

Some Geometry and Combinatorics for the S -Invariant of Ternary Cubics

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In earlier papers [Wilson 04, Totaro 04], the S -invariant of a ternary cubic f was interpreted in terms of the curvature of related Riemannian and pseudo-Riemannian metrics. This is clarified further in Section 3 of this paper. In the case that f arises from the cubic form on the second cohomology of a smooth projective threefold with second Betti number three, the value of the S -invariant is closely linked to the behavior of this curvature on the open cone consisting of Kähler classes. In this paper, we concentrate on the cubic forms arising from complete intersection threefolds in the product of three projective spaces, and investigate various conjectures of a combinatorial nature arising from their invariants.

1. INTRODUCTION

Given a real form $f(x_1, \dots, x_m)$ of degree $d > 2$, there is a pseudo-Riemannian metric, given by the matrix $(g_{ij}) = -(\partial^2 f / \partial x_i \partial x_j) / d(d-1)$, defined on the open subset of \mathbf{R}^m where the determinant $h = \det(g_{ij})$ is nonzero. This metric is referred to by Totaro as the *Hessian metric*, and we study it further in the case that f is a real ternary cubic. Building on previous work in [Wilson 04, Totaro 04], we determine the full curvature tensor of this metric in terms of h and the S -invariant of f (Theorem 3.3).

Motivated by the geometric background, as summarized below, we are led to consider cubic forms associated with complete intersection threefolds in the product of three projective spaces, and from this to study those cubic forms that arise as follows: we choose positive integers d_1, d_2, d_3 and $r \geq 0$ such that $d_1 + d_2 + d_3 = r + 3$, and set

$$P = (x_1 H_1 + x_2 H_2 + x_3 H_3)^3 \prod_{j=1}^r (a_j H_1 + b_j H_2 + c_j H_3),$$

with the a_j, b_j, c_j nonnegative, and such that the cubic $F(x_1, x_2, x_3)$, defined by taking the coefficient of the term in $H_1^{d_1} H_2^{d_2} H_3^{d_3}$ in the above formal product P , is

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nondegenerate. Calculations from [Wilson 04] and the further discussion provided below suggest various conjectures concerning the invariants of such cubics. In this paper, we shall concentrate mainly on Conjecture 4.1, which states that with the S -invariant regarded as a polynomial in the a_j , b_j , and c_j , every coefficient is nonnegative. Extensive computer investigations are described in support of this conjecture.

In Section 5, we consider the cofactors B_{pq} of the Hessian matrix of F given by the matrix of second partial derivatives. In the specific case under consideration, these are polynomials in x_1, x_2, x_3 and the a_j, b_j, c_j . We derive formulas for the coefficients of these polynomials, and deduce that these coefficients are negative if $p = q$ and positive if $p \neq q$ (Theorem 5.1). From this, we deduce that the Hessian determinant H of F , that is, the determinant of the Hessian matrix, has only positive coefficients. This latter result represents a combinatorial version of the Hodge index theorem.

In the final section, we return to a formula for S , given in Section 3, in terms of the cofactors B_{pq} of the Hessian matrix. The fact that for the cubics F being considered we have formulas for the coefficients of monomials in the B_{pq} enables us to produce a useful algorithm for determining the coefficient of a given monomial in S . We run this algorithm for some critical cases, where we check that the conjectured positivity holds.

2. THE GEOMETRIC BACKGROUND

In this preliminary section, the theory and calculations of this paper are set in their geometric context, and motivation is given for the conjectures appearing in Section 4.

For a compact Kähler n -fold X , we can consider the level set in $H^{1,1}(X, \mathbf{R}) \subset H^2(X, \mathbf{R})$ defined by setting the degree- n form $D \mapsto D^n$ (given by the cup product) to be one. The intersection of this level set with the Kähler cone $\mathcal{K} \subset H^{1,1}(X, \mathbf{R})$ gives a manifold \mathcal{K}_1 of dimension $h^{1,1} - 1$, on which there is a natural Riemannian metric. The tangent space to \mathcal{K}_1 at a point D may be identified as $\{L \in H^{1,1} : D^{n-1} \cdot L = 0\}$, and the Riemannian metric specified by

$$(L_1, L_2) \mapsto -D^{n-2} \cdot L_1 \cdot L_2.$$

This is precisely the restriction to \mathcal{K}_1 of the Hessian metric (as defined above) associated with the degree- n form on $H^{1,1}(X, \mathbf{R})$. In [Wilson 04], we initiated the study of this manifold and its curvature, motivated mainly by the implications that any restrictions on this curvature might have concerning the existence and classification of

Calabi–Yau threefolds with a given differentiable structure.

In the cited paper, we showed that if one assumed the existence of limit points in complex moduli corresponding to a certain specified type of degeneration, then the sectional curvatures of \mathcal{K}_1 were bounded between $-\frac{1}{2}n(n+1)$ and 0. In the particular case of complex projective threefolds with second Betti number 3 and $h^{2,0} = 0$, we have a ternary cubic form F on $H^2(X, \mathbf{R})$, and an explicit formula was produced for the curvature of the surface \mathcal{K}_1 , namely

$$-\frac{9}{4} + \frac{1}{4} \frac{6^6 S F^2}{H^2},$$

where S denotes the S -invariant of F (see Section 3) and H the Hessian determinant. From this one notes that if $S \neq 0$ and there exists a point D on the boundary of the Kähler cone at which H vanishes but F doesn't, then the curvature is unbounded on \mathcal{K}_1 .

In the case of a Calabi–Yau threefold, a rational such point D can be seen to correspond to the contraction of a surface on X to a point [Wilson 92]. If D lies in the interior of a codimension-one face of the closure $\bar{\mathcal{K}}$ of the Kähler cone (recalling from [Wilson 92] that away from $F = 0$, the boundary of $\bar{\mathcal{K}}$ is locally rational polyhedral), then in appropriate coordinates the cubic form may be written as $F = ax_1^3 + g(x_2, x_3)$, and in particular, $S = 0$. If, however, D generates an extremal ray of $\bar{\mathcal{K}}$ (i.e., corresponds to a codimension-2 face of $\bar{\mathcal{K}}$), we automatically have that D is rational. Moreover, we may have that S is nonzero, although using the classification of contractions from [Wilson 92], one can show that in this case S must be nonnegative. There exist examples of such Calabi–Yau threefolds with $S > 0$, and hence with the curvature of \mathcal{K}_1 unbounded above.

The simplest examples here are provided by general Weierstrass fibrations over $\mathbf{P}^1 \times \mathbf{P}^1$ or over the Hirzebruch surfaces \mathbf{F}_1 and \mathbf{F}_2 . In the non-Calabi–Yau case, an even simpler example is provided by taking the cone (in \mathbf{P}^4) on a smooth quadric surface in \mathbf{P}^3 , and blowing up in the singular point; the first of the Calabi–Yau examples given above is closely related to this one. In the examples above, the curvature is in fact strictly positive on \mathcal{K}_1 , but there are a number of examples of Calabi–Yau hypersurfaces in weighted projective 4-space, with second Betti number three, where the curvature tends to infinity as one approaches some extremal ray on the boundary, but with it being negative at other points of \mathcal{K}_1 . It follows, however, from the above discussion, at least in the case of Calabi–Yau threefolds with second Betti number 3 and $h^{2,0} = 0$, that the curvature of the

surface \mathcal{K}_1 is bounded below, and in the case in which it is not bounded above, this lower bound will be $-\frac{9}{4}$, or in other words, the S -invariant of the ternary cubic is nonnegative.

The ideas in [Wilson 04] were motivated in the case of Calabi–Yau n -folds by a mirror-symmetry argument relating the curvature on \mathcal{K}_1 to the curvature of the Weil–Petersson metric on the complex moduli space of the mirror. Known results on the curvature of the Weil–Petersson metric in fact involve the Ricci curvature rather than sectional curvatures and only in general provide a lower bound; for 3-folds and 4-folds one can, however, construct from the Weil–Petersson metric and its Ricci curvature an associated metric, the *Hodge metric*, and there are then both upper and lower bounds for the sectional curvatures of this metric [Lu 01, Lu and Sun 04].

Thus one should not perhaps be surprised by the examples given above in which the curvature of \mathcal{K}_1 is positive; this is expected in the mirror to imply that the curvature of the Weil–Petersson metric is not semi-negative (in any neighborhood of the large complex limit point). One should, however, expect a lower bound for the Ricci curvature of the metric on \mathcal{K}_1 , and in the Calabi–Yau case the author conjectures this to be $-(\frac{n}{2})^2(h^{1,1} - 2)$. In fact, using calculations from [O’Neill 83, p. 211] and the interpretation of the Hessian metric in terms of a warped product [Totaro 04, Lemma 2.1], this conjecture may be checked to be equivalent to the semi-Riemannian Hessian metric on \mathcal{K} having *nonnegative* Ricci curvature (compare also with the explicit formula produced in Theorem 3.3). This latter rather attractive conjecture lends itself to being verified by computer, and has been checked by the author to hold for all the standard examples given in [Wilson 04], and also for certain Calabi–Yau threefolds with rather larger values of b_2 , one such example corresponding to a hypersurface of degree 13 in weighted projective space $\mathbf{P}(1, 2, 3, 3, 4)$, a Calabi–Yau threefold with $b_2 = 5$.

Thus for Calabi–Yau threefolds with $b_2 = 3$ and $h^{2,0} = 0$, the above expectation corresponds to \mathcal{K}_1 having curvature bounded below by $-\frac{9}{4}$, or equivalently to the S -invariant of the ternary cubic form being nonnegative. This has been extensively checked against available lists of Calabi–Yau threefolds with $b_2 = 3$. It should be noted that the nonnegativity of S is known to fail in general for complex projective threefolds with $b_2 = 3$ and $h^{2,0} = 0$ [Wilson 04]. However, in the case of complex projective threefolds admitting the specific type of degeneration described in [Wilson 04], the author expects that the lower

bound will again be $-\frac{9}{4}$ rather than the -3 as proved there; for higher values of b_2 , the lower bound of -3 on the *sectional* curvature can be achieved. In the case, for instance, of abelian threefolds, the lower bound of -3 on the *sectional* curvature is attained, although one checks easily that the sharper lower bound of $-9(h^{1,1} - 2)/4$ holds for the Ricci curvature.

For the general case of complete intersection threefolds in the product of three projective spaces, one has $b_2 = 3$ and $h^{2,0} = 0$, and one can degenerate the defining polynomials into products of polynomials on the three factors, and each of these polynomials may be assumed to be the product of distinct linear forms. The author expects (but it will be nontrivial to prove) that the general such degeneration will be of the type described in [Wilson 04], with the product of harmonic two-forms being approximately harmonic. The conjectures introduced in Section 4 will be closely related to the conjecture that the curvature of \mathcal{K}_1 for such threefolds is bounded between $-\frac{9}{4}$ and zero.

The case of complete intersection threefolds in the product of three projective spaces therefore represents an important test case for the above conjectures and speculations. We shall see that they lead to rather striking positivity conjectures of a combinatorial nature, involving the classical invariants of ternary cubic forms, for which extensive computational evidence will be presented.

3. THE S -INVARIANT AND CURVATURE

We consider a general nondegenerate ternary cubic form with real coefficients

$$\begin{aligned} f = & a_{300}x_1^3 + a_{030}x_2^3 + a_{003}x_3^3 + 3a_{210}x_1^2x_2 + 3a_{201}x_1^2x_3 \\ & + 3a_{120}x_1x_2^2 + 3a_{021}x_2^2x_3 + 3a_{102}x_1x_3^2 + 3a_{012}x_2x_3^2 \\ & + 6a_{111}x_1x_2x_3. \end{aligned}$$

Associated with f , we have two basic invariants S and T , one of degree 4 in the coefficients and one of degree 6 [Aronhold 58, Sturmfels 93]. The S -invariant is given explicitly [Sturmfels 93, p. 167] by an expression in the coefficients with 25 terms

$$\begin{aligned} S = & a_{300}a_{120}a_{021}a_{003} - a_{300}a_{120}a_{012}^2 - a_{300}a_{111}a_{030}a_{003} \\ & + a_{300}a_{111}a_{021}a_{012} + \cdots + a_{201}a_{111}a_{102}a_{030} \\ & + a_{120}^2a_{102}^2 - 2a_{120}a_{111}^2a_{102} + a_{111}^4. \end{aligned}$$

As indicated above, this invariant is closely associated with curvature. We define the *index cone* in \mathbf{R}^3 to consist of the points at which f is positive and the indefinite

metric defined by the matrix $f_{ij} = \partial^2 f / \partial x_i \partial x_j$ is of signature (1, 2). The restriction of $g_{ij} = -\frac{1}{6} f_{ij}$ to the level set M given by $f = 1$ in the index cone is then a Riemannian metric, whose curvature at any point is given by the formula

$$-\frac{9}{4} + \frac{1}{4} \frac{Sf^2}{h^2},$$

where $h = \det(g_{ij}) = -H/6^3$, with H denoting the Hessian determinant of f [Wilson 04, Theorem 5.1]. Strictly speaking, we do not need to include the f^2 in this formula, since by definition it has value one on the level set; however for any point in the index cone, the formula given provides the curvature at the unique point of M on the corresponding ray. This formula was both extended to higher degrees and clarified further in [Totaro 04].

Consider now the pseudo-Riemannian metric defined by the matrix $g_{ij} = -\frac{1}{6} f_{ij}$ on a suitable open subset of \mathbf{R}^3 . In the case of cubics, [Totaro 04, Theorem 3.1] reduces to the following statement: if U is an open subset of \mathbf{R}^3 on which the Hessian H is nonzero, and M denotes the level set in U given by $f = 1$, then the sectional curvature of U on the tangent 2-plane to M at a point is just

$$\frac{6^6 Sf}{H^2} = \frac{Sf}{h^2}.$$

This proves again the formula given above for the curvature of the restricted metric to M and generalizes in a natural way to forms f of arbitrary degrees greater than 2 [Totaro 04, Theorem 3.1]. It should be noted here that for ternary cubics f , the Clebsch version $S(f)$ of the S -invariant (as used in Totaro’s paper) is the Aronhold S -invariant (as used in this paper) multiplied by the factor 6^4 .

One point that I wish to emphasize in this section is that once one knows the S -invariant and the Hessian determinant H , the whole curvature tensor of the above pseudo-Riemannian metric is given very simply by Theorem 3.3, thus extending in this case Totaro’s Theorem 3.1 cited above.

Throughout this paper, we shall denote by B the adjoint matrix to $A = (f_{ij})$, with entries the cofactors of A . We shall need the following identity, proved by classical invariant theory.

Lemma 3.1. *With notation as above,*

$$\frac{1}{2} \sum_{p,q} B_{pq} \frac{\partial^2 B_{ij}}{\partial x_p \partial x_q} = 6^4 S x_i x_j. \tag{3-1}$$

Proof: If we apply the Clebsch polarization operator $\sum y_i \partial / \partial x_i$ to f twice, we obtain a mixed concomitant $S^3 V^* \rightarrow S^2 V^* \otimes V^*$ (where V denotes the 3-dimensional real vector space), which in coordinates may be written as

$$f \mapsto \sum_{i,j} y_i y_j \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Passing to the dual quadratic form (scaled by $H(\mathbf{x})$), we obtain a mixed concomitant $S^3 V^* \rightarrow S^2 V^* \otimes S^2 V$, which in coordinates may be written as

$$f \mapsto \sum_{p,q} B_{pq}(\mathbf{x}) \frac{\partial}{\partial y_p} \frac{\partial}{\partial y_q}.$$

Taking a convolution of two such concomitants and contracting out a factor $S^2 V \otimes S^2 V^*$, we obtain a concomitant $S^3 V^* \rightarrow S^2 V^* \otimes S^2 V$, which in coordinates may be written as

$$f \mapsto \sum_{i,j} \left(\sum_{p,q} B_{pq}(\mathbf{x}) \frac{\partial^2 B_{ij}(\mathbf{x})}{\partial x_p \partial x_q} \right) \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j}.$$

We can check easily on the Hesse cubic $x_1^3 + x_2^3 + x_3^3 + 6\lambda x_1 x_2 x_3$ that

$$\frac{1}{2} \sum_{p,q} B_{pq}(\mathbf{x}) \frac{\partial^2 B_{ij}(\mathbf{x})}{\partial x_p \partial x_q} = 6^4 S x_i x_j,$$

where $S = \lambda(\lambda^3 - 1)$ in the S -invariant, and hence we deduce that the two concomitants

$$\sum_{i,j} \frac{1}{2} \left(\sum_{p,q} B_{pq}(\mathbf{x}) \frac{\partial^2 B_{ij}(\mathbf{x})}{\partial x_p \partial x_q} \right) \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j}$$

and

$$\sum_{i,j} 6^4 S x_i x_j \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j}$$

are identical, since clearly they also transform in the same way under the operation of scaling the coordinates. Thus we deduce the result claimed. \square

Remark 3.2. If we now express the cofactors B_{pq} in terms of the f_{ij} , and then operate on both sides of (3-1) by $\partial^2 / \partial x_i \partial x_j$, we get formulas for S analogous to those given in [Aronhold 58, page 116]. From Lemma 3.1, it follows immediately that for any i, j ,

$$\frac{1}{2} \sum_{p,q} \frac{\partial^2 B_{pq}}{\partial x_i \partial x_j} \frac{\partial^2 B_{ij}}{\partial x_p \partial x_q} = 6^4 (1 + \delta_{ij}) S.$$

It is shown in [Totaro 04] that the curvature tensor of the pseudo-Riemannian metric defined above has components

$$R_{ijkl} = -\frac{1}{144} \sum_{p,q} g^{pq} (f_{jlp} f_{ikq} - f_{ilp} f_{jkq}),$$

where (g^{pq}) denotes the inverse matrix to (g_{ij}) . Thus, for instance, if we let $h = \det(g_{ij}) = -H/6^3$, then

$$-4hR_{1212} = 6^{-4} \sum_{p,q} B_{pq} (f_{11p} f_{22q} - f_{12p} f_{12q}).$$

We now observe that

$$\begin{aligned} (f_{11p} f_{22q} + f_{11q} f_{22p} - 2f_{12p} f_{12q}) &= \frac{\partial^2 (f_{11} f_{22} - f_{12}^2)}{\partial x_p \partial x_q} \\ &= \frac{\partial^2 B_{33}}{\partial x_p \partial x_q}, \end{aligned}$$

and so

$$\begin{aligned} -4h6^4 R_{1212} &= \frac{1}{2} \sum_p B_{pp} \frac{\partial^2 B_{33}}{\partial x_p \partial x_p} + \sum_{p < q} B_{pq} \frac{\partial^2 B_{33}}{\partial x_p \partial x_q} \\ &= \frac{1}{2} \sum_{p,q} B_{pq} \frac{\partial^2 B_{33}}{\partial x_p \partial x_q}. \end{aligned}$$

Hence we deduce from Lemma 3.1 that $-4hR_{1212} = Sx_3^2$.

In Lemma 3.1, we can also take $(i, j) = (1, 2)$. Since $B_{12} = f_{13} f_{23} - f_{12} f_{33}$, for any given (p, q) we have

$$\frac{\partial^2 B_{12}}{\partial x_p \partial x_q} = f_{13p} f_{23q} + f_{13q} f_{23p} - f_{12p} f_{33q} - f_{12q} f_{33p}.$$

The formula for curvature then implies that

$$\frac{1}{2} 6^{-4} \sum_{p,q} B_{pq} \frac{\partial^2 B_{12}}{\partial x_p \partial x_q} = 4hR_{1323},$$

and so we deduce from Lemma 3.1 that $Sx_1x_2 = 4hR_{1323} = -4hR_{1332} = -4hR_{3123}$.

Theorem 3.3. *All components of the curvature tensor of the Hessian metric on $U \subset \mathbf{R}^3$, where U is the open subset given by the nonvanishing of H , are given simply in terms the invariant S and the Hessian of f , and are all of the form $\pm \frac{1}{4} S x_i x_j / h$ for appropriate i, j and choice of sign. Moreover, given tangent vectors $\xi = \sum \lambda_i \partial / \partial x_i$ and $\eta = \sum \mu_j \partial / \partial x_j$, the corresponding value of the curvature tensor satisfies*

$$\begin{aligned} -4h R(\xi, \eta, \xi, \eta) &= S (\lambda_1 \mu_2 x_3 + \lambda_2 \mu_3 x_1 + \lambda_3 \mu_1 x_2 \\ &\quad - \lambda_2 \mu_1 x_3 - \lambda_3 \mu_2 x_1 - \lambda_1 \mu_3 x_2)^2. \end{aligned}$$

Proof: Since we have formulas for R_{1212} and R_{1323} , we have the analogous formulas for R_{ijij} and R_{ijkj} . We now use the general fact that the curvature tensor is invariant under exchanging the first pair of indices with the second pair of indices, and is anti-invariant under exchanging the first pair (or second pair) of indices; in our particular case, these symmetries are clear from the above formula for the curvature tensor, taken from [Totaro 04]. In this way, we obtain expressions of the required form for all the components of the curvature tensor. Finally, we deduce that

$$\begin{aligned} -4h R(\xi, \eta, \xi, \eta) &= -4h \sum_{\substack{i < j \\ p < q}} (\lambda_i \mu_j - \lambda_j \mu_i) (\lambda_p \mu_q - \lambda_q \mu_p) R_{ijpq} \\ &= S (\lambda_1 \mu_2 x_3 + \lambda_2 \mu_3 x_1 + \lambda_3 \mu_1 x_2 - \lambda_2 \mu_1 x_3 - \lambda_3 \mu_2 x_1 \\ &\quad - \lambda_1 \mu_3 x_2)^2, \end{aligned}$$

which completes the proof. \square

4. CONJECTURAL POSITIVITY OF S FOR CERTAIN CUBICS

In [Wilson 04, Section 5], we were interested in the cubics that occur as intersection forms for 3-dimensional complete intersections in the product of three projective spaces. We can, however, formalize this into a purely algebraic problem. Suppose a ternary cubic is obtained as follows: We choose positive integers d_1, d_2, d_3 and $r \geq 0$ such that $d_1 + d_2 + d_3 = r + 3$, and set

$$P = (x_1 H_1 + x_2 H_2 + x_3 H_3)^3 \prod_{j=1}^r (a_j H_1 + b_j H_2 + c_j H_3),$$

with the a_j, b_j , and c_j nonnegative, and such that the cubic $F(x_1, x_2, x_3)$ defined by taking the coefficient of the term in $H_1^{d_1} H_2^{d_2} H_3^{d_3}$ in the above formal product P is nondegenerate. To relate this to the geometry, note that if the a_j, b_j, c_j take nonnegative integer values, then we may consider the complete intersection projective threefolds X in $\mathbf{P}^{d_1} \times \mathbf{P}^{d_2} \times \mathbf{P}^{d_3}$ given by r general trihomogeneous polynomials, with tridegrees (a_j, b_j, c_j) for $j = 1, \dots, r$. The cubic we have defined above is then the intersection form on the rank-three sublattice of $H^2(X, \mathbf{Z})$ generated by the pullbacks of hyperplane classes from the three factors; by Lefschetz-type arguments, this is usually the whole of $H^2(X, \mathbf{Z})$.

As in Section 3, we denote the coefficients of the ternary cubic F by a_{ijk} , where $i + j + k = 3$. These coefficients are themselves polynomials in the a_j, b_j, c_j ,

homogeneous of degree r in each such set of variables. We let S denote the S -invariant of F , and H the Hessian determinant of F .

Conjecture 4.1. *Regarding S as a polynomial in the a_j, b_j, c_j , every coefficient of this polynomial is nonnegative.*

Conjecture 4.2. *Regarding $9H^2 - 6^6SF^2$ as a polynomial in the a_j, b_j, c_j and x_1, x_2, x_3 , every coefficient of this polynomial is also nonnegative.*

These conjectures imply their (weaker) geometric counterparts, in the case of X being a complete intersection threefold in the product of three projective spaces, with second Betti number three, and F being its intersection form. Here, we have taken specific nonnegative integral values for the degrees a_j, b_j, c_j . With the notation as in Section 2, these weaker conjectures may be interpreted, for X as given, as saying that the curvature of the surface \mathcal{K}_1 is bounded between $-\frac{9}{4}$ and zero. The previous theoretical and computational evidence for such conjectures to be true was outlined in Section 2. Recall also that the first of these conjectures is equivalent to the statement that the semi-Riemannian Hessian metric on \mathcal{K} associated with F has nonnegative Ricci curvature.

Conjecture 4.3. *The intersection form of X has nonnegative S -invariant.*

Conjecture 4.4. *The polynomial $9H^2 - 6^6SF^2$ in x_1, x_2, x_3 takes nonnegative values on the Kähler cone of X , given by $x_1 > 0, x_2 > 0, x_3 > 0$.*

Considered as a polynomial in the a_j, b_j, c_j , we have that S is homogeneous of degree 4 in any given set (a_j, b_j, c_j) , and hence of total degree $4r = 4(d_1 + d_2 + d_3) - 12$. In fact, by inspection of the given formula for S , we see that S is of degree $4d_1 - 4$ in the variables (a_1, \dots, a_r) , of degree $4d_2 - 4$ in the variables (b_1, \dots, b_r) , and of degree $4d_3 - 4$ in the variables (c_1, \dots, c_r) .

As explained above, the conjectures arose out of the theory developed in [Wilson 04]; moreover, there is now extensive computational evidence in their favor. In particular, Conjecture 4.1 has been checked using Mathematica for all $d_i \leq 5$. One can of course reduce to the case in which all the d_i equal some d , namely the maximum of the d_i , by introducing $3d - d_1 - d_2 - d_3$ extra factors $(a_kH_1 + b_kH_2 + c_kH_3)$ into the product P , and by considering the monomials in S , respectively $9H^2 - 6^6SF^2$,

that are of maximum possible degree 4 (respectively 6) in the appropriate variable a_k, b_k , or c_k (and not therefore involving the other two). For instance, for $d - d_1$ of the extra factors, the monomial considered should be of maximum degree in a_k and not involve b_k or c_k .

To give a flavor of these calculations, I can report that in the case $d = 3$ there are 209,520 nonzero terms in S , all with positive coefficients, and that a simple-minded check of this took some two hours of computer time. However, there is a very large number of symmetries, and taking such symmetries into account, the calculation was reduced to less than a couple of minutes. For larger d , therefore, one should factor out by these symmetries. For $d = 4$ the conjecture was checked in a couple of hours, and for $d = 5$ in about four days. The formula for S given in Section 3 in terms of cofactors turns out to be slightly more efficient computationally than the formula in terms of the coefficients of the cubic.¹ The programs were run on a Sun V880 at the Max-Planck-Institut für Mathematik in Bonn, with eight CPUs and 16 GB of main memory theoretically available (although only a fraction of this would have been used). The case $d = 6$ seems to be beyond the range of standard computers. The author has not carried out as many calculations on Conjecture 4.2, but it has been verified for $d_1 = 3, d_2 = d_3 = 2$, and there are strong theoretical reasons in support of its geometric version, Conjecture 4.4, as outlined in Section 2.

In this paper, however, we shall concern ourselves mainly with the problem of Conjecture 4.1, that S has only nonnegative coefficients, and results closely related to this.

For the case $d = 3$, one can obtain very precise information using Mathematica about the coefficients. The monomials appearing in any of the 25 terms in S all appear in the expansion of a_{111}^4 . There are two types of monomial that appear in a_{111}^4 but not in S (because the coefficients cancel out). Examples of these are $a_1^4b_2^4c_3^4a_4^2b_5^2c_6^2$ and $a_1^4b_2^4c_3^4a_4^2b_4c_5b_5^2c_6^2$. If one considers the exponents as forming a 3×6 matrix, these monomials may be denoted rather more clearly as

$$\begin{bmatrix} 4 & 0 & 0 & 2 & 0 & 2 \\ 0 & 4 & 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{bmatrix}.$$

Matrices differing from each other by permutations of the rows and/or columns are regarded as being of the same type. There are then three types with coefficient 1 in S ,

¹The programs used by the author for these checks may be found on his home page: http://www.dpmms.cam.ac.uk/~pmhw/S_invariant_calculations.

represented by matrices

$$\begin{bmatrix} 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 & 0 & 3 & 0 & 1 \\ 0 & 4 & 0 & 1 & 3 & 0 \\ 0 & 0 & 4 & 0 & 1 & 3 \end{bmatrix},$$

and

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 2 & 2 \\ 0 & 4 & 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 & 2 \end{bmatrix}.$$

A similar feature occurs for higher coefficients, namely that there is rather a small number of types. For instance, the largest coefficients that occur are 356, 280, 214, 176, 164, 128, 106, ..., all of which correspond to only one type. The highest coefficient, 356, corresponds to type

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{bmatrix}.$$

It makes more sense, however, if we ignore all monomials containing fourth powers, on the grounds that these correspond to cases with the d_i smaller. With this convention, the corresponding matrices do not have 4 in any entry. In the case $d_1 = d_2 = d_3 = 3$ as above, the smallest coefficients are then 4, 6, and 9, corresponding (respectively) to matrices

$$\begin{bmatrix} 3 & 0 & 1 & 3 & 0 & 1 \\ 1 & 3 & 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 3 & 0 & 0 & 2 & 0 \\ 1 & 1 & 3 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix},$$

and

$$\begin{bmatrix} 3 & 0 & 1 & 3 & 0 & 1 \\ 1 & 3 & 0 & 0 & 1 & 3 \\ 0 & 1 & 3 & 1 & 3 & 0 \end{bmatrix}.$$

We shall also denote the first of these by

$$(d-1) \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix},$$

where $d = 3$.

If we now move on to the cases $d_1 = d_2 = d_3 = d > 3$, we may ask about the coefficients corresponding to

$$(d-1) \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix};$$

for $d = 4$, the coefficient may be calculated as 40, and for $d = 5$ as 652. For $d = 4$ and 5, a computer check verifies that this is the smallest nonzero coefficient (assuming no fourth powers) and the unique type of monomial corresponding to it, and one would conjecture that a similar

statement is true for arbitrary values of $d > 2$. A formula for this coefficient for arbitrary d will be produced in Section 6.

Computer calculations suggest also a result that the cofactors B_{pq} that appeared in Section 3 satisfy the condition that B_{pq} , considered as a polynomial in the a_i, b_j, c_k and x_1, x_2, x_3 , has only positive coefficients if $p \neq q$, and has only negative coefficients if $p = q$. In the geometric situation of a three-dimensional complete intersection in the product of three projective spaces, with the (a_j, b_j, c_j) being assigned specific nonnegative integral values, the negativity of B_{pp} corresponds to the Hodge index theorem on the surface cut out by $H_p = 0$. We shall prove these properties of the cofactors in the next section.

5. COFACTORS OF THE HESSIAN MATRIX

In this section, we study further the cofactors B_{pq} of the Hessian matrix of our ternary cubic F , where it will be more convenient here to denote the variables by x_1, x_2, x_3 rather than x, y, z . Recall that these cofactors were related to the S -invariant by means of various expressions for S described in Section 3; we shall return to this aspect in Section 6. In particular, for the special type of cubics we have studied in the last two sections, the B_{pq} may be considered as polynomials in the a_j, b_j, c_j and x_1, x_2, x_3 . In this section, we confirm the expectations, mentioned in Section 4, concerning the signs of their coefficients; this in turn will show that the Hessian determinant H has only positive coefficients, Theorem 5.2. This latter fact might be expected because of the Hodge index theorem, which implies the weaker statement that H takes nonnegative values for nonnegative values of its variables.

Theorem 5.1. *The polynomials B_{pp} have only negative coefficients, and the polynomials B_{pq} for $p \neq q$ have only positive coefficients.*

Proof: For the first part, we may consider $B_{33} = f_{11}f_{22} - f_{12}^2$. For a general cubic f , we have

$$\begin{aligned} \frac{1}{36}B_{33} &= (a_{300}x_1 + a_{210}x_2 + a_{201}x_3) \\ &\quad \times (a_{120}x_1 + a_{030}x_2 + a_{021}x_3) \\ &\quad - (a_{210}x_1 + a_{120}x_2 + a_{111}x_3)^2. \end{aligned}$$

The fact that in our particular case this polynomial is nonpositive for all nonpositive values of the variables follows from the Hodge index theorem again. However, we

prove the more precise result that the coefficients are all negative.

Let us consider for instance the term in x_1x_2 ; we prove that its coefficient

$$a_{300}a_{030} - a_{210}a_{120},$$

considered as a polynomial in the a_i, b_j, c_k , has only negative coefficients. Without loss of generality, we can assume that $d_1 = d_2 = d_3 = d$, and we set $s = d - 1$. Then the polynomial in question is of degree 2 in each set of variables (a_j, b_j, c_j) , and is of degree $2s - 1$ in the a_i , degree $2s - 1$ in the b_j , and degree $2s + 2$ in the c_k . On the other hand, a_{300} (respectively, a_{210}) is of degree $s - 2$ (respectively, $s - 1$) in the a_i , degree $s + 1$ (respectively, s) in the b_j , and degree $s + 1$ (respectively, $s + 1$) in the c_k , with analogous statements for a_{030} and a_{120} .

Let us now consider a monomial of the appropriate degrees in the (a_j, b_j, c_j) and ask about its coefficient as a term in $a_{300}a_{030} - a_{210}a_{120}$. We suppose that the monomial in question consists of p_1 , respectively p_2, p_3 , occurrences (for various j) of a_j^2 , respectively b_j^2, c_j^2 , and \tilde{u} , respectively \tilde{v}, \tilde{w} , occurrences of a_jb_j , respectively a_jc_j, b_jc_j . We shall see that only the mixed cases will be of relevance. Note that $2p_1 + \tilde{u} + \tilde{v} = 2s - 1$, $2p_2 + \tilde{u} + \tilde{w} = 2s - 1$, and $2p_3 + \tilde{v} + \tilde{w} = 2s + 2$.

The coefficient of the monomial in $a_{300}a_{030}$ is given by counting the number of ways of expressing it as a monomial in a_{300} times a monomial in a_{030} , and similarly for its coefficient in $a_{210}a_{120}$. To obtain the first factor in the former case involves choosing $s - p_1 - 2 = \frac{1}{2}(\tilde{u} + \tilde{v} - 3)$ of the a_jb_j and a_jc_j appearing for which we choose the a_j , $s - p_2 + 1 = \frac{1}{2}(\tilde{u} + \tilde{w} + 3)$ of the a_jb_j and b_jc_j for which we choose the b_j , and $s - p_3 + 1 = \frac{1}{2}(\tilde{v} + \tilde{w})$ of the a_jc_j and b_jc_j for which we choose the c_j . Note here the necessary parity condition that either \tilde{u} is odd and \tilde{v}, \tilde{w} are even, or the other way round. We shall deal with the first case; the other case follows similarly.

We set $\tilde{u} = 2u + 1$, $\tilde{v} = 2v$, and $\tilde{w} = 2w$. The possible factorizations are then given by choosing k of the $2u + 1$ occurrences of a_jb_j for which we choose the a_j , choosing $u + v - 1 - k$ occurrences of a_jc_j for which we choose the a_j , and finally $w - u + 1 + k$ occurrences of the b_jc_j for which we choose the b_j , the rest then being determined. Thus the number of ways of doing this, and hence the coefficient of the monomial in $a_{300}a_{030}$, is

$$\sum_{k=0}^{2u+1} \binom{2u+1}{k} \binom{2v}{v+u-(k+1)} \binom{2w}{w+u-(k+1)}.$$

Similarly, the coefficient of the monomial in $a_{210}a_{120}$ is seen to be

$$\sum_{k=0}^{2u+1} \binom{2u+1}{k} \binom{2v}{v+u-k} \binom{2w}{w+u-k}.$$

Thus we need to verify the negativity of

$$\sum_{k=0}^{2u+1} \binom{2u+1}{k} \left[\binom{2v}{v+u-(k+1)} \binom{2w}{w+u-(k+1)} - \binom{2v}{v+u-k} \binom{2w}{w+u-k} \right].$$

However, this sum may be rearranged as

$$\begin{aligned} & - \binom{2v}{v+u} \binom{2w}{w+u} + \binom{2v}{v+u+2} \binom{2w}{w+u+2} \\ & - \sum_{k=1}^{2u+1} \binom{2v}{v+u-k} \binom{2w}{w+u-k} \\ & \quad \times \left[\binom{2u+1}{k} - \binom{2u+1}{k-1} \right]. \end{aligned}$$

The first line of this rearranged sum is now clearly non-positive. In the summation, the term $\left(\binom{2u+1}{k} - \binom{2u+1}{k-1} \right)$ is antisymmetric about $u + 1$, and in fact equals

$$\frac{2(u+1-k)}{2u+2} \binom{2u+2}{k}.$$

If we therefore pair these antisymmetric terms, and use the fact that for $j > 0$, we have

$$\binom{2v}{v+j-1} \geq \binom{2v}{v+j+1}$$

and

$$\binom{2w}{w+j-1} \geq \binom{2w}{w+j+1},$$

the claimed inequality follows.

For the term in x_1x_3 , we need to show that the polynomial $a_{300}a_{021} + a_{201}a_{120} - 2a_{210}a_{111}$ has only negative terms. In fact, we prove this for the two polynomials $a_{300}a_{021} - a_{210}a_{111}$ and $a_{201}a_{120} - a_{210}a_{111}$. Let us consider a particular monomial appearing in these polynomials; with the notation as above, the parities on \tilde{u}, \tilde{v} , and \tilde{w} will differ from what they were before. Since $2p_1 + \tilde{u} + \tilde{v} = 2s - 1$, $2p_2 + \tilde{u} + \tilde{w} = 2s$, and $2p_3 + \tilde{v} + \tilde{w} = 2s + 1$, we have either \tilde{v} odd and \tilde{u}, \tilde{w} even, or the other way round. Considering, for instance, the case $\tilde{v} = 2v + 1$, $\tilde{u} = 2u$, and $\tilde{w} = 2w$, we can run through a similar argument to that given above, and find that the

coefficient of the given monomial in $a_{300}a_{021} - a_{210}a_{111}$ is

$$\sum_{k=0}^{2v+1} \binom{2v+1}{k} \binom{2w}{w+v-k} \times \left[\binom{2u}{u+v-(k+1)} - \binom{2u}{u+v-k} \right].$$

We now observe that the bracket in this summation is antisymmetric about $k = v - \frac{1}{2}$, and then pairing off terms proves the result in an analogous way to that given previously. Similarly, the coefficient of the given monomial in $a_{201}a_{120} - a_{210}a_{111}$ is

$$\sum_{k=0}^{2v+1} \binom{2v+1}{k} \binom{2u}{u+v-k} \times \left[\binom{2w}{w+v-(k+1)} - \binom{2w}{w+v-k} \right],$$

and the same argument goes through, switching the roles of u and w .

For the term in x_1^2 , we need to show that the polynomial $a_{300}a_{120} - a_{210}^2$ has only negative terms. For a monomial to appear in this polynomial, we have yet another parity condition, namely that \tilde{u} , \tilde{v} , and \tilde{w} all be even, or all be odd. The reader is left to check the negativity. By symmetry, the only other term we need to consider is that in x_3^2 ; here we need the polynomial $a_{201}a_{021} - a_{111}^2$ to have only negative terms. The parity condition here is the same as for x_1^2 , and the reader is left to verify the details of the negativity.

We now need to consider the cofactors B_{pq} with $p \neq q$. We shall explicitly verify only the x_3^2 terms here, and leave the others to the reader. Note in passing that in the formula for $6^4 Sx_3^2$ from Section 3, we may consider instead the identity given simply by the terms in x_3^2 , and so it will be the x_3^2 terms in the above cofactors that will occur in the algorithm we describe in Section 6. We check these terms for B_{12} and B_{13} , the rest then following from considerations of symmetry. Let us start with $\frac{1}{36}B_{12}$, which is

$$\begin{aligned} & \frac{1}{36}(f_{13}f_{23} - f_{12}f_{33}) \\ &= [(a_{201}x_1 + a_{111}x_2 + a_{102}x_3) \\ & \quad \times (a_{111}x_1 + a_{021}x_2 + a_{012}x_3) \\ & \quad - (a_{210}x_1 + a_{120}x_2 + a_{111}x_3) \\ & \quad \times (a_{102}x_1 + a_{012}x_2 + a_{003}x_3)], \end{aligned}$$

whose term in x_3^2 is

$$a_{102}a_{012} - a_{111}a_{003}.$$

For $\frac{1}{36}B_{13} = \frac{1}{36}(f_{12}f_{23} - f_{13}f_{22})$, we have instead the polynomial

$$a_{111}a_{012} - a_{102}a_{021}.$$

The latter has only positive terms, as we already know from our calculations on the x_1x_3 term for B_{33} , where we saw that the polynomial $a_{201}a_{120} - a_{210}a_{111}$ has only negative terms (simply switch the first and last indices). For a given monomial to appear in the first polynomial, we need parities that \tilde{u} is odd and \tilde{v} , \tilde{w} even, or the other way round. For the monomial to appear in the second polynomial, we need parities that \tilde{v} is odd and \tilde{u} , \tilde{w} even, or the other way round.

For the former, namely $a_{102}a_{012} - a_{111}a_{003}$, the by now familiar calculation shows that the coefficient of our monomial, say in the case $\tilde{u} = 2u + 1$ odd and $\tilde{v} = 2v$, $\tilde{w} = 2w$ even, is the sum

$$\sum_{k=0}^{2u+1} \binom{2u+1}{k} \binom{2v}{v+u-k} \times \left[\binom{2w}{w+u-k} - \binom{2w}{w+u-(k-1)} \right].$$

The bracketed term is now antisymmetric about $k = u + \frac{1}{2}$, and pairing the terms again, we see that the sum is positive. \square

Theorem 5.2. *For the cubics under consideration, the Hessian determinant H is a polynomial in the a_j , b_j , c_j and x_1, x_2, x_3 , all of whose coefficients are positive.*

Proof: Recall that for any $n \times n$ matrix A with $n > 2$, we have $\text{Adj}(\text{Adj} A) = \det(A) A$. Applying this, with $A = (F_{ij})$, we deduce that

$$F_{12}H = -B_{33}B_{12} + B_{23}B_{13}.$$

Theorem 5.1 then implies that $F_{12}H$, a polynomial in the a_j, b_j, c_j and x_1, x_2, x_3 , has only positive coefficients, where we may without loss of generality assume that F_{12} is nontrivial.

However, an easy argument shows that if f, g are polynomials in a finite set of variables, with f nontrivial, such that f and fg have only positive coefficients, then the same is true for g . To see this, choose an order for the variables, and then order the monomials lexicographically. Now pick the largest monomial in f , and the largest monomial (if it exists) whose coefficient in g is negative; the product of these terms would yield a monomial in fg with negative coefficient. Applying this, since F_{12} has only positive coefficients, we deduce that the same holds for H . \square

Remark 5.3. Once we know that H has only positive coefficients, the above argument shows that the same is true for all entries of $\text{Adj}(B)$, for instance $B_{11}B_{22} - B_{12}^2 = F_{33}H$.

6. COMBINATORICS OF THE S -INVARIANT

The fact that we have explicit formulas for the coefficients in both the polynomials $\partial^2 B_{33}/\partial x_p \partial x_q$ and $\partial^2 B_{pq}/\partial x_3^2$ provides an explicit recipe for calculating the coefficients in S directly. From Remark 3.2, we know that

$$\frac{1}{4} \sum_{p,q} \frac{\partial^2 B_{pq}}{\partial x_3^2} \frac{\partial^2 B_{33}}{\partial x_p \partial x_q} = 6^4 S.$$

If we denote the tridegree (in the a_i, b_j and c_k) of the terms in $\partial^2 B_{33}/\partial x_p \partial x_q$ by \mathbf{n}_{pq} , then $\mathbf{n}_{11} = (2s - 2, 2s, 2s + 2)$, $\mathbf{n}_{22} = (2s, 2s - 2, 2s + 2)$, $\mathbf{n}_{33} = (2s, 2s, 2s)$, $\mathbf{n}_{12} = \mathbf{n}_{21} = (2s - 1, 2s - 1, 2s + 2)$, $\mathbf{n}_{13} = \mathbf{n}_{31} = (2s - 1, 2s, 2s + 1)$ and $\mathbf{n}_{23} = \mathbf{n}_{32} = (2s, 2s - 1, 2s + 1)$. The tridegrees of the terms in the entries of the matrix $(\partial^2 B_{pq}/\partial x_3^2)$ are then given by $(4s, 4s, 4s) - \mathbf{n}_{pq}$; for instance the tridegree of the terms in $\partial^2 B_{12}/\partial x_3^2$ is $(2s + 1, 2s + 1, 2s - 2)$.

The recipe for calculating the coefficient of a given allowable monomial M is now clear. Consider all factorizations $M = M_1 M_2$ of M , where the M_i are quadratic in each set of variables (a_j, b_j, c_j) , and where M_2 has one of the tridegrees listed above for $\partial^2 B_{33}/\partial x_p \partial x_q$, with M_1 having the complementary tridegree. The M_i give rise to numbers $\tilde{u}_i, \tilde{v}_i, \tilde{w}_i$, where $i = 1, 2$, from which we have an explicit expression for the coefficient of M_i in the relevant entry of the matrix in question. Adding the products of these two coefficients as we range over the factorizations gives us the coefficient of M in S .

We illustrate this with an example. We consider the case $s = 3t$ (and so $d = 3t + 1$) and M a monomial with matrix of exponents having $4t$ columns of the form $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $4t$ columns of the form $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, and t columns of the form $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$. Note that for all factorizations $M = M_1 M_2$, we have $\tilde{v}_1 = 0 = \tilde{v}_2$. A factorization is determined by specifying for how many of the $a_j^3 b_j$ one takes a_j^2 in M_1 , and for how many of the $b_j c_j^3$ one takes c_j^2 ; if these numbers are denoted by k, l respectively, then $\tilde{u}_1 = 4t - k$, $\tilde{w}_1 = 4t - l$, $\tilde{u}_2 = k$, $\tilde{w}_2 = l$. Note that M_2 then has tridegree $(8t - k, 2t + k + l, 8t - l)$. Thus the only pairs (k, l) of relevance will be $(2t, 2t)$, $(2t, 2t - 1)$, $(2t, 2t - 2)$, $(2t + 1, 2t - 1)$, $(2t + 1, 2t - 2)$, and $(2t + 2, 2t - 2)$. We consider each pair in turn; the fact that $\tilde{v} = 0$ simplifies the algebra considerably. The case $(2t, 2t)$ corresponds

to the x_3^2 term in B_{33} ; the coefficient of the monomial in $B_{33}/36$ is checked to simplify to

$$\binom{2t}{t} \left[\binom{2t}{t-1} - \binom{2t}{t} \right].$$

The case $(2t, 2t - 1)$ corresponds to the $x_2 x_3$ term in B_{33} ; the relevant coefficient is

$$2 \binom{2t-1}{t} \left[\binom{2t}{t-1} - \binom{2t}{t} \right].$$

The case $(2t, 2t - 2)$ corresponds to the x_2^2 term in B_{33} ; the relevant coefficient is

$$\binom{2t-2}{t-1} \left[\binom{2t}{t-1} - \binom{2t}{t} \right].$$

The case $(2t + 1, 2t - 1)$ corresponds to the $x_1 x_3$ term; the relevant coefficient is

$$\binom{2t-1}{t} \left[\binom{2t+1}{t-1} - \binom{2t+1}{t} \right].$$

The case $(2t + 1, 2t - 2)$ corresponds to the $x_1 x_2$ term; the relevant coefficient is

$$\binom{2t-2}{t-1} \left[\binom{2t+1}{t-1} - \binom{2t+1}{t} \right].$$

The case $(2t + 2, 2t - 2)$ corresponds to the x_1^2 term; the relevant coefficient is

$$\binom{2t-2}{t-1} \left[\binom{2t+2}{t} - \binom{2t+2}{t+1} \right].$$

Now we need the corresponding x_3^2 terms in B_{pq} . We already know that $(k, l) = (2t, 2t)$ corresponds to the x_3^2 term in B_{33} with coefficient of the monomial in $B_{33}/36$ being

$$\binom{2t}{t} \left[\binom{2t}{t-1} - \binom{2t}{t} \right].$$

We check that $(2t, 2t - 1)$ corresponds to the x_3^2 term in $B_{23} = f_{12}f_{13} - f_{11}f_{23}$, namely $36(a_{111}a_{102} - a_{201}a_{012})$, and that the coefficient required is

$$\binom{2t+1}{t} \left[\binom{2t}{t} - \binom{2t}{t-1} \right];$$

the pair $(2t, 2t - 2)$ corresponds to the x_3^2 term of $B_{22} = f_{11}f_{33} - f_{13}^2$, namely $36(a_{201}a_{003} - a_{102}^2)$, and the coefficient is

$$\binom{2t}{t+1} \binom{2t+2}{t+2} - \binom{2t}{t} \binom{2t+2}{t+1};$$

the pair $(2t + 1, 2t - 1)$ corresponds to the x_3^2 term of $B_{13} = f_{12}f_{23} - f_{13}f_{22}$, namely $36(a_{111}a_{012} - a_{102}a_{021})$, and the coefficient is

$$\binom{2t-1}{t-1} \binom{2t+1}{t+1} - \binom{2t-1}{t-1} \binom{2t+1}{t} = 0;$$

the pair $(2t + 1, 2t - 2)$ corresponds to the x_3^2 term of $B_{12} = f_{13}f_{23} - f_{12}f_{33}$, namely $36(a_{102}a_{012} - a_{111}a_{003})$, and the coefficient is

$$\binom{2t-1}{t-1} \left[\binom{2t+2}{t+1} - \binom{2t+2}{t+2} \right];$$

finally $(2t + 2, 2t - 2)$ corresponds to the x_3^2 term of $B_{11} = f_{22}f_{33} - f_{23}^2$, namely $36(a_{021}a_{003} - a_{012}^2)$, and the coefficient is

$$\binom{2t-2}{t-1} \left[\binom{2t+2}{t} - \binom{2t+2}{t+1} \right].$$

We now have all the information we need to calculate S from the formula given at the start of the section, where of course for a given (k, l) we shall need to weight the contribution by $\binom{4t}{k} \binom{4t}{l}$. Putting all this together, we get a formula for the relevant coefficient of the S -invariant as a function of t . With the aid of Mathematica, one can then simplify the formula to the surprisingly simple form

$$(4t)!^2 \left(\frac{t!}{(t-1)!(t+1)!} - \frac{1}{t!^2} \right)^4.$$

In particular, one notes that it is positive. Evaluating this formula for t taking values $0, 1, 2, 3, 4, \dots$, one obtains values for the coefficient of the monomial in the S -invariant to be $1, 36, 78400, 533610000, 6363107150400, \dots$. The first two of these values coincide with previously calculated numbers (established using a simple-minded method).

The author has checked positivity of the coefficient for other cases of a similarly general type. Apart from the computer calculations described in Section 4, perhaps the most telling evidence for the positivity of all the coefficients is provided by calculating what was conjectured in Section 4 to be the *smallest* coefficient.

Example 6.1. We consider therefore the case in which $s = d - 1$ and the monomial M has a matrix of exponents

$$(d-1) \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

The coefficient was calculated for $d \leq 5$ in Section 4, and the monomial was conjectured to have the smallest coefficient (assuming no fourth powers) for any given value

of s . For this reason, it is an obvious crucial case in which to verify our main conjecture. In a factorization $M = M_1 M_2$, we suppose that for precisely k of the $a_j^3 b_j$ we have taken a_j^2 in M_1 , for precisely l of the $a_j c_j^3$ we have taken c_j^2 in M_1 , and for precisely m of the $b_j^3 c_j$ we have taken b_j^2 .

Consideration of tridegrees shows that the only pairs $(k - l, m - l)$ of relevance are $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 1)$, $(1, 2)$, and $(2, 2)$. For a given choice of (k, l, m) , the corresponding triple $(\tilde{u}, \tilde{v}, \tilde{w})$ associated with M_2 is just (k, l, m) . Because the \tilde{v} is no longer zero in general, the formula for the coefficient (as a function of s) that we obtain involves triple summations. The rather complicated formula (occupying a page) that results may be found in an appendix to this paper.²

Although Mathematica does not reduce this formula to any simple form, it is nevertheless an explicit formula, which has been checked to give positive values for $s \leq 501$. The proof of positivity for general s presumably follows by suitable rearrangement of the sums that occur in the formula. The values for $s = 1, 2, 3, 4, 5, 6, 7, 8$ are respectively $1, 4, 40, 652, 13174, 308464, 8158021$, and 238306660 ; the first four of these correspond to the previously calculated values.

The fact that the numbers generated tend to have large prime factors (for instance, 8158021 is prime) suggests that there is no simple form of the formula. We should also comment that the cofactor formula for S that we are using expresses the coefficient of the given monomial as the sum of six terms. By taking, for instance, $s = 4$ in this example and evaluating these terms, each of the terms has modulus greater than the sum of the terms; so although the sum is positive, it does involve significant cancellations.

A proof of the positivity of the coefficient for the case of a general monomial still seems some way off, at least using the recipe given above. I restrict myself to the comment that the formulas we derived for the coefficients of monomials in the cofactors can all be expressed as the difference between two reasonably simple hypergeometric series of the form ${}_3F_2$. In some of the special cases worked out, they were the differences of even simpler terms. The theory of hypergeometric series may therefore feature in a proof of the conjectures and in possible alternative proofs of the results from Section 5.

It might be observed that there are other relatively simple formulas that yield S , apart from those in Sec-

²The appendix is available online (<http://www.expmath.org/expmath/volumes/15/15.4/Wilson/WilsonApp.pdf>).

tion 3. By a similar method of proof to that of Lemma 3.1, one can, for instance, show that

$$\frac{1}{2} \sum B_{ij} \frac{\partial^2 H}{\partial x_i \partial x_j} = 6^5 SF.$$

If one could prove positivity of the coefficients for this polynomial, then the desired result would follow for S . With the methods described above, however, the expression for S that we have used is simpler to analyze than this one.

7. CONCLUDING REMARKS

We summarized in Section 2 the theoretical evidence for the geometric Conjectures 4.3 and 4.4 that for complete intersection threefolds in the product of three projective spaces (assuming $b_2 = 3, h^{2,0} = 0$), the curvature of the surface \mathcal{K}_1 is bounded between $-\frac{9}{4}$ and 0. These conjectures were set in the more general context of threefolds admitting certain specific types of degeneration, and for $b_2 \geq 3$ they can be rephrased in an illuminating way in terms of Ricci curvatures. In the Calabi–Yau case, there was further evidence via mirror symmetry from known results on the Weil–Petersson metric on the complex moduli space of the mirror.

However, even if we knew that Conjectures 4.3 and 4.4 held, it is unclear whether this would help in a proof of their combinatorial versions, Conjectures 4.1 and 4.2. An illustration of this is that the geometric version of Theorem 5.2 follows from the Hodge index theorem, but this does not seem to help in a proof of the combinatorial result, or in a proof of Theorem 5.1. If one could produce a proof of Theorem 5.1 that depended less on explicit combinatorial manipulations than the proof given here, I believe that this might suggest alternative approaches to proofs of Conjectures 4.1 and 4.2.

The experimental evidence for Conjecture 4.1 is, I believe, very strong indeed. Not only has it been checked in all cases up to and including $d = 5$, this involving a prodigious amount of calculation, but it has also been checked in the case of the predicted minimum coefficient up to enormous values of d . The computational evidence for Conjecture 4.2 is not as extensive, although still strong, but there is more theoretical evidence in the geometric case for the precise value of the upper bound. Should one want further experimental evidence for Conjecture 4.2, it should be feasible to extend the previous computations at least to include all cases up to and including $d = 3$.

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