices of the initial triangles $(1, 2, 3)$:

Definition 4.4. Two pairs (τ_1, τ_2) and (τ_1', τ_2') are *equivalent* if there exists $\sigma \in S_3$ such that $\tau_1' = \tau_1 \circ \tilde{\sigma}$ and $\tau_2' = \tau_2 \circ \tilde{\sigma}$, where $\tilde{\sigma}(i) = \sigma(i)$ for $i \leq 3$ and $\tilde{\sigma}(i) = i$ for $i > 3$.

We remark that the bijectivity of the PWSs implies that there is a symmetry between the pair (τ_1, τ_2) and the pair (τ_2, τ_1). The inverse map of a similarity is a similarity. More formally, $\mathcal{A}(\tau_1, \tau_2, \Sigma) = \mathcal{A}(\tau_2, \tau_1, \Sigma^{-1})$.

We are now ready to enumerate the piecewise similarities. For every integer n, we can enumerate the combinatorial types of partitions with n triangles using Section 3. Then for each pair of such combinatorial types (τ, τ') and each admissible permutation Σ we can solve the linear system (4–1) to determine the angle vectors in $\mathcal{A}(\tau, \tau', \Sigma)$. We call this set a *solution*. A solution is then an intersection of two n-dimensional simplices. The intersection may be empty, leading to no solution at all, or it may be again a simplex. If it is nonempty, then (τ, τ', Σ) may be called the (combinatorial) type of the corresponding PWS. For n triangles, the number of possible types of PWSs is bounded from above by $n!6^n(3^n(n-1)!)^2 = 54^n n!(n-1)!^2$. This bound is a bit crude, but a better estimation would involve tedious computations unnecessary for our purpose.

Let us now mention a few relevant properties of these solutions.

Definition 4.5. For brevity, the dimension of a solution will be the dimension of the simplex $\mathcal{A}(\tau, \tau', \Sigma)$.

The dimension-zero solutions consist of only one point and will often be called *isolated* in the sequel.

For such an isolated solution $a = (\alpha_1, \ldots, \alpha_{3n})$ with n triangles, all the angles will clearly be rational multiples of π. Moreover, they have the same denominator up to possible simplifications with the numerator. In other words, all the denominators of the angles must be divisors of the same integer: There exists an integer q such that $a \in (\frac{1}{q}\mathbb{Z}\pi)^{3n}$.

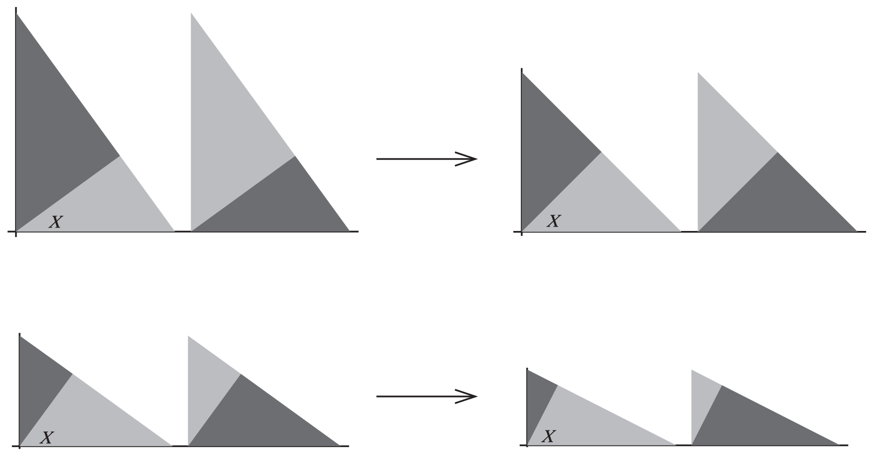

FIGURE 5. The only solution giving piecewise similarities encountered so far; the angles are $(\pi/2 - x, x, \pi/2, \pi/2, \pi/2 - x, x)$, for $0 < x < \pi/2$. The plots show four examples in the family with four increasing values of x. We note that the upper-right figure displays piecewise isometry, thus illustrating the fact that a simplicial solution can contain both piecewise similarities and piecewise isometries.

Since we deal only with maps of the Euclidian plane, we can use the complex numbers to express the vertices of the triangles and the maps themselves. All these quantities can be computed by performing operations in the number field $\mathbb{Q}(e^{i\alpha_1}, \ldots, e^{i\alpha_{3n}})$. Since there exists a q such that $\alpha_i = p_i\pi/q$ for every i, the number field is finitely generated. Its dimension over $\mathbb{Q}$ is the degree of the cyclotomic polynomial of order $2q$, that is, $\phi(2q)$, where ϕ is the Euler function.

Definition 4.6. We define the *degree* of an isolated solution $(\alpha_1, \ldots, \alpha_{3n})$ to be the dimension of the cyclotomic number field $\mathbb{Q}(e^{i\alpha_1}, \ldots, e^{i\alpha_{3n}})$.

A solution with dimension greater than zero will be called a simplicial solution. If the dimension of the set of solutions is p, the solutions of the system can be expressed as a function f of p parameters, defined on a domain of R^p, and valued in R^{3n}, whose coordinates have the following form:

$$f_i(x_1, \ldots, x_p) = d\pi + \sum_{j=1}^{p} a_j x_j, \quad (a_i, d) \in \left(\frac{1}{q}\mathbb{Z}\right)^2,$$

for $i \in 1, \ldots, 3n$ and a unique integer q. The parameters x_i range in intervals whose bounds are of the same form as above.

The following definition is also natural.

Definition 4.7. We say that a solution is *direct* if all the associated transformations have positive determinant. We say that a solution is *reverse* if all the associated transformations have negative determinant, and that it is *mixed* otherwise.

Since the triangles in our lists are oriented, it is clear that if the permutation of the vertices of a given triangle is even, then the resulting similarity will preserve orientation. Thus, if we limit the investigation to direct piecewise similarities, we have $n!3^n$ allowed permutations instead of $n!6^n$, which can save us a significant amount of computational time.

Given an angle vector in a solution and the permutation attached to it, we know all the angles of both partitions. Once we have constructed them, since we know by the permutations which triangle in the first partition is supposed to be mapped onto which one of the second, we can compute the transformations. They are similarities, since they preserve angles and can possibly be isometries. A solution is a PWI if and only if all the associated transformations on all its atoms have their determinants equal to 1 or -1. We cannot say that a simplicial solution has a definite type. In general, it can contain both piecewise similarities and piecewise isometries; see Figure 5. Notice that we still do not have a nice algorithmic way to discriminate PWIs in a simplicial solution containing PWSs.

To avoid some redundancy in the solutions, we will introduce the following definition:

Definition 4.8. We will say that a piecewise similarity is *irreducible* if given any pair of twin triangles, each of them bears a different similarity.

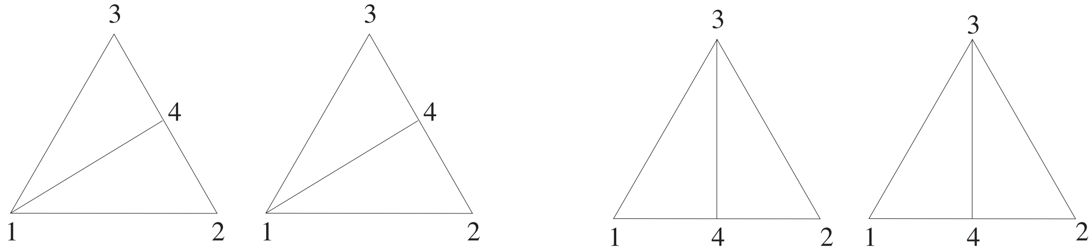

FIGURE 6. The two pairs depicted are equivalent up to a cyclic permutation of the vertices.

Simply stated, this means that the map has the minimum number of atoms. It cannot be reduced by gluing two or more elements to a piecewise similarity on a nice triangulation with fewer triangles.

To find all the piecewise similarities on n triangles, we will first determine all possible combinatorial types on n atoms. There are finitely many of them, indeed fewer than $(n-1)!3^{n-1}$; cf. Section 3. Then we pick all the pairs of combinatorial types and list all the solutions for all admissible permutations.

Clearly, the amount of computation grows dramatically with the number of triangles. In the following sections we will perform an exhaustive enumeration for the cases with two triangles. We will also have a look at some three-triangle cases.

5. FIRST COMPUTATIONS

5.1 Two Triangles

We start with the simplest case, in which we have only two atoms. By the construction described above, given our reference triangle $(1,2,3)$ we have three possible bisections. For each of them, the new vertex 4 will land on a different side. Since we must specify a pair of partitions, we are led to nine possible pairs, many of them equivalent. For instance, the pairs of Figure 6 are clearly equivalent. In fact, each pair can be "rotated" three times, and thus we have only three cases to consider; cf. Figure 7. In the following, we will make an extensive exploration of them.

5.1.1 The "Tower" Case. The first case we will investigate has a structure already encountered in other references. A two-triangle case involving this bisection scheme has been extensively studied in [Goetz 98], where the author shows one of the first examples of self-similar dynamics encountered in the field of piecewise isometries. Generalizations to more than two triangles of this structure in the form of "towers of triangles" have been considered subsequently in [Goetz and Poggiaspalla 04]. Such towers, originally found "by hand," can be retrieved by systematic computations.

In this section and the following, the phase space will be a triangle labeled $(1,2,3)$, with 1 the lower-left corner. All the triangles will be oriented counterclockwise. The bisections of the triangles correspond to the lists $((1,2,4)(1,4,3))$ and $((1,2,4)(4,2,3))$, as shown in Figure 7 (top triangle).

For this model, the computer checked all admissible permutations of the vertices between the two triangles. The solutions are in the six-dimensional open cube $I =]0,\pi[^6$, one dimension for each of the six angles. Their order is based on the list of triangles; that is, for the list $((1,2,4),(1,4,3))$, each component $(a_1,a_2,a_3,a_4,a_5,a_6) \in I$ corresponds to the respective angles

$$\widehat{(124)},\ \widehat{(412)},\ \widehat{(241)},\ \widehat{(143)},\ \widehat{(314)},\ \widehat{(431)}.$$

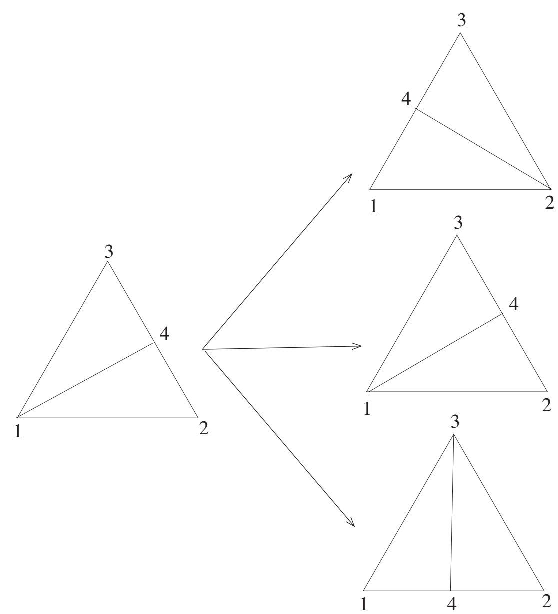

FIGURE 7. The three cases to consider.

Solution	Permutation	Angles
1	(231645)	$(2\pi/5,\ \pi/5,\ 2\pi/5,\ 3\pi/5,\ \pi/5,\ \pi/5)$
2	(312645)	$(3\pi/7,\ 2\pi/7,\ 2\pi/7,\ 5\pi/7,\ \pi/7,\ \pi/7)$

TABLE 1. Isolated solutions for the "Tower" case.

We are thus led to solve the following equations under the constraints of staying in I and for every allowed permutation matrix Σ:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \\ \pi \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \Sigma \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \\ \pi \end{pmatrix}.$$

And moreover, to ensure that the exterior angles coincide,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \Sigma \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}.$$

There is a total of eight solutions, of which two are isolated and direct, and they correspond to two piecewise isometries. The first one is of degree 4, and the second one of degree 6. In the following we will list only the *irreducible* cases. The isolated solutions are listed in Table 1, while the simplicial solutions are listed in Table 2.

The first solution listed in Table 1 is well known. It is the case extensively studied in [Goetz 98]. The second and only other isolated solution has also been investigated by the same author and his collaborator in [Goetz and Sammis 01]. These solutions have proven very interesting and have intricate behavior. With regard to the other solutions, we have conducted an automated investigation, and their behaviors appear to be nearly trivial. All consist of a reflection on one of the atoms and a rotation about the other one, leading to periodic or quasiperiodic motions. Thus, the only cases of interest in that configuration are the isolated solutions listed above.

Let us recall that a first idea of the dynamics of such a map is given by the *mosaic* of the map, i.e., the union of the backward and forward images of the discontinuity lines (for more details about standard tools for the study of the dynamics of PWIs, see, for example, [Poggiaspalla 03]).

5.1.2 The "Symmetric" Case.

In our attempt to list all the possible piecewise isometries, and even all possible piecewise similarities, we mentioned three different pairs of combinatorial types. This section deals with the second case in Figure 7, "symmetric pair" $((1,2,4)(1,4,3))$ and $((1,2,4)(1,4,3))$. Following the same process as before, we will compute all the possible solutions given by all the admissible permutations of the vertices. This time, there is a total of twenty-seven solutions, of which five are direct. We have six isolated solutions, all of them piecewise isometries. The first four are of degree 2 and leave their atoms invariant; thus only the remaining solutions, of degree 4, will be listed in Table 3.

Solutions 3 and 4 do have nontrivial dynamics, but they are topologically conjugate to the well-known dynamics of the case explored by [Goetz 98] already encountered in the previous section. See Figure 8 for the mosaics of their cells.

There are many simplicial solutions, but most of them, Solutions 7 to 21, leave their atoms invariant. Also, Solution 27 is not irreducible. Among the remaining solutions, Solutions 23 to 26 consist of a rotation and a reflection, leading to only periodic or *pseudoperiodic orbits*, that is, the orbits that densely fill a circle or a finite number of circles. The last case to consider, Solution 22, is a family of piecewise similarities. Its partitions for

Solution 4	
Permutation Angles Constraints	(213564) $\left(\frac{\pi}{3} - \frac{x_3}{3},\ x_3,\ \frac{2\pi}{3} - \frac{2x_3}{3},\ \frac{\pi}{3} + \frac{2x_3}{3},\ \frac{\pi}{3} - \frac{4x_3}{3},\ \frac{\pi}{3} + \frac{2x_3}{3}\right)$ $0 < x_3 < \pi/4$
Solution 5	
Permutation Angles Constraints	(213645) $\left(\frac{\pi}{3} + \frac{x_3}{3},\ x_3,\ \frac{2\pi}{3} - \frac{4x_3}{3},\ \frac{\pi}{3} + \frac{4x_3}{3},\ \frac{\pi}{3} - \frac{2x_3}{3},\ \frac{\pi}{3} - \frac{2x_3}{3}\right)$ $0 < x_3 < \pi/2$
Solution 6	
Permutation Angles Constraints	(231645) $\left(\frac{\pi}{2} - \frac{x_3}{2},\ x_3,\ \frac{\pi}{2} - \frac{x_3}{2},\ \frac{\pi}{2} + \frac{x_3}{2},\ \frac{\pi}{2} - \frac{3x_3}{2},\ x_3\right)$ $0 < x_3 < \pi/3$
Solution 7	
Permutation Angles Constraints	(312546) $(\pi - 2x_3,\ x_3,\ x_3,\ \pi - x_3,\ \pi - 3x_3,\ -\pi + 4x_3)$ $\pi/4 < x_3 < \pi/3$

TABLE 2. Simplicial solutions for the Section 5.1.1 case.

Solution	Permutation	Angles
3	(564231)	$(2\pi/5,\ \pi/5,\ 2\pi/5,\ \pi/5,\ 3\pi/5,\ \pi/5)$
4	(645312)	$(3\pi/5,\ \pi/5,\ \pi/5,\ \pi/5,\ 2\pi/5,\ 2\pi/5)$

TABLE 3. Isolated irreducible solutions for the Section 5.1.2 case.

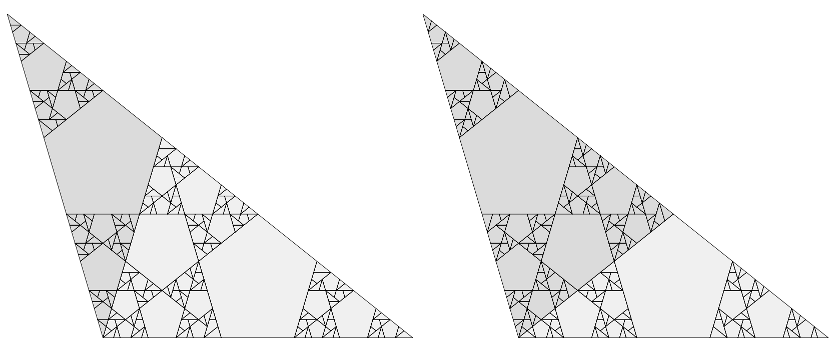

FIGURE 8. Left: mosaic for Solution 3. Right: mosaic for Solution 4. Both are conjugate to Solution 1 of Section 5.1.1.

some value of the parameter are shown in Figure 5. We see that this family contains mostly piecewise similarities, in fact, in every case except for $x = \pi/4$. However, the dynamics of this solution appear to be trivial as well: They consist merely of the exchange of the two atoms.

To conclude, this pair of combinatorial type does not bring anything new, since the only nontrivial solutions are conjugates of some of the solutions from the previous section.

5.1.3 Third Case. The third case to consider, as shown in Figure 7 and given by the lists of triangles $((1,2,4)(1,4,3))$ and $((1,4,3)(4,2,3))$, can be viewed as the inverse of the "tower case" investigated above. Up to a rotation, this is clear, as can be seen in Figure 4.

Since all the solutions must be essentially bijective, we expect the solutions in this section to be the inverses of the solutions of Section 5.1.1. This case thus requires no further investigation, since it brings nothing new in terms of dynamics.

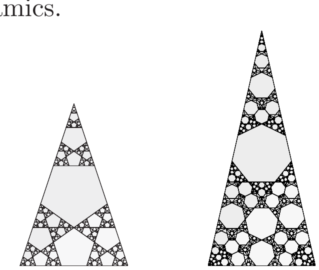

FIGURE 9. Left: mosaic for Solution 1; Right: mosaic for Solution 2.

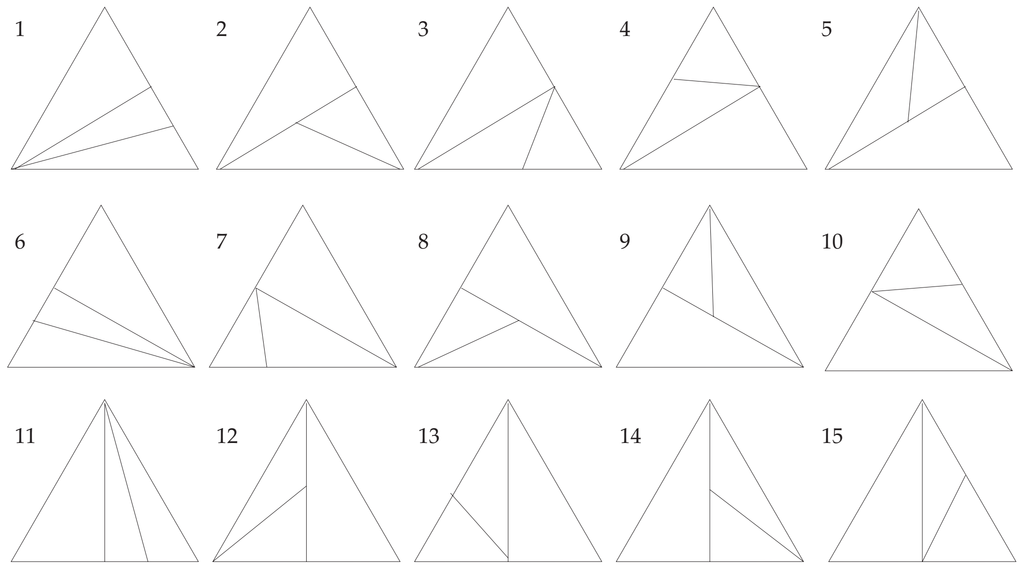

FIGURE 10. The 15 different combinatorial types with three triangles.

From all these computations, we can say that the only nontrivial piecewise isometric dynamics on two triangles are those to be found in Figure 9, both of which are already known in the literature.

5.2 Three Triangles

While the number of cases and the number of solutions in the two-triangle investigation remained relatively low, this is no longer the case with three triangles. First, we have 15 different combinatorial types with three triangles. They are shown in Figure 10. Then there are $15^2 = 225$ pairs. We can as above put an equivalence relation on the pairs to avoid the cases that can be deduced from others by a rotation or a flip. For instance, we have $(1,2) \sim (6,9)$ and $(4,3) \sim (7,10)$.

We are then led to 75 cases. Moreover, we ignore the swapping of combinatorial types inside a pair. Indeed, for example, the pair $(1,2)$ will lead to maps that are the inverses of the maps found for $(2,1)$. This way, we eliminate 20 cases, and only 55 cases are left to consider; they are as follows:

$(1,1)$, $(1,6)$, $(1,7)$, $(1,8)$, $(1,9)$, $(1,10)$, $(1,11)$, $(1,12)$, $(1,13)$, $(1,14)$, $(1,15)$, $(2,2)$, $(2,6)$, $(2,7)$, $(2,8)$, $(2,9)$, $(2,10)$, $(2,11)$, $(2,12)$, $(2,13)$, $(2,14)$, $(2,15)$, $(3,3)$, $(3,6)$, $(3,7)$, $(3,8)$, $(3,9)$, $(3,10)$, $(3,11)$, $(3,12)$, $(3,13)$, $(3,14)$, $(3,15)$, $(4,4)$, $(4,6)$, $(4,7)$, $(4,8)$, $(4,9)$, $(4,10)$, $(4,11)$, $(4,12)$, $(4,13)$, $(4,14)$, $(4,15)$, $(5,5)$, $(5,6)$, $(5,7)$, $(5,8)$, $(5,9)$, $(5,10)$, $(5,11)$, $(5,12)$, $(5,13)$, $(5,14)$, $(5,15)$.

The 3089 solutions took about twenty minutes to compute on a 3-GHz desktop PC using the notebook Complete3TriLt (available as an electronic supplement [Bressaud and Poggiaspalla 06]). The number of solutions for a given pair can vary greatly, from 9 up to 297. We have 449 direct solutions and 810 isolated solutions. Table 4 gives the number of isolated solutions by degree.

Degree	2	4	6	8	10	12
Number of Solutions	130	328	258	6	56	32

TABLE 4. Number of isolated solutions by degree.

Among these solutions we can spot some known cases. For example, the case studied in [Goetz and Poggiaspalla 04] belongs (up to a rotation) to the set of solutions given by the pair $(3,15)$. Similarly, the cases (in fact their inverses) studied in [Adler et al. 01] belong to the set of solutions of the pair $(2,8)$. Since it is impossible to review systematically such a vast number of cases in the present paper, we will focus on only two pairs, which offer a fair number of interesting solutions. The complete listing of all the solutions with a preliminary analysis is available in the electronic supplement [Bressaud and Poggiaspalla 06] to this paper.

Although the choice we made may seem arbitrary, we hope that the great number of solutions it provides as well as the variety of them makes it suitable for our illustrative purpose.

5.2.1 Tower Case Again.

In this section, we will be interested in the pair $(3,15)$. We have been

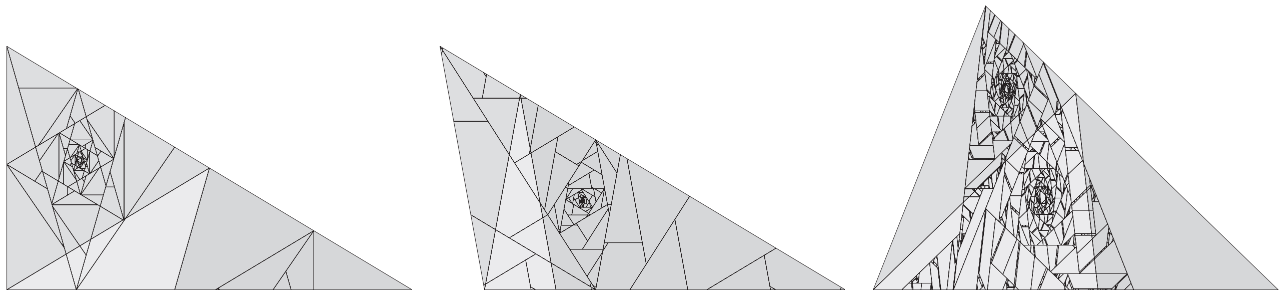

FIGURE 11. The mosaics for Solutions 11, 13, and 34. We clearly see the attracting points in Solutions 11 and 13. Solution 34 has two attracting points and thus slightly more complicated dynamics.

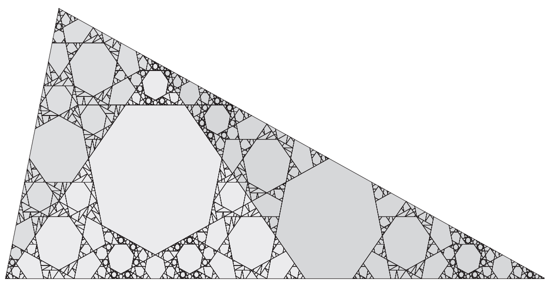

FIGURE 12. The cases from [Goetz and Poggiaspalla 04], found among the solutions given the pairs $(3, 15)$.

working with the lists $((1,5,4)(5,2,4)(1,4,3))$ and $((4,2,5)(4,5,3)(3,1,4))$. Among its 45 isolated and 86 simplices of solutions, we find, up to a rotation, the case studied in [Goetz and Poggiaspalla 04], displayed in Figure 12.

As we said, there are 45 isolated solutions, and some of them have highly nontrivial behavior. Table 5 gives their distribution by degree.

Only 27 of them are piecewise isometries, the rest being piecewise similarities. These solutions display a fair variety of behaviors. Some seem to have nearly trivial behavior, that is, their mosaics of n-cells stabilize at a certain level to a finite partition. Many of them, however, have highly nontrivial behavior. A detailed study of these cases would be impossible here for obvious space reasons and would also be beyond the scope of the paper. Instead, we will merely list the nontrivial cases and display some of the most interesting of them.

Degree	2	4	6	8	10	12
Number of Solutions	2	26	15	0	1	1

TABLE 5. Behavior of the 45 isolated solutions by degree.

We split the list according to the degree of the solutions. We have two degree-2 solutions, both of which have finite mosaics. Among the 26 degree-4 solutions, 10 have finite mosaics. Among the PWS solutions of degree 4, six have a finite number of attracting points, as in the examples of Figure 11, and yield uninteresting dynamics. Table 6 lists the remaining solutions. All these solutions have nontrivial behaviors. Two of them are closely similar to the case studied in [Goetz 98]. In fact, a simple induction leads to the same map.

Looking at Table 6, we notice that an angle vector can appear several times, attached to a different permutation and thus leading to several different dynamics. An interesting example of this phenomenon is given by Solutions 27 and 43, whose mosaics are both displayed in Figure 13. Solution 43 is especially interesting, since it displays unusual features. Indeed, its mosaics seem to be dense and, at least according to the initial conditions we tried, the dynamics seem to be minimal. Such amazing properties would more than justify further investigations in forthcoming works.

We have 15 degree-6 solutions, shown in Table 7. All of them are PWSs, and seven of them lead to simple dynamics. Among the nontrivial cases, Solution 22 is (up to a rotation) precisely the case studied in [Goetz and Poggiaspalla 04]. Solution 20 is interesting because it is a case of a "fake" PWS. Indeed, as we shall see in the next section, inducting on a well-chosen set yields a piecewise isometry, which is enough to describe the whole dynamics. Figure 14 shows three nontrivial cases of piecewise similarities.

We only have two solutions of higher degree. Solutions 23 and 25 have degree 10 and 12, respectively. They are presented in Table 8. Both of them display highly nontrivial dynamics; cf. Figure 15. We will not present any simplicial solutions here. Instead, after a remark on the

Nb	Angles	Permutation	Type	Remark
9	$(3\pi/5, \pi/5, \pi/5, \pi/5, 2\pi/5, 2\pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(231798546)	PWI	conjugate[Goetz 98]
10	$(3\pi/5, \pi/5, \pi/5, \pi/5, 2\pi/5, 2\pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(231798645)	PWI	1 reflection
21	$(\pi/3, \pi/3, \pi/3, \pi/6, 2\pi/3, \pi/6, \pi/2, \pi/12, 5\pi/12)$	(645213987)	PWI	2 reflection
24	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(645312789)	PWI	conjugate[Goetz 98]
26	$(\pi/3, \pi/3, \pi/3, \pi/6, 2\pi/3, \pi/6, \pi/2, \pi/12, 5\pi/12)$	(645321987)	PWI	2 reflection
27	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5)$	(645798123)	PWI	1 reflection
28	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(645798321)	PWI	conjugate[Goetz 98]
30	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(645978321)	PWI	1 reflection
31	$(2\pi/5, \pi/10, \pi/2, \pi/5, 3\pi/5, \pi/5, 3\pi/10, 3\pi/10, 2\pi/5)$	(645987123)	PWS	1 attracting point
37	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(879564321)	PWI	2 reflection
43	$(2\pi/5, \pi/5, 2\pi/5, \pi/5, 3\pi/5, \pi/5, 2\pi/5, \pi/5, 2\pi/5)$	(897564321)	PWI	1 reflection

TABLE 6. Table of degree-4 solutions.

Nb	Angles	Permutation	Type	Remark
14	$(3\pi/7, 3\pi/7, \pi/7, \pi/7, 4\pi/7, 2\pi/7, 4\pi/7, \pi/7, 2\pi/7)$	(312798546)	PWS	
17	$(4\pi/7, \pi/7, 2\pi/7, 2\pi/7, 3\pi/7, 2\pi/7, 3\pi/7, \pi/7, 3\pi/7)$	(321978465)	PWS	
20	$(4\pi/7, \pi/7, 2\pi/7, 3\pi/7, 3\pi/7, \pi/7, 4\pi/7, \pi/7, 2\pi/7)$	(564798321)	PWS	conjugate PWI
22	$(2\pi/7, 2\pi/7, 3\pi/7, \pi/7, 5\pi/7, \pi/7, 3\pi/7, \pi/7, 3\pi/7)$	(645231789)	PWI	cf. [Goetz and Poggiaspalla 04]
38	$(4\pi/7, \pi/7, 2\pi/7, \pi/7, 3\pi/7, 3\pi/7, 2\pi/7, \pi/7, 4\pi/7)$	(879645321)	PWS	
41	$(3\pi/7, 3\pi/7, \pi/7, \pi/7, 4\pi/7, 2\pi/7, 4\pi/7, \pi/7, 2\pi/7)$	(897213546)	PWS	
44	$(7\pi/9, \pi/9, \pi/9, \pi/3, 2\pi/9, 4\pi/9, 4\pi/9, 2\pi/9, \pi/3)$	(978123456)	PWS	
45	$(3\pi/7, \pi/7, 3\pi/7, 2\pi/7, 4\pi/7, \pi/7, 3\pi/7, 2\pi/7, 2\pi/7)$	(987546123)	PWS	

TABLE 7. Degree-6 solutions.

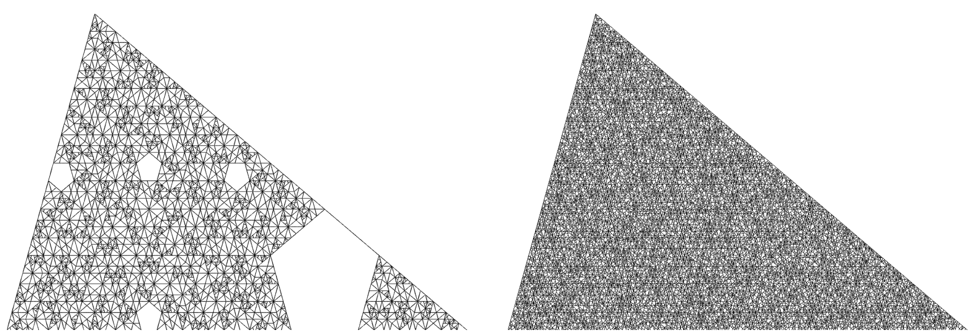

FIGURE 13. The mosaics for Solutions 27 and 43. Though the angles are the same, different permutations yield different dynamics.

behavior of a particular piecewise similarity solution in the next section, we will present a simplicial family belonging to a different pair of types and whose dynamics will be strongly reminiscent of another well-known dynamical family, namely the maps from [Adler et al. 01].

5.2.2 An Example of a "Fake" Piecewise Similarity.

Each of the isolated solutions mentioned above would require a long and detailed study, which we have not done. They would go far beyond the scope of this paper. However, in this section, we are going to present a preliminary study of a piecewise similarity, namely case 11, which is displayed in Figure 16. We call it T, defined on the atoms P_1, P_2, P_3.

The first interesting feature of this map is that in fact, though clearly nonisometric on two of its three atoms, it can be completely described in terms of a standard piecewise isometry. Indeed, ρ_{P_3}, the first return map in

Nb	Angles	Permutation	Type
23	$(2\pi/11, 2\pi/11, 7\pi/11, \pi/11, 9\pi/11, \pi/11, 3\pi/11, 3\pi/11, 5\pi/11)$	(645231978)	PWI
25	$(6\pi/13, \pi/13, 6\pi/13, 3\pi/13, 7\pi/13, 3\pi/13, 4\pi/13, 4\pi/13, 5\pi/13)$	(645312978)	PWI

TABLE 8. Solutions of degrees 10 and 12.

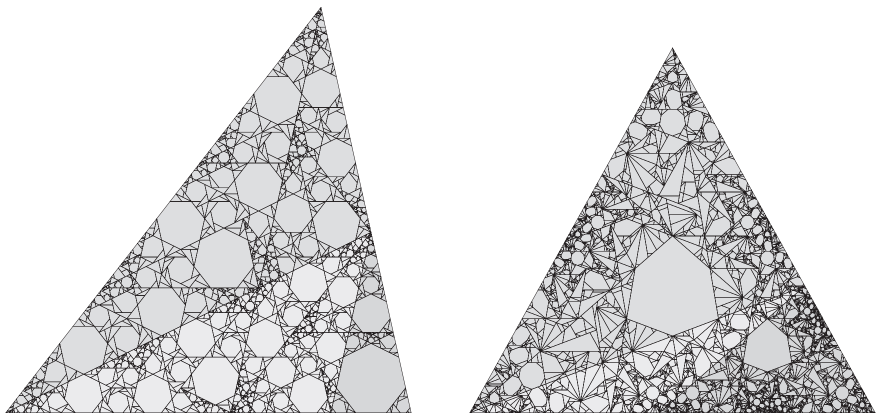

FIGURE 14. Two nontrivial piecewise similarities corresponding to the cases 20 and 44.

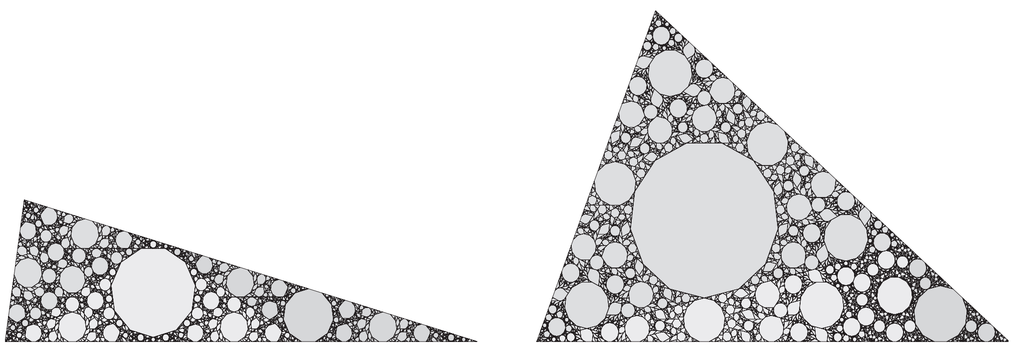

FIGURE 15. Mosaic of Solutions 23 and 25, of order 10 and 12 respectively.

the atom P_3 in the bottom-right corner of Figure 16 (left), is a piecewise isometry, displayed in Figure 17. Moreover, the orbits of all the atoms of the return map cover the whole phase space, except for a finite number of periodic cells, as displayed in Figure 17 (right). Thus, to describe the dynamics of the whole map, it is enough to describe the dynamics of the piecewise isometry ρ_{P_3}.

We already encountered such a feature in the example displayed in Figure 5, but the map was too simple. Indeed, the return map was the identity map. We haven't been able to establish a similar property for other piecewise similarities.

Also based on angles that are multiples of $\pi/7$, this map exhibits some properties reminiscent of the case from [Goetz and Poggiaspalla 04]. It displays nonuniformly bounded return time. Inducting on the lower-right-corner triangular set, highlighted in Figure 18, we are led to the map displayed at the top of the same figure. This map is reminiscent of the unbounded return map encountered in [Goetz and Poggiaspalla 04].

5.2.3 A.K.T.-Like Maps. One of the first (and one of the few) examples of piecewise isometries that have been rigorously studied and whose dynamics are fully understood is found in the work by R. Adler, B. Kitchens, and C. Tresser [Adler et al. 01] (and subsequently by B. Kahng in [Kahng 02]). They describe a continuous one-parameter family of maps consisting of a rotation on

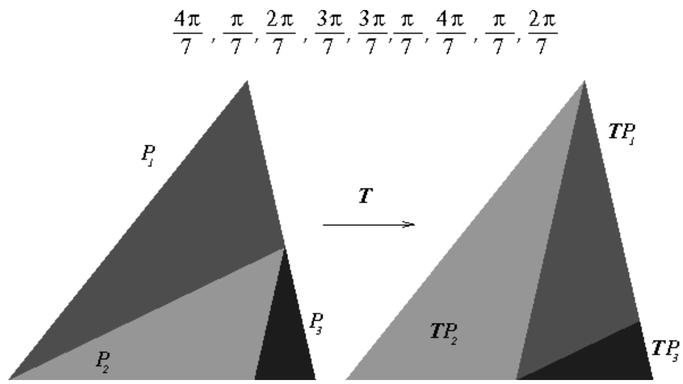

FIGURE 16. Solution 11. It is clear that this is not a piecewise isometry.

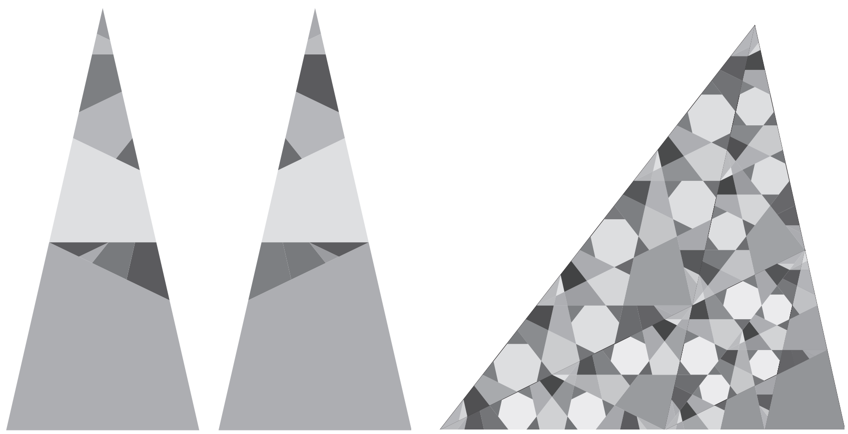

FIGURE 17. On the left, the first return map in the atom P_3 of Solution 11 (Figure 16). On the right, the orbits of its atoms; we can see that the phase space is completely tiled, up to a finite number of periodic cells.

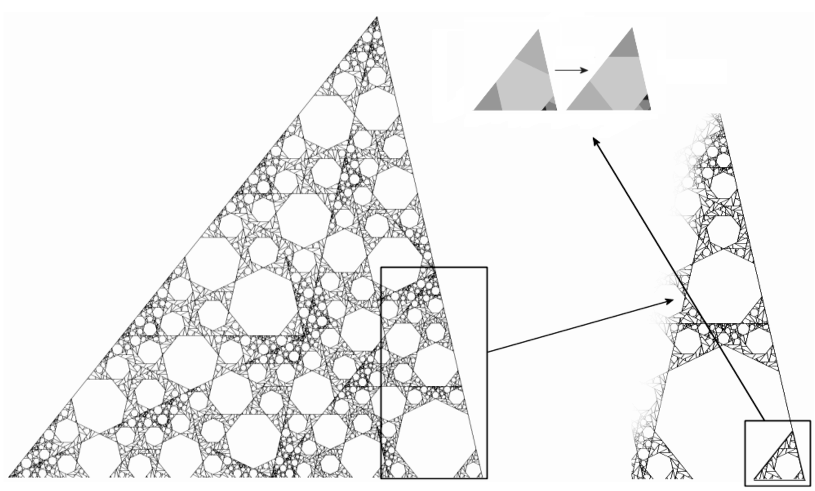

FIGURE 18. The mosaic of Solution 20, with a blowup of the region on which the first return is made. On the lower-right-hand triangle, an induction gives a piecewise isometry with a self-similar partition, which has been already encountered in [Goetz and Poggiaspalla 04].

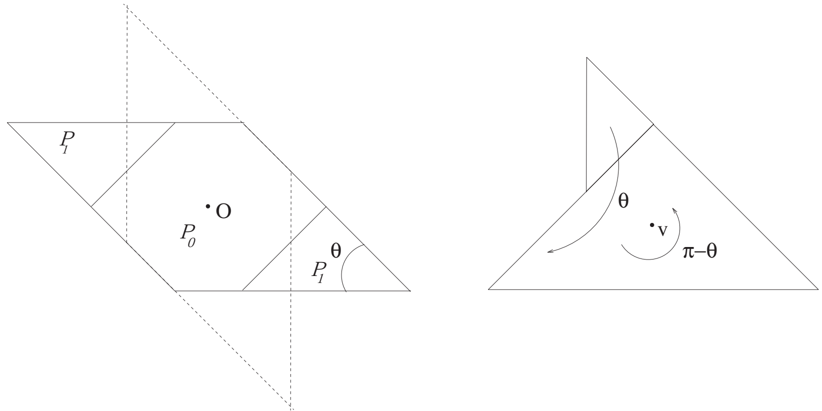

FIGURE 19. On the left, the rhombus has its sides identified to turn it into a torus. The map is then the rotation about the origin with angle $-\theta$. It translates into a piecewise isometry on atoms P_{-1}, P_0, and P_1. On the right, use is made of the symmetry of the dynamics to draw the so-called dart figure on which, essentially, all of the dynamics takes place. The small triangle is rotated by $-\theta$ about the origin of the rhombus, while the big one is rotated about its own center by the angle $\pi - \theta$.

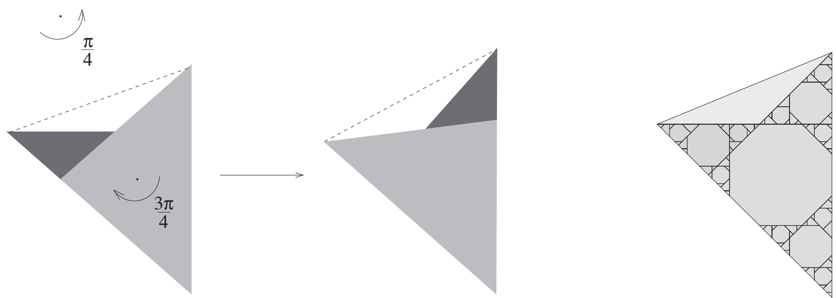

FIGURE 20. The map illustrated on the left is composed of two rotations. The top one, by $\pi/4$, has for fixed point the center of the rhombus, as described in the text. The bottom one has angle $3\pi/4$ and fixed point the center of the bottom triangle. The remaining triangle, below the dashed line, has only reflection defined on it. On the right, we recognize the well-known mosaic of cells that was one of the first to be proven exactly self-similar.

a "tilted" two-dimensional torus, which is equivalent to a piecewise isometry on a rhombus; cf. Figure 19 (left). Its dynamics are described when the rotation angle is equal to $\pi/4$, $\pi/5$, and $2\pi/5$.

The case $\pi/4$ in particular has been extensively studied. This map, and sometimes the whole family, is frequently referred to as the "A.K.T." map (or, shuffling the letters, the "K.A.T" map, by analogy with the Arnold "cat" map).

Regardless of the angle, the dynamics are always symmetric with respect to the center of the rhombus. Thus, using this symmetry we define a new map on a dart-shaped figure (cf. Figure 19 (right)). This shape can be constructed with two bisections of a big triangle. The combinatorial type associated with the dart shape will have three triangles; type 12 in Figure 10 is suitable. The type of the image partition will then be type 5 of Figure 10.

Equations for the pair $(12, 5)$ give a total of 96 solutions, 16 of which are isolated, and 80 simplicial. We have been working with the lists $((1,5,3)(1,4,5)(4,2,3))$ and $((1,5,3)(5,4,3)(1,2,4))$. Among the isolated solutions, six are PWIs. Among the simplicial ones, Solution 35 will particularly focus our interest:

$$\left(\frac{5\pi}{6} - \frac{2x}{3},\ \frac{\pi}{6} - \frac{x}{3},\ x,\ \frac{2\pi}{3} - \frac{4x}{3},\ \frac{\pi}{6} + \frac{2x}{3},\ \frac{\pi}{6} + \frac{2x}{3},\right.$$
$$\left.\frac{\pi}{3} - \frac{2x}{3},\ \frac{\pi}{3} + \frac{4x}{3},\ \frac{\pi}{3} - \frac{2x}{3}\right), \quad 0 < x < \frac{\pi}{2},$$

attached to the angle permutation (132456978). When $x = \pi/8$, the map given by the angles is a PWI and it is precisely the case extensively studied in [Adler et al. 01] up to a flip. It is displayed in Figure 20.

Changing the angle of the rhombus gives birth to a continuous family of piecewise isometries, some of which have been studied in this context [Adler et al. 01, Kahng

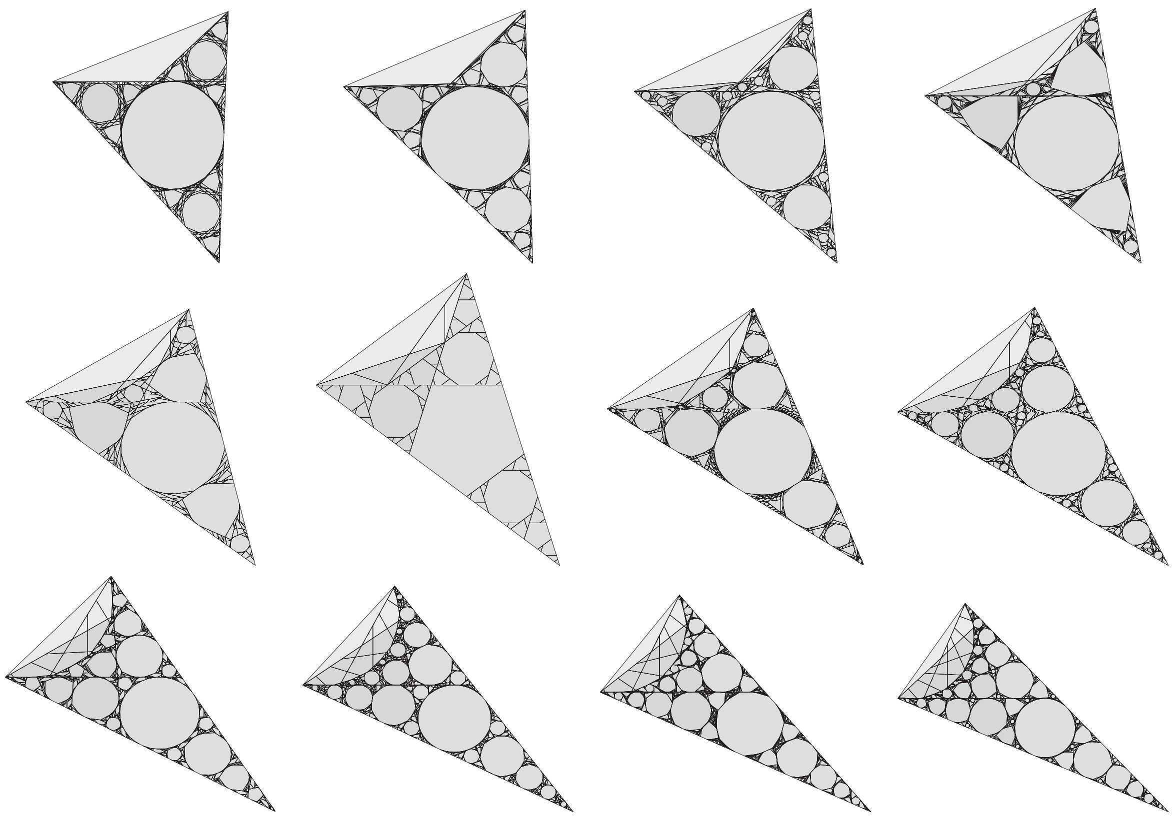

FIGURE 21. From top to bottom and from left to right, 12 mosaics of cells for 12 increasing values of x, their shapes are familiar; they match closely those of [Adler et al. 01] when we change the angles of the rhombus.

02, Kahng 04a, Kahng 04b] or for closely related maps [Lowenstein and Vivaldi 98, Koupstov et al. 03]. This degree of freedom corresponds to the parameter x. Figure 21 shows 12 mosaics for 12 values of x, ranging from $\pi/10$ to $\pi/3$. These plots are quite familiar. Among them we can recognize in particular at the second position of the second row, the case in which the rotation angle is based on $\pi/5$, which has also been described in [Adler et al. 01] (and subsequently in [Kahng 04a]). The mosaic is not exactly the same, however, because of the extra triangle needed in our context to ensure the consistency of the bisection process.

6. PERSPECTIVES

There remains a great deal of work if one is to obtain a better understanding of the underlying geometry of the set of PWSs. Notice that this set, for n triangles, can be viewed as the union of "self-intersections" of an $(n+1)$-dimensional complex embedded in $\mathbb{R}^{3n}$.

By self-intersection we mean intersections of distinct faces of the complex, which is, roughly speaking, the union of the simplices corresponding to the types up to allowed permutations. Self-intersections correspond to nice partitions that can be rearranged to form another nice partition (possibly of a different triangle).

Notice that if we take the closure of the faces, then this set is connected. Moreover, the complex should have a somewhat recursive structure, since its shape for $n+1$ triangles is based on the shape for n triangles. However, up to now we have not been able to extract useful information from these considerations. Nevertheless, the concept is nice and could yield new ideas.

An important point for future research is to understand the "boundaries" of the simplices. In terms of partition types, a boundary corresponds to some angles being zero or π. In particular, the number of triangles in the type is decreasing when some parameters attain a boundary. If the maps in the simplex are all piecewise isometries, we should still have piecewise isometries on the boundary. This is false when the maps are strictly piecewise similarities.

It might be interesting as well to work with the lengths of the sides of the triangles instead of their angles. The difficulty is that the equations linking the parameters would then have polynomial form and hence would be trickier to deal with. The interest, though, would be to discriminate directly the PWIs among the PWSs, especially in the case that the solution is a simplex.

It seems possible to generalize this construction to more than two dimensions. In the tetrahedron we can define a notion corresponding to a partition by bisection. Given a nice triangulation $(P_i)_{1 \leq i \leq n}$ of one face opposite a vertex indexed by k of the tetrahedron, consider the partition of the tetrahedron with tetrahedra of base P_i and vertices k. Since the elements of this partition are tetrahedra, it makes sense to iterate the process as done above.

Another direction for generalizations suggested by our point of view is PWIs in non-Euclidian spaces. The hyperbolic plane and the the sphere have nice isometry groups, and the notion of partitions by bisection makes sense. Some work in this direction is in progress and seems promising.

Our formalism also provides a reasonable context in which to ask whether a given behavior is typical for bijective PWIs. For instance, we have in mind the question raised by Buzzi and Hubert (during the conference Porq'roll 2002) about the genericity of periodic islands among PWIs. Notice that from this point of view, Solution 43 of Section 5.2.1 is striking. If it is true that it has no periodic island, as suggested by the mosaic of Figure 13, it would be highly interesting to understand why.

We have not said anything about the behavior of the boundaries of the triangles of the partitions, although they carry an important part of the dynamics (according to the literature). In particular, it is not even clear whether essentially bijective maps can be extended to the boundaries in a bijective way.

A real breakthrough would be to use this description to induct in a systematic way. Notice that it is by no means clear that it is possible. Indeed, in general, if the partition of T is nice, the partition of T^2 is not. However, we think that there is some hope to find classes of maps for which induction would behave well. If this idea were to prove fruitful, the parameterization of the space of PWSs would give dynamical information on the maps themselves.

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